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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON.

(A.)

FOR THE YEAR MDCCCXCIII.

VOL. 184.

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MDCCCXCIV.

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- *p.* Academy of Natural Sciences.

Madison.
- *p.* Wisconsin Academy of Sciences.

Mount Hamilton (California).
- A. Lick Observatory.

New Haven (Conn.).
- AB. American Journal of Science.
- AB. Connecticut Academy of Arts and Sciences.

New York.
- *p.* American Geographical Society.
- *p.* American Museum of Natural History.
- *p.* New York Academy of Sciences.
- *p.* New York Medical Journal.
- *p.* School of Mines, Columbia College.

Philadelphia.
- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p.* Franklin Institute.
- *p.* Wagner Free Institute of Science.

Rochester (N.Y.).
- *p.* Academy of Science.

St. Louis.
- *p.* Academy of Science.

Salem (Mass.).
- *p.* American Association for the Advancement of Science.
- AB. Essex Institute.

San Francisco.
- AB. California Academy of Sciences.

Washington.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- *p.* United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.
- *p.* United States Department of Agriculture.
- A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)
- AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1893, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to Sir GEORGE GABRIEL STOKES, Bart., F.R.S., for his Researches and Discoveries in Physical Science.

A ROYAL MEDAL to ARTHUR SCHUSTER, F.R.S., for his Spectroscopic Inquiries, and his Researches on Disruptive Discharge through Gases, and on Terrestrial Magnetism.

A ROYAL MEDAL to HARRY MARSHALL WARD, F.R.S., for his Researches into the Life History of Fungi and Schizomycetes.

The DAVY MEDAL to J. H. VAN 'T HOFF and J. A. LE BEL, in recognition of their Introduction of the Theory of Asymmetric Carbon, and its Use in Explaining the Constitution of Optically Active Carbon Compounds.

The Bakerian Lecture, "The Rate of Explosion in Gases," was delivered by Professor HAROLD B. DIXON, F.R.S.

The Croonian Lecture, "The Position of Pathology among Biological Studies," was delivered by Professor RUDOLF VIRCHOW, For.Mem.R.S.

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PHILOSOPHICAL TRANSACTIONS.

I. *Transmission of Sunlight through the Earth's Atmosphere.*

By CAPTAIN W. DE W. ABNEY, *C.B., D.C.L., F.R.S.*

Received April 7,—Read May 5, 1892.

PART II.—LOSS OF LIGHT AT DIFFERENT ALTITUDES.

Introductory.

XXII.—In a previous communication ('Phil. Trans.,' vol. 178, 1887, A., pp. 251–283) the results of the absorption of sunlight by different thicknesses of atmosphere near sea level were found from measurements of the luminosity of the entire spectrum. The method used was that devised by General FESTING and the Author, as detailed in the Bakerian Lecture in 1886. The absorption coefficient for the different thicknesses of atmosphere at sea level was found by measurements made principally at South Kensington, and these were compared with measures taken at the Riffel, above Zermatt, at a height of about 8000 feet. It was shown that when the air thickness is represented by x, the minimum intensity for each ray of the spectrum can be fairly represented by $I' = I\epsilon^{-\cdot 013x\lambda^{-4}}$, and the average intensity by $I' = I\epsilon^{-\cdot 017x\lambda^{-4}}$, I' and I being the transmitted and original intensities, and λ the wave length.

Further, it was shown that even if the absorption of each ray was very different from the above, the integral absorption was very accurately expressed by a^{-z}, z being the air thickness, and a a constant. This corresponds to $a^{-\sec\theta}$ corrected for refraction, when θ is the zenith distance.

This result was arrived at by taking the areas of the curves of luminosity as formed from the actual observations as a measure of the luminosity of the total white light which was decomposed into a spectrum. This was admissible, for in the paper already referred to, it was proved that the sum of luminosities of different rays is equal to their luminosity when compounded.

XXIII.—*Objections to the Use of the Formula.*

The formula involving λ^{-4} is the formula theoretically deduced by Lord RAYLEIGH for the scattering of light by small particles, and its adoption in this research is open to criticism—one being, that if it were absolutely applicable, the light of the sky should probably exhibit greater polarisation in a direction perpendicular to the sun's rays than it does. There is reason for believing, however, that the higher the station at which such observations are made the more complete is the polarisation. In any case, before this can be considered a valid objection, we have to know more than we do at present regarding the condition of the light reflected back from the earth and from the particles themselves.

It must also be admitted that grosser particles exist on some days, and that for these the loss of light must be in the form of $I' = I\epsilon^{-\mu x}$. They would prevent a certain portion of the total light from reaching the observer, whilst the smaller one would selectively reflect, and hence the law would not hold absolutely good, but the small number of gross particles, compared with the fine ones, would not appreciably alter the formula used except the coefficient κ.

Another objection which has been advanced by Mr. S. P. LANGLEY in a private letter against the adoption of the formula is, that if the spectrum were observed with a large dispersion, it will be seen that as the altitude of the sun diminishes the atmospheric lines increase in intensity, and that these must obey the laws of ordinary absorption. This is an objection which at first sight may seem fatal, not to the correctness of the observations, but to the adoption of the law above quoted; but it must be remembered that these special absorptions occupy a very limited area compared with the rest of the spectrum, and that they would practically disappear when the whole loss of light is under consideration, more especially as they would themselves obey the ordinary law of absorption. At any rate, the formula adopted appears to suit the case, and it must be borne in mind, that the results obtained by the integration of the spectrum luminosities bear out the formula which has been universally adopted by astronomers as representing the corrections to be made in star magnitudes when the stars are observed at different altitudes above the horizon.

XXIV.—*The Integration of the Spectrum Luminosities at different Solar Altitudes equivalent to the Luminosity of Monochromatic Light at the same Altitudes.*

In Section XVIII. of the paper of which this is a continuation it was shown that the areas of the curves obtained from the formula $I' = I\epsilon^{-\kappa x \lambda^{-4}}$ being capable of being represented by $I' = I\epsilon^{-\mu x}$ an important deduction could be made.

For the area of the curve is $\epsilon^{\kappa x}\,(a\epsilon^{\lambda_{\prime}^{-4}} + b\epsilon^{-\lambda_{\prime\prime}^{-4}} + c\epsilon^{\lambda_{\prime\prime\prime}^{-4}} +$ &c.). *a*, *b*, *c*, &c., being the original luminosities of the different rays, must then be represented by $I = I\epsilon^{-\kappa x \Lambda^{-4}}$ where Λ is some one ray, that is $\mu = \kappa\Lambda^{-4}$ or $\frac{\mu}{\kappa} = \Lambda^{-4}$; $\frac{\mu}{\kappa}$ is a constant and was found from the observations to be 105 (on the scale used for Λ^{-4}), which is equivalent to λ 5770, or a ray near the place of maximum luminosity. Hence the visual observations of total sunlight at different zenith distances are equivalent to observing the alteration in intensity of one single ray of that wave length in its spectrum,[*] and, therefore, from such observations κ could be at once determined. It was similarly shown that when a photographic silver salt was employed to register the photographic spectrum it was equivalent to measuring the integrated spectrum with a coefficient of absorption μ', and that this was equivalent to observing the alteration in intensity of a ray λ 4540.

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[*] This, of course, is only true within certain limits, but it holds for any thickness of atmosphere through which observations were taken.

XXV.—*Application to the foregoing Results.*

This being the case, if by any photographic means we can measure the total intensity of white light affecting a photographic compound after passing through various air thicknesses, we ought to be able to find the value of μ_1 in the formula $\epsilon^{-\mu_1 z}$, and if we also observe optically the value of light transmitted through various air thicknesses we shall get the value of μ in the formula $\epsilon^{-\mu z}$. As already stated we know that $\mu = 105\kappa$. Similarly, $\mu' = \Lambda_{,}^{-4}.\kappa$, $\Lambda_{,}$ being the equivalent ray observed.

Therefore

$$\frac{\mu'}{\mu} = \frac{\Lambda_{,}^{-4}}{105} \quad \text{or} \quad \Lambda_{,}^{-4} = \frac{105\mu'}{\mu} \quad \text{and} \quad \kappa = \frac{\mu'}{\Lambda_{,}^{-4}},$$

that is, we can find the single ray which is the equivalent of the whole of the spectrum which is impressed on any photographic compound, and having thus found $\Lambda_{,}^{-4}$ we can at once deduce the value of κ—or the coefficient of scattering by the fine particles.

XXVI.—*Sensitive Compound employed.*

The question which presented itself was as to the best form of sensitive salt to use for convenience and accuracy. A process with a silver salt requiring development was almost impracticable. The exposure necessary to give to sunlight would have been so small that accuracy in timing it would have become a very difficult problem. The chloride of silver paper which darkens by sunlight was experimented with, but it had a very serious drawback. ROSCOE and BUNSEN have shown that chloride of silver paper may be prepared, which, when exposed to the same intensity of light for the same time, will always give the same blackness, but it is impossible to keep this paper for more than a few hours, and it would often be inconvenient to prepare it. Further it can never be predicted when a day will be suitable for making observations, and measurements of its blackness would be difficult except at a fixed observing station. There were also two other desiderata which had to be taken into account; one was that the record of the action of light should be as permanent as possible, and the other that it should be *easy to measure* the action produced by the light. Chloride of silver paper fulfilled neither of these desiderata, since fixing the chloride altered all measurement and the darkening was liable to fade even when the paper was fixed, and it was not easy to obtain accurate measures of the darkening owing to its ruddy colour.

In the comparatively new process of platinotype we have, however, a process which is in every way suitable for the purpose. It may be well to call to mind the principle on which it is based. A solution of ferric oxalate is mixed with a proportion of chloroplatinite of potassium and spread over a sized paper and dried. When light acts on this mixture it reduces the iron salt to the ferrous state proportionately to the intensity of the light and to the time during which it acts. When such an exposed paper is placed on a hot solution of potassium oxalate (neutral) the salt reduces the platinum salt, and metallic platinum (platinum black) is deposited. The black or grey produced by this means is most suitable for measuring in the manner which will subsequently be described.

The other desideratum is also found in this process. The prints are absolutely permanent, and the records obtained by it can be referred to at any time, and re-measured if necessary. Further, the paper keeps well before exposure, and after exposure and before development. When kept dry in an air-tight box with calcium chloride in contact with the air, in fact, it will keep unaltered in sensitiveness for several months. As an example of this it may be mentioned that two pieces of paper cut from the same sheet were each exposed to various and similar intensities of light for equal times. One was developed on the potassium oxalate solution immediately after exposure, and the other was not developed till about six weeks had elapsed; on measuring the greys on the two papers their darkness was found to be precisely the same, and the unexposed parts equally white in the two cases.

It was, for these reasons, determined to use the platinum process for the experiments detailed.

XXVII.—*Instrument used in the Observations.*

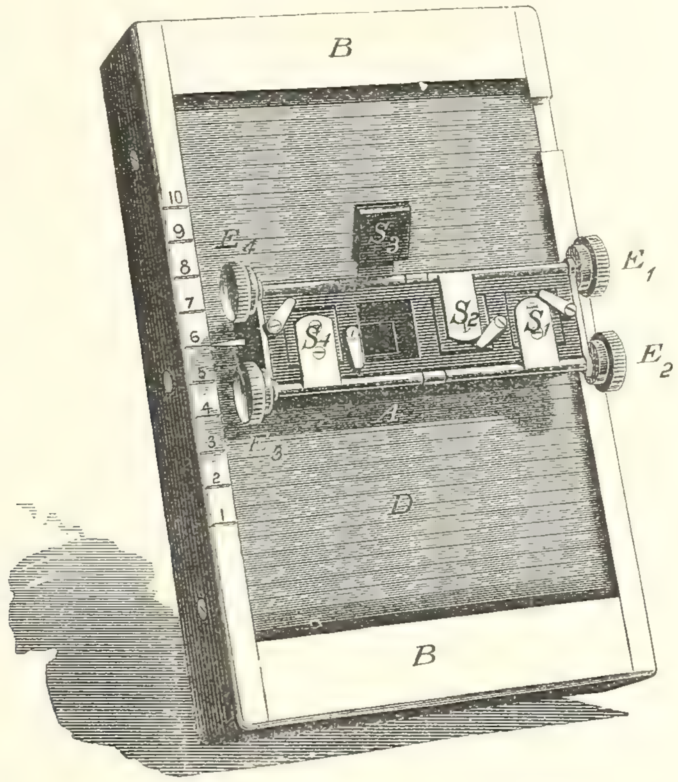
The instrument which was designed for use with this sensitive paper was of the simplest form ; one of the conditions which it had to fulfil was whilst exposing the paper it should allow but a small amount of light from the sky to reach it. The latter light is anything but negligible, for absolute measures, in some cases, show that if the photographic intensity of sunlight be called 100, that of the sky might be as much as 50.

To admit only a practical minimum of sky-light, and such as would be negligible in proportion to the sunlight, the instrument was constructed so that only about 10° of skylight fell upon the paper with the sunlight. The diagram, fig. 1, will give an idea of the instrument. B is a frame, in the back and front of the sides of which are cut deep continuous grooves. A wooden shutter, D, formed of narrow laths, glued on to a leather back, runs in these grooves also at the back of the box. This flexible shutter carries an oblong block A, pierced with four square apertures, as shown, which are closed by covers, S_1, S_2, S_3, S_4. The covers can be opened by turning the mill-headed buttons, E_1, E_2, E_3, E_4, which are connected by long pins with the covers.

In order to confine the photographically active spectrum to smaller limits than would be given if the light were unchecked, two thicknesses of cobalt-blue glass were placed in slots running through the length of A. The light had therefore to pass through these two glasses before it reached the paper beneath them.

Experiment has shown that the aperture can be completely uncovered in less than $\frac{1}{10}$th of a second, and closed in the same time; hence no appreciable error in exposure arises from this cause when the exposure lasts several seconds. The paper is placed on a grid, running the whole length of the box behind the flexible shutter, the bars falling between the aperture and different portions of the paper are exposed by moving the block A into different positions. These positions are indicated by a scale engraved on one side of the frame, a mark engraved on the shutter being moved successively to each division of the scale. As there are ten divisions of the scale, it will be seen that forty different exposures can be obtained on the same paper, ten through each aperture. The frame is closed by a wooden back, lined with velvet, which is pressed against the paper by means of a couple of brass springs working against buttons fixed to the frame.

Fig. 1.



XXVIII.—*Comparison of Results.*

In order to get a scale of blackness, by which to measure the intensity of light acting, it became necessary to have some means of exposing similar paper to light of different known intensities. From the blackness so produced a curve of blackness could be constructed, and the blackness produced by the sun with different exposures could be read off. Now, increase in time of exposure is equivalent to increase in intensity of light acting, at all events, when the exposures are as prolonged as those given, that is, if the unit time of exposure be doubled, the same result will be obtained if the intensity of light acting be doubled. This is not a mere theoretical assumption, but one which the writer has amply proved with very numerous experiments. To obtain such a scale, what is known as SPURGE's sensitometer was employed. This is an instrument which was described in detail before the Photographic Society of Great Britain by the inventor. It is only needful to say that it consists of thirty small chambers, 2 centims. in height and 1 centim. square section, placed in six rows, of five chambers each. A brass plate covers one end of these chambers, and in it holes are pierced, of such dimensions that the area of each hole is exactly $\sqrt[3]{2}$ greater than that of the next to it in one row. One of these holes is pierced in the plate exactly over the centre of each chamber. Thus we have a series of graduated intensities of $1, 2^{\frac{1}{3}}, 2^{\frac{2}{3}}, 2, 2^{\frac{4}{3}}, 2^{\frac{5}{3}}, 4$, &c., falling on the bottom of these chambers when the light falling on them comes from a large and equally illuminated surface.

Paper was exposed in this instrument to the light reflected from a large flat card, uniformly illuminated by sun and sky light, and after development the blackness was measured. The paper used was cut from sheets specially prepared by the kindness of Mr. W. WILLIS, of the Platinotype Company, and the sensitiveness of various parts was tried, and if found to be constant, the rest of the sheet was taken to be comparable with those tried. The care with which this paper is prepared is shown by the fact that with several varieties of paper, procured at intervals of several months, the sensitiveness showed no variation in gradation of tints. Latterly the formula seems to have been a little changed, and the sensitometer curve has slightly varied from that found at first; but this is unimportant, as the results are so reduced as to be comparable.

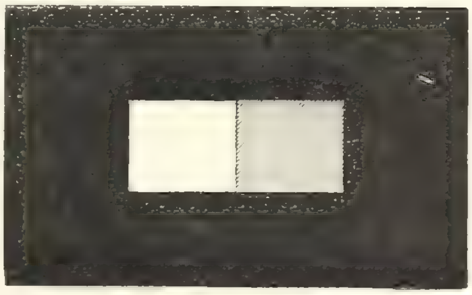
The term "developing" when applied to this paper may be a little misleading. It must not be confounded with the development of plates containing silver salts. In the case of this sensitive paper the amount of platinum deposited is exactly dependent on the amount of the ferric oxalate reduced by light, and if it be exposed to the developing solution after the ferrous salt has reduced the platinum salt in contact with it, no more platinum will be deposited. Hence the deposit of platinum is an exact equivalent of the iron salt reduced, and is a measure of the intensity of light multiplied by the time of exposure. With a silver salt the deposit of silver is increased according to the time during which the developer is in contact with it, and there would, therefore, be a necessity of exposing and developing on the same plate a scale of density together with that produced by the light to be measured.

It is evident that with the platinum process one scale will suffice for a batch of paper which is prepared at one time.

To measure the blackness of the paper a modified RUMFORD method of photometry was employed. A source of light illuminated a small square of white paper (the same as that used in the platinum process to hold the sensitive salt), and one of the blackened squares placed alongside it as in fig. 2.

A rod was so placed that its shadow fell on the white square. A mirror was placed at a little distance to one side of the source of light and the reflected beam was made to illuminate the shadow falling on the white square, and itself to cast a shadow on the dark square. Thus each square was illuminated by the same light, but coming from different points, and the two would be caused to be of equal darkness by placing rotating sectors opening and closing at pleasure in front of one or other of the beams. The amount of light cut off gave a measure of the darkness. Thus, in one case, when both squares were white the sectors had a total aperture of 80° to make them equally luminous; when a grey platinum square was substituted for one of them, the aperture of the sectors was 45°. The grey surface, therefore, only reflected $\frac{45}{80}$ of the light that the white surface reflected. The light to these squares was admitted through an aperture in the front of a closed box, the illumination being judged through an aperture fixed at one corner. The whole apparatus was placed in a darkened and blackened room in which the sole light was that used for the measures.

Fig. 2.

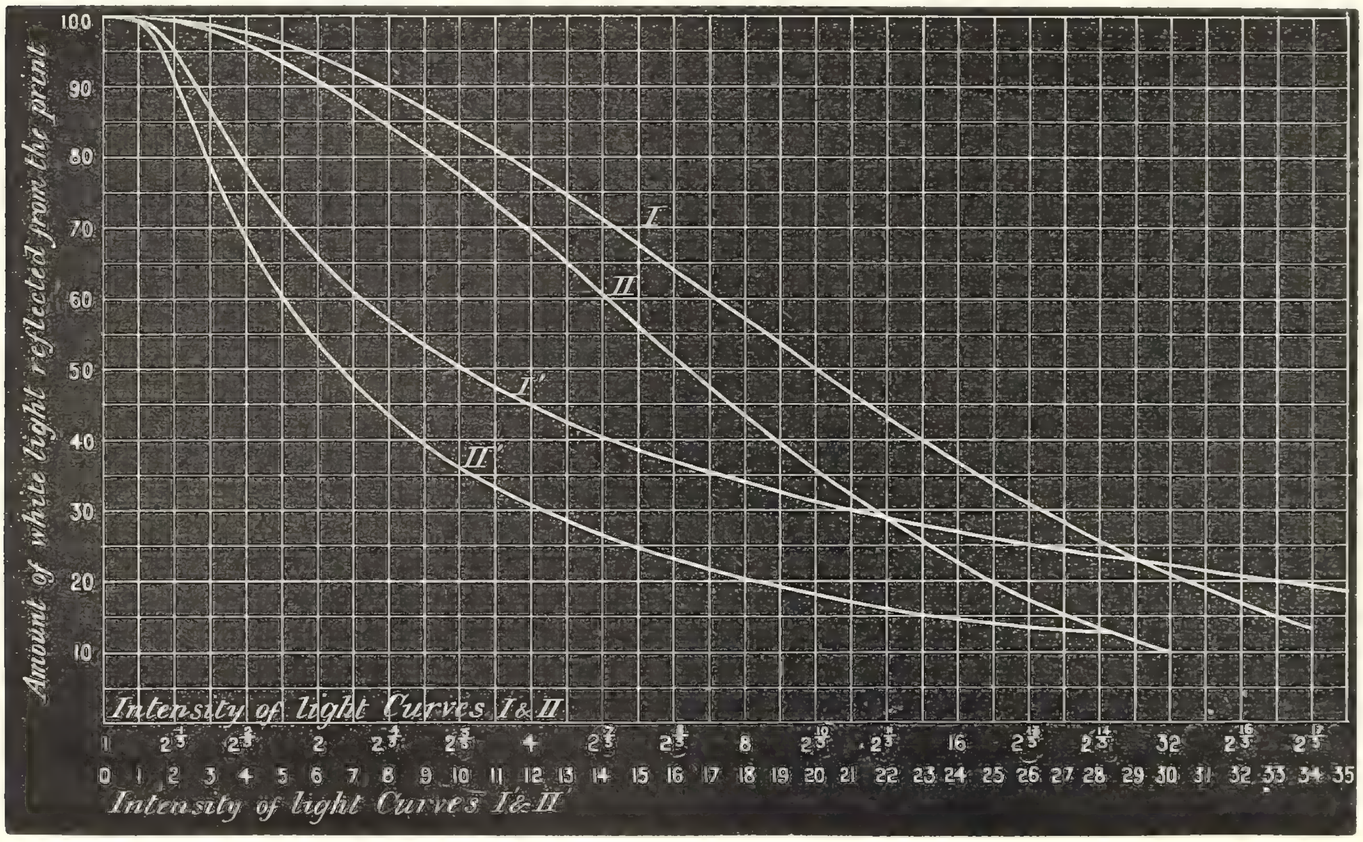


The two following scales of blackness, or, perhaps more accurately, of whiteness, were obtained by this plan, one or other of which will apply to all the measurements given in the paper :—

Comparative intensities of light.	Curve I.	Curve II.
	Amount of white reflected.	Amount of white reflected.
1	100	100
$2^{\frac{1}{3}}$	99·3	99
$2^{\frac{2}{3}}$	97·2	96
2	93·9	91
$2^{\frac{4}{3}}$	89·5	84·5
$2^{\frac{5}{3}}$	84·6	77
4	78	69
$2^{\frac{7}{3}}$	71	60·5
$2^{\frac{8}{3}}$	64	51·7
8	57	43·2
$2^{\frac{10}{3}}$	50	35·5
$2^{\frac{11}{3}}$	43	28·5
16	37	22·7
$2^{\frac{13}{3}}$	31	17·5
$2^{\frac{14}{3}}$	25·5	13·3
32	21	9·8
$2^{\frac{16}{3}}$	17	
$2^{\frac{17}{3}}$	13·5	
64	10·5	

In order to obtain from these measures a scale of the darkening obtained by arithmetical progression in the intensity of light, we have only to recollect that each hole admits $\sqrt[3]{2}$ times the light that the next smaller admits. By easy calculation this scale can be obtained, and is shown in the diagram.

Fig. 3.

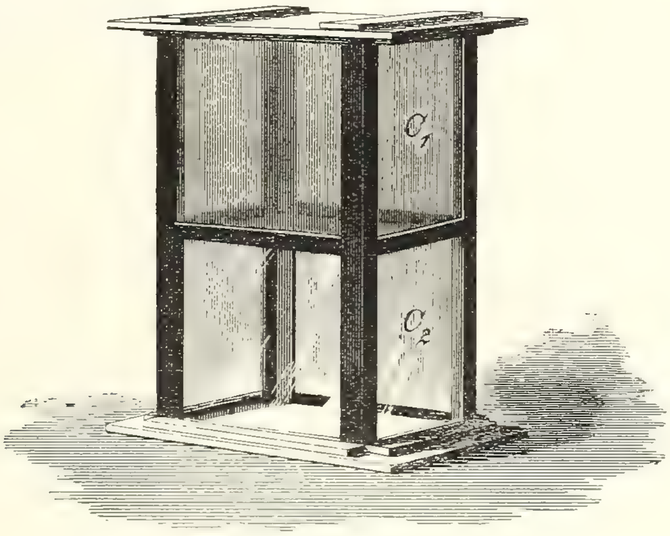


I have shown elsewhere that the curves of blackness on platinum paper can be represented by the formula, $A' = A\epsilon^{-\mu x^2}$ where A' is the amount of reflected white light, and A the amount reflected from the white paper, μ being a coefficient, and x any power of 2. In this case, for curve I., $\mu = \cdot 00302$, and for curve II., $\mu = \cdot 0103$.

XXIX.—*Method of finding* κ.

In the paper on Colour Photometry which appeared in the 'Phil. Trans.,' General FESTING and myself proved that a turbid medium prepared by dropping a solution of mastic dissolved in alcohol into water, obeyed Lord RAYLEIGH's law, as given above. If, therefore, one part of a piece of platinum paper were exposed for a certain time to sunlight after passing through a cell containing pure water, and a simultaneous exposure made on another portion of the paper with the same light after passing through turbid water prepared as above, the measures of the blackened paper would give the value of μ', the photographic coefficient of absorption in the formula $I' = I\epsilon^{-\mu_1\gamma}$, where γ is the thickness of the turbid medium in any unit we please. Further, if the optical values of the light transmitted were compared when passing through the same media, μ could be found and the value of γ be reduced to atmospheric thicknesses. Now, as the ray is known which, if observed, would give the same alteration in luminosity as that of the whole spectrum, it follows that with these data the equivalent ray for the photographic light can be deduced as shown in § XXV.

Fig. 4.



To ensure accuracy several sets of double cells, C_1 C_2, were prepared, as shown in the figure (fig. 4). The top cell, C_1, contained turbid water, the turbidity being caused by suspended mastic. The mastic was very cautiously precipitated and was thus held in suspension. As a matter of fact turbid water of different turbidities was prepared nearly two years before it was employed; and thus any coarse particles had ample time to settle; for the vessels containing it were kept undisturbed during that period and the liquid was syphoned off as required. The bottom cell, C_2, contained pure water. Behind the cell the exposing box (fig. 1), already described, was placed, and in front of C_2 sectors which rotated by means of clock work. The sectors could be clamped at any desired aperture. The apparatus, as shown, was then placed on a stand at such an angle that the sun's rays fell directly on the platinum paper inside the box B, when the exposing apertures were open. The sectors were closed to the extent which it was judged should suffice to make the blackness of the two exposed squares of platinum paper approximately the same, though it will be seen by the experiments to be subsequently described that this was not always attained. Exposures for different lengths of time were given in each experiment, and consequently any small error in measurement, or from any other cause, became insignificant when the mean was taken.

XXX.—*Examples of the Method.*

The following are examples showing the accord between the different values found for Λ_1^{-4}.

Experiment I.—Sunlight—length of cell (inside measures), 3·15 inches; breadth of cell, 1·7 inches. When light passed through length of cell to the sensitive paper a double sector with an opening of 12° rotated in front of the clear cell. When light passed through the breadth of the cell, the sector opening was 36°·13.

For the cells placed lengthways, the value obtained for the light after passing through the clear cell had to be multiplied 180/12 = 15. When the cells were placed breadthways, the value for the light passing through the clear cell had to be multiplied by 180/36·13 = 4·982.

Cell.	1st exposure.			2nd exposure.		
	Mean reading.		Value from scale.	Mean reading.		Value from scale.
	White = 45.	White = 100.		White = 45.	White = 100.	
3·15 in., clear water	23	51	98	33	73·5	47
3·15 in., turbid „	25	55·5	84	35	78	40

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{98}{84} = 1{\cdot}166.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{47}{40} = 1{\cdot}175.$$

$$\text{Mean} = \frac{1{\cdot}166 + 1{\cdot}175}{2} = 1{\cdot}17.$$

The photographic light passing through the clear water = 1·17 × 15 = 17·55 times that passing through the turbid water.

Using the formula $I' = I\epsilon^{-\mu_1 \gamma}$

$$\log 17{\cdot}55 = 2{\cdot}86504 = 3{\cdot}15\mu_1$$

$\mu_1 = {\cdot}911$ for each inch of turbid water.

Cell.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White =45.	White = 100.		White = 45.	White = 100.		White = 45.	White = 100.	
1·7 in., clear water	37	72·5	147	32	63	172	26·5	52	204
1·7 in., turbid „	36	70·5	152	30·5	60	181	25	49	212

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{147}{152} = \frac{1}{1{\cdot}037}.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{172}{181} = \frac{1}{1{\cdot}052}.$$

$$\text{3rd} \quad ,, \quad ,, \quad ,, \quad = \frac{204}{212} = \frac{1}{1{\cdot}055}.$$

$$\text{Mean} = \frac{1}{1{\cdot}05}.$$

The photographic light passing through the clear water $= \frac{4{\cdot}982}{1{\cdot}05} = 4{\cdot}70$ times that passing through the turbid water.

Using the formula $I' = I\epsilon^{-\mu_1\gamma}$

$$\log 4{\cdot}7 = 1{\cdot}54756 = 1{\cdot}7\mu_1$$

$$\mu_1 = {\cdot}910 \text{ for each inch of turbid water.}$$

The optical measures of the light passing through the clear and turbid cells in the two directions were

3·15 inches clear water, 48·5 ; turbid water, 18·2.
1·7 ,, ,, 48·5 ; ,, 28·5.

Taking the measurement of the light passing through the 3·15-inch cell as lying on the curve in the formula $I' = I\epsilon^{-\mu\gamma}$,

$$\log 48{\cdot}5 - \log 18{\cdot}2 = \mu \times 3{\cdot}15,$$

$$\mu = 311.$$

If we take the measure of the 1·7-inch water, we have

$$\log 48{\cdot}5 - \log 28{\cdot}5 = \mu \times 1{\cdot}7,$$

$$\mu = {\cdot}313.$$

$$\text{Mean value} = {\cdot}312.$$

Now mean value of $\mu_1 = {\cdot}9105$. Now

$$\frac{\Lambda_1^{-4}}{\mu_1} = \frac{\Lambda^{-4}}{\mu} = \frac{105}{\mu},$$

therefore

$$\Lambda_1^{-4} = \frac{{\cdot}9015}{{\cdot}312} \times 105 = 306{\cdot}6.$$

Experiment II.—Sunlight, June, 1890, midday. Length of cell (inside measures), 3 inches; breadth, 2·1 inches.

When light passed through length of cell to the sensitive paper a double sector, with an opening 36°, rotated in front of clear cell.

When light passed through the breadth, the sector opening was 67°.

Thus, with cells placed lengthways, the value obtained had to be multiplied by $180°/36° = 5$. When the cells were placed breadthways, the light passing through the clear cell had to be multiplied by $180/67 = 2·686$.

Cells.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
3 in., clear . .	70	89·5	21	68	87	23·5	49	63	46
3 in., turbid . .	55	70·5	39	51	65·5	44	32	41	85

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{21}{39} = ·538.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{23·5}{44} = ·534.$$

$$\text{3rd} \quad ,, \quad ,, \quad ,, \quad = \frac{46}{85} = ·541.$$

$$\text{Mean} = ·538.$$

Photographic light passing through the clear water $= ·538 \times 5 = 2·69$ times that passing through the turbid cell.

Using the formula as before,

$$\mu_1 = \frac{\log 2·69}{3} = \frac{·98954}{3} = ·330.$$

Cell.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
2·1 in., clear .	57	73	36	41	52·5	62	27	34·5	106
2·1 in., turbid .	48	61·5	48	32	41	85	20	26	140

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{36}{48} = {\cdot}750.$$

$$\text{2nd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{62}{85} = {\cdot}729.$$

$$\text{3rd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{140}{106} = {\cdot}757.$$

$$\text{Mean} = {\cdot}745.$$

Photographic light passing through clear water = ·745 × 2·686 = 2·00 times that passing through turbid water.

Using formula as before,

$$\mu_1 = \frac{\log 2{\cdot}00}{2{\cdot}1} = \frac{{\cdot}69315}{2{\cdot}1} = {\cdot}330.$$

$$\text{Mean of } \mu_1 = {\cdot}330.$$

The optical measures of the clear and turbid cells in the direction of the length were—

3 in., clear water, 36·5 ; turbid water, 26.

$$\frac{\text{Clear water}}{\text{Turbid}} = 1{\cdot}4.$$

Using same formula

$$\mu = \frac{\log 1{\cdot}4}{3} = \frac{{\cdot}33647}{3} = {\cdot}1121.$$

The mean value of μ_1 is ·330, and as

$$\Lambda_1^{-4} = \frac{\mu_1}{\mu} \times 105,$$

therefore

$$\Lambda_1^{-4} = \frac{{\cdot}330}{{\cdot}1121} \times 105 = 309{\cdot}1.$$

Experiment III.—Arc electric light; light from crater of positive pole alone used —the arc being cut off by screens.

Length of cell, inside measure, 3 inches; breadth, 1·7 inch. When light passed through length of cells a double sector of $16\frac{1}{2}°$ rotated in front of the cell containing clear water. When passed through the breadth of the cell 28°.

For the cells placed lengthways the value of the light passing through the clear cell had to be multiplied by 180/16·5 = 10·91, when placed breadthways 180/28 = 6·43

Cell.	1st exposure.			2nd exposure.		
	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.	
3 in., clear	40	51·5	63	41	53	61
3 in., turbid	54	69	40	53	70·5	39

$$1\text{st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{63}{40} = 1{\cdot}575.$$

$$2\text{nd} \quad ,, \quad ,, \quad ,, \quad = \frac{61}{39} = 1{\cdot}564.$$

$$\text{Mean} = 1{\cdot}57.$$

Photographic light passing through clear water = 1·57 × 10·91 = 17·1 times that passing through the turbid water.

Taking the formula as before

$$\mu = \frac{\log 17{\cdot}1}{3} = \frac{\overline{2}{\cdot}8390}{3} = {\cdot}946.$$

Cells.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
1·7 in., clear	54	69	40	57	73	36	68	87	22·5
1·7 in., turbid	47	60	50	50	64	45	64	82	28

$$1\text{st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{40}{50} = {\cdot}800.$$

$$2\text{nd} \quad ,, \quad ,, \quad ,, \quad = \frac{36}{45} = {\cdot}800.$$

$$3\text{rd} \quad ,, \quad ,, \quad ,, \quad = \frac{22{\cdot}5}{28} = {\cdot}800.$$

$$\text{Mean} = {\cdot}800.$$

Photographic light passing through clear water = ·8 × 6·43 = 5·058 times that passing through the turbid water.

Using formula as before

$$\mu_1 = \frac{\log 5{\cdot}06}{1{\cdot}7} = \frac{1{\cdot}62136}{1{\cdot}7} = {\cdot}953.$$

The optical measures of the clear and turbid cells—

3 inches, clear water 59, turbid, 22·5,
1·7 inch, ,, 59, ,, 34.

Taking the light through the 3-inch length of the cell as lying on the curve and using the formula

$$\mu = \frac{\log 59 - \log 22{\cdot}5}{3} = \frac{1{\cdot}77495 - {\cdot}81093}{3} = \frac{{\cdot}96402}{3}$$

$$= {\cdot}321.$$

Taking 1·7-inch breadth of the cell

$$\mu = \frac{\log 59 - \log 34}{1{\cdot}7} = \frac{1{\cdot}77495 - 1{\cdot}22377}{1{\cdot}7},$$

$$= \frac{{\cdot}55118}{1{\cdot}7} = {\cdot}324.$$

Mean of these values = ·3225.
Mean of μ^1 = ·950.

$$\Lambda_1^{-4} = \frac{\mu_1}{\mu} \times 105 = \frac{{\cdot}950}{{\cdot}3225} \times 105,$$

$$= 309{\cdot}3.$$

From these three experiments we may derive a very good approximation to the true value of Λ_1. The following are the values obtained for Λ_1^{-4}—

1st experiment	306·6
2nd ,,	309·1
3rd ,,	309·3
Mean	308·3

Therefore

$$\Lambda = 4244 \text{ wave length.}$$

That is, the variation in the intensities of the spectrum, as photographed on platinum paper, is the same as would be observed if a ray of λ 4244 were isolated and the variation in its intensity were observed optically or otherwise.

Further, it must be observed that if we find μ_1 from the photographs, we can at once find μ for the visual intensity; for since

$$\frac{\mu_1}{\mu} = \frac{\Lambda_1^{-4}}{105} = \frac{308}{105},$$

therefore

$$\mu = \frac{105}{308}\mu' = \cdot 341\,\mu' \quad \text{or} \quad \mu = \tfrac{1}{3}\mu' \text{ nearly.}$$

Now

$$\mu = \kappa\Lambda^{-4} \text{ in Lord Rayleigh's formula,}$$

and

$$\mu_1 = \kappa\Lambda_1^{-4},$$

therefore

$$\kappa = \frac{\mu}{105} = \frac{\mu'}{308}.$$

Thus by finding either μ or μ' the absorption of every ray of the spectrum can be determined.

XXXI.—*Absorption of the Blue Glass.*

As the platinum paper was exposed through blue glass, it becomes necessary to know the coefficient of absorption of the glass for the photographically active rays. Experiments have been made with this view, and exposures have been made simultaneously through the blue glass, and without any glass intervening. The results obtained were that the same blackness would be produced if exposure to light through blue glass to that without were as 2·15 to 1. In the tables of exposure, which are annexed, it follows that to obtain the true photographic intensity of total sunlight, the values given must be multiplied by 2·15.

TABLE of Sensitiveness of Platinum Paper to the different parts of the Solar Spectrum when the exposure was made through blue glass, as used in the apparatus with which the observations were made.

Scale number.	Relative sensitiveness.	Scale number.	Relative sensitiveness.
7	0	13	86
8	4	13·5	70
8·5	10	14	56
9	20	14·5	42
10	50	15	30
10·5	66	16	16
11	84	17	6
11·5	95	18	2
12	100	19	1
12·5	97	20	0

H is at 13·8; G at 11·1; Li at 9; and D at 4·3.
Scale No. 12 has a wave length of 4260.

A reference to the spectrum curve, fig. 5, will show that this wave length of 4244 is close to the maximum sensitiveness. This is what was to be expected when it is remembered that the equivalent ray for the optically observed spectrum was close to the place of maximum luminosity.

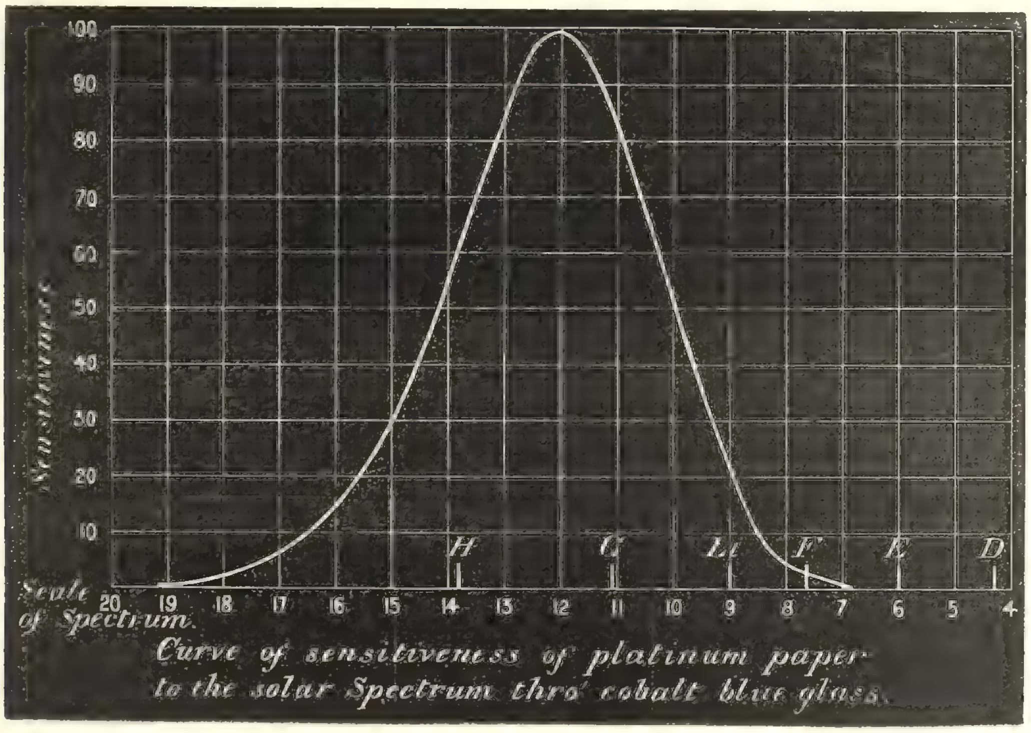
We are now in a position to reduce all the photographically measured coefficients of intensity to those which would have been observed optically.

XXXII.—*Places of Observation.*

It has been the object of these observations to determine, as far as possible, the different coefficients of absorption at different altitudes, and with this object in view the instrument has been taken to various places at different altitudes. Thus, South Kensington, Weybourne, Oxford, Derby, which may be taken as places at sea level. Grindelwald, in the valley, and on the various hills surrounding it, up to 8900 feet on the Faulhorn.

Como, Perarola, Cortina, and Interlaken, also Chamounix and Zermatt, with their surrounding mountains, have also been observing stations.

Fig. 5.



These stations are at various altitudes above the sea, and the results obtained are such as to show that the loss of light varies enormously, according as the altitude of the observing station is increased. This is a result which of course may be looked for. The observations of LANGLEY on Etna, and at Pike's Peak, show such to be the case; but it is necessarily more marked in the case of these observations owing to the part of the spectrum employed being that which suffers most loss by the scattering action of small particles. The blue end of the spectrum for this purpose has evidently a greater advantage over the red end and darker portions, more particularly over the latter, for in it the water absorptions play such an important *rôle* that the law of scattering could hardly be taken into account also. It will be noticed that observations have been made at all times of the year, and that the winter observations, when the surface of the ground at Grindelwald has been covered with snow and the air has been intensely cold (the thermometer sometimes showing as much as 40° F. of frost), give the coefficients of least absorption. The writer has held that most of the scattering particles are water in an extremely fine state of division, and that there is a probability that such is the case is shown by the fact that the most transparent atmosphere is that in which there is most warm aqueous vapour present. A change in the summer to what artists call an "atmospheric landscape" invariably shows a higher coefficient of absorption. Whatever these particles may be, there is no doubt that they are in the same volume fewer in number at high altitudes than they are at lower ones; as the coefficient of absorption is so much less in the former case than in the latter. Any one who has observed the sky at these higher altitudes will have observed that when the sun is brightest and highest, the sky is blackest, that is, that there are fewer scattering particles.

XXXIII.—*Standard of Light.*

It is somewhat difficult to know as to what standard of light to refer the light of the sun. The Author has made many comparisons between it and an amyl acetate lamp, and has come to the conclusion that at midday at sea level, and in the clearest atmosphere, the brightness of the sun exactly overhead would be close upon 7000 such standard lights at 1 foot distant from the screen, which would be equivalent to 5600 standard candles at the same distance. For the purpose of this paper, it is, however, unnecessary to refer the light to any particular standard, since all that was sought for was to obtain a comparison of the losses suffered on any one day by the light after passing through various thicknesses of atmosphere. It will be noticed in the various tables that the calculated brightness at the zenith varies very considerably even at the same altitudes. This must be laid down to one of three causes: (1) either an error in calculating the coefficients; (2) a slight haze of coarse particles intervening; or (3) owing to a varying sensitiveness in the paper used. Where the actual observed intensities throughout a day do not vary much more than those calculated, it may be presumed that the coefficient is not very far from the truth, and, consequently, the variation will in all probability be due to the second cause. As stated before, there was a change made in the paper which occurred in 1889, and it will be found that the intensities are practically the same before and after the above year.

XXXIV.—*Description, Tabulation, and Discussion of Results.*

The results which are tabulated do not show by any means all the observations made. They have been carried on for four years, and only those days are recorded in this paper when the sky has been practically cloudless whilst the observations have been taken. It need scarcely be said that the observations on three out of four days have been incomplete owing to atmospheric conditions not remaining constant. Every one of these has been tabulated, however, in the note books, and a check has thus been introduced in estimating the relative intensities of the light at the same altitudes of the sun above the horizon at the same observing station, for, in many instances, two or three observations under favourable conditions were made, the remainder being useless for the purpose of obtaining the coefficients of absorption.

Even the observations which are recorded are apparently so discordant, that it might appear difficult to arrange them in any order sufficient to get any empirical law which might connect barometric pressure with the exponential coefficient of transparency. As necessarily any approach to a law can only be an approximation, it has not been considered necessary to enter into any refinements such as the varying distance of the sun from the earth at different seasons, since any differences due to difference in atmospheric condition would more than hide any alteration due to that cause.

On certain days remarks are made that the atmosphere is exceptionally clear, and when we group such days together the results are not devoid of regularity.

Taking the observations at Faulhorn when the barometer was 21·5 inches, and when the day was noted as exceptionally fine, we get two values, ·231 and ·235 μ'. These are winter observations. We may take ·233 as the mean exponential coefficient at this barometric height in the clearest weather.

We also have on similar days—

	Bar.	μ_1.
At Derby	29·6	·437
Above Zweilutchinen . .	26·6	·356
„ Grindelwald . .	24·4	·307 and ·307
„ „ . .	23·3	·275 „ ·273
Faulhorn	21·5	·231 „ ·233

In addition to the observations given in the tables others were made at higher altitudes up to 12,000 feet at Zermatt on suitable days. It was not practicable to take a whole series of readings at these high altitudes, as it was not possible under ordinary circumstances to spend sufficient time at such elevations and that to get a difference in zenith distance sufficient to give a variation in air-thickness of such a magnitude as would give a reliable coefficient. As, however, the expeditions were made at such a time of the day as enabled a return to a station where the lapse of a few hours in the afternoon sufficed to obtained a large variation in the air-thickness, it became possible, by calculating first the coefficient of this station, and thence calculating the readings which would have been obtained at the time when the observations at the higher altitudes were made, to calculate the coefficient for the latter, assuming, of course, that no alteration in the general condition of the sky had taken place. Thus, at the Riffel house it was calculated that the coefficient at a barometric pressure of 21·2 inches, was ·222; and from observations made with the barometer at 18·9, it was calculated the coefficient would be ·182. Similarly, with the barometer at 19·9, it was calculated to be ·196.

It will be seen, if we take the Faulhorn observations as a standard, that we get what appears to be a very close approximation to the law that the exponential coefficient varies as the square of the barometric pressure.

The following is a table calculated on this assumption:—

30 inches	·453
29 ,,	·423
28 ,,	·394
27 ,,	·367
26 ,,	·340
25 ,,	·314
24 ,,	·289
23 ,,	·266
22 ,,	·244
21 ,,	·222
20 ,,	·201
19 ,,	·182

It will be seen that the coefficients obtained on these exceptionally clear days fall extremely close to the calculated values. The calculation for

29·6 is 441, 26·6 is 356, 24·4 is 299,
23·4 is 276, 21·5 is 232.

On the day when the Faulhorn observation gave a value to the coefficient of ·261, the sky was fairly good. If we take this as a starting point and calculate as before, we find the following table:—

30 inches	·509
29 ,,	·477
27 ,,	·412
26 ,,	·381
25 ,,	·353
24 ,,	·325
23 ,,	·297
21·5 ,,	·261

THE following coefficients were obtained from observations made on days which were classed as ordinarily bright.

Barometer.		Observed.	Calculated.
21·5	Faulhorn	·261	·261
23·4	Above Grindelwald . .	·290	·306
25·6	Montanvert	·384	·370
26·6	Grindelwald	·400	·400
29·7	Derby	·514	·497

It will be seen that on such days the same order of change in the coefficients is to be found.

Other coefficients were obtained on moderately bright days, which are higher than the above for the varying heights of the barometer. These observations are given to show the great variations which may occur owing to atmospheric conditions.

It thus appears that the light acting on platinum paper at any altitude up to those observed can be expressed by

$$I' = I\epsilon^{-\mu_1 h^2 x/30},$$

where h is the height of the barometer and μ_1 is the coefficient of absorption.

In the case of the most transparent atmosphere $\mu_1 = {\cdot}453$ at a barometer of 30 inches in height.

Therefore

$$I' = I\epsilon^{-{\cdot}0005033h^2}.$$

The coefficient for the visual absorption is $\cdot341\mu'$.

$$A' = A\epsilon^{-{\cdot}0001716h^2},$$

where A and A′ are the original and transmitted lights.

We thus arrive at the following results:—

Barometer.	μ'	a'	u	a
30	·453	·639	·154	·853
29	·423	·654	·144	·866
28	·344	·671	·134	·875
27	·367	·689	·124	·884
26	·340	·708	·115	·891
25	·314	·730	·107	·899
24	·289	·746	·098	·908
23	·266	·763	·090	·915
22	·244	·787	·083	·922
21	·222	·800	·075	·928
20	·201	·819	·068	·934
19	·182	·833	·062	·940

μ' and μ are the exponential coefficients as before. a' and a are the constants in the formula, $\mathrm{I}a^z = \mathrm{I}'$, z being the air thickness, and I and I′ the intensities before and after transmission.

We are now in a position to determine the constant κ in the formula

$$\mathrm{I}' = \mathrm{I}\epsilon^{-\kappa x\lambda^{-2}}$$

at sea-level with the barometer at 30 in. and in a very clear sky, for as

$$\kappa\Lambda^{-4} = \mu_1,$$

therefore

$$\kappa = \frac{\mu'}{\Lambda^{-4}} = \frac{\cdot 453}{308} = \cdot 00146\,;$$

in the case of fairly clear skies,

$$\kappa = \frac{\cdot 497}{\cdot 308} = \cdot 00161.$$

In Part I. of this paper the minimum value of κ was found to be ·0013, and a mean value about ·0017, so that these observations are fairly accordant.

κ may be taken to be a measure of the number of particles the rays encounter, and thence it may be concluded that the number of particles at any thin layer of the atmosphere is ah. The formula, therefore, for the scattering of a ray of any wave-length at any altitude becomes

$$\mathrm{I}' = \mathrm{I}\epsilon^{-ch^2x\lambda^{-4}}$$

where c is a constant, h the height of the barometer, and x the air thickness, those of the zenith being "unity."

In comparing the comparative scattering of a ray at the same zenith distance, but at different altitudes, λax are constants, and

$$\mathrm{I}' = \mathrm{I}\epsilon^{-mh^2}$$

where m is a constant and h the variable. This formula and that of the law of error are identical.

XXXV.—*Conclusions.*

In conclusion, it should be remarked that the loss of light as light from transmission through the atmosphere is, and must be, very different to that of the heating effect of the solar radiation. The latter is not principally dependent on scattering by small particles, but on the absorption of aqueous vapour, which is a very different matter. LANGLEY has shown that the heating effect diminishes much more rapidly as the barometric pressure is diminished than is usually supposed, and this is not to be wondered at. For it must be recollected that the absorption by aqueous vapour takes place principally in the infra-red region of the spectrum, and, if it were entirely confined to that region, the presence or absence of the aqueous vapour would have no practical effect on the luminosity of the visible rays transmitted.* The absorption by this vapour where it takes place is large and follows the ordinary laws, as already stated, being principally in the infra-red of the spectrum, the heating effect will therefore diminish much more rapidly than the total illuminating value of sunlight as increased thickness of atmosphere is penetrated. In other words, the amount of light transmitted from the sun bears no comparison with that of its total heating effect.

It should be remarked that the portion of the spectrum used in these observations is almost free from any absorption by aqueous vapour, and, consequently, the scattering effect of the small particles in the atmosphere is probably almost entirely the cause of the loss of light from it, and enables a factor for such scattering to be arrived at without much difficulty. The measurement of the amount of light transmitted through different thicknesses of atmosphere has hitherto been almost entirely made on star and moon light, and, for reasons given in Part I. of this paper, the coefficients arrived at are probably slightly too large for the transmission of sunlight.

The results of these observations made at different altitudes, combined with others made in the laboratory, point to a probability that the particles which selectively scatter light are due to water. Their dimensions are probably very closely, if not exactly, the same, and the "mist" particles are large compared with them.

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* See a paper in No. 224 of the 'Proceedings of the Royal Society,' 1883, by General FESTING and the Author.

Grindelwald, 16th May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	—	6.48	26·4	2·49				100	97	*u* = ·408. Fairly clear sky.
2	18				33·2	182	101			
3	40				16·5	400	100			
4	10	7.45		1·90	45	120	120	120	124	
5	20				29·7	210	105			
6	25				21·7	300	120			
7	5	9.55		1·26	62·5	67	134	164	162	
8	10				36·2	160	160			
9	9				39	146	162			
10	19				21	310	163			
11	5	P.M. 1.55		1·24	56·5	82	164	164	163	
12	10				36·2	168	168			
13	30				14·5	480	160			
14	15	3.16		1·50	30·5	204	136	132	146	
15	40				12·2	512	128			
16	10	6.0		4·04	68	57	57	51	51	
17	30				37·5	154	51			
18	60				16	310	51			
19	30				37·5	154	51			

ZWEILÜTCHINEN, 2nd May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	12.0	26·4	1·17	70·5	52	104	130	135	μ = ·356.
2	10				42·5	130	130			
3	20				—	—	—			
4	5	1.25		1·23	63·5	65	130	131	131	Very clear.
5	10				41·5	132	132			
6	30				89·2	360	120			
7	—									
8	5	2.18		1·35	76	43	86	83	126	
9	20				36·5	160	80			
10	20				—	—	—			
11	5	4.8		1·98	69	51	102	102	100	
12	10				50	102	102			
13	20				29·5	207	103			
14	—									
15	5	4.50		2·51	84·5	31	62	83	83	
16	10				56·5	82	82			
17	20				35	170	85			
18	5	5.30		3·53	88	27	54	52	56·7	
19	20				50·5	100	50			
20	20	6.25		7·8	90	24	12	12	12·6	

Montanvert, 10th August, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	12.55	25·6	1·19	68	45	45	48	52·7	$\mu = {}$·384.
2	20				42	90	45			
3	20				—	96	48			
4	30				28·7	142	48			Fairly bright.
5	10				69	44	44			Sky clear.
6	15	3.0		1·5	50	72	48	47	47	
7	10				67·5	46	46			
8	24				35	112	46·6			
9	10				64	49	49			
10	22				37·5	106	47·2			
11	10	4.0		1·89	75	38	38	37·5	40·4	
12	15				60	55	37			
13	20				47·5	76	38			
14	20				48·5	74	37			
15	20	5.0		2·72	66	47	23·5	26·5	29·6	
16	30				47·5	76	25·3			
17	40				37	108	27			
18	45	5.56		4·8	55·5	61	13·5	13·2	13·2	
19	75				35·5	112	13			
20	120	6.5		7	44	78	6·5	6·5	5·7	

ABOVE Grindelwald, 23rd December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	1.15	23·4	3·23	91·3	19	19	17·5	17·5	$\mu = ·273$. Very clear sky, sun very hot, and light excellent.
2	20				75	32	16			
3	30				53	53	17·6			
4	10	2.15		4·2	96			13·1	13·5	
5	20				82	27	13·5			
6	30				67·3	38	13			
7	10				94					
8	20				84	26	13			
9	30				70	37	12·6			
10										
11	10	2.40		5·03	96			10·8	10·7	
12	20				86	22	11			
13	30				77	32	10·6			

ABOVE Grindelwald, 2nd January, 1891.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	17	1.10	23·4	3·1	72·7	34	20	19·5	19·3	$\mu = {}$·290.
2	30				47·4	60	20			
3	45				35	88	19·6			
4	30				48·7	59	18·3			Sky deep.
5	17	2.45		5·11	93·5	18	10·6	11·1	10·7	Very good sun.
6	30				72	34	11·3			
7	45				56·2	48	10·6			
8	60				44	66	11			
9	80				35	88	11			
10	—									
11	120	3.14		7·04	38·6	76	6·3	6·2	6·2	
12	120				40	73	6·1			

FAULHORN, 13th January, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	20	9.50	24·56		33	91	45·5	54		$\mu = \cdot 235$.
2	30				32	162	54			Splendid sky for depth, sun very bright.
3	30				21	162	54			
4	20	12.13	21·6	2·6	23	144	72	72	72	
5	20				24·5	137	68·5			
6	30				15·5	216	72			
7	20	2.13	21·6	3·5	28	116	58	57	58	
8	20				28·5	114	57			
9	30				21·5	160	53·3			
10	30				19·5	177	59			
11	30				21	162	54			
12	30				18·5	185	61·6			
13	20	3.8	21·6	5	38	72	36	40	40	
14	30				27	122	41			
15	30				27·5	120	40			

FAULHORN, 18th January, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	8.55	25		60	24	24	30		$\mu = \cdot 231$.
2	20				43	58	28			
3	30				33	92	31			
4	30				33	91	30			Same as January 13.
5	10	12.0	21·9	2·5	33	91	91	92·5	92·5	
6	20				18·5	185	92·5			
7	20				18·5	185	92·5			
8	30				12·6	280	93·3			
9	20	1.45	21·7	3	21	162	81	81	82·3	
10	10				37	68	68			
11	20				23	146	73			
12	20				21	162	81			
13	30				14·5	243	81			
14	30				14·5	243	81			
15	10	3.10	21·7	4·9	44	52	52	53	53	
16	20				31	100	50			
17	30				21	164	55			

GRINDELWALD, 12th August, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	20	6.48	26·6	2·84	38	94	47	47	47	μ = 400. Fair sky.
2	40				17·5	197	44			
3	30				27·5	132	44			
4	60				12	276	49			
5	5	7.40			56·5	25	50	48		
6	20				35	100	50			
7	30				28	130	43			
8	20				37	96	48			
9	25				28·5	128	51			
10	15	8.2			37·5	96	64	68	68	
11	15				32·7	102	68			
12	20				25	144	72			
13	10	8.58			42·5	81	81	81	80	
14	15				30	122	81			
15	25				19·7	182	73			
16	13	10.24			32	114	89	89	89	
17	20				20·5	172	86			
18	20				18·7	178	89			

GRINDELWALD, 23rd May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	11.40	26·6		59·5	74	74	67		$\mu = \cdot 381$.
2	20				41	136	68			Very fair sky ; good sun.
3	10				66	60	60			
4	20				41	136	68			
5	10				63	66	66			
6	10	1.33		1·24	58·5	76	76	75	75	
7	20				39·7	143	71·5			
8	20				37·5	154	77			
9	10	2.48		1·41	62	68	68	70	70·4	
10	20				39·5	144	72			
11	10	4.30		2·02	74	46	46	57·5	56·5	
12	20				47	112	56			
13	30				34	178	59			
14	15	5.20		2·64	70·5	52	35	41	43·9	
15	30				45	119	40			
16	45				32	189	42			
17	40	5.52		3·31	39	146	36·5	35·8	34·7	
18	60				29	212	35·3			
19	60	6.18		4·46	42·5	130	22	22	22	
20	60				42·5	130	22			

GRINDELWALD, 7th July, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	—	12.40	26·6	1·075				125	125	$\mu = \cdot 71$.
2	10				43·5	126	126			
3	15				33	184	123			
4	5	2.42		1·26	66	60	120	111	110	Poor sky; milky.
5	10				47·5	110	110			
6	20				27·8	224	112			
7	5	5.40		2·63	80·5	37	37	36·7	37·5	
8	10				70·5	51	35			
9	20				51·5	96	48			
10	30				46·2	115	38			
11	15	6.10		3·34	79	39	26	25	25	
12	20				71·5	51	25·5			
13	30				59·5	74	24·7			
14	40				50	100	25			
15	20	6.35		4·28	86	29	14·5	13·3	13·6	
16	25				82·5	34	13·6			
17	40				72·5	49	12			
18	20	7.0		5·5	99	11	5·5	5·5	5·4	
19	40				93·5	21	5·2			
20	60				82	34	5·7			

COMO, 28th May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	10.45	29·1	1·142	64	64	128	120	120	$\mu = \cdot 55$.
2	10				46	115	115			
3	20				27	232	116			Blue sky but rather milky.
4	5	12.30			62	68	132	128	123	
5	10				41·5	134	134			
6	20				25·2	255	128			
7	5	3.23			75	50	100	100	100	
8	10				50	101	101			
9	20				31	100	100			

Rosenlaui, 4th July, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	7.5	26·5	2·19	72	50	100	98	98	$\mu = \cdot 435$.
2	10				50	100	100			
3	20				32	190	95			Fairly good sky. Clouded over about 1.30.
4	5	8.50		1·4	53	92	184	143	143	
5	10				39	140	146			
6	20				23	280	140			
7	5	11.15		1·02	52	94	188	163	163	
8	10				36·5	160	160			
9	15				25·5	250	167			
10	20				19·2	326	163			

GRINDELWALD, 24th June, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	12.0	26·2	1·09	77	34	68	68	68	$\mu = ·707$.
2	10				49	68	68			
3	20				27	134	67			Milky sky, cloudy in morning, bright after 11.
4	5	1.10		1·125	76	33	66	65	66·5	
5	10				51·5	64	64			
6	20				28	130	65			
7	5	4.25		1·842	84	18	36	39·6	40	
8	10				69·5	40	40			
9	20				40	87	43			

SOUTH KENSINGTON, 25th September, 1889.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	12.40	29·6	1·75	75·5	34	34	34	34	$\mu = \cdot 627$.
2	20				48·5	68	34			
3	40				28·2	130	32·5			Poor sky, but same tint maintained.
4	10	2.0		1·91	85	25	25	26	30·5	
5	20				64	45	22·5			
6	40				35	104	26			
7	15	3.0		2·48	79·5	30	20	21·5	21·4	
8	30				56·5	57	19·5			
9	40				41·2	88	22			
10	10	3.55			98	14	14			
11	15				90	21	14			
12	30				71	38	12·7			
13	10	4.2		3·4	96	—		12	12	
14	20				86	24	12			
15	40				65·5	44	11			
16	10				98	14	14			
17	20				85	25	12·5			
18	40				65	44	11			

ABOVE Grindelwald towards Faulhorn, 25th December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint, White, =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	15	12.0	24·4	2·92	82·5	26	17·3	17·7		$\mu = \cdot 307$. Beautifully deep sky; sun very bright.
2	30				53·3	53	17·6			
3	30				53·3	53	17·6			
4	45				36·7	82	18·2			
5	30	12.25		3	52	55	18·3	18·5	18·5	
6	15				80	28	18·6			
7	10	1.30		3·4	95	17	17	16·5	16·5	
8	20				73·5	33	16·5			
9	30				52·7	54	18			
10	20				72·5	32	16			
11	20	2.0		3·8	77·5	30	15·	14·5	14·5	
12	40				49·5	58	14·5			
13	60				36	86	14·5			
14										
15	30	2.40		5	79	29	9·7	9·7	9·7	
16	45				60	45	10			
17	60				50·7	57	9·5			
18	30	3.30		7·6	97	15	5	4·7	4·5	
19	60				85·5	28	4·7			
20	60				87	27	4·5			

ABOVE Grindelwald towards Faulhorn, 27th December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	15	12·0		2·9	83	27	18	16·3		μ = ·275.
2	30				58·5	47	15·7			
3	45				42·5	68	15·1			
4	15	12.25	24·4	3	81	28	18·6	18·3	18·3	
5	30				52·5	54	18			
6	15	1.45	23·4	3·6	86·5	23	15·3	15·3	15·3	μ = ·306.
7	30				57	48	16			
8	—									
9	45				44	66	14·7			Beautiful black sky; sun very bright.
10	30	2.35	23·4	4·9	72	34	11·3	10·7	10·7	
11	60				47	61	10·2			
12	60	3.34	24·4		95·5	15	2·5	2·5	2·5	
13	120				84	26	2·2			

FAULHORN, 8th July, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	15	A.M. 5.10	21·4	6·7	68·5	56	42	40	40	μ = ·261.
2	40				38	151	38			
3	80				19	350	41			
4	10	5.40		4·55	65	62	62	70	70	Good sky, sun rose yellow.
5	20				44·5	122	61			
6	60				16·5	420	70			Sky good at top and at Grindelwald.
7	10	6.40		2·55	50	101	101	114	116	
8	20				29	210	105			
9	35				17	400	114			
10	11	8.10		1·6	36·2	161	146	147	147	
11	9				42	132	147			
12	15				28	222	147			
13	—									
14	5	9.45		1·24	54·5	87	174	160	163	
15	10				36·5	160	160			
16	20				20·5	320	160			
17	10	10.15		1·18	34	170	170	170	170	
18	15				25	255	170			
19	10	1.55		1·20	47	112	112	112	—	
20	15				34·5	172	113			

DERBY, 27th August, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	3.17	29·6	1·90	65·4	61	61	56	56·1	$\mu = \cdot514$.
2	20				47·2	109	54·5			
3	30				34·7	171	57			
4	20				48·6	104	52			Sky rather milky but sun good.
5	10	4.9		2·41	75	44	44	42	44·5	
6	20				54·2	86	43			
7	30				43	126	42			
8	40				34·7	165	41			
9	—									
10	20	4.46		3·00	66·7	58	29	32·3	31·9	
11	30				49·3	102	34			
12	10				82	34	34			
13	20	5.15		3·85	80·5	36	18	18·2	20·6	
14	40				61·1	69	17·3			
15	60				47·2	109	18·2			
16	30				75	44	14·3			
17	60	5.40		5·07	61·1	69	11·5	11	11	
18	90				51·4	94	10·4			
19	80	5.58		6·59	77·7	40	5	5	5	
20	120				68	56	4·6			

DERBY, 28th August, 1887.

No. of exposure.	Expo-sure in seconds.	Time.	Barometer. pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec exposure.	Calculated intensity.	Remarks.
1	—		29·6							$\mu = \cdot 437$.
2	10	12.25		1·37	54·5	87	87	86	87	Exceptionally clear. Saw the Malvern Hills distinctly. *The* one day of the year probably.
3	20				35·5	166	83			
4	15				42	132	88			
5	10	1.3		1·42	52·5	93	93	88	88	
6	15				42·5	130	86			
7	20				34·5	174	87			
8	10	2.4		1·53	48·5	107	107	83·5	83·8	
9	20				35·5	168	84			
10	12				50·5	99	83			
11	10	3.15		1·89	59	74	74	72	71·6	
12	20				40·5	139	70			
13	8	4.13		2·47	44·5	41	51	51	55·6	
14	20				51·5	99	49			
15	40				30·5	204	51			
16	20	4.45		3·04	54	88	44	44	43·3	
17	80	5.16		3·96	30·5	234	29	29	29	
18	40				46·5	114	28·5			

II. *The Potential of an Anchor Ring.*

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Introduction.

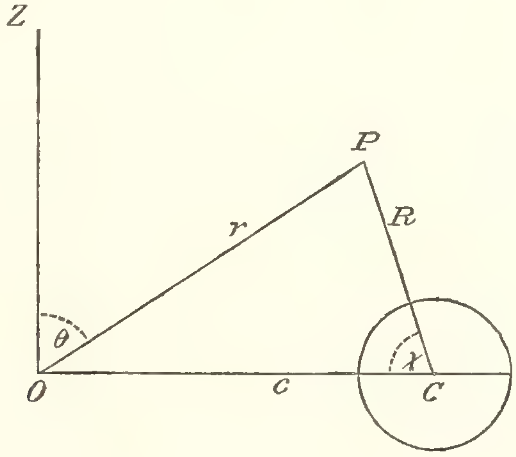
In this Paper I have developed a method of dealing with questions connected with Anchor Rings.

If r, θ, ϕ be the coordinates of any point outside an anchor ring, whose central circle is of radius c, then

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

is a solution of Laplace's equation, finite at all external points and vanishing at infinity. Let this be called I. Then $d\mathrm{I}/dz$ is another solution; and two sets of solutions may be found by differentiating I and $d\mathrm{I}/dz$ any number of times with respect to c. These solutions are symmetrical with respect to the axis of the ring. In the first set z is involved only in even powers; in the second set in odd powers.

Take a section through the axis Oz of the ring and the point P, (r, θ) cutting the central circle of the ring in C.



Let CP = R and $\angle$ OCP = χ.

When R is less than c, the integral

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

is expanded in ascending powers of R/c, and the expansions of the integrals, obtained by differentiating this with respect to c, are deduced. Section I. is devoted to the discussion of these functions, and some similar ones, needed in hydrodynamic applications.

In Section II. the potential of an anchor ring at all external points is found in a very convergent series of integrals. The expansions of Section I. are not needed; but the first few terms are reduced to elliptic integrals. The equipotential surfaces are drawn for the ratios $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, 1 of the thickness of the ring to its mean diameter.

In Section III. the potential of a conductor, in the form of an anchor ring, is found at external points; the surface density at any point of the ring and the charge are also determined. Section IV. consists of a discussion of the motion of an anchor ring in an infinite fluid; the velocity potential or stream line function for motion parallel to the axis, perpendicular to the axis, &c., being first determined. The kinetic energy is determined in the several cases; and in the case of the cyclic motion through the ring, the linear momentum. In this last case, the solid angle subtended by a circle at a point near a circumference, is incidentally found.

In Section V. the annular form of rotating fluid is discussed, when the thickness of the annulus is small compared with its mean radius.

It is shown that the form of the cross section may be represented by $R = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi +$, &c.), where β_2, β_3, &c., are of the second, third, &c., order in a/c. Their values are found as far as $(a/c)^4$.

To the second order

$$\frac{\omega^2}{\pi\rho} = \left(\frac{a}{c}\right)^2 \left(\log\frac{8c}{a} - \frac{5}{4}\right),$$

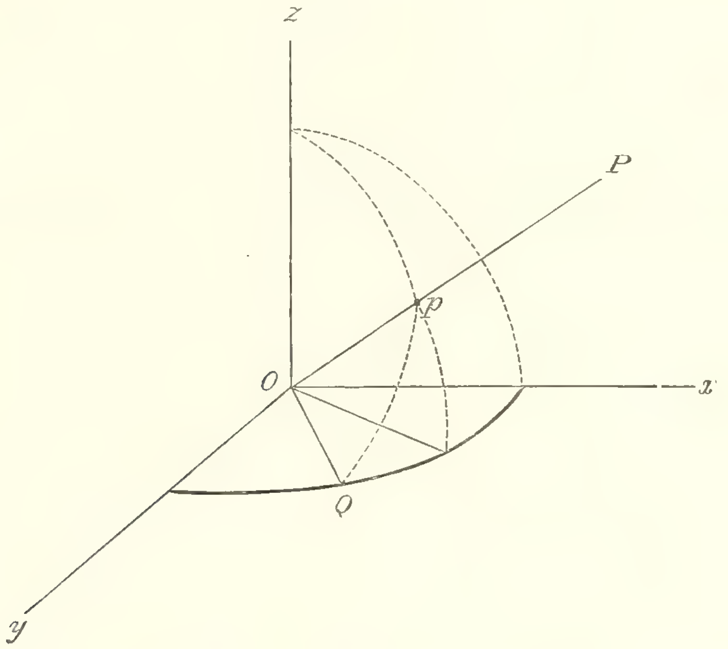
$$\beta_2 = \frac{5}{8}\left(\frac{a}{c}\right)^2 \left(\log\frac{8c}{a} - \frac{17}{12}\right).$$

The method employed throughout the paper has not the analytical elegance of Mr. Hicks' Toroidal Functions, but it has many advantages. The potential of an attracting ring takes a very simple form. The boundary conditions to be satisfied in hydrodynamical applications are very simple. The results are obtained in terms of R and χ, the quantities most obviously connected with a ring.

I am greatly indebted to Mr. Herman, Fellow and Assistant Tutor of Trinity College, Cambridge, for the careful manner in which he has read over much of the work, and the many errors he has corrected. In consequence, the paper will, I think, be found free from any serious mistakes.

Section I.—*Preliminary Analysis.*

§ 1. Take a fine ring in the plane of x, y, centre at the origin, and consisting of attracting matter of density $k \cos n\,\phi'$, where ϕ' is the azimuth of any point.



The potential of this ring at a point P, whose coordinates are r, θ, ϕ, is

$$\int_0^{2\pi} \frac{k \cos n\phi' \, c d\phi'}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos(\phi - \phi')\}}}.$$

Put $\phi' = \phi + \psi$, then the potential

$$= \int_0^{2\pi} \frac{ck\,(\cos n\phi \cos n\psi - \sin n\phi \sin n\psi)\, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

$$= \cos n\phi \int_0^{2\pi} \frac{ck \cos n\psi \, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

as the second integral vanishes between the limits.

Therefore

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi \, d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

or

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

is a solution of LAPLACE'S equation.

Similarly

$$\sin n\phi \int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

is a solution.

Let V stand for either of these integrals. Then

$$\frac{d^{p+q}\,\mathrm{V}}{dc^p \, dz^q}$$

is also a solution of LAPLACE'S equation.

For different values of p and q the solutions are not all independent: as it is easily seen that

$$\int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

satisfies a linear partial differential equation of the second order in c and z.

But two independent sets of solutions are obtained by giving different values to p in $\frac{d^p\mathrm{V}}{dc^p}$ and $\left(\frac{d}{dc}\right)^p . \left(\frac{d\mathrm{V}}{dz}\right)$.

The only cases considered in this paper are when $n = 0$ or $n = 1$. That is the solutions of the forms

$$\left(\frac{d}{dc}\right)^p \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} ;$$

$$\left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} ;$$

$$\cos \phi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + c^2)}} ;$$

and

$$\cos \phi \left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} ;$$

as these cover all the cases to which the functions are applied in this paper.

It is easily seen that

$$\varpi \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

and, consequently,

$$\varpi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

are solutions of the equation

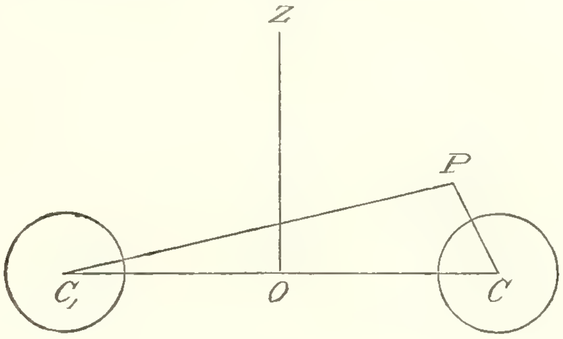
$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

For putting $\psi = \varpi\phi$, this equation becomes

$$\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\Phi}{d\varpi} - \frac{1}{\varpi^2}\Phi = 0,$$

that is, $\Phi \cos \phi$ is a solution of LAPLACE'S equation.

§ 2. *Expansion of the Functions.*—Let the plane through the axis of the ring and a point P, cut the ring in the two circles whose centres are C and C_1.



Let

$$CP = R; \quad C_1P = R_1; \quad OC = c;$$

and the angle

$$OCP = \chi.$$

Also, for convenience, put

$$s = \frac{R}{c},$$

and

$$l = \log\frac{8c}{R} - 2.$$

When R is less than c the above integrals may be expanded in ascending powers of R/c or s.

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}}$$

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \frac{1^2 3^2}{2^2 4^2}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{2}{3.4}\right) + \ldots\right\}.$$

[CAYLEY, 'Ell. Func.,' p. 54.]

$$\int_0^{\pi} \frac{\cos\phi\, d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{(2\cos^2\phi - 1)\, d\phi}{\sqrt{\left\{1 - \frac{R_1^{\,2} - R^2}{R_1^{\,2}}\cos^2\phi\right\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{2\left(\cos^2\phi - \frac{R_1^{\,2}}{R_1^{\,2} - R^2}\right) - 1 + \frac{2R_1^{\,2}}{R_1^{\,2} - R^2}}{\sqrt{\left\{1 - \frac{R_1^{\,2} - R^2}{R_1^{\,2}}\cos^2\phi\right\}}}\, d\phi,$$

$$= \frac{2}{R_1}\left\{\frac{2}{1 - \frac{R^2}{R_1^{\,2}}}\left[\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^{\,2} - R^2}{R_1^{\,2}}\sin^2\phi\right\}}} - \int_0^{\frac{\pi}{2}} \sqrt{\left\{1 - \frac{R_1^{\,2} - R^2}{R_1^{\,2}}\sin^2\phi\right\}}\, d\phi\right]\right.$$
$$\left. - \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^{\,2} - R^2}{R_1^{\,2}}\sin^2\phi\right\}}}\right\}$$

$$= \frac{4}{R_1\left(1 - \frac{R^2}{R_1^{\,2}}\right)}\left[\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^{\,2}}\left(\log\frac{4R_1}{R} - \frac{2}{1\,.\,2}\right) + \ldots\right\}\right.$$
$$\left. - \left\{1 + \frac{1}{2}\frac{R^2}{R_1^{\,2}}\left(\log\frac{4R_1}{R} - \frac{1}{1\,.\,2}\right) + \frac{1^2\,.\,3}{2^2\,.\,4}\frac{R^4}{R_1^{\,4}}\left(\log\frac{4R_1}{R} - \frac{2}{1\,.\,2} - \frac{1}{3\,.\,4}\right) + \ldots\right\}\right]$$
$$- \frac{2}{R'}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^{\,2}}\left(\log\frac{4R_1}{R} - \frac{2}{1\,.\,2}\right) + \frac{1^2\,.\,3^2}{2^2\,.\,4^2}\frac{R^4}{R_1^{\,4}}\left(\log\frac{4R_1}{R} - \frac{2}{1\,.\,2} - \frac{2}{3\,.\,4}\right) + \ldots\right\}$$

[CAYLEY, 'Ell. Func.,' p. 54.]

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} - 2 + \frac{R^2}{R_1^{\,2}}\frac{5\log\frac{4R_1}{R} - 7}{4} + \frac{R^4}{R_1^{\,4}}\frac{162\log\frac{4R_1}{R} - 225}{128} + \&c.\right\}.$$

Now

$$R_1^{\,2} = 4c^2 - 4cR\cos\chi + R^2$$
$$= 4c^2\left(1 - \frac{R}{c}\cos\chi + \frac{R^2}{4c^2}\right)$$
$$= 4c^2\left(1 - s\cos\chi + \frac{s^2}{4}\right).$$

Therefore

$$\log\frac{4R_1}{R} = \log\frac{8}{s} - \frac{s}{2}\cos\chi - \frac{s^2}{4}\frac{\cos 2\chi}{2} - \frac{s^3}{8}\frac{\cos 3\chi}{3} - \&c.$$
$$= l + 2 - \frac{s}{2}\cos\chi - \frac{s^2}{8}\cos 2\chi - \frac{s^3}{24}\cos 3\chi - \&c.$$

$$\frac{R^2}{4R_1^{\,2}} = \frac{s^2}{16} + \frac{s^3}{16}\cos\chi + \frac{s^4}{64}(1 + 2\cos 2\chi) + \&c.$$

Substituting in the above expansions

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\Big\{l + 2 - s\frac{\cos\chi}{2} + s^2\left(\frac{l+1}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{2l+1}{32}\cos\chi - \frac{\cos 3\chi}{24}\right)$$

$$+ s^4\left(\frac{50l + 15}{2048} + \frac{4l+1}{128}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right) + \ldots$$

$$\int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\Big\{l - s\frac{\cos\chi}{2} + s^2\left(\frac{5l+3}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{10l+1}{32}\cos\chi - \tfrac{1}{24}\cos 3\chi\right)$$

$$+ s^4\left(\frac{322l + 35}{2048} + \frac{20l-3}{128}\cos 2\chi - \tfrac{1}{64}\cos\chi\right) + \ldots\Big\}.$$

Now

$$\frac{2}{\mathrm{R}_1} = \frac{1}{c}\left(1 - s\cos\chi + \frac{s^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{c}\Big\{1 + \frac{s\cos\chi}{2} + s^2\left(\frac{1}{16} + \frac{3}{16}\cos 2\chi\right) + s^3\left(\frac{3}{64}\cos\chi + \frac{5}{64}\cos 3\chi\right)$$

$$+ s^4\left(\frac{9}{1024} + \frac{20}{1024}\cos 2\chi + \frac{35}{1024}\cos 4\chi\right) + \ldots\Big\}.$$

Therefore

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\Big\{l + 2 + s\frac{l+1}{2}\cos\chi + s^2\left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)$$

$$+ s^3\left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)$$

$$+ s^4\left(\frac{105l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right) + \text{\&c.}\Big\}.$$

And

$$\int_0^\pi \frac{\cos\phi d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\Big\{l + s\frac{l-1}{2}\cos\chi + s^2\left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)$$

$$+ s^3\left(\frac{33l+1}{64} + \frac{15l-23}{192}\cos 2\chi\right)$$

$$+ s^4\left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right) + \text{\&c.}\Big\}.$$

§ 3. To obtain the expansion of $\frac{d}{dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$ in ascending powers of $\frac{\mathrm{R}}{c}$, the corresponding expansion of $\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$ must be differentiated with respect to c, R and χ being considered as functions of r, θ, and c.

Now

$$\left.\begin{array}{l} \mathrm{R} \cos\chi = c - r \sin\theta = c - \varpi \\ \mathrm{R} \sin\chi = r \cos\theta = z \end{array}\right\}.$$

Therefore

$$\left.\begin{array}{l} \cos\chi \dfrac{\partial \mathrm{R}}{\partial c} - \mathrm{R} \sin\chi \dfrac{\partial\chi}{\partial c} = 1 \\ \sin\chi \dfrac{\partial \mathrm{R}}{\partial c} + \mathrm{R} \cos\chi \dfrac{\partial\chi}{\partial c} = 0 \end{array}\right\}.$$

Solving these equations

$$\frac{\partial \mathrm{R}}{\partial c} = \cos\chi$$

$$\mathrm{R}\frac{\partial\chi}{\partial c} = -\sin\chi.$$

Again

$$\left.\begin{array}{l} \cos\chi \dfrac{\partial \mathrm{R}}{\partial z} - \mathrm{R} \sin\chi \dfrac{\partial\chi}{\partial z} = 0 \\ \text{and} \\ \sin\chi \dfrac{\partial \mathrm{R}}{\partial z} + \mathrm{R} \cos\chi \dfrac{\partial\chi}{\partial z} = 1 \end{array}\right\}.$$

Solving these equations

$$\frac{\partial \mathrm{R}}{\partial z} = \sin\chi\,; \quad \mathrm{R}\frac{\partial\chi}{\partial z} = \cos\chi.$$

From these formulæ it follows that

$$\left.\begin{array}{l} \dfrac{d}{dc}(\mathrm{R}^n \cos n\chi) = n\mathrm{R}^{n-1} \cos(n-1)\chi \\ \text{and} \\ \dfrac{d}{dc}\left(\dfrac{\cos n\chi}{\mathrm{R}^n}\right) = -n\dfrac{\cos(n+1)\chi}{\mathrm{R}^{n+1}} \end{array}\right\}$$

and that

$$\left.\begin{array}{l} \dfrac{d}{dz}(\mathrm{R}^n \cos n\chi) = -n\mathrm{R}^{n-1} \sin(n-1)\chi \\ \text{and} \\ \dfrac{d}{dz}\left(\dfrac{\cos n\chi}{\mathrm{R}^n}\right) = -\dfrac{n\sin(n+1)\chi}{\mathrm{R}^{n+1}} \end{array}\right\}$$

Let us write

$$\mathrm{I}_1 = \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

I_1 has been expanded as a function of R, χ, and c.

The expansion of dI_1/dc may therefore be found by differentiating this by the rule

$$\frac{d}{dc} = \frac{\partial}{\partial c} + \cos\chi \frac{\partial}{\partial R} - \frac{\sin\chi}{R}\frac{\partial}{\partial\chi}.$$

It is rather simpler to use the formulæ for $d/dc\ (R^n \cos n\chi)$ proved above.

Take, for example, the term

$$\frac{1}{c}\frac{9l+3}{64} s^3 \cos\chi.$$

This

$$= \frac{9l+3}{64}\frac{R^2}{c^4} R\cos\chi.$$

$$\frac{d}{dc}\left(\frac{9l+3}{64}\frac{R^2}{c^4} R\cos\chi\right) = \frac{(9l+3)\,R^2}{64c^4}\frac{d}{dc}(R\cos\chi) + R\cos\chi\frac{d}{dc}\left\{\frac{(9l+3)\,R^2}{64c^4}\right\}$$

$$= \frac{(9l+3)\,R^2}{64c^4} + \frac{R\cos^2\chi}{64c^4}\frac{\partial}{\partial R}\{(9l+3)\,R^2\} + \frac{R^3\cos\chi}{64}\frac{\partial}{\partial c}\left(\frac{9l+3}{c^4}\right).$$

In performing the partial differentiations with respect to R and c, it must be remembered that $l = \log 8c/R - 2$, and is a function of R and c.

Let

$$I_1 = \frac{l+2}{c} + \frac{l+1}{2}\frac{R\cos\chi}{c^2} + \left\{\frac{(2l+1)\,R^2}{16c^3} + \frac{(3l+2)\,R^2\cos 2\chi}{16c^3}\right\}$$
$$+ \left\{\frac{(9l+3)\,R^3\cos\chi}{64c^4} + \frac{(15l+7)\,R^3\cos 3\chi}{192c^4}\right\} + \&c.$$

$$\frac{dI_1}{dc} = -\frac{\cos\chi}{cR} - \frac{1+\cos 2\chi}{4c^2} + \left\{\frac{(8l-3)\,R\cos\chi}{32c^3} - \frac{3R\cos 3\chi}{32c^3}\right\}$$
$$+ \left\{\frac{(18l-3)\,R^2}{128c^4} + \frac{(18l-8)\,R^2\cos 2\chi}{128c^4} - \frac{5R^2\cos 4\chi}{128c^4}\right\} + \ldots$$
$$+ \frac{l+1}{2c^2} + \frac{(3l+2)\,R\cos\chi}{8c^3} + \left\{\frac{(9l+3)\,R^2}{64c^4} + \frac{(15l+7)\,R^2\cos 2\chi}{64c^4}\right\} + \ldots$$
$$- \frac{l+1}{c^2} - \frac{(2l+1)\,R\cos\chi}{2c^3} - \left\{\frac{(6l+1)\,R^2}{16c^4} + \frac{(9l+3)\,R^2\cos 2\chi}{16c^4}\right\} + \ldots$$

$$= -\frac{\cos\chi}{cR} - \left(\frac{2l+3}{4c^2} + \frac{\cos 2\chi}{4c^2}\right) - \left\{\frac{(20l+11)\,R\cos\chi}{32c^3} + \frac{3R\cos 3\chi}{32c^3}\right\}$$
$$- \left\{\frac{(12l+5)\,R^2}{128c^4} + \frac{(12l+9)\,R^2\cos 2\chi}{64c^4} + \frac{5R^2\cos 4\chi}{128c^4}\right\} - \&c.$$

From this $d^2\mathrm{I}_1/dc^2$ might be obtained, and then $d^3\mathrm{I}_1/dc^3$, &c., but in the question of an attracting ring there is a special advantage in using functions of the form

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\mathrm{I}_1.$$

Call this I_n. Thus

$$\mathrm{I}_n = \left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^{\pi}\frac{d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}.$$

Also denote

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^{\pi}\frac{\cos\phi\,d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}$$

by J_n.

The values of the functions are collected here for the sake of reference.

(A)

$$\mathrm{I}_1 = \frac{1}{c}\left\{l + 2 + \frac{l+1}{2}\cos\chi\, s + \left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)s^2\right.$$
$$+ \left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)s^3$$
$$\left. + \left(\frac{108l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right)s^4 + \ldots\right\}$$

$$\mathrm{I}_2 = \frac{1}{c^2\mathrm{R}}\left\{\cos\chi + \left(\frac{2l+3}{4} + \frac{\cos 2\chi}{4}\right)s + \left(\frac{12l+11}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2\right.$$
$$\left. + \left(\frac{12l+5}{128} + \frac{12l+9}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3 + \ldots\right\}$$

$$\mathrm{I}_3 = \frac{1}{c^3\mathrm{R}^2}\left\{\cos 2\chi + \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s\right.$$
$$\left. + \left(\frac{36l+51}{32} + \tfrac{3}{4}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2 + \ldots\right\}$$

$$\mathrm{I}_4 = \frac{2!}{c^4\mathrm{R}^3}\left\{\cos 3\chi + \left(\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi\right)s + \ldots\right\}$$

$$\mathrm{I}_5 = \frac{3!}{c^5\mathrm{R}^4}\left\{\cos 4\chi + \ldots\right\}.$$

(B)

$$\frac{d\mathrm{I}_1}{dz} = -\frac{1}{c\mathrm{R}}\left\{\sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(\frac{4l+5}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2\right.$$
$$+ \left(\frac{3l+3}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3$$
$$\left. + \left(\frac{6l+3}{256}\sin\chi + \frac{120l+101}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \ldots\right\}$$

$$\begin{aligned}\frac{d\mathrm{I}_2}{dz} = -\frac{1}{c^2\mathrm{R}^2}\Big\{&\sin 2\chi + (\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi)\,s + (\tfrac{3}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi)\,s^2 \\ &+ \left(\frac{12l+13}{64}\sin\chi + \tfrac{27}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right)s^3 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\frac{d\mathrm{I}_3}{dz} = -\frac{2\,!}{c^3\mathrm{R}^3}\{&\sin 3\chi + (\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi)\,s \\ &+ (\tfrac{15}{16}\sin\chi + \tfrac{15}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi)\,s^2 + \ldots\}\end{aligned}$$

$$\frac{d\mathrm{I}_4}{dz} = -\frac{3\,!}{c^4\mathrm{R}^4}\{\sin 4\chi + (\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi)\,s + \ldots\}$$

$$\frac{d\mathrm{I}_5}{dz} = -\frac{4\,!}{c^5\mathrm{R}^5}\{\sin 5\chi + \ldots\}.$$

(C)

$$\begin{aligned}\mathrm{J}_1 = \frac{1}{c}\Big\{&l + \frac{l-1}{2}\cos\chi\,s + \left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)s^2 \\ &+ \left(\frac{33l+1}{64}\cos\chi\ \frac{15l-23}{192}\cos 3\chi\right)s^3 \\ &+ \left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right)s^4 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{J}_2 = \frac{1}{c^2\mathrm{R}}\Big\{&\cos\chi + \left(\frac{2l-1}{4} + \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{4l+21}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2 \\ &+ \left(\frac{12l+5}{128} - \frac{12l+19}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3 \\ &+ \left(\frac{18l+1}{256}\cos\chi - \frac{35l+32}{256}\cos 3\chi + \tfrac{35}{2048}\cos 5\chi\right)s^4 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{J}_3 = \frac{1}{c^3\mathrm{R}^2}\Big\{&\cos 2\chi + (\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi)\,s \\ &+ \left(\frac{52l-21}{32} + \tfrac{1}{2}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2 \\ &+ \left(-\frac{20l+127}{64}\cos\chi + \tfrac{21}{128}\cos 3\chi + \tfrac{5}{128}\cos 5\chi\right)s^3 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{J}_4 = \frac{2!}{c^4\mathrm{R}^3}\{&\cos 3\chi + (\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi)\,s \\ &+ (\tfrac{81}{16}\cos\chi + \tfrac{21}{32}\cos 3\chi + \tfrac{3}{32}\cos 5\chi)\,s^2 + \ldots\}\end{aligned}$$

$$\mathrm{J}_5 = \frac{3!}{c^5\mathrm{R}^4}\{\cos 4\chi + (\tfrac{35}{12}\cos 3\chi + \tfrac{1}{4}\cos 5\chi)\,s + \ldots\}$$

$$\mathrm{J}_6 = \frac{4!}{e^6\mathrm{R}^5}\{\cos 5\chi + \ldots\}.$$

(D)

$$\begin{aligned}\frac{d\mathrm{J}_1}{dz} = -\frac{1}{c\mathrm{R}}\Big\{&\sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(-\frac{12l+11}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2 \\ &+ \left(-\frac{9l+5}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3 \\ &+ \left(-\frac{30l+9}{256}\sin\chi - \frac{360l+123}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\frac{d\mathrm{J}_2}{dz} = -\frac{1}{c^2\mathrm{R}^2}\Big\{&\sin 2\chi + (\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi)\, s + (\tfrac{1}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi)\, s^2 \\ &+ \left(-\frac{36l+31}{64}\sin\chi + \tfrac{3}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right) s^3 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\frac{d\mathrm{J}_3}{dz} = -\frac{2!}{c^3\mathrm{R}^3}\{&\sin 3\chi + (\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi)\, s \\ &+ (\tfrac{17}{16}\sin\chi + \tfrac{11}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi)\, s^2 + \ldots\}\end{aligned}$$

$$\frac{d\mathrm{J}_4}{dz} = -\frac{3!}{c^4\mathrm{R}^4}\{\sin 4\chi + (\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi)\, s + \ldots\}$$

$$\frac{d\mathrm{J}_5}{dz} = -\frac{4!}{c^5\mathrm{R}^5}\{\sin 5\chi + \ldots\}.$$

The following formulæ are useful for questions involving the stream-line function. They are obtained by multiplying the expansions just given for J_1, J_2, &c., by ϖ or $c(1 - s\cos\chi)$.

(E)

$$\begin{aligned}\mathrm{J}_1\varpi = l - \frac{l+1}{2}\cos\chi\, s &+ \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\chi\right)s^2 + \left(\frac{3l+5}{64}\cos\chi - \frac{3l-1}{192}\cos 3\chi\right)s^3 \\ &+ \left(\frac{12l+11}{2048} + \frac{12l+17}{768}\cos 2\chi - \frac{15l-8}{3072}\cos 4\chi\right)s^4 + \ldots\end{aligned}$$

$$\begin{aligned}\mathrm{J}_2\varpi = \frac{1}{c\mathrm{R}}\Big\{&\cos\chi + \left(\frac{2l-3}{4} - \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{20l+17}{32}\cos\chi - \tfrac{1}{32}\cos 3\chi\right)s^2 \\ &+ \left(\frac{20l+47}{128} - \frac{8l+1}{64}\cos 2\chi - \tfrac{1}{128}\cos 4\chi\right)s^3 \\ &+ \left(\frac{18l+29}{256}\cos\chi - \frac{11l+1}{256}\cos 3\chi - \tfrac{5}{2048}\cos 5\chi\right)s^4 + \ldots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{J}_3\varpi = \frac{1}{c^2\mathrm{R}^2}\Big\{&\cos 2\chi + (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)\, s + \left(\frac{52l-57}{32} - \tfrac{3}{4}\cos 2\chi - \tfrac{1}{32}\cos 4\chi\right)s^2 \\ &+ \left(-\frac{124l+101}{64}\cos\chi - \tfrac{17}{128}\cos 3\chi - \tfrac{1}{128}\cos 5\chi\right)s^3 + \ldots\Big\}\end{aligned}$$

$$J_4\varpi = \frac{2!}{c^3R^3}\{\cos 3\chi + (2\cos 2\chi - \tfrac{1}{4}\cos 4\chi)\,s + (\tfrac{61}{16}\cos\chi - \tfrac{23}{32}\cos 3\chi - \tfrac{1}{32}\cos 5\chi)\,s^2 + \ldots\}$$

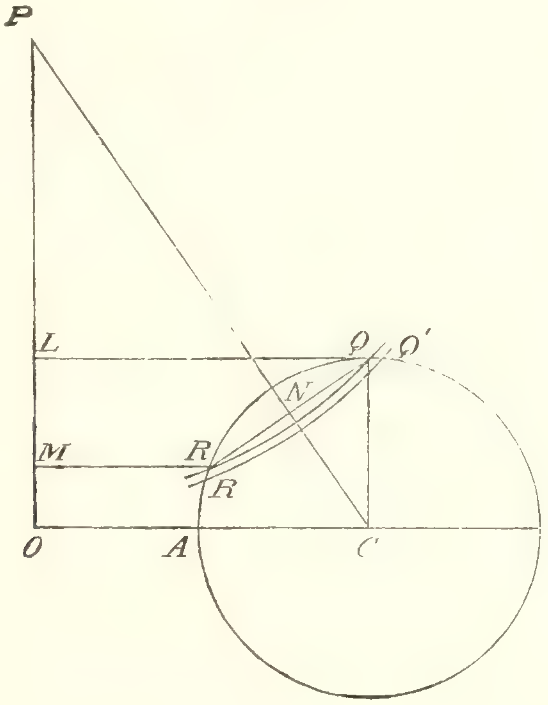
$$J_5\varpi = \frac{3!}{c^4R^4}\{\cos 4\chi + (\tfrac{29}{12}\cos 3\chi - \tfrac{1}{4}\cos 5\chi)\,s + \ldots\}$$

$$J_6\varpi = \frac{4!}{c^5R^5}\{\cos 5\chi + \ldots\}.$$

These five sets of formulæ will be referred to as (A), (B), (C), (D), (E).

Section II.—*The Potential of an Anchor Ring at an External Point.*

§ 4. The potential of an anchor ring at a point on its axis may be easily found in several ways. One simple method is to divide the ring into elements by spheres, having the given point as centre.



Let QAR be a circle which by revolution round OP generates an anchor ring. Let C be its centre, and let OC be perpendicular to OP.

With centre P describe circles, dividing the circle QAR into elements; let QR and Q′R′ be two of these circles.

By revolution of the figure round OP we obtain an anchor ring divided into elements.

Let

$$AC = a. \qquad OC = c. \qquad PC = R. \quad PQ = \rho.$$
$$\angle\, PCQ = \psi \qquad \angle\, OCP = \alpha.$$

Then

$$\rho^2 = \mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi.$$

Therefore

$$\rho\, d\rho = a\mathrm{R}\sin\psi\, d\psi.$$

The volume formed by the revolution of QRR′Q′ is

$$2\pi\rho\, d\rho . \mathrm{LM}.$$

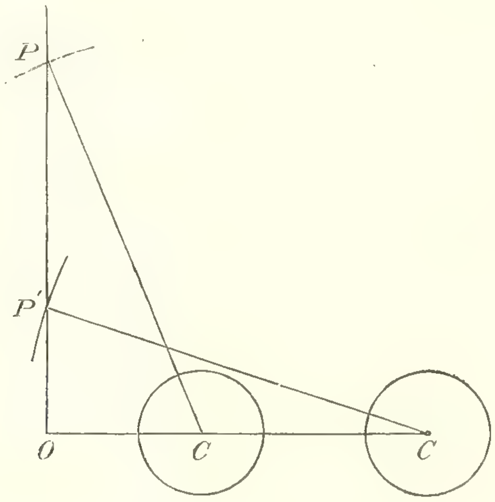
The potential of this at P

$$\begin{aligned} &= \frac{2\pi\rho\, d\rho . \mathrm{LM}}{\rho} \\ &= 2\pi \frac{a\mathrm{R}\sin\psi . d\psi 2a\sin\psi . \cos\alpha}{\sqrt{\{\mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi\}}} \\ &= 4\pi a^2 c \frac{\sin^2 . \psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}} . \end{aligned}$$

Therefore the potential of the whole ring

$$= 2\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}},$$

where M is the mass of the ring.



Let there be two anchor rings of different radii, but of equal generating circles, having the same axis and centre.

Let C and C′ be the centres of the generating circles.

Let P and P′ be points on the axis such that CP = C′P′.

If the densities of the rings be inversely proportional to the distance of the centres of their generating circles from the axis of revolution, that is, if the densities be made such that the masses of the rings are equal, then the above formula shows that the potential of the C ring at P is equal to the potential of the C′ ring at P′.

§ 5. The integral

$$\int_0^{\pi} \frac{\sin^2 \psi \, d\psi}{\sqrt{\{R^2 - 2aR \cos \psi + a^2\}}}$$

may be reduced to elliptic functions.

Let it be called I. Then

$$\begin{aligned} I &= - \int_0^{\pi} \frac{\sin \psi \, d(\cos \psi)}{\sqrt{\{R^2 - 2aR \cos \psi + a^2\}}} \\ &= - \frac{1}{aR} \int_0^{\pi} \cos \psi \sqrt{\{R^2 - 2aR \cos \psi + a^2\}} \, d\psi \\ &= - \frac{1}{aR} \int_0^{\pi} \frac{(R^2 + a^2) \cos \psi - 2aR\,(1 - \sin^2 \psi)}{\sqrt{\{R^2 - 2aR \cos \psi + a^2\}}} \, d\psi. \end{aligned}$$

Therefore

$$3I^* = 2 \int_0^{\pi} \frac{d\psi}{\sqrt{\{R^2 - 2aR \cos \psi + a^2\}}} - \left(\frac{R}{a} + \frac{a}{R}\right) \int_0^{\pi} \frac{\cos \psi \, d\psi}{\sqrt{\{R^2 - 2aR \cos \psi + a^2\}}} .$$

Writing α for a/R,

$$I = \frac{2}{3R} \int_0^{\pi} \frac{d\psi}{\sqrt{\{1 - 2\alpha \cos \psi + \alpha^2\}}} - \frac{1}{3R}\left(\alpha + \frac{1}{\alpha}\right) \int_0^{\pi} \frac{\cos \psi \, d\psi}{\sqrt{\{1 - 2\alpha \cos \psi + \alpha^2\}}} .$$

These integrals may be transformed by putting

$$\sin(\phi - \psi) = \alpha \sin \phi.$$

This gives

$$\begin{aligned} d\psi &= d\phi \left\{1 - \frac{\alpha \cos \phi}{\surd \cos(\phi - \psi)}\right\} \\ &= d\phi \left\{1 - \frac{\alpha \cos \phi}{\sqrt{(1 - \alpha^2 \sin^2 \phi)}}\right\}. \end{aligned}$$

Also

$$\sin \phi \, (\cos \psi - \alpha) = \cos \phi \sin \psi.$$

Therefore

$$\sin^2 \phi \, (\cos \psi - \alpha)^2 = \cos^2 \phi \sin^2 \psi,$$

or

$$\cos^2 \psi - 2\alpha \cos \psi + \alpha^2 = \cot^2 \phi - \cot^2 \phi \cos^2 \psi.$$

Therefore

$$\cos^2 \psi \operatorname{cosec}^2 \phi - 2\alpha \cos \psi + \alpha^2 = \cot^2 \phi,$$

or

$$\begin{aligned} (\cos \psi - \alpha \sin^2 \phi)^2 &= \cos^2 \phi - \alpha^2 \sin^2 \phi + \alpha^2 \sin^4 \phi \\ &= \cos^2 \phi - \alpha^2 \sin^2 \phi \cos^2 \phi. \end{aligned}$$

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* This method of reduction was given by one of the Referees.

Therefore

$$\cos\psi = \alpha\sin^2\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}.$$

Therefore

$$1 - 2\alpha\cos\psi + \alpha^2 = 1 + \alpha^2 - 2\alpha^2\sin^2\phi - 2\alpha\cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}$$
$$= [\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi]^2.$$

Therefore

$$\sqrt{(1-2\alpha\cos\psi+\alpha^2)} = \{\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi\}.$$

Thus

$$\frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}}$$

and

$$\frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{\alpha\sin\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}}{\sqrt{(1-\alpha^2\sin^2\phi)}}\, d\phi,$$

giving

$$\int_0^\pi \frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = 2\mathrm{F}$$

$$\int_0^\pi \frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\alpha\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = \frac{2}{\alpha}(\mathrm{F}-\mathrm{E}).$$

Therefore

$$\int_0 \frac{\sin^2\psi\, d\psi}{\sqrt{\{1-2\alpha\cos\psi+\alpha^2\}}} = \frac{2}{3}\left\{4\mathrm{F} - \frac{2}{\alpha}\left(\alpha+\frac{1}{\alpha}\right)(\mathrm{F}-\mathrm{E})\right\}$$
$$= \frac{4}{3}\left\{\mathrm{F}\left(1-\frac{1}{\alpha^2}\right) + \mathrm{E}\left(1+\frac{1}{\alpha^2}\right)\right\}.$$

Therefore the potential

$$\mathrm{V} = \frac{4\mathrm{M}}{3\pi\mathrm{R}}\left\{\left(1-\frac{1}{\alpha^2}\right)\mathrm{F} + \left(1+\frac{1}{\alpha^2}\right)\mathrm{E}\right\}.$$

Now

$$\mathrm{F} = \frac{\pi}{2}\left\{1 + \frac{1^2}{2^2}\alpha^2 + \frac{1^2.3^2}{2^2.4^2}\alpha^4 + \&c.\right\}$$

and

$$\mathrm{E} = \frac{\pi}{2}\left\{1 - \frac{1.1}{2^2}\alpha^2 - \frac{1^2.3}{2^2.4^2}\alpha^4 - \&c.\right\}.$$

Substituting, we find

$$\mathrm{V} = \frac{\mathrm{M}}{\mathrm{R}}\left\{1 - \frac{1}{8}\alpha^2 - \frac{1}{64}\alpha^4 - \frac{5}{1024}\alpha^6 - \&c. - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\alpha^{2n} - \&c.\right\}$$
$$= \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&c. - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&c.\right\}.$$

This series is less than the series whose general term is

$$\frac{1}{2n^2}\frac{a^{2n}}{\mathrm{R}^{2n+1}},$$

and is therefore convergent if $a = \mathrm{R}$ or $a < \mathrm{R}$.

§ 6. To find the potential of an anchor ring at any external point.

When $a = 0$, that is when the generating circle of the ring is infinitesimal, the potential at the point r, θ is

$$\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}.$$

When $\theta = 0$, and $r = z$, *i.e.*, for a point on the axis, the expression reduces to $\mathrm{M}/\pi\ \pi/\mathrm{R}$, *i.e.*, to M/R.

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

satisfies LAPLACE'S equation, and at a point on the axis of the ring becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2)}} = (-1)^n.\pi\, 1.\ 3.\ 5\ \ldots\ (2n-1)\frac{1}{(z^2+c^2)^{n+\frac{1}{2}}}$$

$$= (-1)^n.\pi\frac{1.\ 3.\ 5\ldots(2n-1)}{\mathrm{R}^{2n+1}}.$$

Now at any point on the axis of the ring

$$\mathrm{V} = \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&\text{c.} - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&\text{c.}\right\}.$$

Therefore, at any external point r, θ,

$$\begin{aligned}\mathrm{V} = \frac{\mathrm{M}}{\pi}\Bigg\{&\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \frac{a^2}{8}\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}\\ &- \frac{a^4}{192}\left(\frac{d}{c\,dc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\text{c.}\\ &+ (-1)^{n+1}\frac{2a^{2n}}{2n+2}\,\frac{1.\ 3.\ 5\ldots(2n-3)}{2^2.4^2.6^2\ldots(2n)^2}\left(\frac{d}{c\,dc}\right)^n\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\text{c.}\Bigg\}.\end{aligned}$$

For this series is finite at all external points, satisfies LAPLACE'S equation, vanishes at infinity, and agrees with the value of V on the axis. The series is very convergent, as is seen by the next paragraph.

§ 7. The integral

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

takes a simple form at points in the plane of the central circle. For, putting $\theta = \pi/2$, it becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}}.$$

Now

$$\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}} = \int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}}$$

when $r > c$, and

$$= \int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}}$$

when $r < c$.

Therefore at points in the plane of x, y, further from the axis than the ring,

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}} + \frac{a^2}{8}\int_0^\pi \frac{\sin^2\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{\frac{3}{2}}} - \&\mathrm{c}.\right.$$

$$\left. + (-1)^n \frac{1^2.3^2 \ldots (2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2 \ldots (2n-2)^2\,(2n)^2\,(2n+2)} \int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} + \&\mathrm{c}.\right\}.$$

And at a point in the plane of xy between the axis and the ring

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}} - \frac{a^2}{8}\int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{\frac{3}{2}}} - \&\mathrm{c}.\right.$$

$$\left. - \frac{1^2.3^2 \ldots (2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2 \ldots (2n-2)^2\,(2n)^2\,(2n+2)} \int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{n+\frac{1}{2}}} - \&\mathrm{c}.\right\}.$$

[[*] The series in its general form must converge at about the same rate at which these two particular cases do. Their convergency is easily discussed. For

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{(r^2 - c^2)^{n+\frac{1}{2}}},$$

and r is $> a + c$.

Thus

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}\,(2c + a)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}\,(2c)^{n+\frac{1}{2}}}.$$

Hence at all points in the plane of the central circle, beyond the ring, the series is more convergent than the series whose general term is

$$(-1)^n \frac{2a^{2n}}{2n\,(2n+2)}\,\frac{1}{2^n a^n c^n},$$

i.e., than the series whose n^{th} term is

$$\frac{1}{n\,(n+1)}\left(\frac{a}{2c}\right)^n.$$

Similarly at points within the ring, in the plane of the central circle, the series is more convergent than the series whose general term is

$$\frac{1}{n\,(n+1)}\left(\frac{a}{2c - a}\right)^n.]$$

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[*] This consideration of the convergency was inserted at the suggestion of one of the Referees, August, 1892.

§ 8. For the purposes of calculation, it is convenient to transform the integrals of § 6 by LANDEN's theorem.

$$\int_0^{\pi} \frac{d\phi}{\sqrt{\{r^2 - 2cr \sin\theta \cos\phi + c^2\}}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{\mathrm{R}_1{}^2 - (\mathrm{R}_1{}^2 - \mathrm{R}^2)\sin^2\phi\}}},$$

$$= \frac{2}{\mathrm{R}_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{\mathrm{R}_1{}^2 - \mathrm{R}^2}{\mathrm{R}_1{}^2}\sin\phi\right\}}},$$

$$= \frac{4\mathrm{F}}{\mathrm{R} + \mathrm{R}_1};$$

where

$$\mathrm{F} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \mu^2 \sin^2\phi}},$$

and

$$\mu = \frac{\mathrm{R}_1 - \mathrm{R}}{\mathrm{R}_1 + \mathrm{R}},$$

R and R_1 being the least and greatest distances of the point r, θ, from the circular axis of the ring.

Now,

$$\left.\begin{aligned} \frac{d\mathrm{F}}{c\,dc} &= \frac{d\mathrm{F}}{d\mu}\frac{d\mu}{c\,dc} = \frac{\mathrm{E} - \mu'^2\mathrm{F}}{\mu\mu'^2}\frac{d\mu}{c\,dc} \\ \text{and} \quad \frac{d\mathrm{E}}{c\,dc} &= \frac{d\mathrm{E}}{d\mu}\frac{d\mu}{c\,dc} = \frac{\mathrm{E} - \mathrm{F}}{\mu}\frac{d\mu}{c\,dc} \end{aligned}\right\}.$$

[CAYLEY, 'Ell. Func.,' p. 48.]

But

$$\mathrm{R}^2 = r^2 + c^2 - 2cr\sin\theta.$$

Therefore

$$\frac{d\mathrm{R}}{dc} = \frac{c - r\sin\theta}{\mathrm{R}} = \frac{4c^2 + \mathrm{R}^2 - \mathrm{R}_1{}^2}{4c\mathrm{R}}.$$

Similarly

$$\frac{d\mathrm{R}_1}{dc} = \frac{c + r\sin\theta}{\mathrm{R}} = \frac{4c^2 + \mathrm{R}_1{}^2 - \mathrm{R}^2}{4c\mathrm{R}}.$$

Therefore

$$\frac{d\mu}{dc} = \frac{(\mathrm{R}_1 + \mathrm{R})\left(\frac{d\mathrm{R}_1}{dc} - \frac{d\mathrm{R}}{dc}\right) - (\mathrm{R}_1 - \mathrm{R})\left(\frac{d\mathrm{R}_1}{dc} + \frac{d\mathrm{R}}{dc}\right)}{(\mathrm{R}_1 + \mathrm{R})^2},$$

$$= 2\frac{\mathrm{R}\frac{d\mathrm{R}_1}{dc} - \mathrm{R}_1\frac{d\mathrm{R}}{dc}}{(\mathrm{R}_1 + \mathrm{R})^2},$$

$$= \frac{\mathrm{R}_1 - \mathrm{R}}{\mathrm{R}_1 + \mathrm{R}}\frac{\mathrm{R}^2 + \mathrm{R}_1{}^2 - 4c^2}{2c\mathrm{R}\mathrm{R}_1},$$

$$= \frac{\mu}{c}\cos\psi;$$

where ψ is the angle between R and R_1.

Hence

$$\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2-2cr\sin\theta\cos\phi+c^2\}}} = 4\,\frac{d}{c\,dc}\left(\frac{\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\right)$$

$$= \frac{1}{c^2}\left\{\mathrm{E}\,\frac{\mathrm{R}+\mathrm{R}_1}{\mathrm{RR}_1}\cos\psi - \frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\cos^2\frac{\psi}{2}\right\}.$$

Differentiating again we shall find

$$\left(\frac{d}{c\,dc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{\{r^2-2cr\sin\theta\cos\phi+c^2\}}}$$

$$= \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{c^4\mathrm{RR}_1}\left\{2\cos^3\psi - 4\cos\psi + \frac{2c^2}{\mathrm{RR}_1}\cos 2\psi\right\}$$

$$+ \frac{\mathrm{F}}{c^4\,(\mathrm{R}+\mathrm{R}_1)}\left\{5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{\mathrm{RR}_1}\cos 2\psi\right\}.$$

Therefore, at any external point,

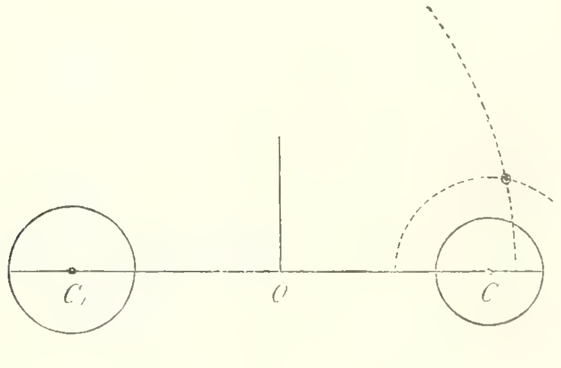
$$\mathrm{V} = \frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2} - \tfrac{1}{768}\frac{a^4}{c^4}\left[5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{\mathrm{RR}_1}\cos 2\psi\right]\right.$$

$$\left. + \&\text{c.}\right\}$$

$$+ \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{\mathrm{RR}_1}\left\{\frac{a^2}{8c^2}\cos\psi - \tfrac{1}{192}\frac{a^4}{c^4}\left[2\cos^2\psi - 4\cos\psi + \frac{2c^2}{\mathrm{RR}_1}\cos 2\psi\right] + \&\text{c.}\right\},$$

where ψ is the angle between R and R_1, and the modulus of the elliptic functions is

$$\frac{\mathrm{R}_1-\mathrm{R}}{\mathrm{R}_1+\mathrm{R}}.$$



I have calculated the value of V, retaining only the terms

$$\frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2}\right\} + \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{\mathrm{RR}_1}\frac{a^2}{8c^2}\cos\psi,$$

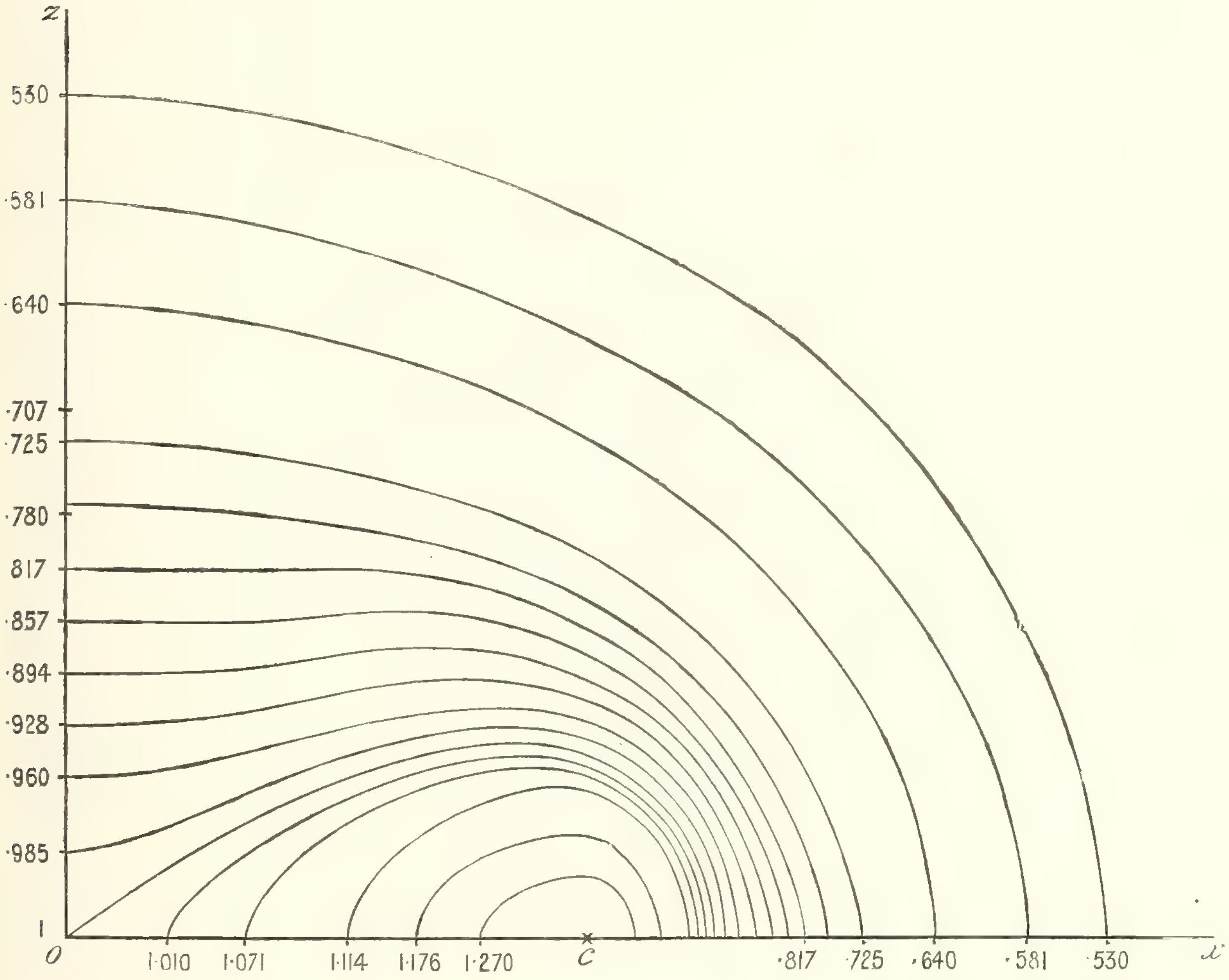
for the values

$$a = 0,\ \frac{c}{5},\ \frac{2c}{5},\ \frac{3c}{5},\ \frac{4c}{5},\ c,$$

and drawn the equipotential surfaces. The method consisted in drawing circles of known radii, with centres, C and C_1; and finding the value of V at their points of intersection.

In the most unfavourable case, where $a = c$, the value of the potential is not more than three per cent. in error at any point, the introduction of the terms in a^4/c^4 reduces the error to less than one and a half per cent. When $a < c$, the approximation is considerably nearer.

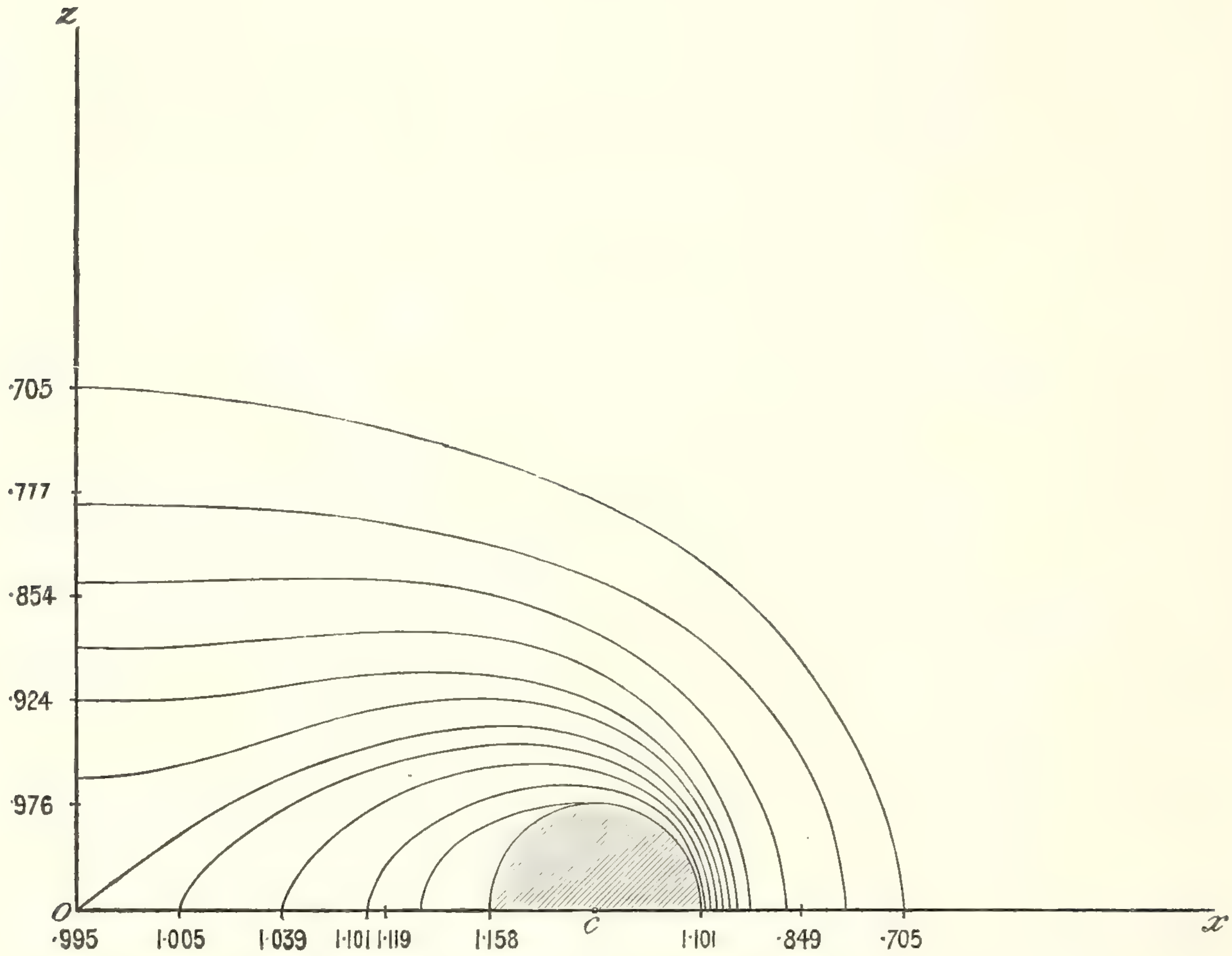
Fig. 1.



Equipotential surfaces of ring formed by the revolution of the point C about Oz.

$\left.\begin{array}{r}\text{If Mass} = \text{M}\\ \text{and OC} = c\end{array}\right\}$ Then $\dfrac{\text{M}}{c}$ is taken $= 1$

Fig. 2.



Section of Equipotential surfaces of the ring formed by revolution of shaded ⊙ round Oz.
Mass = M. OC = c.

and $\frac{M}{c}$ is taken = 1.

Fig. 3.

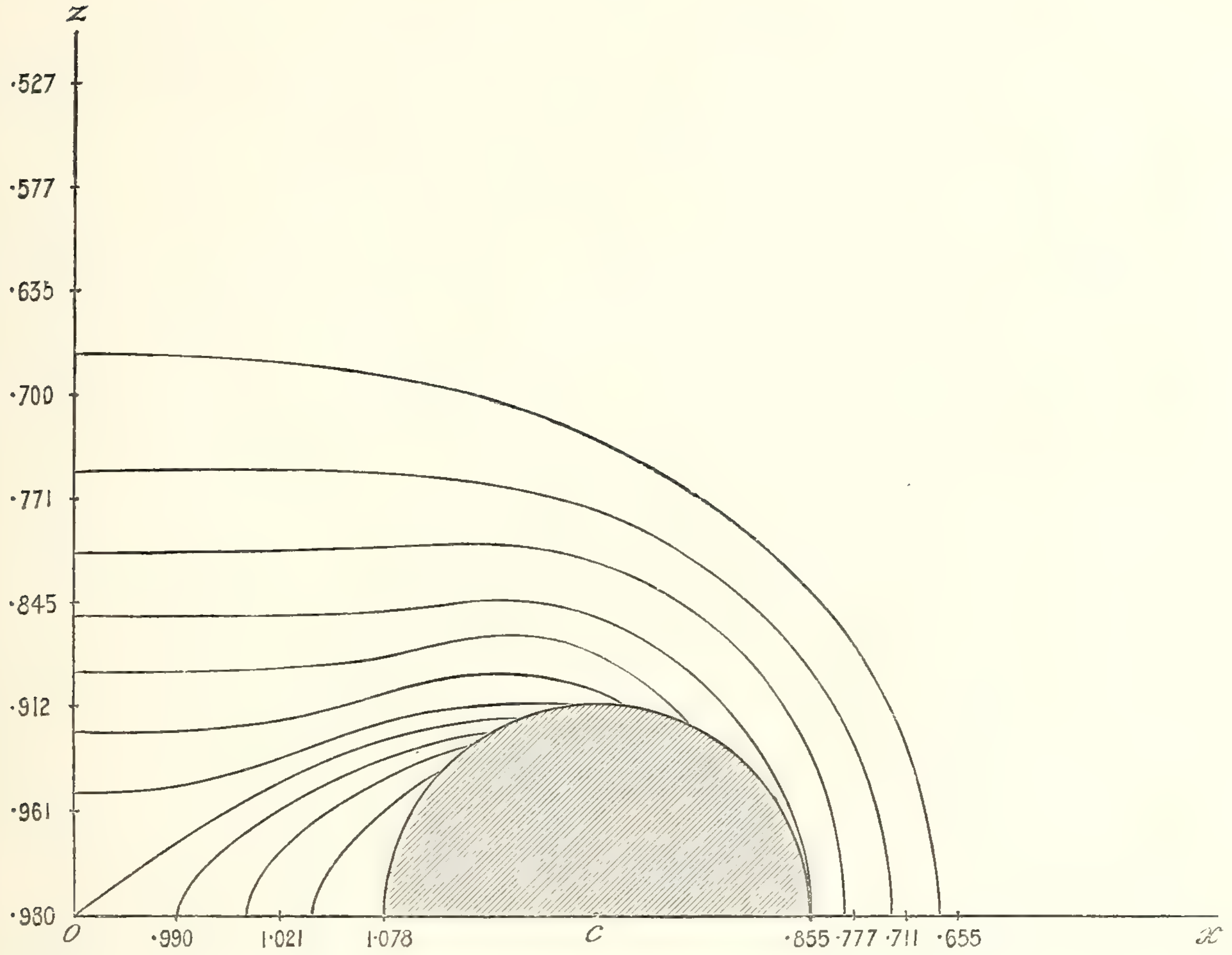

z
·527
·577
·635
·700
·771
·845
·912
·961
·980
O
·990
1·021
1·078
C
·855
·777
·711
·655
x

Fig. 4.

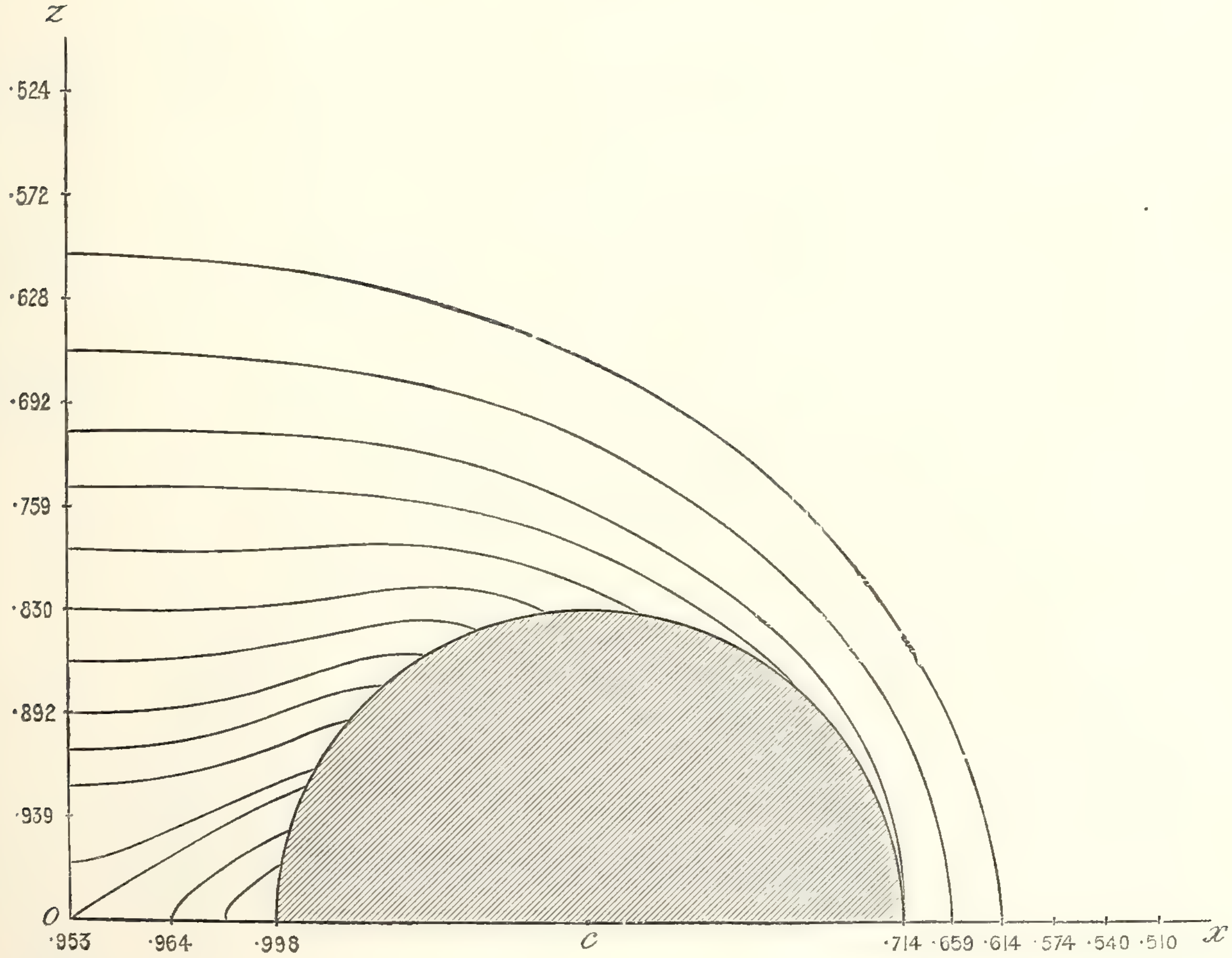

z
·524
·572
·628
·692
·759
·830
·892
·939
O
·953
·964
·998
C
·714
·659
·614
·574
·540
·510
x

Fig. 5.

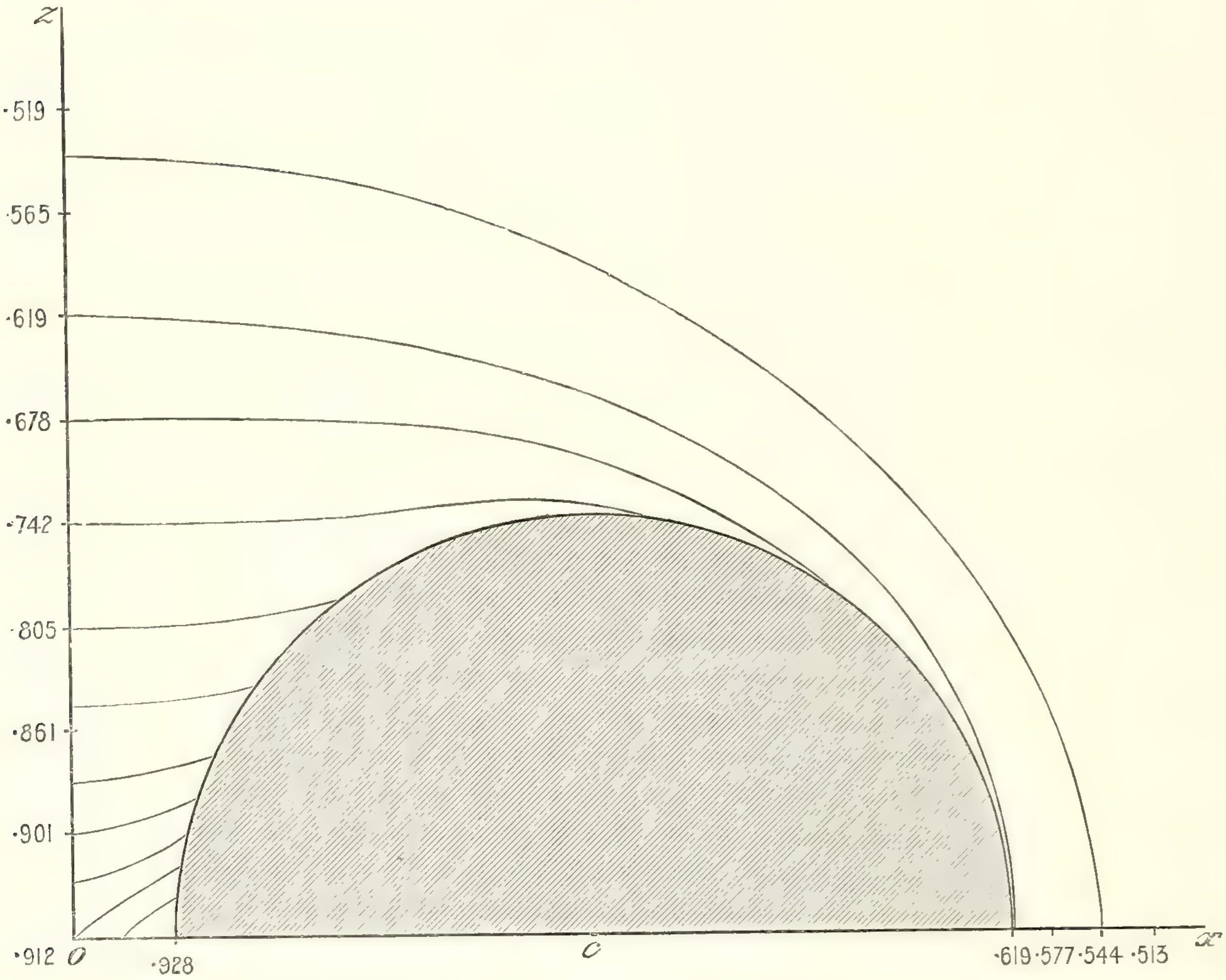
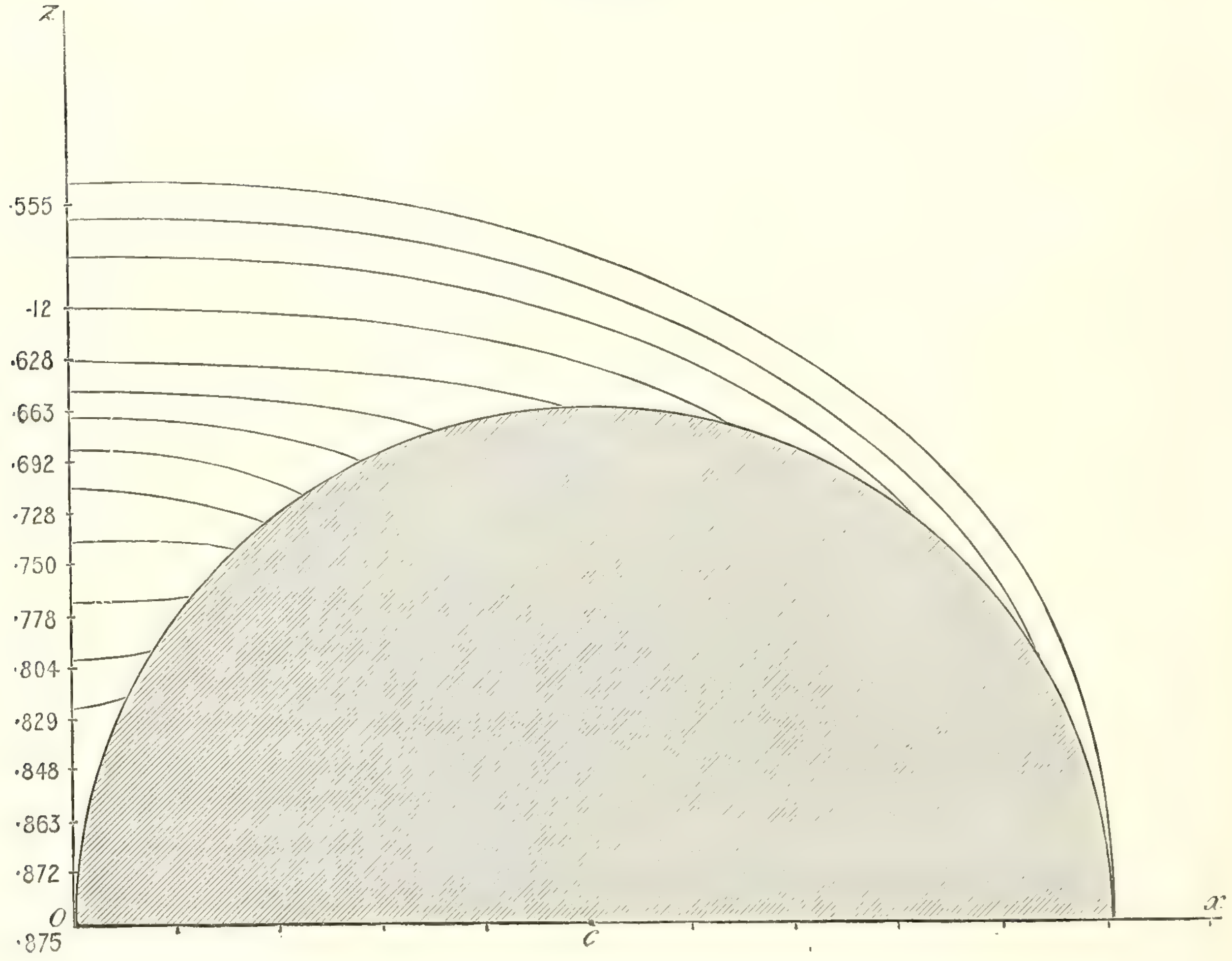

z
·519
·565
·619
·678
·742
·805
·861
·901
·912
O
·928
C
·619
·577
·544
·513
x

Fig. 6.


z
·555
·12
·628
·663
·692
·728
·750
·778
·804
·829
·848
·863
·872
O
·875
C
x

Section III.—*Electric Potential of an Anchor Ring.*

§ 9. The potential at any external point may be taken to be

$$\mathrm{A}_1\mathrm{I}_1 + a^2\mathrm{A}_2\mathrm{I}_2 + a^4\mathrm{A}_3\mathrm{I}_3 + \&c.$$

For this is a solution of LAPLACE's equation, finite at all points outside the ring, and vanishing at infinity.

The constants are to be determined by the condition that at the surface of the ring the potential is the constant V_0.

Using the expansions of § 3 (A.), we have that at any external point

$$\begin{aligned}\mathrm{V}c = {} & \mathrm{A}_1 \left\{ l + 2 + \frac{2l+1}{16} s^2 + \frac{108l+27}{2048} s^4 \right\} + \mathrm{A}_2\sigma^2 \left\{ \frac{2l+3}{4} + \frac{12l+5}{128} s^2 \right\} + \mathrm{A}_3\sigma^4 \frac{36l+51}{32} \\ & + \cos\chi \left\{ \mathrm{A}_1 \left(\frac{l+1}{2} s + \frac{9l+3}{64} s^3 \right) + \mathrm{A}_2 \frac{\sigma^2}{s} \left(1 + \frac{12l+11}{32} s^2 \right) + \mathrm{A}_3\sigma^4 \frac{9}{4s} \right\} \\ & + \cos 2\chi \left\{ \mathrm{A}_1 \left(\frac{3l+2}{16} s^2 + \frac{20l+4\frac{1}{3}}{256} s^4 \right) + \mathrm{A}_2\sigma^2 \left(\tfrac{1}{4} + \frac{12l+9}{64} s^2 \right) \right. \\ & \qquad \left. + \mathrm{A}_3\sigma^4 \left(\frac{1}{s^2} + \tfrac{3}{4} \right) + \mathrm{A}_4\sigma^6 \frac{5}{s^2} \right\} \\ & + \cos 3\chi \left\{ \mathrm{A}_1 \frac{5l+2\frac{1}{3}}{64} s^3 + \mathrm{A}_2\sigma^2 \frac{3s}{32} + \mathrm{A}_3\sigma^4 \frac{1}{4s} + \mathrm{A}_4\sigma^6 \frac{2}{s^3} \right\} \\ & + \cos 4\chi \left\{ \mathrm{A}_1 \frac{35l+11\frac{1}{3}}{1024} \sigma^4 + \mathrm{A}_2\sigma^2 \tfrac{5}{128} s^2 + \mathrm{A}_3\sigma^4 \tfrac{3}{32} + \mathrm{A}_4\sigma^6 \frac{1}{2s^2} + \mathrm{A}_5\sigma^8 \frac{6}{s^4} \right\}.\end{aligned}$$

All terms have been retained which will, when R is put $= a$ or s put $= \sigma$, be of the order σ^4.

$$\frac{d\mathrm{V}}{dn} = \frac{d\mathrm{V}}{d\mathrm{R}} = \frac{d\mathrm{V}}{a\,ds},$$

s being put $= \sigma$, after differentiation.

Therefore

$$\begin{aligned} ac\frac{d\mathrm{V}}{dn} = {} & \left\{ \mathrm{A}_1 \left(-1 + \tfrac{\lambda}{4}\sigma^2 + \frac{27\lambda}{128}\sigma^4 \right) + \mathrm{A}_2 \left(-\frac{\sigma^2}{2} + \frac{12\lambda-1}{64}\sigma^4 \right) - \tfrac{9}{8}\mathrm{A}_3\sigma^4 \right\} \\ & + \sigma\cos\chi \left\{ \mathrm{A}_1 \left(\frac{\lambda}{2} + \frac{27\lambda}{64}\sigma^2 \right) + \mathrm{A}_2 \left(-1 + \frac{12\lambda-1}{32}\sigma^2 \right) - \tfrac{9}{4}\mathrm{A}_3\sigma^2 \right\} \\ & + \sigma^2\cos 2\chi \{ \mathrm{A}_1 \frac{6\lambda+1}{16} + \frac{10\lambda-\frac{1}{3}}{32}\sigma^2) + \mathrm{A}_2 \frac{12\lambda+3}{32}\sigma^2 - 2\mathrm{A}_3 - 10\mathrm{A}_4\sigma^2 \} \\ & + \sigma^3\cos 3\chi \left\{ \mathrm{A}_1 \frac{15\lambda+2}{64} + \mathrm{A}_2 \tfrac{3}{32} - \mathrm{A}_3 \tfrac{1}{4} - \mathrm{A}_4\, 6 \right\} \\ & + \sigma^4\cos 4\chi \left\{ \mathrm{A}_1 \frac{140\lambda+10\frac{1}{3}}{1024} + \mathrm{A}_2 \tfrac{5}{64} - \mathrm{A}_4 - 24\mathrm{A}_5 \right\}.\end{aligned}$$

The density at any point is $-\dfrac{1}{4\pi}\dfrac{dV}{dn}$.

The charge

$$\begin{aligned}
&= -\frac{1}{4\pi}\iint \frac{dV}{dn}\,ds \\
&= -\int_0^{2\pi}\int_0^{2\pi}\frac{1}{4\pi}\frac{dV}{dn}(c - a\cos\chi)\,a\,d\chi\,d\phi \\
&= -\frac{ac}{2}\int_0^{2\pi}\frac{dV}{dn}(1 - \sigma\cos\chi)\,d\chi \\
&= \pi A_1.
\end{aligned}$$

This value is evidently correct, for the potential at a great distance is A_1J_1 or

$$A_1\int_0^{\pi}\frac{d\phi}{\sqrt{r^2 + c^2 - 2cr\sin\theta\cos\phi}} = \frac{\pi A_1}{r}.$$

Putting $s = \sigma$ in the expression for V, we find

$$\begin{aligned}
A_2 &= -A_1\left(\frac{\lambda + 1}{2} - \frac{12\lambda^2 + 23\lambda + 8}{64}\sigma^2\right) \\
A_3 &= -A_1\left(\frac{\lambda}{16} - \frac{12\lambda^2 + 21\lambda - 1}{256}\sigma^2\right) \\
A_4 &= -A_1\frac{\lambda - \frac{2}{3}}{128} \\
A_5 &= -A_1\frac{5\lambda - 6}{6144}.
\end{aligned}$$

Substituting these values we find that

$$V_0 = \frac{A_1}{c}\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4, \text{\&c.}\right\}$$

and

$$-\frac{1}{4\pi}\frac{dV}{dn} = \frac{A_1}{4\pi ac}\left\{1 - \left(\frac{2\lambda + 1}{2} - \frac{24\lambda^2 + 7}{64}\sigma^2\right)\left(\frac{\sigma^2}{2} + \sigma\cos\chi\right)\right.$$
$$\left. - \left(\frac{8\lambda - 1}{16} - \frac{36\lambda^2 + \lambda + 13}{128}\sigma^2\right)\sigma^2\cos 2\chi - \frac{16\lambda - 3}{64}\sigma^3\cos 3\chi - \frac{11(8\lambda - 1)}{1024}\sigma^4\cos 4\chi\right\}.$$

[The capacity (q)

$$\begin{aligned}
&= \pi A_1/V \\
&= \pi c\Big/\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4\right\}.
\end{aligned}$$

When

$$\sigma = 1 \qquad \lambda = 2{\cdot}3820 \qquad \frac{\pi c}{q} = 4{\cdot}3820 - {\cdot}0286 + {\cdot}0003$$

$$\sigma = 2 \qquad \lambda = 1{\cdot}6889 \qquad \frac{\pi c}{q} = 3{\cdot}6889 - {\cdot}0798 + {\cdot}0022$$

$$\sigma = 3 \qquad \lambda = 1{\cdot}2834 \qquad \frac{\pi c}{q} = 3{\cdot}2834 - {\cdot}1229 + {\cdot}0073$$

$$\sigma = 4 \qquad \lambda = {\cdot}9957 \qquad \frac{\pi c}{q} = 2{\cdot}9957 - {\cdot}1691 + {\cdot}0152$$

$$\sigma = 5 \qquad \lambda = {\cdot}7726 \qquad \frac{\pi c}{q} = 2{\cdot}7726 - {\cdot}1818 + {\cdot}0260$$

The numbers are given which arise from each term, as they serve to indicate roughly the convergency of the series.

They give

$$\begin{aligned} q &= {\cdot}7216c && \text{for } \sigma = 1, \\ q &= {\cdot}8700c && \text{for } \sigma = 2, \\ q &= {\cdot}9917c && \text{for } \sigma = 3, \\ q &= 1{\cdot}106c && \text{for } \sigma = 4, \\ q &= 1{\cdot}200c && \text{for } \sigma = 5, \end{aligned}$$

which agree with some figures given by Mr. HICKS in the 'Phil. Trans.,' 1881. Aug., 1892.]

Section IV.—*Motion of an Anchor Ring in an Infinite Fluid.*

§ 10. All cases of motion of an anchor ring in a fluid may be found by compounding

(i.) Linear motion parallel to the axis of the ring.
(ii.) Linear motion perpendicular to the axis of the ring.
(iii.) Rotation round a diameter of the central circle.
(iv.) Cyclic motion through the ring.

Let Φ be the velocity potential in cases (i.), (ii.), or (iii.), and Ψ be STOKES' stream-line function in cases (i.) or (iv.).

Then (*a*) Φ and Ψ are single-valued functions;

(*b*) They and their differential coefficients are finite and continuous at all points of the fluid, and vanish at infinity;

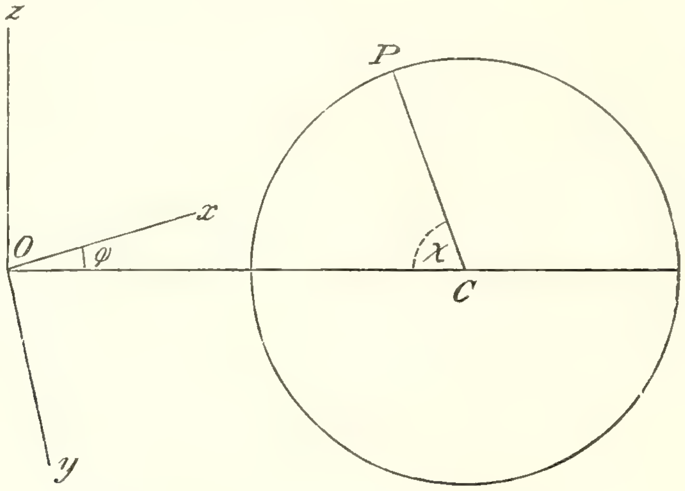
(*c*) Φ satisfies the equation $\nabla^2\Phi = 0$, and Ψ satisfies the equation

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0\,;$$

(*d*) $d\Phi/dn$ has a definite value at the surface of the ring;
Ψ has a definite value at the surface of the ring.

The different cases must be considered separately.

Let the figure represent a section of the anchor ring through the axis Oz, at an azimuth ϕ.



Let P be a point on the surface and let $\angle$ OCP $= \chi$.

(i.) When the ring moves parallel to Oz, with velocity w;

$$\frac{d\Phi}{dn} = w \sin \chi.$$

(ii.) When it moves parallel to Ox with velocity u,

$$\frac{d\Phi}{dn} = - u \cos \chi \cos \phi.$$

(iii.) When it rotates round Oy with angular velocity ω_2,

$$\frac{d\Phi}{dn} = - \mathrm{C}\omega_2 \sin \chi \cos \phi.$$

The stream function for motion parallel to Oz, with velocity w, satisfies at the surface the equation

$$\text{(i.)} \quad \Psi = \mathrm{C} + \tfrac{1}{2} w \, (c - a \cos \chi)^2.$$

For the cyclic motion,

$$\text{(iv.)} \quad \Psi = \mathrm{C}' \text{ at the surface of the ring.}$$

The formulæ of Section I. show that we may take

$$\Phi = \left\{ \mathrm{A}_1 \frac{d\mathrm{I}_1}{dz} + \mathrm{A}_2 a^2 \frac{d\mathrm{I}_2}{dz} + \mathrm{A}_3 a^4 \frac{d\mathrm{I}_3}{dz} + \&\text{c.} \right\} \text{ in case (i.)}$$

$$\Phi = \{ \mathrm{A}_1 \mathrm{J}_1 + \mathrm{A}_2 a^2 \mathrm{J}_2 + \mathrm{A}_3 a^4 \mathrm{J}_3^2 + \&\text{c.} \} \cos \phi \text{ in case (ii.)}$$

$$\Phi = \left\{ \mathrm{A}_1 \frac{d\mathrm{J}_1}{dz} + \mathrm{A}_2 a^2 \frac{d\mathrm{J}_2}{dz} + \mathrm{A}_3 a^4 \frac{d\mathrm{J}_3}{dz} + \&\text{c.} \right\} \cos \phi \text{ in case (iii.),}$$

and that we may also take

$$\Psi = r \sin\theta \{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3 + \&c.\} \text{ in cases (i.) and (iv.).}$$

A_1, A_2, &c., are constants which must be found separately in each case.

§ 11. *Motion Parallel to the Axis of the Ring.*

Let the ring move with velocity w parallel to its axis.

The value of the normal velocity is $w \sin\chi$ at the surface of the ring.

At the surface of the ring $R = a$.

Let $\sigma = a/c$ = value of s or R/c at the surface of the ring.

Let $\lambda = \log 8c/a - 2$ = value of l or $\log 8c/R - 2$ at the surface of the ring.

Let us assume that

$$\Phi = A_1 \frac{dI_1}{dz} + A_2 a^2 \frac{dI_2}{dz} + A_3 a^4 \frac{dI_3}{dz} + \&c.$$

Then A_1, A_2, &c., are to be chosen, so that

$$\frac{d\phi}{dn} = w \sin\chi,$$

i.e.,

$$\frac{d\phi}{ds} = cw \sin\chi.$$

Differentiating the formulæ (B) of § 3, we find

$$\frac{d}{ds}\left(\frac{dI_1}{dz}\right) = \frac{1}{c^2 s^2}\left\{\sin\chi - \left(\frac{4l+1}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2 - \left(\frac{6l+3}{32}\sin 2\chi + \tfrac{5}{64}\sin 4\chi\right)s^3 \right.$$
$$\left. - \left(\frac{18l+3}{256}\sin\chi + \frac{360l+183}{2048}\sin 3\chi + \tfrac{105}{2048}\sin 5\chi\right)s^4 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_2}{dz}\right) = \frac{1}{c^4 s^3}\left\{2\sin 2\chi + (\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi)s \right.$$
$$\left. - \left(\frac{12l+1}{64}\sin\chi + \tfrac{27}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right)s^3 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_3}{dz}\right) = \frac{2!}{c^6 s^4}\left\{3\sin 3\chi + (\tfrac{5}{2}\sin 2\chi + \tfrac{1}{2}\sin 4\chi)s \right.$$
$$\left. + (\tfrac{15}{16}\sin\chi + \tfrac{15}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi)s^2 + \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_4}{dz}\right) = \frac{3!}{c^8 s^5}\left\{4\sin 4\chi + (\tfrac{21}{4}\sin 3\chi + \tfrac{3}{4}\sin 5\chi)s + \ldots\right\}$$

$$\frac{d}{ds}\left(\frac{dI_5}{dz}\right) = \frac{4!}{c^{10} s^6} 5\sin 5\chi.$$

At the surface of the ring $s = \sigma = a/c$: and we have the following equations to determine the constants:—

$$\left.\begin{aligned}
&\mathrm{A}_1\left(1 - \frac{4\lambda + 1}{32}\sigma^2 - \frac{18\lambda + 3}{256}\sigma^4\right) + \mathrm{A}_2\left(\tfrac{3}{4}\sigma^2 - \frac{12\lambda + 1}{64}\sigma^4\right) + \mathrm{A}_3\,\tfrac{15}{8}\sigma^4 = a^2cw \\
&\mathrm{A}_1\left(-\frac{6\lambda + 3}{32}\sigma^2\right) + 2\mathrm{A}_2 + \mathrm{A}_3\,5\sigma^2 = 0 \\
&\mathrm{A}_1\left(-\tfrac{3}{32} - \frac{360\lambda + 183}{2048}\sigma^2\right) + \mathrm{A}_2\left(\tfrac{1}{4} - \tfrac{27}{128}\sigma^2\right) + \mathrm{A}_3\left(6 + \tfrac{15}{16}\sigma^2\right) + \mathrm{A}_4\,\tfrac{63}{2}\sigma^2 = 0 \\
&\quad -\tfrac{5}{64}\mathrm{A}_1 + \mathrm{A}_3 + 24\mathrm{A}_4 = 0 \\
&\quad -\tfrac{105}{2048}\mathrm{A}_1 - \tfrac{5}{128}\mathrm{A}_2 + \tfrac{3}{16}\mathrm{A}_3 + \tfrac{9}{2}\mathrm{A}_4 + 120\mathrm{A}_5 = 0
\end{aligned}\right\}$$

These equations give

$$\mathrm{A}_1 = \left\{1 + \frac{4\lambda + 1}{32}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{1024}\sigma^4\right\}a^2cw.$$

$$\mathrm{A}_2 = \frac{12\lambda + 1}{128}\sigma^2a^2cw.$$

$$\mathrm{A}_3 = \frac{1}{64}\left(1 + \frac{360\lambda + 57\frac{2}{3}}{64}\sigma^2\right)a^2cw.$$

$$\mathrm{A}_4 = \frac{1}{384}a^2cw.$$

$$\mathrm{A}_5 = \frac{5}{2^{14}}a^2cw.$$

Thus

$$\Phi = a^2cw\left\{\left(1 + \frac{4\lambda + 1}{32}\sigma^2 + \ldots\right)\frac{d\mathrm{I}_1}{dz} + \frac{12\lambda + 1}{128}\sigma^2a^2\frac{d\mathrm{I}_2}{dz} + \ldots\right\}.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2\mathrm{T} &= -\rho\iint\Phi\frac{d\Phi}{dn}d\mathrm{S} \\
&= -\rho w\,2\pi\int_0^{2\pi}(c - a\cos\chi)\sin\chi\,\Phi\,ad\chi \\
&= -2\pi\rho acw\int_0^{2\pi}\left(\sin\chi - \frac{\sigma}{2}\sin 2\chi\right)\Phi\,d\chi \\
&= 2\pi^2\rho\frac{ac}{c^2\sigma}w\left\{\mathrm{A}_1\left(1 + \frac{4\lambda + 1}{32}\sigma^2 - \frac{6\lambda + 9}{256}\sigma^2\right) + \mathrm{A}_2\sigma^2\left(\tfrac{1}{4} + \frac{12\lambda + 1}{64}\sigma^2\right) + \mathrm{A}_3\sigma^4\tfrac{5}{8}\right\} \\
&= \mathrm{M}w^2\left\{1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4 + \ldots\right\},
\end{aligned}$$

where M is the mass of the fluid displaced.

§ 12. *Velocity Potential for a Ring moving at Right Angles to its Axis.*—The velocity potential Φ is of the form

$$\{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3 + \ldots\} \cos\phi.$$

The constants are to be determined from the condition that the normal velocity at the surface of the ring is $-u\cos\chi\cos\phi$.

Thus

$$\frac{d\Phi}{dR} = -u\cos\chi\cos\phi$$

$$\frac{d\Phi}{ds} = -cu\cos\chi\cos\phi.$$

Now the formulæ (C) of § 3 give on differentiation

$$\begin{aligned}\frac{dJ_1}{ds} = \frac{1}{cs}\Big\{ & -1 + \frac{l-2}{2}s\cos\chi + \left(\frac{3l-1}{4} + \frac{6l-11}{16}\cos 2\chi\right)s^2 \\ & + \left(\frac{99l-30}{64}\cos\chi + \frac{15l-28}{64}\cos 3\chi\right)s^3 \\ & + \left(\frac{135l-27}{128} + \frac{240l-83}{192}\cos 2\chi + \frac{420l-809}{3072}\cos 4\chi\right)s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_2}{ds} = \frac{1}{c^3s^2}\Big\{ & -\cos\chi - \frac{s}{2} - \left(\frac{4l+}{32}\cos\chi - \tfrac{3}{32}\cos 3\chi\right)s^2 \\ & + \left(\frac{12l-1}{64} - \frac{12l+13}{32}\cos 2\chi + \tfrac{5}{64}\cos 4\chi\right)s^3 \\ & + \left(\frac{54l-15}{256}\cos\chi - \frac{105l+61}{256}\cos 3\chi + \tfrac{105}{2048}\cos 5\chi\right)s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_3}{ds} = \frac{1}{c^5s^3}\Big\{ & -2\cos 2\chi - \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s - \tfrac{13}{8}s^2 \\ & + \left(-\frac{20l+107}{64}\cos\chi + \tfrac{21}{128}\cos 3\chi + \tfrac{5}{128}\cos 5\chi\right)s^3 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_4}{ds} = \frac{1}{c^7s^4}\{ & -6\cos 3\chi - (10\cos 2\chi + \cos 4\chi)s \\ & - (\tfrac{81}{8}\cos\chi + \tfrac{21}{16}\cos 3\chi + \tfrac{3}{16}\cos 5\chi) + \ldots\}.\end{aligned}$$

$$\frac{dJ_5}{ds} = \frac{1}{c^9s^5}\{-24\cos 4\chi - (\tfrac{105}{2}\cos 3\chi + \tfrac{9}{2}\cos 5\chi)s - \ldots\}.$$

$$\frac{dJ_6}{ds} = \frac{1}{c^{11}s^6}\{-120\cos 5\chi - \ldots\}.$$

Therefore, the equations giving A_1, A_2, &c., are

$$\left.\begin{aligned}
&A_1\left(-1+\frac{3\lambda-1}{4}\sigma^2+\frac{135\lambda-27}{128}\sigma^4\right)+A_2\sigma^2\left(-\tfrac{1}{2}+\frac{12\lambda-1}{64}\sigma^2\right)-A_3\tfrac{13}{8}\sigma^4=0.\\
&A_1\left(\frac{\lambda-2}{2}+\frac{99\lambda-30}{64}\sigma^2\right)+A_2\left(-1-\frac{4\lambda+17}{32}\sigma^2+\frac{54\lambda-15}{256}\sigma^4\right)\\
&\qquad\qquad+A_3\sigma^2\left(-\tfrac{9}{4}-\frac{20\lambda+107}{64}\sigma^2\right)-\tfrac{81}{8}A_4\sigma^4=-c^2u.\\
&A_1\left(\frac{6\lambda-11}{16}+\frac{240\lambda-83}{192}\sigma^2\right)+A_2\sigma^2\left(-\frac{12\lambda+13}{32}\right)-2A_3-10\sigma^2A_4=0.\\
&A_1\frac{15\lambda-28}{64}+A_2\left(+\tfrac{3}{32}-\frac{105\lambda+61}{256}\sigma^2\right)+A_3\left(-\tfrac{1}{4}+\tfrac{21}{128}\sigma^2\right)\\
&\qquad\qquad+A_4\left(-6-\tfrac{21}{16}\sigma^2\right)-\tfrac{105}{2}\sigma^2A_5=0.\\
&A_1\frac{420\lambda-809}{3072}+\tfrac{5}{64}A_2-A_4-24A_5=0,\\
&\qquad\tfrac{105}{2048}A_2+\tfrac{5}{128}A_3-\tfrac{3}{16}A_4-\tfrac{9}{2}A_5-120A_6=0.
\end{aligned}\right\}$$

These equations give

$$A_1=-\frac{\sigma^2}{2}\left(1-\frac{\sigma^2}{4}\right)c^2u.$$

$$A_2=\left(1-\frac{12\lambda+1}{32}\sigma^2+\frac{48\lambda^2+344\lambda+159}{1024}\sigma^4\right)c^2u.$$

$$A_3=-\frac{18\lambda+7}{64}\sigma^2c^2u.$$

$$A_4=\left(\frac{1}{64}-\frac{24\lambda+25}{384}\sigma^2\right)c^2u.$$

$$A_5=\frac{1}{384}c^2u.$$

$$A_6=\frac{5}{2^{14}}c^2u.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T&=-\iint\Phi\frac{d\Phi}{dn}\,dS\\
&=\int_0^{2\pi}\int_0^{2\pi}\{A_1J_1+A_2a^2J+\ldots\}\cos\phi\,u\cos\chi\cos\phi\,(c-a\cos\chi)\,a\,d\chi\,d\phi\\
&=\pi acu\int_0^{2\pi}\{A_1J_1+\ldots\}\left\{\cos\chi-\frac{\sigma}{2}-\frac{\sigma}{2}\cos 2\chi\right\}d\chi.
\end{aligned}$$

This is evaluated by picking out the coefficients of cos χ, &c., in the expansions of J_1, J_2, &c., given in § 3 (C), and

$$= \pi^2 au \left\{ A_1 \left(- \frac{\lambda + 1}{2} \sigma + \frac{3\lambda + 5}{64} \sigma^3 \right) + A_2 \left(\sigma - \frac{20\lambda + 17}{32} \sigma^3 + \frac{18\lambda + 29}{256} \sigma^5 \right) \right.$$

$$\left. + A_3 \tfrac{7}{4} \sigma^3 + A_4 \tfrac{61}{8} \sigma^5 \right\}$$

$$= \frac{Mu^2}{2} \left\{ 1 - \frac{12\lambda + 5}{16} \sigma^2 + \frac{144\lambda^2 + 24\lambda + 57}{512} \sigma^4 \right\}.$$

§ 13. *Ring rotating about a Line through its Centre in the Plane of its Central Circle.*—The velocity potential Φ may be taken to be of the form

$$\left\{ A_1 \frac{dJ_1}{dz} + A_2 a^2 \frac{dJ_2}{dz} + \ldots \right\} \cos \phi.$$

The constants A_1, A_2, &c., are determined by the boundary condition

$$\frac{d\Phi}{dR} = - c\omega_2 \sin \chi \cos \phi,$$

or

$$\frac{d\Phi}{ds} = - c^2 \omega_2 \sin \chi \cos \phi.$$

Differentiating the formulæ (D) of § 3,

$$\frac{d}{ds} \left(\frac{dJ_1}{dz} \right) = \frac{1}{c^2 s^2} \left\{ \sin \chi + \left(\frac{12l - 1}{32} \sin \chi - \tfrac{3}{32} \sin 3\chi \right) s^2 \right.$$

$$+ \left(\frac{18l + 1}{32} \sin 2\chi - \tfrac{5}{64} \sin 4\chi \right) s^3$$

$$\left. + \left(\frac{90l - 3}{256} \sin \chi + \frac{1080l + 9}{2048} \sin 3\chi - \tfrac{105}{2048} \sin 5\chi \right) s^4 \right\}$$

$$\frac{d}{ds} \left(\frac{dJ_2}{dz} \right) = \frac{1}{c^3 s^3} \left\{ 2 \sin 2\chi + \left(\tfrac{3}{4} \sin \chi + \tfrac{1}{4} \sin 3\chi \right) s \right.$$

$$\left. + \left(\frac{35l - 5}{64} \sin \chi - \tfrac{3}{128} \sin 3\chi - \tfrac{5}{128} \sin 5\chi \right) s^3 + \ldots \right\}$$

$$\frac{d}{ds} \left(\frac{dJ_3}{dz} \right) = \frac{1}{c^4 s^4} \{ 6 \sin 3\chi + (5 \sin 2\chi + \sin 4\chi) s$$

$$+ \left(\tfrac{17}{8} \sin \chi + \tfrac{11}{16} \sin 3\chi + \tfrac{3}{16} \sin 5\chi \right) s^2 + \ldots \}$$

$$\frac{d}{ds} \left(\frac{dJ_4}{dz} \right) = \frac{1}{c^5 s^5} \{ 24 \sin 4\chi + \left(\tfrac{63}{2} \sin 3\chi + \tfrac{9}{2} \sin 5\chi \right) s + \ldots \}$$

$$\frac{d}{ds} \left(\frac{dJ_5}{dz} \right) = \frac{1}{c^6 s^6} 120 \sin 5\chi + \ldots$$

The equations to determine A_1, A_2, &c., are, therefore,

$$\left.\begin{aligned}
&A_1\left(1+\frac{12\lambda-1}{32}\sigma^2+\frac{90\lambda-3}{256}\sigma^4\right)+A_2\sigma^2\left(\tfrac{3}{4}+\frac{36\lambda-5}{64}\sigma^2\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+A_3\sigma^4\,\tfrac{17}{8}=-a^2c^2\omega_2\\
&A_1\,\frac{18\lambda+1}{32}\sigma^2+2A_2+5A_3\sigma^2=0\\
&A_1\left(-\tfrac{3}{32}+\frac{1080\lambda+9}{2048}\sigma^2\right)+A_2\left(\tfrac{1}{4}-\tfrac{3}{128}\sigma^2\right)+A_3\left(6+\tfrac{11}{16}\sigma^2\right)+A_4\,\tfrac{63}{2}\sigma^2=0\\
&-\tfrac{5}{64}A_1+A_3+24A_4=0\\
&-\tfrac{105}{2048}A_1-\tfrac{5}{128}A_2+\tfrac{3}{16}A_3+\tfrac{9}{2}A_4+120\,A_5=0
\end{aligned}\right\}$$

These equations give

$$A_1=-\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)a^2c^2\omega_2\,;$$

$$A_2=+\frac{36\lambda+7}{128}\sigma^2.\,a^2c^2\omega_2\,;$$

$$A_3=-\left(\tfrac{1}{64}-\frac{42\lambda+7\frac{1}{24}}{512}\sigma^2\right)a^2c^2\omega_2\,;$$

$$A_4=-\tfrac{1}{384}\,a^2c^2\omega_2\,;$$

$$A_5=-\frac{5}{2^{14}}\,a^2c^2\omega_2.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T&=-\iint\Phi\,\frac{d\Phi}{dn}\,dS\\
&=\int_0^{2\pi}\!\!\int_0^{2\pi}c\omega_2\sin\chi\,(c-a\cos\chi)\left(A_1\frac{dJ_1}{dz}+A_2a^2\frac{dJ_2}{dz}+\ldots\right)\cos^2\phi\,a\,d\chi\\
&=\pi ac^2\omega_2\int_0^{2\pi}\left(\sin\chi-\frac{\sigma}{2}\sin\chi\right)\left(A_1\frac{dJ_1}{dz}+\ldots\right)d\chi\\
&=\pi^2a^2c^3\omega_2{}^2\left\{\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)\left(1-\frac{12\lambda+15}{32}\sigma^2+\frac{6\lambda+11}{256}\sigma^4\right)\right.\\
&\qquad\qquad\left.-\frac{36\lambda+7}{128}\sigma^2\,\frac{\sigma^2}{4}+\tfrac{1}{64}\cdot\tfrac{7}{8}\,\sigma^4\right\}\\
&=M\frac{c^2\omega_2{}^2}{2}\left\{1-\frac{12\lambda+7}{16}\sigma^2+\frac{18\lambda^2-3\lambda+5}{64}\sigma^4-\ldots\right\}.
\end{aligned}$$

§ 14.—*Stream Line Function for a Ring moving parallel to its Axis, and for the Cyclic Motion through the Ring.*—Let the velocity of the ring be w, and the circulation be κ.

Let ψ be the stream-line function.

Then

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0$$

at all points of the fluid.

At the surface of the ring

$$\psi - \tfrac{1}{2}w\varpi^2 = \text{const.},$$

or

$$\psi = \mathrm{A}\kappa + wc^2\left(\mathrm{B} - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right).$$

We may therefore assume ψ to be of the form

$$\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2a^2\mathrm{J}_2\varpi + \mathrm{A}_3a^4\mathrm{J}_3\varpi + \ldots\}.$$

Using the values of $\mathrm{J}_1\varpi$, $\mathrm{J}_2\varpi$, &c., given in § 4 (E),

The equations giving the constants A_1, A_2, &c., are

$$\left.\begin{aligned}
&\mathrm{A}_1\left(\lambda + \frac{2\lambda+5}{16}\sigma^2 + \frac{12\lambda+11}{2048}\sigma^4\right) + \mathrm{A}_2\sigma^2\left(\frac{2\lambda-3}{4} + \frac{20\lambda+47}{128}\sigma^2\right)\\
&\qquad\qquad + \mathrm{A}_3\sigma^4\frac{52\lambda-57}{32} = \mathrm{A}\kappa + \mathrm{B}wc^2\\
&\mathrm{A}_1\left(-\frac{\lambda+1}{2} + \frac{3\lambda+5}{64}\sigma^2\right) + \mathrm{A}_2\left(1 - \frac{20\lambda+17}{32}\sigma^2\right) + \mathrm{A}_3\tfrac{7}{4}\sigma^2 = -wc^2\\
&\mathrm{A}_1\left(-\frac{\lambda}{16} + \frac{12\lambda+17}{768}\sigma^2\right) + \mathrm{A}_2\left(-\tfrac{1}{4} - \frac{8\lambda+1}{64}\sigma^2\right) + \mathrm{A}_3\left(1 - \tfrac{3}{4}\sigma^2\right) + 2\mathrm{A}_4\sigma^2 = \frac{wc^2}{4}\\
&\mathrm{A}_1\frac{3\lambda-1}{192} - \frac{\mathrm{A}_2}{32} - \frac{\mathrm{A}_3}{4} + 2\mathrm{A}_4 = 0\\
&-\mathrm{A}_1\frac{15\lambda-8}{3072} - \frac{\mathrm{A}_2}{128} - \frac{\mathrm{A}_3}{32} - \frac{\mathrm{A}_4}{2} + 6\mathrm{A}_5 = 0.
\end{aligned}\right\}$$

Another equation is found by using the fact that the circulation is κ.

Integrating round a circle concentric with the crosss-ection (radius $< c$), we have

$$\kappa = \int_0^{2\pi}\left(-\frac{1}{\varpi}\frac{d\psi}{d\mathrm{R}}\right)\mathrm{R}\,d\chi$$

$$= -\int_0^{2\pi}\frac{d}{d\mathrm{R}}\left\{\mathrm{A}_1\,\mathrm{J}_1\varpi + \mathrm{A}_2a^2\,\mathrm{J}_2\varpi + \ldots\right\}\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi}.$$

Differentiating the equations of § 3 (E),

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi) = \frac{1}{\mathrm{R}}\left\{-1 - \frac{l}{2}s\cos\chi + \left(\frac{l+2}{4} - \frac{2l-1}{16}\cos 2\chi\right)s^2\right.$$

$$\left. + \left(\frac{9l+12}{64}\cos\chi - \frac{3l-2}{64}\cos 3\chi\right)s^3 + \left(\frac{3l+2}{128} + \frac{6l+7}{96}\cos 2\chi - \frac{60l-47}{3072}\cos 4\chi\right)s^4\right\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi) = \frac{1}{c\mathrm{R}^2}\left\{-\cos\chi - \frac{s}{2} - \left(\frac{20l-3}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2\right.$$

$$\left. + \left(\frac{20l+37}{64} - \frac{3l-1}{16}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right)s^3 + \ldots\right\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_3\varpi) = \frac{1}{c^2\mathrm{R}^3}\{-2\cos 2\chi - (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)s - \tfrac{13}{8}s^2 \ldots\}.$$

Also

$$\frac{1}{c - \mathrm{R}\cos\chi} = \frac{1}{c}(1 - s\cos\chi)^{-1}$$

$$= \frac{1}{c}\left\{1 + \frac{s^2}{2} + \frac{3s^4}{8} + 2\cos\chi\frac{s}{2}\left(1 + \frac{3s^2}{4} + \frac{5s^4}{8}\right)\right.$$

$$\left. + 2\cos 2\chi\frac{s^2}{4}(1 + s^2) + 2\cos 3\chi\frac{s^3}{8} + \ldots\right\}.$$

Therefore

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c}\left\{\left(1 + \frac{s^2}{2} + \frac{3s^4}{8} + \ldots\right)\left(1 - \frac{l+2}{4}s^2 - \frac{3l+2}{128}s^4\right)\right.$$

$$\left. + \frac{s^2}{4}\left(1 + \frac{3s^2}{4}\right)\left(l - \frac{9l+12}{32}s^2\right) + \frac{2l-1}{64}s^4 + \ldots\right\}$$

$$= \frac{2\pi}{c}$$

Similarly

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c^3}$$

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = 3\,\frac{2\pi}{c^5}$$

&c.

Therefore

$$\kappa = \frac{2\pi}{c}\left\{\mathrm{A}_1 + \frac{\mathrm{A}_2 a^2}{c^2} + 3\,\frac{\mathrm{A}_3 a^4}{c^4} + \ldots\right\}$$

$$= \frac{2\pi}{c}\{\mathrm{A}_1 + \mathrm{A}_2\sigma^2 + 3\mathrm{A}_3\sigma^4 + \&\mathrm{c.}\}.$$

Thus we have, in addition to the equations above, for the determinations of A_1, A_2, &c.,

$$A_1 + A_2\sigma^2 + 3A_3\sigma^4 + \&c. = \frac{\kappa c}{2\pi}\cdot$$

Solving the equations

$$A_1 = wc^2\sigma^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(1 - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 17\lambda + 6}{64}\sigma^4\right)$$

$$A_2 = -\,wc^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(\frac{\lambda + 1}{2} + \frac{4\lambda^2 - 19\lambda - 18}{64}\sigma^2\right)$$

$$A_3 = -\,wc^2\sigma^2\,\frac{12\lambda - 1}{128} + \frac{\kappa c}{2\pi}\left(\frac{3\lambda + 2}{16} + \frac{12\lambda^2 + \lambda + 1}{256}\sigma^2\right)$$

$$A_4 = -\,wc^2\,\tfrac{1}{64} + \frac{\kappa c}{2\pi}\,\frac{9\lambda + 10}{384}$$

$$A_5 = -\,wc^2\,\tfrac{1}{256} + \frac{\kappa c}{2\pi}\,\frac{81\lambda + 56}{18432}.$$

Substituting these values we obtain

$$A = \frac{c}{2\pi}\left\{\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right\}$$

$$B = \sigma^2\left\{\frac{2\lambda + 3}{4} + \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right\}.$$

The kinetic energy is given by the equation

$$\begin{aligned}
T &= -\,\pi\rho\int_0^{2\pi}\left[\frac{\psi}{\omega}\cdot\frac{d\psi}{dR}\right]_{R=a}\cdot\; a\,d\chi \\
&= -\,\pi\rho\int_0^{2\pi}\frac{A\kappa + wc^2\left(B - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right)}{c - a\cos\chi}\,\frac{d\psi}{dR}\,a\,d\chi \\
&= \pi\rho\int_0^{2\pi}\left\{A\kappa + wc^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\right\}\left(-\frac{1}{\varpi}\frac{d\psi}{dR}\right)a\,d\chi \\
&\qquad\qquad - \frac{\pi\rho acw}{2}\int_0^{2\pi}(1 - \sigma\cos\chi)\,\frac{d\psi}{dR}\,d\chi \\
&= \pi\rho A\kappa^2 + \pi\rho c^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\kappa\,w \\
&\qquad + \pi^2\rho wc\left\{A_1\left(1 - \frac{\lambda + 1}{2}\sigma^2 + \frac{3\lambda + 5}{64}\sigma^4\right) - A_2\,\frac{20\lambda + 17}{32}\sigma^4 + \tfrac{3}{4}A_3\sigma^4\right\} \\
&= \pi\rho\left\{A\kappa^2 + c^2\kappa w\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4} + \tfrac{1}{2} - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right)\right. \\
&\qquad\qquad \left. + \,w^2c^3\sigma^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2\right)\right\}.
\end{aligned}$$

The coefficient of κw vanishes, as it necessarily should (BASSET, 'Hydrodynamics,' vol. 1, art. 171), and the kinetic energy of the forward motion and the cyclic motion is given by

$$2\mathrm{T} = \mathrm{M}w^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \ldots\right)$$
$$+ \rho c\,\kappa^2\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4 + \ldots\right).$$

The first part of this result agrees, as far as it goes, with the result of § 11.

§ 15. As the cyclic motion through a ring is of considerable interest, it seems worth while to give other proofs of some of the above results.

To find the circulation we may integrate the velocity along the axis of the ring from $-\infty$ to ∞, and then along a semicircle from ∞ to $-\infty$.

$$\psi = \{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\,\mathrm{J}_2\varpi + \ldots\}.^*$$

The velocity of the fluid at any point on the axis is the limit of $\frac{1}{\varpi}\frac{d\psi}{d\varpi}$, where ϖ is indefinitely small.

Consider the part arising from the term $\mathrm{A}_1\mathrm{J}_1\varpi$ or

$$\mathrm{A}_1\int_0^\pi \frac{\varpi\cos\phi\,d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}.$$

$$\text{It} = \text{limit of } \frac{\mathrm{A}_1}{\varpi}\frac{d}{d\varpi}\int_0^\pi \frac{\varpi\cos\phi\,d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}$$

$$= \text{limit of } \int_0^\pi \frac{\mathrm{A}_1\,(z^2 + c^2 - \varpi c\cos\phi)}{\varpi\,(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)^{\frac{3}{2}}}\cos\phi\,d\phi$$

$$= \text{limit of } \int_0^\pi \frac{\mathrm{A}_1}{\varpi\,(z^2 + c^2)^{\frac{1}{2}}}\left(1 - \frac{\varpi c\cos\phi}{z^2 + c^2}\right)\left(1 + \frac{3\varpi c\cos\phi}{z^2 + c^2}\right)\cos\phi\,d\phi$$

$$= \frac{\pi\mathrm{A}_1 c}{(z^2 + c^2)^{\frac{3}{2}}}.$$

We may notice that at the centre of the ring this

$$= \frac{\pi\mathrm{A}_1}{c^2}.$$

---

[*] As we are only considering the cyclic motion, only the parts of A_1, A_2, &c., which involve κ, are denoted by these letters now.

Again, the part of the velocity of the fluid arising from the term $A_1J_1\varpi$ in the stream function along an arc of a very great circle whose centre is at the origin, is the limit of $-\frac{1}{\varpi}\frac{d}{dr}(A_1J_1\varpi)$

$$= \text{limit of } -\frac{A_1}{r\sin\theta}\int_0^\pi \frac{\sin\theta\,(c^2 - cr\sin\theta\cos\phi)}{(r^2 + c^2 - 2cr\sin\theta\cos\phi)^{\frac{3}{2}}}\cos\phi\,d\phi$$

$$= \frac{\pi}{2}\frac{A_1 c\sin\theta}{r^3}.$$

Therefore, the part of the circulation rising from the term $A_1J_1\varpi$ of the stream function is

$$\int_{-\infty}^{\infty}\frac{\pi A_1 c}{(z^2 + c^2)^{\frac{3}{2}}}dz + \int_0^\pi \frac{\pi}{2}\frac{A_1 c\sin\theta}{r^3}r\,d\theta$$

$$= \frac{2\pi}{c}A_1.$$

Now the part arising from $A_2a^2J_2\varpi_2$ or $-A_2a^2\frac{d}{c\,dc}(J_1\varpi)$, is obtained by differentiation with respect to c, and

$$= A_2a^2\frac{2\pi}{c^3}.$$

Therefore, the circulation is given by

$$\kappa = \frac{2\pi}{c}\left\{A_1 + A_2\sigma^2 + 1 \,.\, 3\, A_3\sigma^4 + 1 \,.\, 3 \,.\, 5 \,.\, A_4\sigma^6 + \ldots\right\},$$

agreeing with the result already obtained.

We also see that the velocity of the fluid at the centre of the ring is

$$\frac{\pi A_1}{c^2} - A_2a^2\frac{d}{c\,dc}\left(\frac{\pi}{c^2}\right) + \ldots = \frac{\pi}{c^2}\{A_1 + 2\sigma^2A_2 + 8\sigma^4A_3 + \ldots\}.$$

The kinetic energy of the cyclic motion may be obtained simply in the following way—

$$2T = \rho\iint\kappa\frac{d\Phi}{dn}dS,$$

the integral being taken over a barrier.

This

$$= \rho\iint\kappa\frac{1}{\varpi}\frac{d\psi}{d\varpi}\varpi\,d\varpi\,d\phi$$

$$= 2\pi\rho\kappa\int\frac{d\psi}{d\varpi}d\varpi$$

$$= 2\pi\rho\kappa\,(\psi_B - \psi_A),$$

where ψ_R denotes the value of ψ at the surface of the ring and ψ_A at the axis. This result applies to the cyclic motion through a circular ring of any cross-section.

We have taken ψ_A to be zero, and called ψ_R, A : so that we have $T = \pi\rho\kappa A$.

The Linear Momentum of the Cyclic Motion is parallel to the axis of the ring and

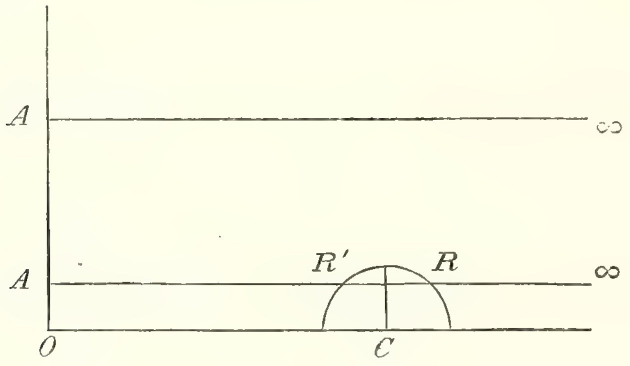
$$= \rho \iiint \frac{1}{\varpi} \frac{d\psi}{d\varpi} \varpi \, d\varpi \, dz \, d\phi :$$

the integral being taken all over the fluid.

This integral

$$= 4\pi\rho \int_0^\infty dz \int \frac{d\psi}{d\varpi} d\varpi$$

$$= 4\pi\rho \int_0^a (\psi_\infty - \psi_R + \psi_{R'} - \psi_A) \, dz + 4\pi\rho \int_a^\infty (\psi_\infty - \psi_A) \, dz.$$



We have taken $\psi_A = 0$.

ψ is constant on the ring, so that

$$\psi_{R'} - \psi_R = 0.$$

Therefore, the resultant linear momentum

$$= 4\pi\rho \int_0^\infty \psi_\infty \, dz,$$

where ψ_∞ means the value of ψ at an infinite distance from the axis.

Consider first the part of ψ_∞ arising from the term $A_1 J_1 \varpi$.

This is the limit of

$$A_1 \int_0^\pi \frac{\varpi \cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2\varpi c \cos\phi + \varpi^2)}}$$

where ϖ is infinite.

This

$$= A_1 \int_0^\pi \frac{\varpi}{\sqrt{(\varpi^2 + z^2)}} \left\{ 1 + \frac{\varpi c \cos\phi}{\sqrt{(z^2 + c^2)}} \right\} \cos\phi \, d\phi$$

$$= A_1 \frac{\pi}{2} \frac{c\varpi^2}{(\varpi^2 + z^2)^{\frac{3}{2}}} .$$

Therefore the momentum of the cyclic motion is

$$4\pi\rho\,\frac{\pi}{2}\left(\mathrm{A}_1c-\mathrm{A}_2\frac{a^2}{c}-\mathrm{A}_3\frac{a^4}{c^3}-1.3.\mathrm{A}_4.\frac{a^6}{c^5}+\ldots\right)$$

$$=2\pi^2\rho c\,\{\mathrm{A}_1-\mathrm{A}_2\,\sigma^2-\mathrm{A}_3\,\sigma^4-1.3.\mathrm{A}_4\,\sigma^6-1.3.5.\mathrm{A}_5\,\sigma^8+\ldots\}$$

$$=\pi^2\kappa\rho c\left\{1-(\lambda+1)\,\sigma^2-\frac{4\lambda^2+5\lambda-2}{32}\,\sigma^4-\&\text{c.}\right\}.$$

§ 16. The cyclic motion might have been started by applying an impulsive pressure $\kappa\rho$ at all points of the aperture of the ring, this would produce an impulsive pressure at all points of the ring. To determine this, the velocity potential of the cyclic must be found at these points. It is easy to find an expression for the velocity potential at points *not far from the surface of the ring.*

For

$$\frac{d\Phi}{\mathrm{R}d\chi}=-\frac{1}{\varpi}\,\frac{d\psi}{d\mathrm{R}}$$

$$=-\frac{1}{\varpi}\,\frac{d}{d\mathrm{R}}\,\{\mathrm{A}_1\mathrm{J}_1\varpi+\mathrm{A}_2a^2\,\mathrm{J}_2\varpi+\mathrm{A}_3a^4\,\mathrm{J}_3\varpi+\ \ldots\}$$

$$
\begin{aligned}
-\ &\frac{\mathrm{A}_1}{c\mathrm{R}\,(1-s\cos\chi)}\Big\{1+\frac{l}{2}\,s\cos\chi-\left(\frac{l+2}{4}-\frac{2l-1}{16}\cos 2\chi\right)s^2\\
&\qquad-\left(\frac{9l+12}{64}\cos\chi-\frac{3l-2}{64}\cos 3\chi\right)s^3\\
&\qquad-\left(\frac{3l+2}{128}+\frac{6l+7}{96}\cos 2\chi-\frac{60l-47}{3072}\cos 4\chi\right)s^4\ \ldots\Big\}\\
+\ &\frac{\mathrm{A}_2a^2}{c^2\mathrm{R}^2\,(1-s\cos\chi)}\Big\{\cos\chi+\frac{s}{2}+\left(\frac{20l-3}{32}\cos\chi+\tfrac{1}{32}\cos 3\,\chi\right)s^2\\
&\qquad-\left(\frac{20l+37}{64}-\frac{3l-1}{16}\cos 2\chi-\tfrac{1}{64}\cos 4\chi\right)s^3,\ \&\text{c.}\Big\}\\
+\ &\frac{\mathrm{A}_3a^4}{c^3\mathrm{R}^3\,(1-s\cos\chi)}\,\{2\cos 2\chi+(\tfrac{7}{4}\cos\chi-\tfrac{1}{4}\cos 3\chi)\,s-\tfrac{13}{8}\,s^2\,\}\\
+\ &\frac{\mathrm{A}_4a^6}{c^4\mathrm{R}^4\,(1-s\cos\chi)}\,\{6\cos 3\chi+(8\cos 2\chi-\cos 4\chi)\,s\}+\&\text{c}
\end{aligned}
$$

Expanding $\dfrac{1}{1-\sigma\cos\chi}$ in ascending powers of σ, and multiplying, $d\Phi/d\chi$ is easily expressed in cosines of multiples of χ

The term independent of χ is $\kappa/2\pi$

On integration, we obtain

$$
\begin{aligned}
\Phi = \text{const} &+ \frac{\kappa}{2\pi}\chi \\
&+ \frac{\mathrm{A}_1}{c}\left\{\left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi\right. \\
&\qquad \left. + \frac{5l+4}{32}s^3\sin 3\chi + \frac{35l+20\frac{1}{12}}{1024}s^4\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_2\sigma^2}{cs}\left\{\left(1 + \frac{20l+37}{32}s^2\right)\sin\chi + \left(\frac{s}{4} + \frac{16l+21}{64}s^3\right)\sin 2\chi\right. \\
&\qquad \left. + \tfrac{3}{32}s^2\sin 3\chi + \tfrac{5}{128}s^3\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_3\sigma^4}{c^2s^2}\left\{\tfrac{11}{4}s\sin\chi + \left(1 + \tfrac{7}{8}s^2\right)\sin 2\chi + \tfrac{1}{4}s\sin 3\chi + \tfrac{3}{32}s^3\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}^4\sigma^6}{c^3s^3}\left\{\tfrac{11}{2}s\sin 2\chi + 2\sin 3\chi + \frac{s}{2}\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_5\sigma^8}{c^4s^3}4\sin 4\chi.
\end{aligned}
$$

This holds only at points not far from the surface of the ring.

At the surface of the ring $s = \sigma$. Substituting the values of the constants given in § 14, we find that here

$$
\Phi = \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right)\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi\right. \\
\left. + \frac{57\lambda+49}{192}\sigma^3\sin 3\chi + \&\text{c.}\right\}.
$$

This gives the impulsive pressure at any point on the surface of the ring.

The momentum of the cyclic motion may be deduced.

It

$$
\begin{aligned}
&= \pi(c-a)^2\kappa\rho - \iint \Phi\sin\chi\, d\mathrm{S} \\
&= \pi\kappa\rho(c-a)^2 - 2\pi a\int_0^{2\pi}\Phi\sin\chi\,(c - a\cos\chi)\,d\chi \\
&= \pi\kappa\rho(c-a)^2 - \frac{\kappa\rho}{2\pi}2\pi ac\int_0^{2\pi}\left(\sin\chi - \frac{\sigma}{2}\sin 2\chi\right) \\
&\quad \left(\chi + \left\{\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right\}\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi\right)d\chi \\
&= \pi\kappa\rho c^2\left\{1 - 2\sigma + \sigma^2 + \left(2\sigma - \frac{\sigma^2}{2}\right) - \sigma^2\left(\frac{2\lambda+3}{2} + \frac{8\lambda^2+26\lambda+11}{64}\sigma^2\right) + \frac{16\lambda+15}{64}\sigma^4\right\} \\
&= \pi\kappa\rho c^2\left\{1 - (\lambda+1)\sigma^2 - \frac{4\lambda^2+5\lambda-2}{32}\sigma^4\right\},
\end{aligned}
$$

which agrees with the result already obtained.

If the cross-section of the ring is very small

$$A_1 = \frac{\kappa c}{2\pi}$$

and

$$\Phi = \text{const} + \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15+7}{384}s^4\right)\sin 2\chi \right.$$
$$\left. + \frac{5l+4}{32}s^3 \sin 3\chi + \frac{35l + 20\frac{1}{12}}{1024}s^4 \sin 4\chi + \ldots\right\}.$$

But in this case Φ is proportional to Ω the solid angle subtended by the ring at any point.

Hence the solid angle subtended by a circular ring at a point near it is

$$2\left\{\pi - \chi - \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi - \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi - \&c.\right\}$$

§ 17. The kinetic energy of the fluid has been calculated for the different motions of the ring. To find the kinetic energy of the solid ring, its moments of inertia round the axis, and round a perpendicular to the axis through its centre must be found

Let ρ' be the density of the ring, and M′ be the mass of the ring.

The moment of inertia round the axis is

$$\int_0^a\int_0^{2\pi} 2\pi\rho'(c - R\cos\chi)^3 R\,dR\,d\chi$$
$$= 4\pi^2\rho'\int_0^a (c^3 + \tfrac{3}{2}cR^2)R\,dR$$
$$= M'(c^2 + \tfrac{3}{4}a^2).$$

The moment of inertia round a perpendicular axis is

$$\int_0^a\int_0^{2\pi} 2\pi\rho'(c - R\cos\chi)\left\{\frac{(c - R\cos\chi)^2}{2} + R^2\sin^2\chi\right\}R\,dR\,d\chi$$
$$= M'\left(\frac{c^2}{2} + \tfrac{3}{8}a^2\right) + M'.\frac{a^2}{4}$$
$$= M'\left(\frac{c^2}{2} + \tfrac{5}{8}a^2\right).$$

§ 18. Therefore, ρ being the density of the fluid, M and M′ the masses of the fluid displaced by the ring, and of the ring, respectively, the kinetic energy is given by

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + K\kappa^2,$$

where

$$P = \frac{M}{2}\left(1 - \frac{12\lambda + 1}{16}\sigma^2 + \frac{144\lambda^2 + 24\lambda + 57}{512}\sigma^4\right) + M',$$

$$R = M\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4\right) + M',$$

$$A = M\frac{c^2}{2}\left(1 - \frac{12\lambda + 7}{16}\sigma^2 + \frac{18\lambda^2 - 3\lambda + 5}{64}\sigma^4\right) + M'\frac{c^2}{2}(1 + \tfrac{5}{4}\sigma^2),$$

$$C = M'c^2(1 + \tfrac{3}{4}\sigma^2),$$

$$K = \rho c\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right),$$

Also the linear momentum of the cyclic motion is given by

$$Z = \pi c^2 \kappa\rho\left\{1 - (\lambda + 1)\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{32}\sigma^4\right\}.$$

§ 19. The motion of a ring in an infinite fluid is discussed in BASSET'S 'Hydrodynamics,' vol. 1, pp. 196–208.

It is there shown that if a ring be set rotating with angular velocity ω about a diameter of its circular axis, it will make complete revolutions if

$$\omega > Z\sqrt{\frac{R}{AP(R - P)}},$$

but will oscillate if

$$\omega < Z\sqrt{\frac{R}{AP(R - P)}}.$$

Putting in the values of the constants, a very fine ring will make complete revolutions if

$$\omega > \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}},$$

and will oscillate if

$$\omega < \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}}.$$

A possible steady motion of the ring occurs when it moves with velocity w parallel to the axis, and has also an angular velocity Ω round its axis. It is shown that this motion will be stable if

$$(Rw + Z)\,[(R - P)\,w + Z] + \frac{\Omega^2 c^2}{4A}P$$

is positive.

Substituting, we see that the motion will be stable for any value of Ω if

$$\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c \rho}{2\pi a^2 (\rho + \rho')}\right) > 0.$$

Therefore the motion is stable for all positive values of w, and for all negative values numerically less than

$$\frac{\kappa c}{\pi a^2}\frac{\rho}{2(\rho + \rho')},$$

or numerically greater than

$$\frac{\kappa c}{\pi a^2}.$$

The motion is stable for values of w between these values if

$$\Omega^2 > -\frac{2\rho(\rho + \rho')^2}{\rho'^2(\rho + 2\rho')}\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c}{\pi a^2}\frac{\rho}{2(\rho + \rho')}\right).$$

The greatest value the right-hand side can have is

$$\frac{\kappa^2 c^2}{\pi^2 a^4}\frac{\rho(3\rho + 2\rho')}{\rho'^2}.$$

Therefore the motion is always stable if

$$\Omega > \frac{\kappa c}{\pi a^2}\sqrt{\frac{\rho(3\rho + 2\rho')}{\rho'^2}}.$$

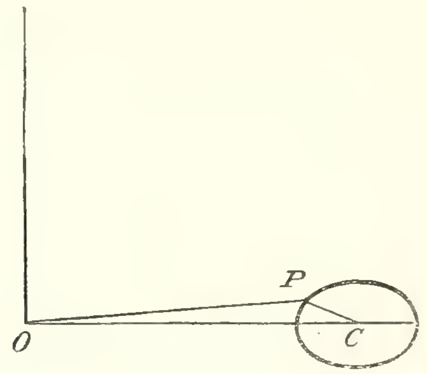
Another possible steady motion of the ring is for it to move round in a circle, as if it were a rigid body attached to an axis in the plane of the ring. Mr. BASSET shows that if T be the time of a complete oscillation, and r the radius of the circle described by the centre of the ring, then $r = \mathrm{T}/r\pi \,.\, \mathrm{Z}/\mathrm{R}$. When the ring is very fine this becomes

$$r = \frac{\mathrm{T}}{16\pi^2}\frac{c\kappa}{a^2}\frac{\rho}{\rho + \rho'}.$$

Section V.—*Annular Form of Rotating Fluid.*

§ 20. In order that a surface may be a possible figure of equilibrium of rotating fluid, it is necessary that $\mathrm{V} + \frac{1}{2}\omega^2 r^2 \sin^2\theta$ should be constant over the boundary.

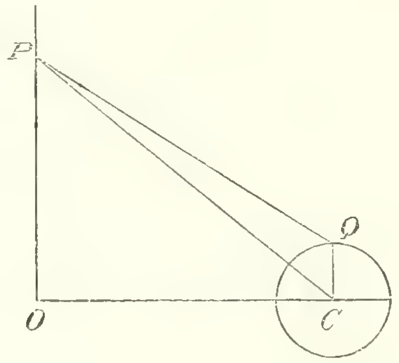
Let us assume that $\rho = a(1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \ldots)$ is the equation to the cross-section of the annulus, and that β_1, β_2, &c., are small quantities.



Then, by making C the centre of gravity of the cross-section, we obtain the equation $\beta_1 + \beta_1\beta_2 + \beta_2\beta_3 + \ldots = 0$; so that β_1 vanishes compared with β_2.

We shall show that β_2 is of order σ^2, β_3 of order σ^3, &c., where σ denotes a/c, and is taken fairly small.

To the first order of the small quantities β_2, β_3, β_4, the potential of the ring is the potential of the solid ring $r = a$, together with the potential of a distribution on the ring of surface density $a\ (\beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$, it will be most convenient to find the potential in this way, and take account of the terms arising from $\beta_2{}^2$ separately.



§ 21.—*To find the Potential of a Surface Distribution* $a\beta_2 \cos 2\chi$ *on the Ring at a Point on the Axis.*

Let

$$\angle \mathrm{OCQ} = \chi : \angle \mathrm{OCP} = \alpha : \angle \mathrm{PCQ} = \phi.$$
$$\mathrm{OC} = c : \mathrm{CQ} = a : \mathrm{CP} = \mathrm{R}.$$

Then

$$\mathrm{V} = 2\pi a^2 \int_0^{2\pi} \frac{(c - a\cos\chi)\,\beta_2 \cos 2\chi}{\sqrt{(\mathrm{R}^2 - 2a\mathrm{R}\cos\phi + a^2)}}\,d\phi$$

$$= \frac{2\pi a^2 c\,\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2\,(\phi + \alpha) - \frac{a}{2c}\cos(\phi + \alpha) - \frac{a}{2c}\cos 3\,(\phi + \alpha)\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a}{\mathrm{R}^2}\mathrm{P}_2 + \&c.\right\} d\phi$$

$$= \frac{2\pi a^2 c\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2\phi \cos 2\alpha - \frac{\sigma}{2}\cos\phi\cos\alpha - \frac{\sigma}{2}\cos 3\phi\cos 3\alpha\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a^2}{\mathrm{R}^2}\mathrm{P}_2 + \ldots\right\} d\phi,$$

since such integrals as

$$\int_0^{2\pi} \sin 2\phi \, \mathrm{P}_n (\cos \phi) \, d\phi$$

vanish. Therefore

$$\begin{aligned} \mathrm{V} = & \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos 2\alpha \int_0^{2\pi} \left\{ \frac{a^2}{\mathrm{R}^2} \cos 2\phi \, \mathrm{P}_2 + \frac{a^4}{\mathrm{R}^4} \cos 2\phi \, \mathrm{P}_4 + \frac{a^6}{\mathrm{R}^6} \cos 2\phi \, \mathrm{P}_6 + \ldots \right\} d\phi \\ & + \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos \alpha \int_0^{2\pi} \left\{ -\frac{\sigma}{2} \frac{a}{\mathrm{R}} \cos \phi \, \mathrm{P}_1 - \frac{\sigma}{2} \frac{a^3}{\mathrm{R}^3} \cos \phi \, \mathrm{P}_1 - \ldots \right\} d\phi \\ & + \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos 3\alpha \int_0^{2\pi} \left\{ -\frac{\sigma}{2} \frac{a^3}{\mathrm{R}^3} \cos 3\phi \, \mathrm{P}_3 - \frac{\sigma}{2} \frac{a^5}{\mathrm{R}^5} \cos 3\phi \, \mathrm{P}_5 \right\} d\phi \end{aligned}$$

$$\begin{aligned} = & \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \cos 2\alpha \left\{ \tfrac{3}{4} \frac{a^2}{\mathrm{R}^2} + \tfrac{5}{16} \frac{a^4}{\mathrm{R}^4} + \tfrac{105}{512} \frac{a^6}{\mathrm{R}^6} + \ldots \right\} \\ & - \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \frac{\sigma}{2} \cos \alpha \left\{ \frac{a}{\mathrm{R}} + \tfrac{3}{8} \frac{a^3}{\mathrm{R}^3} + \ldots \right\} \\ & - \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \frac{\sigma}{2} \cos 3\alpha \left\{ \tfrac{5}{8} \frac{a^3}{\mathrm{R}^3} + \tfrac{35}{128} \frac{a^5}{\mathrm{R}^5} + \ldots \right\}, \end{aligned}$$

the integrals being found at once by using the expansions of P_1, P_2, &c., in cosines of multiples of ϕ.

Substituting now $\frac{c}{\mathrm{R}}$ for $\cos \alpha$, $\frac{2c^2}{\mathrm{R}^2} - 1$ for $\cos 2\alpha$, $\frac{4c^3}{\mathrm{R}^2} - \frac{3c}{\mathrm{R}}$ for $\cos 3\alpha$: we find for the value of the potential

$$\mathrm{V} = 2\pi a^2 c \beta_2 \left\{ -\tfrac{5}{4} \frac{a^2}{\mathrm{R}^3} + \left(\tfrac{3}{2} a^2 c^2 + \tfrac{7}{16} a^4\right) \frac{1}{\mathrm{R}^5} - \tfrac{5}{8} a^4 c^2 \frac{1}{\mathrm{R}^7} - \tfrac{35}{256} \frac{a^6 c^2}{\mathrm{R}^9} + \text{\&c.} \right\}.$$

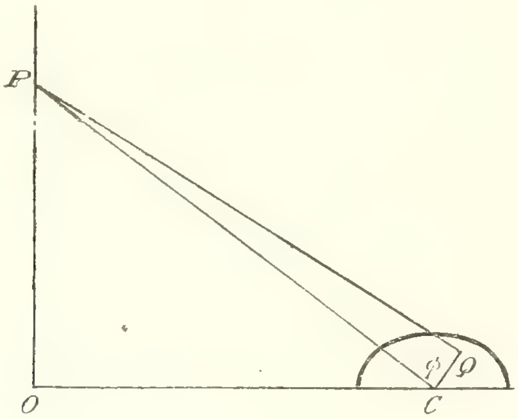
Similarly, the surface density $a\beta_3 \cos 3\chi$ gives

$$\mathrm{V} = 2\pi a^2 c \beta_3 \left\{ -\tfrac{21}{8} \frac{a^3 c}{\mathrm{R}^5} + \tfrac{5}{2} \frac{a^3 c^3}{\mathrm{R}^7} - \tfrac{35}{32} \frac{a^5 c^3}{\mathrm{R}^9} + \ldots \right\},$$

and the surface density $a\beta_4 \cos 4\chi$ gives

$$\mathrm{V} = 2\pi a^2 c \beta_4 \left\{ \tfrac{35}{8} \frac{a^4 c^4}{\mathrm{R}^9} \right\}.$$

To find the part of the potential involving β_2^2.



The potential of the ring at P is given by

$$V = \int_0^{2\pi}\int_0^{\rho} 2\pi \frac{(c - r\cos\chi)\, r\, dr\, d\chi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}}$$

$$= 2\pi \int_0^{2\pi} \left\{ \left(\frac{c\rho^2}{2} - \frac{\rho^3}{3}\cos\chi\right)\frac{1}{R} + \left(\frac{c\rho^3}{3} - \frac{\rho^4}{4}\cos\chi\right)\frac{\rho_1(\cos\phi)}{R^2} + \ldots \right\} d\chi.$$

Now

$$\rho = a(1 + \beta_2 \cos 2\chi + \ldots),$$

therefore

$$\rho^2 = a^2 \left\{ 1 + 2\beta_2 \cos 2\chi + \ldots + \frac{\beta_2^2}{4} + \frac{\beta_2^2}{4}\cos 4\chi + \ldots \right\},$$

$$\rho^3 = a^3 \left\{ 1 + 3\beta_2 \cos 2\chi + \ldots + \frac{3\beta_2^2}{2} + \frac{3\beta_2^2}{2}\cos 4\chi + \ldots \right\},$$

&c.

Therefore, the terms which will give rise to terms of highest order in β_2^2 at surface of ring are

$$2\pi a^2 c \beta_2^2 \int_0^{2\pi} \left\{ \frac{1}{4R} + \tfrac{5}{4}\frac{a^4}{R^5}\cos 4\chi\, P_4(\cos 4\phi) \right\} d\phi$$

$$= 2\pi^2 a^2 c \beta_2^2 \left\{ \frac{1}{2R} + \tfrac{5}{4} \cdot \tfrac{35}{64} \cos 4\alpha \frac{a^4}{R^5} \right\}$$

$$= 2\pi^2 a^2 c \beta_2^2 \left\{ \frac{1}{2R} + \tfrac{175}{32}\frac{a^4 c^4}{R^9} \right\}.$$

Collecting these results, the potential of the ring at P is given by

$$\frac{V}{2\pi a^2 c} = \frac{1}{R}\left(1 + \frac{\beta_2^2}{2}\right) + \frac{ac}{R^3}\left(-\frac{\sigma}{8} - \tfrac{5}{4}\beta_2\sigma\right) + \frac{a^2c^2}{R^5}\left\{-\frac{\sigma^2}{64} + \beta_2\left(\tfrac{3}{2} + \tfrac{7}{16}\sigma^2\right) - \tfrac{21}{8}\sigma\beta_3\right\}$$

$$+ \frac{a^3c^3}{R^7}\left\{-\frac{5\sigma^3}{1024} - \tfrac{5}{8}\sigma\beta_2 + \tfrac{5}{2}\beta_3\right\} + \frac{a^4c^4}{R^9}\left\{-\tfrac{35}{2^{14}}\sigma^4 - \tfrac{35}{256}\sigma^2\beta_2 - \tfrac{35}{32}\sigma\beta_3 + \tfrac{35}{8}\beta_4 + \tfrac{175}{32}\beta_2^2\right\}$$

+ &c.

§ 22. The potential at any point external to the ring whose polar coordinates are r, θ, is found by writing

$$\frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - cr \sin\theta \cos\phi)}}, \text{ \&c.},$$

instead of $1/\mathrm{R}$, $1/\mathrm{R}^3$, &c., as in Section II., § 5.

Instead of $1/\mathrm{R}$, $1/\mathrm{R}^3$, $1/\mathrm{R}^5$, &c., we write

$$\frac{\mathrm{I}_1}{\pi}, \quad \frac{\mathrm{I}_2}{\pi}, \quad \frac{\mathrm{I}_3}{3\pi}, \quad \frac{\mathrm{I}_4}{3.5\pi}, \quad \frac{\mathrm{I}_5}{3.5.7\pi}.$$

Therefore, the potential at any point outside the ring is given by

$$\frac{\mathrm{V}}{2\pi a^2 c} = \mathrm{A}_1\mathrm{I}_1 + \mathrm{A}_2\, ac\, \mathrm{I}_2 + \mathrm{A}_3 a^2c^2\, \mathrm{I}_3 + \text{\&c.},$$

where

$$\begin{aligned}
\mathrm{A}_1 &= 1 + \frac{\beta_2{}^2}{2} \\
\mathrm{A}_2 &= -\frac{\sigma}{8} - \tfrac{5}{4}\sigma\beta_2 \\
\mathrm{A}_3 &= -\frac{\sigma^2}{192} + \beta_2\left(\tfrac{1}{2} + \tfrac{7}{48}\sigma^2\right) - \tfrac{7}{8}\sigma\beta_3 \\
\mathrm{A}_4 &= \tfrac{1}{3}\left(-\frac{\sigma^3}{1024} - \tfrac{1}{8}\sigma\beta_2 + \tfrac{1}{2}\beta_3\right) \\
\mathrm{A}_5 &= \tfrac{1}{3}\left(-\frac{\sigma^4}{2^{14}} - \frac{\sigma^2\beta_2}{256} - \frac{\sigma\beta_3}{32} + \frac{\beta_4}{8} + \frac{5\beta_2{}^2}{32}\right).
\end{aligned}$$

Using the expansions given in § 3 (A), the potential at a point R, χ near the surface of the ring is given by

$$\begin{aligned}
\frac{\mathrm{V}}{2\pi a^2} = {} & \mathrm{A}_1\left(l + 2 + \frac{2l+1}{16}s^2 + \frac{108l+27}{2048}s^4\right) + \mathrm{A}_2\sigma\left(\frac{2l+3}{4} + \frac{12l+5}{128}s^2\right) + \mathrm{A}_3\sigma^2\frac{36l+51}{32} \\
& + \cos\chi\left\{\mathrm{A}_1\left(\frac{l+1}{2}s + \frac{9l+3}{64}s^3\right) + \mathrm{A}_2\sigma\left(\frac{1}{s} + \frac{12l+11}{32}s\right) + \mathrm{A}_3\sigma^2\frac{9}{4s}\right\} \\
& + \cos 2\chi\left\{\mathrm{A}_1\left(\frac{3l+2}{16}s^2 + \frac{20l+4\frac{1}{3}}{256}s^4\right) + \mathrm{A}_2\sigma\left(\tfrac{1}{4} + \frac{12l+9}{64}s^2\right)\right. \\
& \qquad\qquad\qquad\qquad \left. + \mathrm{A}_3\sigma^2\left(\frac{1}{s^2} + \tfrac{3}{4}\right) + \mathrm{A}_4\sigma^3\frac{5}{s^2}\right\} \\
& + \cos 3\chi\left\{\mathrm{A}_1\frac{5l+2\frac{1}{3}}{64}s^3 + \mathrm{A}_2\sigma\tfrac{3}{32}s + \mathrm{A}_3\sigma^2\frac{1}{4s} + \mathrm{A}_4\sigma^3\frac{2}{s^3}\right\} \\
& + \cos 4\chi\left\{\mathrm{A}_1\frac{35l+11\frac{1}{3}}{1208}s^4 + \mathrm{A}_2\sigma\tfrac{5}{128}s^2 + \mathrm{A}_3\sigma^2\tfrac{3}{32} + \mathrm{A}_4\sigma^3\frac{1}{2s^2} + \mathrm{A}_5\sigma^4\frac{6}{s^4}\right\}
\end{aligned}$$

where as before, l stands for $\log(8c/\mathrm{R}) - 2$, s for R/c, and σ for a/c.

§ 23. At the surface of the ring

$$\mathrm{V} + \frac{\omega^2 \varpi^2}{2}$$

is constant. Therefore

$$\mathrm{V} + \frac{\omega^2}{2} (c - \mathrm{R} \cos \chi)^2$$

is constant at the surface of the ring, or

$$\frac{\mathrm{V}}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2} (1 - s \cos \chi)^2$$

is constant, when

$$s = \sigma (1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi).$$

Let

$$\frac{\mathrm{V}}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2} (1 - s \cos \chi)^2$$

be called $f(s)$.

Then

$$f(s) = f(\sigma) + f'(\sigma)\, \sigma (\beta_2 \cos 2\chi + \ldots) + \tfrac{1}{2} f''(\sigma)\, \sigma^2 \beta_2{}^2 \frac{1 + \cos 4\chi}{2}.$$

Now

$$\begin{aligned} f'(s) = {} & \mathrm{A}_1 \left(-\frac{1}{s} + \frac{l}{4} s \right) - \mathrm{A}_2 \sigma \frac{1}{2s} + \frac{\omega^2 c^2}{4\pi a^2} s \\ & + \cos \chi \left\{ \mathrm{A}_1 \frac{l}{2} - \frac{\mathrm{A}_2 \sigma}{s^2} - \frac{\omega^2 c^2}{2\pi a^2} \right\} \cos \chi \\ & + \cos 2\chi \left\{ \mathrm{A}_1 \frac{6l + 1}{16} s - \frac{2\mathrm{A}_3 \sigma^2}{s^3} + \frac{\omega^2 c^2}{4\pi a^2} s \right\} \end{aligned}$$

and

$$f''(s) = \left(\frac{\mathrm{A}_1}{s^2} + \frac{\omega^2 c^2}{4\pi a^2} \right) + \frac{\omega^2 c^2}{4\pi a^2} \cos 2\chi.$$

Therefore, $\sigma f'(\sigma) (\beta_2 \cos 2\chi + \&c.)$

$$\begin{aligned} = {} & \left(\mathrm{A}_1 \frac{6\lambda + 1}{32} \sigma^2 - \mathrm{A}_3 + \frac{\omega^2 c^2}{4\pi a^2} \cdot \frac{\sigma^2}{2} \right) \beta_2 \\ & + \left(\mathrm{A}_1 \frac{\lambda}{4} \sigma - \frac{\mathrm{A}_2}{2} - \frac{\omega^2 c^2}{4\pi a^2} \sigma \right) \beta_2 \cos \chi \\ & + \left\{ \left[\mathrm{A}_1 \left(-1 + \frac{\lambda}{4} \sigma^2 \right) - \frac{\mathrm{A}_2}{2} \sigma + \frac{\omega^2 c^2}{4\pi a^2} \right] \beta_2 + \mathrm{A}_1 \left[\frac{\lambda}{4} \sigma - \frac{\mathrm{A}_2}{2} - \frac{\omega^2 c^2}{4\pi a^2} \sigma \right] \beta_3 \right\} \cos 2\chi \\ & + \left\{ \left(\mathrm{A}_1 \frac{\lambda}{4} \sigma - \frac{\mathrm{A}^2}{2} - \frac{\omega^2 c^2}{4\pi a^2} \sigma \right) \beta_2 - \mathrm{A}_1 \beta_3 \right\} \cos 3\chi \\ & + \left\{ \left(\mathrm{A}_1 \frac{6\lambda + 1}{32} \sigma^2 - \mathrm{A}_3 + \frac{\omega^2 c^2}{4\pi a^2} \frac{\sigma^2}{2} \right) \beta_2 + \left(\mathrm{A}_1 \frac{\lambda}{4} \sigma - \frac{\mathrm{A}_2}{2} - \frac{\omega^2 c^2}{4\pi a^2} \sigma \right) \beta_3 - \mathrm{A}_1 \beta_4 \right\} \cos 4\chi, \end{aligned}$$

and

$$\frac{\sigma^2}{2} f''(\sigma)\beta_2^2 \frac{1+\cos 4\chi}{2} = \frac{A_1}{4}\beta_2^2 + \frac{A_1}{4}\beta_2^2 \cos 4\chi.$$

All terms which are of order higher than σ^4 have been neglected; and throughout, it has been assumed that $\omega^2c^2/4\pi a^2$ is of zero order in σ, β_2 of the second order, β_3 of the third, and so on—an assumption which will be justified by the result.

The value of

$$\frac{V}{2\pi a^2} + \frac{\omega^2c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

at the surface of the ring is

$$\begin{aligned}
&A_1\left(\lambda + 2 + \frac{2\lambda+1}{16}\sigma^2 + \frac{108\lambda+27}{2048}\sigma^4 + \frac{6\lambda+1}{32}\sigma^2\beta_2 + \tfrac{1}{4}\beta_2^2\right) \\
&\qquad + A_2\left(\frac{2\lambda+3}{4}\sigma + \frac{12\lambda+5}{128}\sigma^3\right) + A_3\left(\frac{36\lambda+51}{32}\sigma^2 - \beta_2\right) + \frac{\omega^2c^2}{4\pi a^2}\left(1 + \frac{\sigma^2}{2} + \beta_2\frac{\sigma^2}{2}\right) \\
&+ \cos\chi\left\{A_1\left(\frac{\lambda+1}{2}\sigma + \frac{9\lambda+3}{64}\sigma^3 + \frac{\lambda}{4}\sigma\beta_2\right) + A_2\left(1 + \frac{12\lambda+11}{32}\sigma^2 - \frac{\beta_2}{2}\right)\right. \\
&\qquad\qquad \left. + A_3\tfrac{9}{4}\sigma + \frac{\omega^2c^2}{4\pi a^2}(-2\sigma - \sigma\beta_2)\right\} \\
&+ \cos 2\chi\left\{A_1\left(\frac{3\lambda+2}{16}\sigma^2 + \frac{20\lambda+4\frac{1}{3}}{256}\sigma^4 - \beta_2 + \frac{\lambda}{4}\sigma^2\beta_2 + \frac{\lambda}{4}\sigma\beta_3\right)\right. \\
&\qquad + A_2\left(\frac{\sigma}{4} + \frac{12\lambda+9}{64}\sigma^3 - \frac{\sigma}{2}\beta_2 - \tfrac{1}{2}\beta_3\right) + A_3(1 + \tfrac{3}{4}\sigma^2) \\
&\qquad\qquad \left. + A_4\,5\sigma + \frac{\omega^2c^2}{4\pi a^2}\left(\frac{\sigma^2}{2} + \beta_2 - \sigma\beta_3\right)\right\} \\
&+ \cos 3\chi\left\{A_1\left(\frac{5\lambda+2\frac{1}{3}}{64}\sigma^3 + \frac{\lambda}{4}\sigma\beta_2 - \beta_3\right) + A_2(\tfrac{3}{32}\sigma^2 - \tfrac{1}{2}\beta_2)\right. \\
&\qquad\qquad \left. + A_3\frac{\sigma}{4} + 2A_4 - \frac{\omega^2c^2}{4\pi a^2}\sigma\beta_2\right\} \\
&+ \cos 4\chi\left\{A_1\left(\frac{35\lambda+11\frac{1}{3}}{1024}\sigma^4 + \frac{6\lambda+1}{32}\sigma^2\beta_2 + \frac{\lambda}{3}\sigma\beta_3 - \beta_4 + \frac{\beta_2^2}{4}\right)\right. \\
&\qquad + A_2(\tfrac{5}{128}\sigma^3 - \tfrac{1}{2}\beta_3) + A_3(\tfrac{3}{32}\sigma^2 - \beta_2) \\
&\qquad\qquad \left. + A_4\frac{\sigma}{2} + 6A_5 + \frac{\omega^2c^2}{4\pi a^2}\left(\frac{\sigma^2}{2}\beta_2 - \sigma\beta_3\right)\right\}.
\end{aligned}$$

Equating the coefficients of $\cos\chi$, $\cos 2\chi$, $\cos 3\chi$, $\cos 4\chi$ to zero, we obtain four equations to find

$$\frac{\omega^2c^2}{4\pi a^2},\ \beta_2,\ \beta_3,\ \text{and}\ \beta_4.$$

Substituting the values of A_1, A_2, &c., these equations are

$$\left.\begin{aligned}
&-\frac{\omega^2c^2}{2\pi a^2}\left(1+\frac{\beta_2}{2}\right)+\frac{4\lambda+3}{8}+\frac{12\lambda-1}{128}\sigma^2+\frac{4\lambda-1}{16}\beta_2=0,\\
&\frac{3\lambda+\frac{17}{12}}{16}\sigma^2+\frac{7\lambda-\frac{19}{24}}{128}\sigma^4+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{8}+\beta_2\left\{-\tfrac{1}{2}+\frac{4\lambda+1}{16}\sigma^2+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{4}\right\}\\
&\qquad\qquad\qquad\qquad+\beta_3\left\{(\lambda-\tfrac{1}{12})\frac{\sigma}{4}-\frac{\omega^2c^2}{\pi a^2}\frac{\sigma}{4}\right\}=0,\\
&\frac{\lambda+\frac{35}{24}}{64}\sigma^3+\left(\lambda+\tfrac{5}{12}-\frac{\omega^2c^2}{\pi a^2}\right)\frac{\sigma}{4}\beta_2-\tfrac{2}{3}\beta_3=0,\\
&\frac{35(\lambda+\frac{19}{120})}{1024}\sigma^4+\frac{3\lambda+\frac{7}{8}}{16}\sigma^2\beta_2+\frac{\lambda+\frac{1}{3}}{4}\sigma\beta_3-\tfrac{3}{4}\beta_4+\frac{\beta_2{}^2}{16}+\frac{\omega^2c^2}{\pi a^2}\left(\frac{\sigma^2}{8}\beta_2-\frac{\sigma}{4}\beta_3\right)=0.
\end{aligned}\right\}$$

These equations give

$$\left.\begin{aligned}
\frac{\omega}{\pi}&=(\lambda+\tfrac{3}{4})\sigma^2-\tfrac{1}{8}(\lambda+\tfrac{19}{12})\sigma^4\\
\beta_2&=\frac{\frac{5}{8}(\lambda+\frac{7}{12})\sigma^2+\frac{3}{64}(\lambda-\frac{11}{12})\sigma^4}{1-(\lambda+\frac{1}{2})\sigma^2}\\
\beta_3&=\tfrac{5}{128}(\lambda-\tfrac{7}{24})\sigma^3\\
\beta_4&=\frac{75\lambda^2+80\lambda+21}{256}\sigma^4,
\end{aligned}\right\}$$

where $\sigma = a/c$ and $\lambda = \log_e(8c/a) - 2$.

The result $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$ is given by MME. KOWALEWSKI in a paper on Saturn's rings in the 'Astronomische Nachrichten' for 1885. She finds $\beta_2 = \frac{1}{2}(\lambda + \frac{3}{4})\sigma^2$. M. POINCARÉ also gives $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$, and finds $\beta_2 = \frac{3}{4}(\lambda + \frac{3}{4})\sigma_2$. Both papers are given in TISSERAND's 'Mécanique Céleste,' vol. 2. The value found above has been kindly verified for me by Mr. HERMAN, Fellow of Trinity College, Cambridge.

The numerical values of ω^2/π, β_2, &c., are given below when $\sigma = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$.

$$\sigma = \cdot 1 \qquad \lambda = 2\cdot 3820 \qquad \beta_2 = \cdot 0189 \qquad \beta_3 = \cdot 0001 \qquad \beta_4 = \cdot 0003 \qquad \frac{\omega^2}{\pi\rho} = \cdot 0313$$

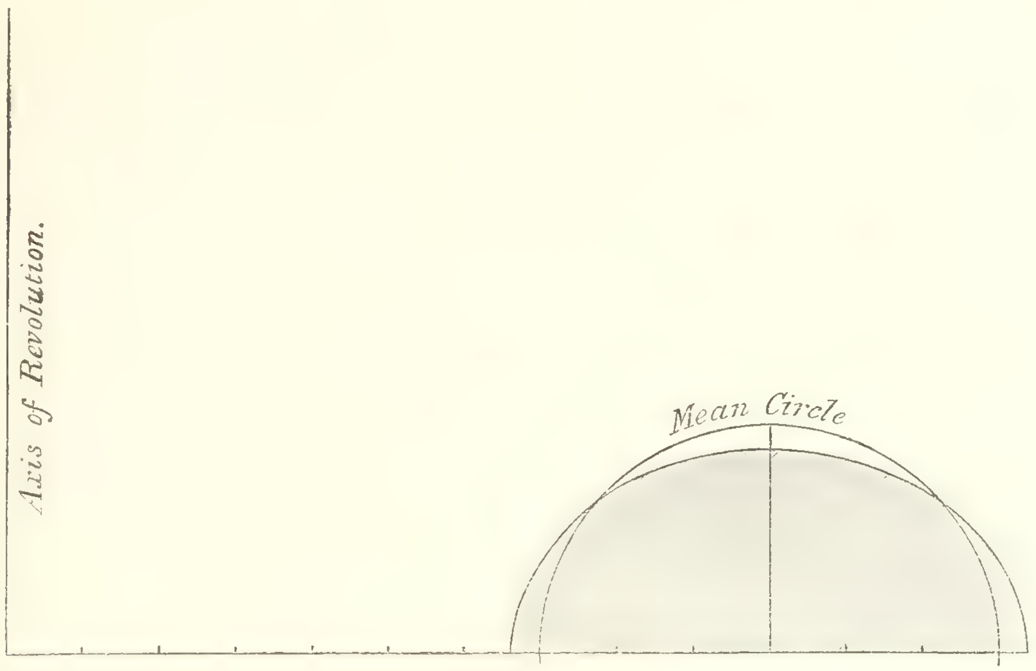
$$\sigma = \cdot 2 \qquad \lambda = 1\cdot 6889 \qquad \beta_2 = \cdot 0570 \qquad \beta_3 = \cdot 0004 \qquad \beta_4 = \cdot 0023 \qquad \frac{\omega^2}{\pi\rho} = \cdot 0970$$

$$\sigma = \cdot 3 \qquad \lambda = 1\cdot 2834 \qquad \beta_2 = \cdot 1268 \qquad \beta_3 = \cdot 0010 \qquad \beta_4 = \cdot 0078 \qquad \frac{\omega^2}{\pi\rho} = \cdot 1836.$$

A figure is given for the case of $\sigma = \cdot 3$.

The cross-section is roughly an ellipse whose major axis is perpendicular to the axis of the ring. The eccentricity of this ellipse increases with the angular velocity.

[When the paper was read, numerical results were given for larger values of σ. The above method cannot, however, be employed in such cases. For, as the eccentricity of the cross-section increases, the expression given for the potential at external points will become divergent at the extremity of the minor axis of the cross-section. A similar result will occur if the potential of an elliptic cylinder be found at external points, considering it as an approximation to a circular cylinder. In this case it is easy to show that the eccentricity must be $< 1/\sqrt{2}$. Assuming the same result for the case of a ring, it will be seen that $\beta_2 = \cdot 2076$, the value the above method gives for $\sigma = \cdot 4$, gives the cross-section of too great eccentricity for the series to be convergent. That the series were divergent for large values of β_2 was also suggested by one of the Referees. Aug., 1892.]



III. BAKERIAN LECTURE.—*The Rate of Explosion in Gases.*

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[PLATE 1.]

CAP. I.—INTRODUCTION.

THE experiments described in this memoir were undertaken with two objects : in the first place, to obtain information concerning the course of chemical change pursued by reacting gases ; and, secondly, to examine the nature of the "explosion-wave" in gaseous mixtures discovered by M. BERTHELOT.

The idea of using the rate of explosion as a means of determining the course of a chemical reaction occurred to me in 1877, when investigating the influence of steam on the union of carbonic oxide and oxygen. If steam acts as a carrier of oxygen to the carbonic oxide by a series of alternate reductions and oxidations, an increase in the amount of steam present, beyond that required to initiate the reaction, should be accompanied by an increase in the rate of combination up to a certain limit. Attempts were therefore made to detect such an increase by measuring the velocity of the flame in a tube.* But while the difference in the rate of explosion between the nearly dry and the moist gases was well marked, the attempts to directly measure the rate of the explosion of the moist gases failed, owing to the great rapidity of the flame. In the spring of 1881 I attempted to measure the rate of explosion of carbonic oxide and oxygen with varying quantities of steam by photographing on a moving plate the flashes at the beginning and end of a closed tube 20 feet long. The two flashes appeared to be *simultaneous* to the eye, but no record of the rate was obtained, for the apparatus was broken to pieces by the violence of the explosion. Shortly after this attempt was made the first of the brilliant series of papers by MM. BERTHELOT and VIEILLE, and by MM. MALLARD and LE CHATELIER, was read before the French Academy of Sciences. The work of these French chemists has opened a new era in the theory of explosions.

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* 'Phil. Trans.,' 1884, Pt. II., p. 635.

Thirty-five years ago BUNSEN* described a method of measuring the rapidity of the flame in gas explosions. Passing a mixture of explosive gases through an orifice at the end of a tube and igniting the gases as they issued into the air, he determined the rate at which the gases must be driven through the tube to prevent the flame passing back through the opening. By this method he found that the rate of propagation of the ignition of hydrogen and oxygen was 34 metres per second, while the rate of ignition of carbonic oxide and oxygen was less than 1 metre per second. BUNSEN *applied these results to the rate of explosion of gases in closed vessels:* his results were accepted without cavil for four and twenty years.

By 1880 facts began to accumulate which seemed inconsistent with BUNSEN's conclusions. For instance, between 1876–80 I had several times observed that the flame produced by igniting a mixture of moist carbonic oxide and oxygen travelled in a long eudiometer too quickly to be followed by the eye. Again, Mr. A. V. HARCOURT,† in his investigation of an explosion of coal-gas and air, which happened in a large gas main near the Tottenham Court Road in 1880, was led to the conclusion that the flame travelled at a rate exceeding 100 yards per second. In the winter of 1880–1 I was startled by the rapid increase of velocity and violence as a flame of carbon bisulphide with nitric oxide travelled down a long glass vessel.

In July, 1881, two papers appeared in the 'Comptes Rendus,' one by M. BERTHELOT, the other by MM. MALLARD and LE CHATELIER. Both papers announced the discovery of the enormous velocity of explosion of gaseous mixtures. Other papers quickly followed by the same authors. M. BERTHELOT made the important discovery that the rate of explosion rapidly increases from its point of origin until it reaches a maximum which remains constant, however long the column of gases may be. Each mixture of gases has a definite maximum velocity of explosion. The rate of explosion thus forms a new physico-chemical constant, having important theoretical and practical bearings. The name "L'Onde Explosive" is given by BERTHELOT to the flame when propagated through an explosive mixture of gases at the maximum velocity.

While BERTHELOT, associated with VIEILLE, was measuring the rate of the "explosion-wave" for various mixtures of gases, MALLARD and LE CHATELIER continued the study of the preliminary phenomena of explosion which precede the formation of the "wave." They showed by photographing on a revolving cylinder:—(1) that when a mixture such as nitric oxide and carbon bisulphide is ignited at the open end of a tube, the flame travels a certain distance (depending on the diameter and length of the tube) at a uniform velocity; (2) that at a certain point in the tube vibrations are set up, which alter the character of the flame, and that these vibrations become more intense, the flame swinging backwards and forwards with oscillations of increasing amplitude; and (3) that the flame either goes out altogether, or the rest of the gas detonates with extreme velocity. Again, when a mixture of gases was fired near the closed end of the tube, they found the velocity of the flame regularly increased, as far as their instruments were able to record the rapidly increasing pace.

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* 'Gasometrische Methoden,' 1857.

† Report to the Board of Trade.

Mixtures of coal-gas with air, and of fire-damp with air, show phenomena of the first and second kind. Ignited at the open end of a tube these mixtures burn at a uniform rate for a certain distance, and then the flame begins to vibrate. The vibrations acquire greater or less velocity according to the nature of the mixture and the conditions of the experiment; but the third *régime* of uniform maximum velocity is not set up. In narrow tubes the explosion soon dies out.

Cap. II.—Berthelot's Experiments.

Berthelot's experiments were made partly with a tube of lead, and partly with a tube of thick caoutchouc, usually 5 mm. in internal diameter. The rate of explosion was determined by making the flame break two strips of thin tin stretched across the tube, each carrying a current. The interval between the interruption of the two circuits was measured by a Le Boulengé chronograph. The gases were fired by an electric spark near one of the interrupters. To ensure the fracture of the strips of tin a grain of fulminate was placed in a fold of the metal. The tube was 40 metres long, and was supported on a wooden screen in horizontal layers.

The following are the chief conclusions reached by Berthelot concerning the propagation of the explosion-wave in gases:—

(1) The propagation is uniform. Measurements made in tubes of 20, 30, and 40 metres length show the same velocity for the same mixture.

(2) The velocity is independent of the material of the tube: the explosion travels at the same rate in tubes of lead as in tubes of caoutchouc.

(3) The velocity is independent of the diameter of the tube above a small limit: the explosion travels at the same rate in a tube of 5 mm. diameter as in a tube of 15 mm. diameter.

(4) The velocity is independent of the pressure: the explosion travels at the same rate when the gas is under a pressure of 560 mm. as it does under a pressure of 1580 mm.

(5) The velocity of the explosion equals, or approximates closely to, the mean velocity of translation of the molecules at the moment of combination, on the supposition that they retain all the heat developed in the reaction. The rates of explosion of some twenty different mixtures agree with the theoretical rate calculated by the formula of Clausius:—

$$\theta = 29 \cdot 354 \sqrt{\frac{T}{d}},$$

where T is the absolute temperature reached in the explosion, and d the density of the products of combustion referred to air.

In the following tables the rates of explosion of simple mixtures as measured by Berthelot are compared with the theoretical velocity of the products of combustion calculated from Clausius' formula. The temperature T is calculated on the supposition (1) that the gases are heated *at constant pressure*, and (2) that the specific heat of a compound gas is the sum of the specific heats of its constituents.

Table I.—Berthelot's Experiments. Combustible Gases with Oxygen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen	2831	2810
Carbonic oxide	1940	1090
Marsh gas	2427	2287
Ethane	2483	2363
Ethylene	2517	2210
Acetylene	2660	2482
Cyanogen	2490	2195

Table II.—Combustible Gases with Nitrous Oxide.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen	2250	2284
Carbonic oxide	1897	1106
Cyanogen	2198	2036

Berthelot observes that the formula does not apply to mixtures of carbonic oxide either with oxygen or with nitrous oxide. *It is to be noticed that* BERTHELOT *worked with dry gases.*

TABLE III.—Mixtures of Combustible Gases with Oxygen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen and carbonic oxide.		
(1) $H_2 + CO + O_2$	2236	2008
(2) $3H_2 + 2CO + O_5$ . . .	2321	2170
Ethylene and hydrogen.		
(1) $C_2H_4 + H_2 + O_7$ . . .	2551	2417
(2) $C_2H_4 + 2H_2 + O_8$ . . .	2588	2579
Ethane and hydrogen.		
$C_2H_6 + H_2 + O_8$	2522	2250

With mixtures of hydrogen and carbonic oxide the formula is found to hold good. BERTHELOT explains this by saying that "the hydrogen communicates to the carbonic oxide a law of detonation analogous to its own."

The general concordance between the observed velocities and the calculated rate of translation of the molecules shows, according to BERTHELOT, that dissociation plays but a small part in these phenomena, perhaps because of the high pressure developed: a result which is confirmed by the fact that the velocities are found to be independent of the pressure.

A comparison is next made between the rates of explosion of "isomeric mixtures," *i.e.*, mixtures of different gases which yield the same products of combustion. For instance, the rate of explosion of marsh gas and oxygen is compared with the rate of explosion of a mixture of ethylene, hydrogen, and oxygen yielding identical products; the object being to eliminate the influence of the individual gases burnt. The same conclusions are drawn.

TABLE IV.—Combustible Gases with Oxygen and Nitrogen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
$H_2 + O + N_2$	1935	2121
$H_2 + O + N_4$ (Air)	1820	1439
$CO + O + N_2$	1661	1000 ?
$CO + O + N_4$	1236	Detonation not propagated
$CH_4 + O_4 + N_4$	2002	1858
$CH_4 + O_4 + N_8$	1744	1151
$CH_4 + O_4 + N_{15}$ (Air)	1450	Detonation not propagated
$C_2N_2 + O_4 + N_2$	2334	2044
$C_2N_2 + O_4 + N_4$	2152	1203
$C_2N_2 + O_4 + N_8$	1920	Detonation not propagated

When the explosive gases are mixed with an inert gas, nitrogen, which takes no part in the reaction, the same law holds good—except when the nitrogen is added in excess. Before the gases are diluted sufficiently to stop the explosion, there is found a marked falling off in the velocity. The formula gives the theoretically highest rate the explosion can attain—a maximum reached in few cases only, but approached in a large number.

Berthelot's Conclusions.

These results show, according to BERTHELOT, that the velocity of the explosion-wave constitutes, for each inflammable mixture, a true specific constant. The wave is propagated by the impact of the products of combustion of one layer upon the unburnt gases in the next layer, and so on to the end of the tube at the rate of movement of the products of combustion themselves. In a word, the mean velocity of translation of the gaseous molecules retaining the total *vis viva* which corresponds to the heat developed in the reaction *may be regarded as a limit representing the maximum rate of propagation of the explosion-wave* ('Sur la Force des Matières Explosives,' 1, p. 159).

If this theory is true, it accounts not only for the extreme rapidity of explosion of gaseous mixtures, and gives us the means of calculating the maximum velocity obtainable with any mixture of gases, but it also affords us information on the specific heats of gases at very high temperatures, and it explains the phenomena of *detonation* whether of gases or of solid or liquid explosives.

Cap. III.—Repetition of Berthelot's Experiments.

§ 1. *Objects in View.*

A consideration of Berthelot's results, published in the 'Annales de Chimie'[*] in 1883, led me to think it would be useful to repeat and extend these experiments. The close coincidence between the rates of explosion of hydrogen, both with oxygen and with nitrous oxide, and the calculated velocities of the products of combustion showed that the formula held good for gases which could readily be prepared in a pure state; and, again, the great discordance between the found and calculated rates for carbonic oxide, both with oxygen and nitrous oxide, was what I should have expected from my own experiments on the part taken by steam in the oxidation of carbonic oxide. On the other hand, Berthelot's contention that the gases are heated at constant pressure appeared improbable, and his results obtained with the addition of an inert gas seemed to vary capriciously. The chief objects I had in view in continuing these experiments were:—

(1) To determine as accurately as possible the rate of the explosion-wave for some simple mixtures under varying conditions, *e.g.*, diameter of tube, initial pressure, initial temperature.

(2) To measure the rate of the explosion-wave in carbonic oxide and oxygen with different quantities of steam.

(3) To compare the effect of inert gases and of excess of one or other of the reacting gases on the rate of explosion.

§ 2. *Methods and Apparatus used.*

The mixtures of hydrogen, carbonic oxide, and marsh gas with oxygen were prepared in a graduated 5-feet iron gas-holder over water; the mixtures containing ethylene, acetylene, cyanogen, and nitrous oxide were prepared in a 1-foot iron holder over mercury; the mixtures of hydrogen and chlorine were passed directly from the generating and purifying apparatus into the explosion tube.

The gases were driven from the holders through drying vessels into the explosion tube by placing weights upon the holders. When all air-traps were avoided in the drying tubes and connections, very little diffusion was found to occur in driving out the air by the explosive mixture. One "Drechsel" washing bottle and three towers packed with pumice, all containing boiled oil of vitriol, were usually employed as drying vessels. When the explosive mixture contained either ethylene or acetylene the gas was driven from the mercury holder through one washing bottle containing strong aqueous potash and through two towers packed with lumps of caustic potash. The cyanogen mixtures were passed direct from the mercury holder into the explosion tube, the gases having been dried in preparation.

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[*] 'Ann. Chim. et Phys.,' [V.] vol. 28; also 'Sur la force des Matières Explosives,' vol. 1., chap. 7.

The explosion tube was, in most of the experiments, a leaden pipe, 100 metres long and 9 mm. in diameter. With some of the more violent explosives—*e.g.*, cyanogen and acetylene, a leaden tube of 6·5 mm. was used; and in the chlorine experiments straight wrought-iron pipes lined with glass were employed.

The leaden tube was, in most cases, coiled on a drum, 2 feet in diameter, which was immersed in an iron vessel containing water. By heating the water and pumping dry air through the tube, the apparatus could be quickly dried after an explosion in which water was formed. For determinations of the rate of explosion of gases at 100° C. this arrangement was also suitable. Two inconveniences arose from this disposition of the pipe: (1) the difficulty of determining its exact length, and (2) the uncertainty whether the explosion-wave was propagated along the axis of the coiled pipe, or whether it followed a shorter path nearer to the inside wall.

It was found impossible to coil a small leaden pipe without lengthening it appreciably; the outside of the tube was therefore measured after each coil was wound on the drum, and the length of the axis of the pipe calculated. After a series of experiments the coils were then unwound, by rolling the drum along a corridor, and the length was measured directly. The length so obtained did not vary more than an inch, or at most two, from that calculated on winding.

To determine whether the flame was propagated centrally, or whether it took a shorter cut, measurements of the rate of explosion of samples of oxygen and hydrogen from the same mixture were made alternately, in a leaden pipe 9 mm. in diameter (1) coiled on the drum, and (2) lying straight on the floor. The experiments were thrice repeated, and no appreciable difference in the rate could be detected; from which it may be concluded that in a pipe of this bore the wave-front travels quickest along the axis. Possibly in the experiments with tubes of a larger diameter the curvature may have slightly affected the results.

In the experiments made under reduced pressure, the tube was filled from the holder in the ordinary way. The steel stopcock[*] near the second bridge was closed, and the gas was then sucked from the tube by a powerful pump, maintaining a vacuum of about 26 inches of mercury. A gauge showed when the desired reduction of pressure was reached. The other stopcock was then closed, and the spark immediately passed. In the experiments made under increased pressure, the gases were driven into the tube from a strong reservoir, either by water or by mercury, according to the nature of the gaseous mixture. In all cases the gases were fired by an electric spark near one end of the tube; about four feet from the firing point, the flame broke a piece of silver foil stretched across the tube, the moment of rupture being marked on a moving plate by an electro-magnetic style. On reaching the further end of the tube, the flame broke a second strip of silver, the moment of rupture being recorded by a second style. To eliminate errors due to retardation of the electro-magnets, a blank experiment was made immediately before firing the gases. The plate of the chronograph passed over two breaks, which could be adjusted so as to be broken by it at the same instant. In the preliminary experiment, the circuits of the two magnets were completed through these two breaks, and were thus broken simultaneously, the two styles registering their marks on the moving plate. The magnets, without their position being altered, were then connected with the silver bridges across the explosion tube, and the gases were fired by the moving plate striking one or other of the two breaks. The flame travelling down the tube broke the two silver bridges in turn, releasing the two styles, which again registered their marks on the moving plate. The two marks made by the first style gave the interval of time between the spark and the breaking of the first bridge, independently of the error of the electro-magnet; for both marks were equally affected by any error of retardation. In the same way, the two marks made by the second style gave the interval of time between the spark and the breaking of the second bridge, independently of the error of the electro-magnet. The difference between the intervals was the time taken by the flame to travel between the two bridges.

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[*] These steel stopcocks, made for me by Mr. J. J. HICKS, of Hatton Garden, have stood many hundreds of explosions without leaking. The gun-metal stopcocks employed at first were indented by the explosions and soon leaked.

This method of cutting out the errors of the chronograph is only valid when the current flowing through the electro-magnets is constant. A small resistance coil was interposed in the preliminary experiment in each chronograph circuit, to match the resistance of the silver-bridge and connections used in the actual determination. The current was supplied by a storage cell (ELWELL and PARKER) to each electro-magnet, and the time between the preliminary and the final experiment was never more than one minute. In all the later portion of the work a confirmatory experiment was made after the gases were fired, the connections being made as in the preliminary. In most cases the styles passed absolutely over the preliminary marks—merely deepening the lines, thus showing that nothing had been displaced. In some cases the preliminary and confirmatory marks were just visibly separated; in these cases the mean position of the two was used to calculate the rate. In the few instances in which the confirmatory did not closely agree with the preliminary marks the experiment was rejected. In order further to eliminate errors of the instrument a second experiment was always made as soon as possible after the first, the connections of the chronograph being all reversed—the first style being connected with the second bridge, and *vice versâ*. The mean result of the two experiments is taken as one determination.

The chief differences between my apparatus and that employed by BERTHELOT were the following:—

a. The explosion tube was longer and wider.

b. The interrupters or "bridges" were of silver foil, and no fulminate was used.

c. A longer space was allowed between the firing spark and the first "bridge" where the record of the rate was begun. This was found essential in several cases, especially when the mixture was fired under reduced pressure, or in presence of inert gases, in which cases the explosion does not reach its maximum rate for some feet.

§ 3. *On the Constant Velocity of the Explosion Wave.*

Experiments on the rate of explosion of hydrogen and oxygen in a leaden tube, 55 metres long and 8 mm. in internal diameter, gave a mean rate of 2817 metres per second. In a tube of the same diameter and 100 metres long, the mean rate was 2821 metres; and in a tube 100 metres long and 13 mm. diameter the mean rate was 2819.

The general mean of these experiments—viz., 2819, is in close agreement with the mean result obtained by BERTHELOT in a shorter tube—viz., 2810.

The constancy of the rate of the explosion-wave, under ordinary conditions of temperature and pressure, was, therefore, fully confirmed.

In the course of this investigation I have made many measurements of the rate of explosion of hydrogen and oxygen under ordinary conditions. Some of these were made with more precautions than others, so that in arriving at the most probable value of the rate, I have given greater weight to some sets than to others. The sets, each of which consists of from eight to ten distinct measurements, are lettered in the following table A, B, C, etc., in the order in which they were made:—

TABLE V.—General Mean of $H_2 + O$ Rates.

Date.	Sets.	Mean.
1884, 1885	A + B + C	2819
1888	D	2822
1889	E + F + G	2824
1890	H	2818
		2821

The details of these experiments will be found in the Appendix.

§ 4. *The Influence of Pressure on the Velocity of the Explosion Wave.*

Preliminary experiments on the rate of explosion of hydrogen and oxygen under 500 mm. and 1000 mm. pressure having shown an appreciable difference, careful measurements of the rate were made at pressures varying from 200 mm. to 1500 mm. at a temperature close to 10° C. In the experiments at low pressures it was, of course, essential that the firing tube should be absolutely gas-tight; the tube was therefore tested before each experiment. An important fact concerning the development of the explosion-wave was detected in these experiments. As the pressure was reduced the flame was found to travel a greater distance before its rate became constant; so that, although four feet start was allowed before the first measurement was taken, this distance was found insufficient. Accordingly a tube 20 feet long was interposed between the firing point and the first bridge, after it had been shown that the flame acquired its maximum velocity in this distance under the lowest pressure used. Electrolytic gas can be exploded under lower pressures than 200 mm., but the flame travels irregularly. At 100 mm. pressure the flame went out in the tube, and at 150 mm. the rates found were not constant.

TABLE VI.—Pressure Experiments. $H_2 + O$.

Pressure.	200 mm.	300 mm.	500 mm.	760 mm.	1100 mm.	1500 mm.
Mean rate . .	2627	2705	2775	2821	2856	2872

These figures show that the rate of explosion increases rapidly with increase of pressure from 200 mm.; that the rate of increase diminishes, and that the velocity becomes nearly constant at two atmospheres pressure. This effect of pressure on the rate is plainly seen when the results are expressed graphically (see Plate 1).

BERTHELOT'S conclusion that the explosion-wave is independent of the initial pressure of the gases is, therefore, not strictly accurate. At lower pressures the rate falls off, but above a certain *crucial pressure*, which, in the case of hydrogen and oxygen, seems to be about two atmospheres, the velocity is independent of the pressure.

It will be observed that the mean rate of explosion of hydrogen and oxygen at 1500 mm. pressure is appreciably higher than BERTHELOT'S θ—the calculated rate of translation of the steam molecules. But BERTHELOT, in his calculation, has not taken into account the fact that the gases are not at absolute zero to start with. Assuming that he worked at about 13° C., the theoretical rate would be 2900 metres per second.

Analogous results were found in experiments with other mixtures under varying pressures. With hydrogen and nitrous oxide the rate is slower at 500 mm. than at 760 mm., but no increase is found on raising the pressure to 1000 mm. The crucial pressure is in this instance below the atmospheric pressure. The same was found to be the case with marsh gas, ethylene, acetylene, and cyanogen exploded with oxygen.

In the following table some of these results are shown :—

TABLE VII.—Pressure Experiments.

$H_2 + N_2O$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2094	2307	2302

$CH_4 + O_4$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2280	2322	2319

$CH_4 + O_3$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2418	2470	2488

$C_2N_2 + O_2$ (in 5 mm. tube).

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2536	2677	2671

§ 5. *Influence of Temperature on the Velocity of the Explosion-Wave.*

BERTHELOT has published no experiments on the influence of initial temperature on the explosion-wave; but, if the wave is propagated in a manner analogous to a sound wave, it might be anticipated that an increase in the initial temperature of the gases would increase the velocity. The reverse effect is found in the explosion of hydrogen and oxygen. Experiments made at 10° C. and 100° C. showed a small, but distinct, lowering of the rate at the higher temperature. The experiments were made alternately at the high and low temperature, in order to eliminate the effect which might be produced by any mistake in the mixture or error in the chronograph. The comparative experiments made at 10° agreed with those previously obtained under the same conditions, so that the results found at 100° must be accepted.* A similar lowering of the rate at 100° was found to occur at low and high pressures.

Table VIII.

Temperature Experiments. $H_2 + O$.

Temperature.	10°.	100°.
Mean rate . .	2821	2790

$H_2 + O$ at 100° C. at different pressures.

Pressure.	390 mm.	500 mm.	760 mm.	1000 mm.	1450 mm.
Mean rate . .	2697	2738	2790	2828	2842

When the rates at 100° and varying pressures are compared graphically with the rates found at 10° and varying pressures, it is seen that the two curves run parallel one with the other (see fig. 1 on Plate 1).

The explosion-rates of ethylene and oxygen at 10° C. and 100° C. were also compared. Samples of the same mixture were exploded alternately at the low and at the high temperature. Five concordant determinations at 10° C. gave a mean velocity of 2581 metres per second; four concordant determinations at 100° C. gave a mean velocity of 2538 metres per second. Similar experiments were made with cyanogen.

Table IX.

Rate of Explosion of Ethylene and Oxygen. $C_2H_4 + 2O_2$.

Temperature.	10° C.	100° C.
Mean rate . .	2581	2538

Rate of Explosion of Cyanogen and Oxygen. $C_2N_2 + O_2$.

Temperature.	10° C.	100° C.
Mean rate . .	2728	2711

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* The expansion of the metal tube is taken into account in calculating the rates at 100° C.

The small differences in the rate of explosion of hydrogen and oxygen, and of ethylene and oxygen at 10° and at 100° show that the effect of ordinary atmospheric changes of temperature may be neglected.

§ 6. *Comparison of the Rate of Explosion of different Gaseous Mixtures with* BERTHELOT'S *results.*

The results obtained, under ordinary conditions of pressure and temperature, with hydrogen and oxygen, with hydrogen and nitrous oxide, and with marsh gas and oxygen, in exact proportions for complete combustion, were in close accordance with the mean results of BERTHELOT; for ethylene, acetylene, and cyanogen my numbers differed appreciably, but in no case differed by more than 7 per cent. from the rates observed by BERTHELOT. In the following table our measurements for the same mixtures are compared :—

TABLE X.—Mean Velocity of Explosion in Metres per Second.

		BERTHELOT.	DIXON.
Hydrogen and oxygen	H_2+O	2810	2821
Hydrogen and nitrous oxide	H_2+N_2O	2284	2305
Marsh gas and oxygen	CH_4+O_4	2287	2322
Ethylene and oxygen	$C_2H_4+O_6$	2210	2364
Acetylene and oxygen	$C_2H_2+O_5$	2482	2391
Cyanogen and oxygen	$C_2N_2+O_4$	2195	2321

The general agreement between these measurements leaves, I think, no room for doubt about the substantial accuracy of the results.

CAP. IV.—THE INFLUENCE OF STEAM ON THE RATE OF EXPLOSION OF CARBONIC OXIDE AND OXYGEN.

The results already described show that the formula proposed by BERTHELOT expresses with a close degree of approximation the rates of explosion of several gaseous mixtures. The formula fails for the explosion of carbonic oxide with oxygen or with nitrous oxide. This was to be expected if, in the *detonation* of carbonic oxide in a long tube, the oxidation is effected indirectly by means of steam—as it is in the ordinary combustion of the gas. Measurements of the rate of explosion of carbonic oxide and oxygen in a long tube showed that the rate increased as steam was added to the dry mixture, until a maximum velocity was attained when between 5 and 6 per cent. of steam was present.

The experiments were conducted in the following way:—In the "dry" experiments the coil of pipe was heated and a current of dry air drawn through it. The mixture of carbonic oxide and oxygen was passed into the dry coil through drying apparatus, consisting of two wash-bottles and three pumice towers containing boiled oil of vitriol, and then two long tubes packed with anhydrous phosphoric acid. The mixture under these conditions is called "well-dried"; it readily transmits the explosion. In the next experiments the phosphoric acid tubes were omitted; the mixture is called "dried." For saturating the gases with steam the mixture was led through water in three small wash-bottles before entering the coil. The first wash-bottle was kept about 5° C., the second about 3° C., and the third exactly 1° above the temperature at which the mixture was to be exploded. The last bottle and the connections between it and the coil were kept entirely surrounded by water. The gas, on its passage through the wash-bottles, carried over steam, which was partly condensed in the cooler wash-bottles and in the coil. While the gas was being admitted to the coil the water jacket was kept about ·5° above the final temperature required. When the tube was full the source of heat was removed, and the water in the jacket well stirred. When the exact temperature was reached the stop-cocks were closed and the gases fired. This arrangement ensured that the gases were *saturated* with moisture, and prevented any accumulation of liquid in the coil which might have retarded the explosion.

In the following table the *mean* results obtained with the dried and with the moistened gases are given:—

TABLE XI.—Rate of Explosion of Carbonic Oxide and Oxygen saturated with Steam at Different Temperatures.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Well dried	..	1264
Dried	..	1305
Saturated at 10° C.	1·2	1676
,, 20° C.	2·3	1703
,, 28° C.	3·7	1713
,, 35° C.	5·6	1738
,, 45° C.	9·5	1693
,, 55° C.	15·6	1666
,, 65° C.	24·9	1526
,, 75° C.	38·4	1266

The maximum rate was obtained when the mixture was saturated at 35° C.—*i.e.*, contained 5·6 per cent. of steam. Excess of steam slowly retards the rate. A few experiments were made above and below the atmospheric pressure, which showed that the maximum rate always occurred when the same percentage of steam was present. Under 1100 mm. pressure the fastest rate was found when the mixture was saturated at 43°—*i.e.*, contained 5·6 per cent. of steam; and under 400 mm. pressure the fastest rate was found when the mixture was saturated at 25°—*i.e.*, contained 5·6 per cent. of steam. Increase of pressure increased, and diminution of pressure diminished, the rate of explosion of moist carbonic oxide and oxygen. Under 300 mm. pressure the explosion-wave was not propagated in the mixture of carbonic oxide and oxygen saturated at 10° C. In one experiment the flame reached the end of the tube, taking about 30 seconds to travel 100 metres. Under 400 mm. pressure the explosion-wave was propagated.

TABLE XII.—Rate of Explosion of Carbonic Oxide and Oxygen.

1. UNDER 1100 mm. pressure.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Saturated at 26° C.	2·3	1737
,, 43° C.	5·6	1782
,, 53° C.	9·5	1742

2. UNDER 400 mm. pressure.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Saturated at 10° C.	2·3	1576
,, 25° C.	5·6	1616
,, 33° C.	9·5	1570

These results are shown graphically by curves on fig. 2, Plate 1, in which the ordinates are the rates of explosion, and the abscissæ the percentages of steam present in the mixture.

These curves show the very marked increase of the rate of explosion on adding 1 or 2 per cent. of steam to the dried gases. Now the addition of steam to other dry gaseous mixtures, such as hydrogen and oxygen, ethylene and oxygen, cyanogen and oxygen, has been found to *lower* the rate of explosion. I think these results, therefore, are strong evidence that *at the extreme temperatures of the explosion-wave, as well as in ordinary combustion, carbonic oxide is oxidised by the steam, and not directly by the oxygen.*

CAP. V.—THE MODE OF BURNING OF GASEOUS CARBON.

§ 1. The fact that carbonic oxide is not acted on directly by oxygen, even in the explosion-wave, points to the possibility that the carbon in gaseous hydrocarbons, and in other volatile carbon compounds, is not oxidised directly to carbonic acid when these gases are burnt with an excess of oxygen. When cyanogen, for instance, is exploded with oxygen, the carbon may burn in one or in two stages. Each atom may either combine directly with two atoms of oxygen to form carbonic acid, or each atom may combine first with a single atom of oxygen to form carbonic oxide, which afterwards combines with a second atom of oxygen to form carbonic acid. If the oxidation is effected in *two* stages, the chemical change first occurring, viz., the oxidation of the carbon to carbonic oxide, must take place in the presence of an excess of oxygen. Will this excess of oxygen influence the velocity with which the *initiation* of the chemical change is propagated, *i.e.*, the rate of explosion? The interest of tracing the *course* of any chemical change, and the scientific importance of this particular reaction, led me to study the influence of the presence of an excess of oxygen, and of an inert gas, on the rate of explosion of gaseous mixtures—with a view to obtain evidence on their mode of burning.

§ 2. *The Retardation of the Wave caused by Inert Gases.*

When electrolytic gas is mixed with oxygen, nitrogen, or steam, the explosion-wave is retarded, the oxygen having most effect, the steam least. For instance, the addition of one volume of steam to three volumes of electrolytic hydrogen and oxygen reduced the rate of explosion to 2494 metres per second in experiments made at 65° C. At the same temperature the addition of one volume of nitrogen to three of electrolytic gas reduced the rate to 2402 metres per second. According to BERTHELOT's formula a retardation of this order was to be expected—the greater specific heat of the steam being more than counterbalanced by the greater density of the nitrogen. Experiments made with hydrogen and oxygen at ordinary temperatures, with successive additions of oxygen and of nitrogen respectively, showed that, so long as the explosion-wave was propagated in both cases, the retarding influence of nitrogen was less than that of an equal volume of oxygen. Three volumes of electrolytic gas can be exploded with five volumes of nitrogen, but with seven volumes of nitrogen the explosion is not propagated through a long tube; whereas three volumes of electrolytic gas can still be exploded when mixed with eight of oxygen. But so long as the volume of nitrogen added is not sufficient to stop the explosion, the addition of this inert gas, incapable of taking part in the chemical change, produces less effect than the addition of oxygen, one of the reacting substances.

Some experiments published by BERTHELOT in 1885[*] seem to be at variance with this statement. When electrolytic gas, largely diluted with oxygen, was exploded in a bomb, the flame traversed the bomb more quickly than when an equal volume of nitrogen was used as a diluent. But the rate measured in this case was not that of the explosion-wave; it was the mean rate of the flame from the point of inflammation to a point about 250 mm. distant. On the other hand, with a smaller volume of diluent gas, BERTHELOT found the rate of explosion to be less retarded by nitrogen than by oxygen:—

	Time taken by flame to travel 250 mm.
$H_2 + O$	2·14 thousandths of a second.
$H_2 + O + 3O_2$	16·04 ,, ,,
$H_2 + O + 3N_2$ . . .	24·45 ,, ,,
$H_2 + O + O_2$	8·16 ,, ,,
$H_2 + O + N_2$	6·87 ,, ,,

I have found that the explosion-wave is sooner initiated when electrolytic gas is largely diluted with oxygen than when it is diluted with the same volume of nitrogen; a fact which explains the apparent anomaly in BERTHELOT's experiments.

On repeating my measurements of the rate of explosion of electrolytic gas with large volumes of diluent gases, some discrepancies were found, which could not at first be explained. A tube of larger diameter (13 mm.) was then substituted for the smaller tube (9 mm.). The results were more regular. It was then discovered that the explosion frequently died out near the end of the smaller tube, whilst transmitting a sound-wave sufficiently powerful to break the silver foil composing the "second bridge." A record was thus obtained on the chronograph plate, indicating a smaller velocity than the true rate of explosion. By inserting a glass tube just before the second bridge, and watching for the flame, the observers made sure that the explosion-wave reached the end of the pipe. A second precaution, which must also be observed in experiments on largely diluted gases, is to make sure that the explosion-wave has been set up before the flame reaches the first bridge. M. BERTHELOT does not state the free run he allowed his flame before beginning the measurements; but, from an inspection of the figure of his apparatus ('Sur la Force des Matierès Explosives,' vol. 1, p. 138) it would appear that a few inches only were allowed between the firing spark and his first "interrupter." No doubt, the use of a grain of fulminate in the interrupter hastened the establishment of the explosion-wave in his experiments, but I think it possible that in some of his measurements he began to record the rate before the wave was started, and so obtained a rate below the true one. In all my later experiments on diluted gases, some pure electrolytic gas was passed into the long firing piece, just before the explosion, so as to fill it for a length of two or three feet.

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[*] 'Ann. Chim. Phys.' [VI.], vol. 4, p. 44.

The explosion-wave, rapidly started in this mixture, was communicated to the dilute mixture beyond, without break of continuity, before the first bridge was reached.

A comparison between the few experiments made by BERTHELOT with diluted mixtures and the similar ones made by me will, I think, show the need of these precautions. To three volumes of electrolytic gas BERTHELOT added two volumes of nitrogen : the rate of explosion was found to be 2121 metres per second. To three volumes of electrolytic gas he added four volumes of nitrogen : the rate of explosion was only 1439 metres per second. By interpolation* I find from my own experiments with one, three, and five volumes of nitrogen (added to three volumes of electrolytic gas) the rates 2200 and 1930, for the explosion of electrolytic gas with two volumes and four volumes of nitrogen respectivly. The first agrees fairly with BERTHELOT's number, the second is far higher than his. I imagine in the second case the interrupter was broken before the explosion-wave was established in his experiment. Similarly, in two experiments with cyanogen, BERTHELOT found a rapid falling off of the rate which I did not observe. On firing a mixture of cyanogen with twice its volume of oxygen and one volume of nitrogen, he found the rate to be 2044 metres; on adding another volume of nitrogen the rate fell to 1203 metres. By interpolation I find from my experiments 2125 and 1960 as the rates of these two mixtures. The first agrees fairly well with, the second is far higher than, BERTHELOT's number.

The *mean* results obtained in a number of experiments on the retarding effect of oxygen and nitrogen are given in the following table :—

TABLE XIII.—Electrolytic Gas with excess (1) of Oxygen and (2) of Nitrogen.

(1.)

Mixture.	2 vols. hydrogen } 1 vol. oxygen. } $H_2 + O$	With addition of 1 vol. oxygen. $H_2 + O + O$	With addition of 3 vols. oxygen. $H_2 + O + O_3$	With addition of 5 vols. oxygen. $H_2 + O + O_5$	With addition of 7 vols. oxygen. $H_2 + O + O_7$
Mean rate . .	2821	2328	1927	1707	1281

(2.)

Mixture.	2 vols. hydrogen } 1 vol. oxygen. } $H_2 + O$	With addition of 1 vol. nitrogen. $H_2 + O + N$	With addition of 3 vols. nitrogen. $H_2 + O + N_3$	With addition of 5 vols. nitrogen. $H_2 + O + N_5$	With addition of 7 vols. nitrogen. $H_2 + O + N_7$
Mean rate . .	2821	2426	2055	1822	—

These results are shown graphically in fig. 3, Plate 1, where the ordinates are volumes of nitrogen (or oxygen) added to three volumes of electrolytic gas, and the abscissæ are the rates of explosion of the mixtures in metres per second.

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* See fig. 3, Plate 1.

These curves plainly show that the retardation of the explosion-wave depends upon the density of the gas added as well as upon its volume—a result in accordance with BERTHELOT's theory.

I think it a fair inference from these facts to conclude, when the addition of a gas to an explosive mixture retards the rate of explosion by an amount which depends upon its volume and density, that such added gas is inert as far as the propagation of the wave is concerned, and that any change which it may undergo takes place after the wave-front has passed by—in other words, is a *secondary* change.

§ 3. *Oxygen Inert to Carbonic Oxide in Explosion of Carbon Compounds.*

This principle can be applied to determine whether, in the combustion of gaseous carbon, the oxidation to carbonic acid is effected in one or two stages. If, for instance, the carbon is first burnt to carbonic oxide, which subsequently is burnt to carbonic acid, the rate of the explosion-wave should correspond with the carbonic oxide reaction retarded by an excess of oxygen ; whereas, if the carbon burns to carbonic acid directly, in one stage, then the rate of the explosion-wave should correspond with the complete and unretarded reaction.

Now, if we adopt BERTHELOT's theory as a working hypothesis, we can calculate the theoretical rates of explosion of marsh gas, ethylene, or cyanogen : (1) on the supposition that, with sufficient oxygen, the carbon burns directly to carbonic acid, and (2) on the supposition that the carbon burns first to carbonic oxide, and the further oxidation is a subsequent or secondary reaction. On the first supposition, if 100 represents the rate of explosion of these three gases with oxygen only sufficient to burn them to carbonic oxide, the addition of the oxygen required to burn the gases to carbonic acid should *increase* the rate of explosion :—

	Marsh gas.	Ethylene.	Cyanogen.
Calculated rate of explosion when burnt to CO_2 .	104	103	107

Whereas, if these gases really burn first to carbonic oxide, and the extra oxygen is inert in propagating the explosion-wave, then the addition of this inert oxygen would *diminish* the rate of explosion :—

	Marsh gas.	Ethylene.	Cyanogen.
Calculated rate of explosion when burnt to CO with inert oxygen present	92	88	87

The experiments show that if 100 be taken as the rate of explosion when the oxygen is only sufficient to burn the carbon to carbonic oxide, the following are the rates found when oxygen is added sufficient to burn the carbon to carbonic acid :—

	Marsh gas.	Ethylene.	Cyanogen.
Rates found	94	92	84

The results are, therefore, in favour of the view that, in the explosion of these gases, the carbon is first burnt to carbonic oxide.*

But stronger evidence on this point is obtained by comparing the explosion rates of these gases (1) when fired with oxygen sufficient to burn the carbon in them to carbonic acid, and (2) when nitrogen is substituted for the oxygen in excess of that required to burn the carbon to carbonic oxide. We have seen that oxygen, added to electrolytic gas, hinders the explosion more than nitrogen. In precisely the same way oxygen, added to a mixture of equal volumes of cyanogen and oxygen, hinders the explosion more than the same volume of nitrogen. The conclusion we must come to is, that the oxygen, added to the mixture expressed by the formula $C_2N_2 + O_2$, is as inert (so far as the propagation of the explosion-wave is concerned) as oxygen added to the mixture expressed by the formula $H_2 + O$. The same phenomena occur in the explosion of marsh gas, ethylene, and acetylene.

The following tables contain the results of my last experiments on this subject. The rates come out on the whole somewhat higher than in my earlier experiments. This is mainly due to the later explosions having been made in a tube of larger diameter—9 mm. instead of 5 mm.

TABLE XIV.—The Rate of Explosion of Cyanogen (1) with Oxygen, and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	1 vol. cyanogen 1 vol. oxygen $C_2N_2+O_2$	With addition of 1 vol. oxygen. $C_2N_2+2O_2$	With addition of 2 vols. oxygen. $C_2N_2+3O_2$
Mean rate	2728	2321	2110

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* [It might be urged that, with defect of oxygen, the reactions really are—

1. $CH_4 + O_3 = CO_2 + H_2O + H_2$
2. $C_2H_4 + O_4 = 2CO_2 + 2H_2$
3. $C_2N_2 + O_2 = CO_2 + C + N_2$

If this were true, and 100 represented the observed rates of explosion in each case, then on adding oxygen sufficient for complete combustion, the rates should be 101, 99, and 107 respectively.—Jan., 1893.]

(2.)

Mixture.	1 vol. cyanogen 1 vol. oxygen. $C_2N_2+O_2$	With addition of 1 vol. nitrogen. $C_2N_2+O_2+N_2$	With addition of 2 vols. nitrogen. $C_2N_2+O_2+2N_2$
Mean rate . .	2728	2398	2165

Cyanogen will explode when mixed with four times its volume of oxygen. When three volumes of nitrogen are added to one volume of cyanogen and one of oxygen, the mixture will not explode, but burns at the mouth of the tube. So long, however, as the nitrogen does not prevent the explosion, its addition to the mixture has less retarding effect than the addition of an equal volume of oxygen.

TABLE XV.—The Rate of Explosion of Marsh Gas (1) with Oxygen and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	2 vols. methane 3 vols. oxygen CH_4+O_3	With addition of 1 vol. oxygen. CH_4+O_4	With addition of 3 vols. oxygen. CH_4+O_6	With addition of 5 vols. oxygen. CH_4+O_8
Mean rate . .	2470	2322	2146	1963

(2.)

Mixture.	2 vols. methane 3 vols. oxygen CH_4+O_3	With addition of 1 vol. nitrogen. CH_4+O_3+N	With addition of 3 vols. nitrogen. $CH_4+O_3+N_3$	With addition of 5 vols. nitrogen. $CH_4+O_3+N_5$
Mean rate . .	2470	2349	2154	1880 (goes out sometimes)

The addition of one and of three volumes of nitrogen to the mixture of marsh gas and oxygen ($CH_4 + O_3$) is found to retard the rate less than the addition of the same volumes of oxygen. When, however, five volumes of nitrogen are added, the retardation is greater than with oxygen : this mixture is near the limit of explosion, and once or twice the flame died out before reaching the end of the tube.*

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* Many attempts were made to explode mixtures of marsh gas and air in a tube. A vigorous explosion, initiated by firing electrolytic gas down the first few feet of the tube, soon dies out in the mixtures of marsh gas and air. In a metal tube 1 inch in diameter the flame travelled a considerable distance, but in a slow and irregular way. Similar results were obtained with mixtures of coal-gas and air. On the other hand, mixtures of ethylene and air explode readily down a narrow tube.

TABLE XVI.—The Rate of Explosion of Ethylene (1) with Oxygen, and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	1 vol. ethylene 2 vols. oxygen $C_2H_4+2O_2$	With addition of 1 vol. oxygen. $C_2H_4+3O_2$	With addition of 2 vols. oxygen. $C_2H_4+4O_2$	With addition of 4 vols. oxygen. $C_2H_4+6O_2$	With addition of 6 vols. oxygen. $C_2H_4+8O_2$	With addition of 8 vols. oxygen. $C_2H_4+10O_2$
Mean rate .	2581	2368	2247	2118	1980	1856

(2.)

Mixture.	1 vol. ethylene 2 vols. oxygen $C_2H_4+2O_2$	With addition of 1 vol. nitrogen. $C_2H_4+2O_2+N_2$	With addition of 2 vols. nitrogen. $C_2H_4+2O_2+2N_2$	With addition of 4 vols. nitrogen. $C_2H_4+2O_2+4N_2$	With addition of 6 vols. nitrogen. $C_2H_4+2O_2+6N_2$	With addition of 8 vols. nitrogen. $C_2H_4+2O_2+8N_2$
Mean rate .	2581	2413	2211	2024	1878	1734

In the case of ethylene, the retarding action of oxygen, which is greater than that of nitrogen for the first volume added to the mixture ($C_2H_4+2O_2$), becomes less and less as a large excess of oxygen is added; so that the rate of explosion of ethylene and air would be considerably increased by the substitution of oxygen for the nitrogen of the air. With a large excess of oxygen the *secondary* reaction, whereby carbonic acid is formed, influences the result, for the conditions are eminently favourable for the interaction of the carbonic oxide, steam, and oxygen.

TABLE XVII.—The Rate of Explosion of Acetylene (1) with Oxygen, and (2) with Oxygen and Nitrogen.*

(1.)

Mixture.	2 vols. acetylene 3 vols. oxygen $C_2H_2+O_3$	With addition of 2 vols. oxygen. $C_2H_2+O_5$
Mean rate . .	2716	2391

(2.)

Mixture.	2 vols. acetylene 3 vols. oxygen $C_2H_2+O_3$	With addition of 2 vols. nitrogen. $C_2H_2+O_3+N_2$
Mean rate . .	2716	2414

These experiments show that in all the cases examined, viz., the combustion of cyanogen, marsh gas, ethylene, and acetylene, the substitution of nitrogen for the oxygen required to burn the carbon from carbonic oxide to carbonic acid *increases* the velocity of the explosion. *These facts seem only consistent with the view that the carbon burns directly to carbonic oxide, and the formation of carbonic acid is an after-occurrence.*

§ 4. BERTHELOT'S *Experiments bearing on the mode of Combustion of Carbon.*

M. BERTHELOT found that the pressures produced in the explosion of cyanogen were greater for the *incomplete* than for the *complete* combustions.†

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* No doubt with a large excess of oxygen the explosion of acetylene would be less retarded than with a large excess of nitrogen, as in the case of ethylene. Owing to the expense of preparing the acetylene, experiments with a large excess of oxygen were not made.

† 'Ann. Chim. et Phys.' [VI.], vol. 4, p. 38.

Mixtures.	Pressures.
$C_2N_2 + O_2$	25·11 atmospheres.
$C_2N_2 + 2O_2$	20·96 ,,
$C_2N_2 + 2N_2O$	26·02 ,,
$C_2N_2 + 4N_2O$	22·66 ,,
$C_2N_2 + 2NO$	23·34 ,,
$C_2N_2 + 4NO$	16·92 ,,

It is difficult to reconcile these numbers with the received view as to the burning of carbon; they are readily explained on the hypothesis that carbon burns in two stages.

Berthelot has also made some observations of the initial rate of explosion of cyanogen when burnt to carbonic oxide and when burnt to carbonic acid. The time taken by the flame in travelling a distance of about 130 mm. from the point of inflammation was *less* for the incomplete than for the complete combustion; a result which agrees with my measurements of the rate of the explosion-wave in the two cases. When a mixture of carbonic oxide, oxygen and nitrogen, having a composition corresponding to the first stage of the combustion of cyanogen, was fired in the same apparatus, the rate of the flame was found by Berthelot to be far slower than in the complete combustion of cyanogen:—

	Time taken by flame to travel 130 mm.
$C_2N_2 + O_2$	1·05 thousandths of a sec.
$C_2N_2 + 2O_2$	1·55 ,, ,,
$2CO + O_2 + N_2$	17·78 ,, ,,

M. Berthelot adds:—“D'après ces nombres, il ne paraît pas que la combustion totale du cyanogène s'effectue en deux temps, en formant d'abord en totalité de l'oxyde de carbone qui brûlerait ensuite; car la combustion totale est beaucoup plus rapide que la somme de ces deux effets séparés.”*

On first reading this passage I understood M. Berthelot to mean that in the complete combustion of cyanogen there was no intermediate formation of carbonic oxide. But I am led to think that this interpretation cannot be correct for two reasons. First, because the conditions of temperature and pressure are so entirely different in the two cases compared, that no argument drawn from one can be applied to the other. In one case, a reaction is initiated in a mixture of carbonic oxide, oxygen, and nitrogen at the atmospheric temperature and pressure; in the other, the same gases are at an enormously high temperature and pressure when the reaction begins. Secondly, because M. Berthelot, in the same paper, has drawn from analogous facts an argument in an exactly opposite sense. Having determined the initial rate of explosion of hydrogen and oxygen, and of carbonic oxide and oxygen in a bomb, he found the rate of explosion of the mixture of the two to be less than the mean of the two separately :—

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* ‘Ann. Chim. et Phys.,’ [VI.] vol. 4, p. 43.

	Time.
$H_2 + O$	1·04
$CO + O$	12·86
$H_2 + CO + O_2$	3·88

"La vitesse de combustion n'est, dans aucun cas, la moyenne de celle des composants mélangés. Mais les deux gaz paraissent tendre à brûler séparément, chacun avec sa vitesse propre."

The meaning of the passage quoted, concerning the combustion of cyanogen, would therefore appear to be : "In the complete combustion of cyanogen there are not two flames propagated from the point of inflammation with different velocities—the quicker due to the burning of cyanogen to carbonic oxide, the slower to the burning of carbonic oxide to carbonic acid." With this literal interpretation of the passage I entirely agree : there are not two flames or two explosions, but the burning of each molecule takes places in two stages, the second stage prolonging the duration of the flame due to the first.

§ 5. *Other Experiments on the burning of Carbon.*

It is usually assumed that carbon, in all its states, burns to carbonic acid, and then the excess of carbon, if it is present, reduces the carbonic acid, first formed, to carbonic oxide. This view, no doubt, originated from the observation that solid carbon burns in oxygen, without apparent flame, to form carbonic acid ; and that carbonic acid is readily reduced to carbonic oxide when passed over heated carbon. The experiments, however, of Mr. BRERETON BAKER ('Phil. Trans.,' 1888) show that purified charcoal may be heated to redness in a current of dry oxygen without visible combustion taking place, but with the formation of carbonic oxide—although the oxygen is in large excess. It would appear, from Mr. BAKER'S observations, that carbonic oxide is the first product of the action of oxygen on solid carbon at a high temperature ; the formation of carbonic acid being a secondary change due to the interaction of carbonic oxide, steam and oxygen. Both solid and gaseous carbon, therefore, appear to undergo oxidation in two stages ; the second stage being facilitated by the presence of steam.

The evidence, above described, led me, two years ago, to hazard an explanation of the appearance of a cyanogen flame burning in air. The crimson inner zone might be due to the primary reaction—the burning of the cyanogen to carbonic oxide and nitrogen, with the limited air supply that reaches it ; the blue outer zone might be due to the burning of the carbonic oxide (formed in the crimson zone), as it meets with a plentiful supply of air outside. A few experiments I made showed the presence of carbonic oxide in abundance on the edge of the crimson zone; but I have left the further investigation of the structure of the cyanogen flame to Professor A. SMITHELLS, who has devised a most ingenious method of separating the outer and inner cones of flame, formed when a mixture of a combustible gas and air is burnt in a Bunsen burner. His experiments show that the cyanogen is completely destroyed in the inner red flame, and carbonic oxide is the main product of its oxidation there. This carbonic oxide burns above, on reaching the air, with its well-known blue flame[*] ('Chem. Soc. Journ.,' 1892, vol. 1, p. 215).

CAP. VI.—THE REACTIONS OF CARBONIC OXIDE IN THE FLAME.

If it be true that in a flame gaseous carbon burns to carbonic oxide, and that carbonic oxide reacts with steam (if present) to a limited extent, and is completely oxidised in presence of steam and oxygen, two questions arise: (1) What happens to the carbonic oxide first formed in the combustion of hydrocarbons or cyanogen with excess of oxygen? and (2) How do these secondary reactions affect the rate of explosion? In the case of cyanogen we should expect, if the gases are fairly dry, that the formation of carbonic acid would be comparatively slow, since no steam is formed in the reaction. We find, in accordance with this expectation, that so long as the mixtures will explode, successive additions of oxygen retard the rate more than the additions of nitrogen—just as in the case of electrolytic gas. This means that in the cyanogen explosion the excess of oxygen and carbonic oxide are as inert towards each other (as far as the propagation of the wave is concerned) as the excess of oxygen and steam are in the explosion of electrolytic gas. On the other hand, the carbonic oxide formed in the explosion of hydrocarbons, with excess of oxygen, meets with the steam also formed in the flame. This is the condition necessary for its most rapid oxidation. We should expect, therefore, to find in the explosions of hydrocarbons that the retarding action of large quantities of oxygen is partly counteracted by this secondary reaction, for the oxygen in presence of the steam is not inert towards the carbonic oxide. As a matter of fact, we find that the first additions of oxygen to the hydrocarbon mixtures retard more than nitrogen, but as the additions are continued the oxygen has less and less retarding influence compared with the nitrogen.

When carbonic oxide, which has the same density as nitrogen, is added to electrolytic gas, the retarding effect on the explosion is very nearly the same as when an equal volume of nitrogen is added:—

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[*] The greenish-grey tint, observable in the outer flame, may be due to some oxide of nitrogen formed in the inner flame. A little peroxide of nitrogen gives a similar tint to an ordinary Bunsen gas flame; and peroxide of nitrogen is formed when cyanogen is burnt under certain conditions (*vide* 'Chem. Soc. Journ.,' 1886, vol. 1, p. 390).

TABLE XVIII.—Rate of Explosion of Electrolytic Gas (1) with Nitrogen, and (2) with Carbonic Oxide. (K.)

(1.)

Mixture.	2 vols. hydrogen and 1 vol. oxygen. H_2+O	With addition of 1 vol. nitrogen. H_2+O+N	With addition of 3 vols. nitrogen. H_2+O+N_3
Mean rate .	2821	2426	2055

(2.)

Mixture.	2 vols. hydrogen and 1 vol. oxygen. H_2+O	With addition of 1 vol. carbonic oxide. H_2+O+K	With addition of 3 vols. carbonic oxide. H_2+O+K_3
Mean rate .	2821	2455	2080

It is true that carbonic acid and free hydrogen are found in the tube after the explosion ; but these are due to a *secondary* reaction, which need not necessarily affect the rate ; in other words, the carbonic acid may be formed in the heated gases after the wave-front has passed by.

This similarity in the retarding action of carbonic oxide and nitrogen can be explained in two ways. First, the carbonic oxide may be for the most part unaffected in the initial change occurring in the wave-front, and be, consequently, as inert as an equal volume of nitrogen. Secondly, the carbonic oxide—inert towards the hydrogen and oxygen—may react with the steam produced both in the wave-front and behind it. The reaction between the steam and carbonic oxide *in the wave-front* would slightly quicken the explosion by developing heat. The experiments show that nitrogen retards the wave to an extent just appreciably greater than carbonic oxide.

To test the correctness of these views concerning the *rôle* played by carbonic oxide in the flame, an experiment was tried with electrolytic gas diluted (1) with oxygen and nitrogen, and (2) with oxygen and carbonic oxide. We have seen that electrolytic gas, when diluted with carbonic oxide, explodes at nearly the same rate as when diluted with nitrogen. Now, if the views expressed above are correct, we may replace some of the diluent nitrogen by oxygen, and thereby make the explosion slower ; but if the diluent carbonic oxide is partly replaced by oxygen, the explosion should be considerably faster. The rates found in the two cases agree with this prediction :—

TABLE XIX.

(1.)

Mixture.	2 vols. hydrogen 1 vol. oxygen H_2+O	With addition of 3 vols. nitrogen. H_2+O+N_3	With addition of 2 vols. nitrogen and 1 vol. oxygen. $H_2+O+O+N_2$
Mean rate.	2821	2055	2003

(2.)

Mixture.	2 vols. hydrogen 1 vol. oxygen H_2+O	With addition of 3 vols. carbonic oxide. H_2+O+K_3	With addition of 2 vols. carb. ox. and 1 vol. oxygen. $H_2+O+O+K_2$
Mean rate .	2821	2080	2143

A study of these several tables brings us to the conclusion that secondary reactions, whereby carbonic oxide is oxidised in the flame, may affect the velocity of the wave to a smaller extent when the explosion is rapid, to a greater extent when the explosion is slow.

CAP. VII.—EXAMINATION OF BERTHELOT'S THEORY.

The measurements of the rate of explosion of the various gaseous mixtures described in Cap. V. permitted an extended comparison between the theoretical velocities calculated by BERTHELOT'S formula (θ) and the actual rates (ν). This comparison is made in the following tables for electrolytic gas, diluted with oxygen and with nitrogen; for cyanogen burning to carbonic oxide, and when this mixture is diluted with nitrogen; for marsh gas, ethylene, and acetylene, all burning to carbonic oxide and steam, and for the same gases diluted with nitrogen. In calculating θ, I made a correction for the gases being at the ordinary temperature (13° C.) before explosion, and not at absolute zero; the theoretical velocities are, therefore, rather higher than those given by BERTHELOT. This correction raises the theoretical velocity of explosion of electrolytic gas from 2831 metres per second (BERTHELOT) to 2900.

TABLE XX.—Electrolytic Gas with excess of Oxygen added.

Mixture.	2 vols. hydrogen 1 vol. oxygen. H_2+O	With addition of 1 vol. oxygen. H_2+O+O	With addition of 3 vols. oxygen. H_2+O+O_3	With addition of 5 vols. oxygen. H_2+O+O_5
Theory, θ . .	2900	2252	1730	1476
Found, ν . .	2821	2328	1927	1707

TABLE XXI.—Electrolytic Gas with excess of Nitrogen added.

Mixture.	H_2+O	H_2+O+N	H_2+O+N_3	H_2+O+N_5
Theory, θ . .	2900	2321	1814	1558
Found, ν . .	2821	2426	2055	1822

These measurements show that BERTHELOT'S formula, which gives a calculated rate 3 per cent. too high for pure electrolytic gas, also gives a calculated rate which is found to be 16 and 17 per cent. *too low* for the same gas when largely diluted with oxygen and with nitrogen respectively.

Again, in the explosion of cyanogen to carbonic oxide with its own volume of oxygen, and also when the same mixture is diluted with nitrogen, the theoretical rates calculated by BERTHELOT'S formula are far too low :—

TABLE XXII.—Cyanogen and Oxygen with excess of Nitrogen added.

Mixture.	1 vol. cyanogen 1 vol. oxygen. $C_2N_2+O_2$	With addition of 1 vol. nitrogen. $C_2N_2+O_2+N_2$	With addition of 2 vols. nitrogen. $C_2N_2+O_2+2N_2$
Theory, θ . .	2361	2083	1877
Found, ν . .	2728	2397	2166

The observed rates of explosion are 15 per cent. higher than the calculated rates. Similar differences are found between the observed and the calculated rates when the cyanogen mixture is diluted with oxygen.

When marsh gas and ethylene are exploded with oxygen sufficient to burn them to carbonic oxide and steam, the observed rates are higher than those given by the formula. Again, when these mixtures are diluted with nitrogen the rates come out higher than the formula predicts.

TABLE XXIII.—Marsh Gas and Oxygen with excess of Nitrogen added.

Mixture.	2 vols. methane } 3 vols. oxygen. } CH_4+O_3	With addition of 1 vol. nitrogen. CH_4+O_3+N	With addition of 3 vols. nitrogen. $CH_4+O_3+N_3$
Theory, θ . .	2387	2211	1958
Found, ν . .	2470	2349	2154

For pure marsh gas and oxygen, the calculated velocity is 3½ per cent. too low; when 3 volumes of nitrogen are added, the calculated velocity is 10 per cent. too low. The mixture cannot be detonated with a large excess of nitrogen.

TABLE XXIV.—Ethylene and Oxygen with excess of Nitrogen added.

Mixture.	$C_2H_4+2O_2$	$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
Theory, θ .	2487	2234	2050	1796	1624	1498
Found, ν .	2581	2413	2211	2024	1878	1734

The divergence between the calculated and observed rates varies with the dilution from 4 per cent. to 16 per cent. The calculated velocities are all too low. In the following table are given the rates of explosion of acetylene burning to carbonic oxide and steam, alone, and with successive additions of nitrogen.

TABLE XXV.—Acetylene and Oxygen with excess of Nitrogen added.

Mixture.	2 vols. acetylene 3 vols. oxygen. $C_2H_2+O_3$	$C_2H_2+O_3+N_2$	$C_2H_2+O_3+3N_2$	$C_2H_2+O_3+4N_2$	$C_2H_2+O_3+5N_2$	$C_2H_2+O_3+6N_2$
Theory, θ .	2749	2397	1983	1848	1739	1650
Found, ν .	2716	2414	2209	2116	2019	1908

Here, as in the case of electrolytic gas, the rate for the pure mixture is in close agreement with the calculated velocity. Taken by itself, this experiment (not tried by BERTHELOT) would have offered strong confirmation of the correctness of the theory. But as successive additions of nitrogen are made to the mixture, the calculated velocities fall *below* the observed rates, until, with a large excess of nitrogen, the calculated rate is 16 per cent. too low.

The gradual divergence between the observed and the calculated rates of explosion for different gaseous mixtures, as successive quantities of nitrogen were added to the pure explosive gas, led me to try the effect of diluting electrolytic gas with hydrogen. According to BERTHELOT's formula a slight increase in the rate should occur; actually a very marked increase in the rate was observed.

TABLE XXVI.—Rate of Explosion of Electrolytic Gas with excess of Hydrogen added.

Mixture.	2 vols. hydrogen 1 vol. oxygen. H_2+O	With addition of 2 vols. hydrogen. H_2+O+H_2	With addition of 4 vols. hydrogen. H_2+O+H_4	With addition of 6 vols. hydrogen. H_2+O+H_6
Theory, θ . .	2900	3055	3061	3028
Found, ν . .	2821	3268	3527	3532

To compare with this series of experiments, and with those given in Table XXI., a mixture of hydrogen and nitrous oxide was exploded (1) by itself, (2) with excess of hydrogen, and (3) with excess of nitrogen. A similar divergence between the calculated and observed rates was found both when the mixture was diluted with hydrogen, and when it was diluted with nitrogen.

TABLE XXVII.—Rate of Explosion of Hydrogen and Nitrous Oxide, with excess of (1) Hydrogen, (2) Nitrogen.

(1.)

Mixture.	2 vols. hydrogen 2 vols. nitrous oxide. H_2+N_2O	With addition of 2 vols. hydrogen. $H_2+N_2O+H_2$	With addition of 4 vols. hydrogen. $H_2+N_2O+H_4$	With addition of 5 vols. hydrogen. $H_2+N_2O+H_5$
Theory, θ . .	2319	2374	2396	2400
Found, ν . .	2305	2545	2705	2732

(2.)

Mixture.	$H_2 + N_2O$	With addition of 2 vols. nitrogen. $H_2 + N_2O + N_2$	With addition of 3 vols. nitrogen. $H_2 + N_2O + N_3$
Theory, θ . .	2319	1912	1782
Found, ν . .	2305	1991	1880

Now M. BERTHELOT especially insists that the formula gives a *maximum* velocity which may be reached, but not surpassed, by the actual explosion-wave. If, then, the rates of explosion of the diluted mixtures had fallen below the calculated rates, such divergence might be explained on the supposition that the diluent gases interfere with the normal propagation of the wave, an explanation which is indeed advanced by BERTHELOT to account for the falling off in the rate of certain diluted mixtures observed by him. But since the theory assumes that the explosion-wave travels at the rate of the gaseous molecules themselves while they still retain all the heat developed by the reaction, the presence of inert molecules could not possibly *increase* this rate of motion.*

Since the amount and regularity of the divergence between the found and calculated rates precluded the idea of experimental error being its sole cause, I was driven to conclude either that the hypothesis was incorrect, or that the formulas used failed to express the hypothesis with exactness.

M. BERTHELOT'S theory is that the explosion does not travel with the velocity of *sound* in the heated gases, but with a velocity equal to the mean rate of translation of the molecules produced in the explosion.

Let us examine the mode in which BERTHELOT calculates the theoretical velocity, *i.e.*, the mean rate of translation of the products of combustion at the temperature of the explosion. In CLAUSIUS' formula

$$\nu = 29{\cdot}354 \sqrt{\frac{T}{d}}$$

M. BERTHELOT calculates the absolute temperature of the explosion by dividing the quantity of heat developed in the complete reaction by the specific heat of the products of combustion *taken at constant pressure.* He argues that each layer of gas, in transmitting the explosion, is heated under constant pressure. I cannot follow his reasoning.

"The combustion," he says, "in propagating itself from layer to layer, is preceded by the compression of the gaseous layer which it is about to transform.

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* It will be shown in the sequel that BERTHELOT'S formula fails to express the rates of explosion of some undiluted explosive mixtures, viz., hydrogen and chlorine, ammonia and nitrous oxide.

. . . The combustion of each layer produces both heat and at the same time the work necessary to compress the following layer—that is to say, the layer loses on this score just as much heat as it gained by its own compression. The whole proceeds, as far as the elevation of temperature is concerned, precisely as if we had operated under constant pressure." On the facts of the case there is no dispute. The gas is exploded in a closed vessel. Each layer is compressed before being fired; after firing it compresses the layer beyond it. Now, as regards this preliminary compression, each layer, in turn, expends the same energy as was previously communicated to it, and, therefore, it does no work of its own. But a gas heated under conditions where it does no work is raised to the same temperature as it would be had its volume remained constant. M. BERTHELOT admits that it would appear at first sight as if the gases were heated at constant volume; he adds that the concordance of the calculated with the observed numbers supports his explanation of the phenomena.

Again, the fact that each layer is fired by compression involves the preliminary heating of that layer, and this heating must be added to the heat developed by its burning. The temperature of the burning layer must, therefore, be greater than that obtained by dividing the heat developed in the chemical change by the specific heat of the products of combustion.

When a sound-wave alone is transmitted through a gas its velocity shows that each layer of gas forming the wave-front is heated by compression, and there is no reason why this should not happen when the compression is accompanied by a chemical change.

To what extent is each layer heated before combustion? MM. MALLARD and LE CHATELIER* state that the explosion-wave will be propagated when each layer is brought by compression to its own temperature of inflammation. For hydrogen and oxygen, they find this temperature to be about 550° C., and calculate that a pressure of 30 atmospheres must be exerted upon the gas to raise it to this point. This temperature may be regarded as the *lowest limit* of the preliminary heating of each layer before combustion. But having regard to the fact that the individual molecules of hydrogen and oxygen in a yet unburnt layer may, in the interval of time between their meetings, be heated above the point at which they combine with each other, we cannot affirm that this limit of temperature may not be greatly surpassed in the explosion.

If we regard the transmission of the explosion to be due to the collisions of the molecules, and assume that molecules which are chemically inactive towards each other act as elastic bodies when they come into collision, and that molecules which combine chemically lose energy of translation and gain energy of vibration, then it must happen that unburnt molecules come into collision with burnt molecules and take up their energy in the form of motion. For instance, in the explosion of hydrogen and chlorine, the energy of the hydrogen chloride formed will be communicated by collision to molecules of hydrogen and chlorine;[*] these *heated* molecules, moving forwards, will meet *unheated* molecules moving backwards, when combination will occur between those of opposite kind. Heated hydrogen will thus combine with cool chlorine, and heated chlorine with cool hydrogen. The heat of combustion is thus communicated to a molecule of hydrogen or chlorine which *shares it* with a molecule of the opposite kind; each hydrogen chloride molecule formed in turn will therefore have, on the average, a temperature corresponding to the heat of chemical combination *plus* half the heat of a molecule previously formed. According to this view the temperature reached by each successive layer would increase until it was *double* that due to the chemical change alone. The temperature of the explosion would then remain constant, and the wave would advance at a uniform rate as long as it met the same mixture of gases.

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* 'Recherches sur la combustion des Mélanges Gazeux Explosifs,' p. 88-91.

My conception may perhaps be illustrated by an analogy, although an imperfect one. Imagine a kilogram of water at 0° falling *in vacuo* through 425 metres into a vessel filled with an equal mass of water at 0°: its motion is stopped, but an equal mass of water is displaced. Imagine that the heat developed (which would raise the fallen water to 1°) be divided equally between the water which fell and that which was displaced. The displaced kilogram of water, now at ·5°, falls into a similar vessel 425 metres below, where it divides its heat with, and displaces, a third kilogram of water, and so on. By successive divisions of heat, the temperature of the *falling* water will approach 1°, beyond which it cannot rise. But if the heat developed were not divided, and remained in the fallen water, the *falling* water would be always at 0°, and the *fallen* water at 1°. The latter case is the analogue of the condition imagined by M. BERTHELOT, where the heat of the burnt layer is not imparted to the unburnt layer next it; the former case is the analogue of the condition which I imagine holds in the propagation of the explosion-wave.

Again, M. BERTHELOT considers the velocity of the wave to be solely governed by the mean rate of translation of the *products* of combustion. If, however, we accept the hypothesis that the explosion is propagated by molecular collisions, we have just seen that the movement of the products of combustion is communicated to the unburnt molecules in front. It will, therefore, follow that the rate of the advancing explosion will depend not only on the rate of translation of the products of combustion, but also on the rate of translation of the heated *but yet uncombined molecules.* If the burnt molecules communicate their rise of temperature without loss to the molecules in front, no difference in the mean rate of motion will be caused by this transference so long as the burnt and the unburnt gases are of the same average density. But if a change in density is produced by the combustion, the average velocity of the molecules in the burnt layer will differ from the average velocity of the molecules in the heated but yet unburnt layer, and this difference must be taken into account. Lastly, there is a correction to be applied for changes of density in the explosion which in part counteracts the foregoing correction. In the oxidation of hydrogen and of carbonic oxide there is a contraction from three volumes to two; in the burning of cyanogen to carbonic oxide there is an expansion of two volumes to three.* If each layer of the reacting gases is heated at constant volume, a change in the number of molecules must affect the temperature reached.

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[*] [I make the assumption that a very thin layer of gas (comparable in thickness with the mean free path of the molecules) may be raised by collisions to *nearly* the same temperature as the highly heated layer next it.—Jan., 1893.]

If the number of molecules is increased by the chemical change the temperature from which the velocity of the wave is calculated will be higher than that directly deduced from calorimetric observations; and *vice versâ*. In allowing for this change we may suppose that the new formed molecules have been suddenly compressed or expanded, without loss of heat, into the volume previously occupied by the initial molecules, in which case the temperature would be altered according to RANKINE's formula

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma - 1}$$

where T_1 and T_2 are the absolute temperatures, V_1 and V_2 the volumes before and after the reaction, and γ the ratio of the specific heats at constant pressure and at constant volume.

CAP. VIII.—COMPARISON OF THE RATE OF EXPLOSION TO THAT OF A SOUND-WAVE.

The criticisms which I have ventured to make on BERTHELOT's method of calculating the mean rate of translation of the gaseous molecules concerned in the propagation of the explosion-wave, tend to show that the rate so calculated must be too small. But in attempting to apply corrections suggested by these criticisms, one is beset with difficulties. As an approximation I assume (1) that the explosion-wave is carried forward by the movements of molecules of density intermediate between that of the products of combustion and that of the unburnt gas; (2) that the temperature of the gas propagating the wave is double that due to the chemical reaction alone; (3) that the temperature is increased when the chemical volume of the products is larger, and is diminished when the chemical volume of the products is smaller than that of the initial gases; (4) that the gases are heated at constant volume, and their specific heats remain constant at high temperatures. On calculating the mean rate of translation of the molecules on these assumptions one arrives at numbers greatly in excess of any of the observed rates of explosion; but some of the observed rates agree with *the velocity of sound* in a gas of the temperature and density so calculated. For instance, when one volume of cyanogen is exploded with an equal volume of oxygen, two volumes of carbonic oxide are formed, and one volume of nitrogen :—

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* M. BERTHELOT rightly takes this change into account in interpreting BUNSEN's experiments on the temperature produced in explosions. 'Ann. Chim. et Phys.,' [V.], vol. 12, p. 302.

$$C_2N_2 + O_2 = 2CO + N_2.$$

Taking the quantity of heat evolved as 126,100 calories, and the specific heat at constant volume of the products of combustion as 4·8 × 3 = 14·4, the temperature produced by the chemical change is 8694° C. If the gases were initially at 13°, or 286° degrees above absolute zero, the chemical reaction will raise the temperature to 8980°. But, since three molecules are formed where two previously existed, the temperature is further raised by the heat developed in compressing three volumes to two. This will raise the temperature to 10,595°. At *double* this temperature the mean rate of translation of a molecule of the mean density of the burnt and unburnt gases would be 3892 metres per second. If the formula for the velocity of sound under ordinary conditions held good in the explosion, the velocity of the sound wave would be 2670 metres per second—a rate which is about 2 per cent. less than the observed velocity of the explosion-wave. Now, the theoretical velocity of sound is calculated on the assumption that the disturbance is very small; if the displacements are large the velocity of sound should be higher. Direct measurements of the velocity of sound-waves of great intensity have confirmed this anticipation. Under ordinary conditions the rate of the sound-wave is to the mean rate of the molecules as ·688 : 1. If we take the ratio in an explosion as ·7 : 1 the velocity of the sound-wave agrees with the observed rate of explosion in this particular case.

We may now compare the rate of the sound-wave so calculated with the velocity of explosion of cyanogen with oxygen (1) in presence of excess of oxygen, (2) in presence of excess of nitrogen ; (3) of cyanogen with nitrous oxide alone, and (4) in presence of excess of nitrogen ; and lastly (5) of cyanogen with nitric oxide.

Taking the data furnished by Berthelot's experiments, we have, for the quantities of heat evolved, the expansion on explosion, the specific heat of the products, and for the mean density of the burnt and unburnt gases :—

TABLE XXVIII.—Explosion of Cyanogen with Oxygen.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2N_2 + O_2$ . .	. .	126,100 Cals.	2 : 3	4·8 × 3	1·213
" . .	O_2	"	3 : 4	4·8 × 4	1·172
" . .	$2O_2$	"	4 : 5	4·8 × 5	1·154
" . .	$3O_2$	"	5 : 6	4·8 × 6	1·144
EXPLOSION of Cyanogen with Oxygen and Nitrogen.					
$C_2N_2 + O_2$ . .	N_2	126,100 Cals.	3 : 4	4·8 × 4	1·132
" . .	$2N_2$	"	4 : 5	4·8 × 5	1·092
EXPLOSION of Cyanogen with Nitrous Oxide and Nitrogen.					
$C_2N_2 + 2N_2O$.	. .	167,300 Cals.	3 : 5	4·8 × 5	1·294
" .	N_2	"	4 : 6	4·8 × 6	1·213
" .	$2N_2$	"	5 : 7	4·8 × 7	1·165
EXPLOSION of Cyanogen with Nitric Oxide.					
$C_2N_2 + 2NO$.	. .	169,300 Cals.	3 : 4	4·8 × 4	1·132

TABLE XXIX.—The Rate of Explosion of Cyanogen Mixtures compared with the calculated Velocity of Sound (Σ).

1. CYANOGEN with Oxygen.

Mixture . . .	$C_2N_2 + O_2$.	$C_2N_2 + 2O_2$.	$C_2N_2 + 3O_2$.
Mean rate . .	2728	2321	2110
Σ	2725	2310	2066

2. CYANOGEN with Oxygen and Nitrogen.

Mixture . . .	$C_2N_2 + O_2$.	$C_2N_2 + O_2 + N_2$.	$C_2N_2 + O_2 + 2N_2$.
Mean rate . .	2728	2397	2163
Σ	2725	2397	2166

3. Cyanogen with Nitrous Oxide and Nitrogen.

Mixture.	$C_2N_2 + 2N_2O$	$C_2N_2 + 2N_2O + N_2$	$C_2N_2 + 2N_2O + 2N_2$
Mean rate .	2454	2283	2098
Σ	2416	2237	2093

4. Cyanogen with Nitric Oxide.

Mixture.	$C_2N_2 + 2NO$
Mean rate . . .	2760
Σ	2763

This comparison shows that the formula, which makes the velocity of sound in cyanogen burning to carbonic oxide equal to the observed rate of explosion, also gives for the velocity of sound in the same gases diluted with oxygen or nitrogen numbers closely concordant with the observed rates of explosion. The same formula holds for the explosion of cyanogen with nitrous oxide and when this mixture is diluted with nitrogen. Lastly, by firing a cartridge of fulminate in a steel cylinder filled with a mixture of cyanogen and nitric oxide I succeeded in propagating the explosion from the cylinder through a long leaden tube filled with the gases. The rate of explosion so obtained agreed exactly with the calculated sound-wave. I believe this was the first time that a mixture of cyanogen and nitric oxide had been exploded in a tube. Berthelot, who fired this mixture in a bomb, states that the *wave* is not propagated in these gases. I found that neither a strong spark, nor an initial explosion of hydrogen and nitric oxide, would set up the explosion-wave in cyanogen and nitric oxide. The details of these experiments are given in the Appendix.

The formula that I have given was thus found to agree with all the cyanogen explosions. It is therefore, at all events, an empirical expression which can be applied to calculate the rates of explosion of cyanogen burning to carbonic oxide and nitrogen under a fairly wide range of conditions. It was accordingly a matter of considerable interest to apply the formula to the rates found when electrolytic gas was exploded by itself and when diluted with hydrogen, oxygen, and nitrogen. In the following table the rates for electrolytic gas with excess of hydrogen and of oxygen are compared with the numbers calculated from Berthelot's formula (θ), and with the sound wave (Σ) calculated as before. In calculating Σ the *contraction* which occurs on the union of hydrogen and oxygen is taken into account, but not the existence (which is possible) of free *atoms* of oxygen :—

TABLE XXX.

Explosive gases.	Inert gases.	Heat developed.	Contraction.	Specific heat of products.	Mean density.
$H_2 + O$	—	59,000 Cals.	3 : 2	4·8 × 1·5	·520
$H_2 + O$	H_6	,,	9 : 8	4·8 × 4·5	·1964
$H_2 + O$	O_5	,,	8 : 7	4·8 × 4	·910
$H_2 + O$	N_5	,,	8 : 7	4·8 × 4	·8302

THE Rate of Explosion of Electrolytic Gas with excess of Hydrogen and with excess of Oxygen compared with Calculated Velocities.

Mixture.	$H_8 + O$	$H_6 + O$	$H_4 + O$	$H_2 + O$	$H_2 + O_2$	$H_2 + O_4$	$H_2 + O_6$
BERTHELOT'S θ.	3028	3061	3055	2900	2252	1730	1476
Rate of explosion . .	3532	3527	3268	2821	2328	1927	1707
Σ	3516	3571	3585	3416	2650	2024	1718

A glance at this table reveals the fact that the sound-wave calculated by my formula does not agree in velocity with the explosion-wave of pure electrolytic gas; but as the electrolytic gas is more and more diluted with hydrogen or with oxygen, the observed and calculated velocities come nearer together, until, not far from the two limits of regular explosion, they are in close agreement. Conversely we have seen that BERTHELOT'S formula gives the correct rate for pure electrolytic gas, but diverges more and more from the observed rates as the gas is diluted. Where BERTHELOT'S "θ" fails to represent the facts, the "Σ" I have calculated does so; and *vice versâ*.

I advance the following hypothesis to account for this divergence. *At the high temperature of the explosion-wave the combination of hydrogen and oxygen is not complete; or, in other words, steam is partly dissociated under these conditions.* The temperature of the wave front is therefore less than that calculated. As more and more inert gases are added to electrolytic gas, the temperature of the explosion is lowered by the division of the heat between the diluent gases; more and more of the explosive gases are thus able to combine. As the gases are diluted, therefore, the temperature of the wave front comes nearer and nearer to that calculated.

A comparison of the sound-wave, Σ, with the rates of explosion of electrolytic gas diluted with nitrogen, and of hydrogen with nitrous oxide and nitrogen, brings out similar divergences which admit of the same explanation. It is to be noted in the case of hydrogen and nitrous oxide that no alteration of volume occurs in the chemical change; no correction in temperature need, therefore, be made in calculating the sound-wave, and the question of the density of the molecules concerned does not arise.

TABLE XXXI.—The Rate of Explosion of Electrolytic Gas with excess of Nitrogen compared with Calculated Velocities.

Mixture.	H_2+O	H_2+O+N	H_2+O+N_3	H_2+O+N_5
BERTHELOT'S θ . . .	2900	2321	1814	1558
Rate of explosion . .	2821	2426	2055	1822
Σ	3416	2731	2122	1813

TABLE XXXII.—The Rate of Explosion of Hydrogen and Nitrous Oxide, with excess of Hydrogen, and of Nitrogen, compared with Calculated Velocities.

Mixture.	H_7+N_2O	H_6+N_2O	H_4+N_2O	H_2+N_2O	$H_2+N_2O+N_2$	$H_2+N_2O+N_3$
BERTHELOT'S θ . . .	2400	2396	2374	2319	1912	1782
Rate of explosion . .	2732	2705	2545	2305	1991	1880
Σ	2776	2781	2765	2706	2227	2067

When electrolytic gas is diluted with excess of nitrogen, the observed rate of explosion closely agrees with the calculated sound-wave. With hydrogen and nitrous oxide the temperature cannot be diminished to the same extent by dilution. When three volumes of hydrogen, or two volumes of nitrogen, are added to the mixture consisting of one volume of hydrogen and one of nitrous oxide, the explosion-wave is not propagated with regularity; consequently the temperature cannot be brought down by dilution below the dissociation-point of steam. As the mixture is diluted the rate of explosion approaches Σ, but agreement is not reached at the limits of dilution at which the explosion-wave will still travel.

The rates of explosion of mixtures of ammonia with oxygen and with nitrous oxide were also determined. The gases were mixed in an iron holder standing over mercury. The rate of explosion of ammonia and oxygen is faster than the velocity calculated by

BERTHELOT's formula, but falls far below the velocity of sound "Σ." On the other hand, the rate of explosion of ammonia and nitrous oxide is close to the calculated sound-wave and is far above "θ."

TABLE XXXIII.

	Explosive gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
1.	$4NH_3+3O_2$	305,200	7 : 8	4·8 × 11	·761
2.	$2NH_3+3N_2O$	193,800	5 : 7	4·8 × 8·5	·987

(1.)

Mixture.	$4NH_3+3O_2$.
BERTHELOT's θ . . .	2300
Rate of explosion . .	2390
Σ	2665

(2.)

Mixture.	$2NH_3+3N_2O$.
BERTHELOT's θ . . .	1953
Rate of explosion . .	2200
Σ	2223

Mixtures of oxygen with the hydrocarbons ethylene and acetylene will explode when largely diluted with nitrogen. For instance, the explosion-wave is propagated in a mixture of one volume of ethylene with two of oxygen and eight of nitrogen. The temperature reached in the explosion is comparable with that produced when three volumes of electrolytic gas are fired with five volumes of nitrogen. It is found both with ethylene and acetylene that the rate of explosion exactly agrees with the calculated velocity of the sound-wave when the mixtures are largely diluted with nitrogen, but not when the gases are exploded without dilution :—

TABLE XXXIV.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2H_4+O_4$	——	185,000	3 : 4	4·8 × 5	·930
$C_2H_4+O_4$	$8N_2$	185,000	11 : 12	4·8 × 13	·954

Rate of Explosion of Ethylene and Oxygen with excess of Nitrogen compared with Calculated Velocities.

Mixture.	$C_2H_4+2O_2$	$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+4N_4$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
BERTHELOT'S θ . .	2487	2234	2050	1796	1624	1498
Rate of explosion .	2581	2413	2211	2024	1878	1734
Σ	2856	2575	2367	2073	1873	1727

TABLE XXXV.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2H_2+O_3$	—	171,700	5 : 6	4·8 × 3·5	·9406
$C_2H_2+O_3$	$6N_2$	,,	17 : 18	4·8 × 9·5	·9594

Rate of Explosion of Acetylene and Oxygen with excess of Nitrogen compared with Calculated Velocities.

Mixture.	$C_2H_2+O_3$	$C_2H_2+O_3+N_2$	$C_2H_2+O_3+N_6$	$C_2H_2+O_3+N_8$	$C_2H_2+O_3+N_{10}$	$C_2H_2+O_3+N_{12}$
BERTHELOT'S θ . . .	2749	2397	1983	1848	1739	1650
Rate of Explosion .	2716	2414	2209	2116	2019	1908
Σ	3187	2784	2303	2144	2016	1910

Since steam is known to be partially dissociated under atmospheric pressure at the temperature of the oxy-hydrogen flame, it does not seem improbable that dissociation would also occur at the higher pressures and higher temperatures of the explosion-wave. The researches of MM. BERTHELOT and VIEILLE, and those of MM. MALLARD and LE CHATELIER, on the pressures registered in an explosion of gases, have led these investigators to the conclusion that the specific heat of steam rapidly rises with the temperature. The deficiency of "available" pressure, which BUNSEN first observed in the explosion of gases and attributed to incomplete combustion, they consider to be due to an increase of specific heat. Such an increase in the specific heat of steam with rise of temperature would explain the divergence between the observed and calculated rates of explosion of hydrogen with oxygen, and with nitrous oxide. And conversely, it appears to me that the results of the French experimenters would be equally well explained by the temporary dissociation of steam in their explosions.

One fact appears, at first sight, to be opposed to the hypothesis of incomplete combustion. The addition of oxygen, one of the products of the dissociation of steam, retards the explosion of electrolytic gas more than the addition of an equal volume of nitrogen—an inert gas. WURTZ found, in the case of phosphoric chloride, that the amount of dissociation was greater when the chloride was volatilised in air—an inert gas, than when it was volatilised in the presence of the lower chloride—one of the products of its decomposition. Arguing from this observation, one might have expected that excess of oxygen would tend to hinder dissociation, and therefore, if the combustion were really incomplete, that the rate should be faster when oxygen was used as a diluent instead of nitrogen. But the condition of equilibrium under which WURTZ determined the vapour density of phosphoric chloride is not the condition of the wave-front of an explosion. If we imagine in the wave-front a molecule A dashing against a molecule B with such extreme velocity that the compound AB is not formed, the two molecules may be supposed to rebound, exchanging energies, and the motion of B may be communicated to the molecules in front continuing the explosion. This is a case where dissociation of AB occurs in the wave-front. Now if a number of B molecules are present in excess, A will shortly combine with one of them, behind the wave-front, and the dissociation will not be permanent. On the other hand, if there are a number of inert molecules C present, dissociation may occur as before in the wave-front, but the uncombined molecules may be prevented from combining behind the wave by the inert molecules present. The propagation of the wave does not depend on what occurs after it has passed; the rate of the wave will therefore be independent of the permanency of the dissociation occurring in it.

That the combustion of pure electrolytic gas is not *wholly* complete in the explosion-wave has been proved by collecting the residue and exploding it.* In the propagation of the wave the cooling due to expansion is so rapid that some molecules of hydrogen and oxygen, which are unburnt in the wave-front, have not time to combine before they are cooled below the temperature of combination. In a leaden tube, 9 mm. in diameter and 100 metres long, about 1 per cent. of electrolytic gas was found uncombined after the explosion. That this incompleteness of combustion was not due to the cooling effect of the walls was shown by making comparative experiments in tubes 4 mm. and 19 mm. in diameter. Nearly the same percentage of unburnt residue was found in all the tubes, and also when the gases were detonated in an iron bomb 100 mm. in diameter. We have, therefore, positive evidence that in the explosion-wave the combustion of electrolytic gas is incomplete; it seems, therefore, not unreasonable to assume that in the wave-front—*i.e.*, at the highest temperature—a considerable proportion of hydrogen and oxygen is uncombined, and the propagation of the wave is retarded accordingly.

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* "Incompleteness of Combustion in Gaseous Explosions," by H. B. DIXON and H. W. SMITH. 'Manchester Memoirs' [IV.], vol. 2. 1888.

The residual gases, left after the explosion of electrolytic gas, were examined for peroxide of hydrogen; the water condensed on the tube was also washed out and examined. In neither case could peroxide of hydrogen be detected. If the view were correct that hydrogen and oxygen first form peroxide on explosion, it would seem likely that, in the rapid cooling, some of the peroxide would remain undecomposed, and be found in the residue.

In the explosion of carbonic oxide and oxygen a similar residue of unburnt gas is found—indicating the dissociation of carbonic acid at the temperature of the wave. No trace of peroxide of hydrogen could be detected in the tube. These observations give no support to the view put forward by MORITZ TRAUBE[*] that steam promotes the oxidation of carbonic oxide by dividing with it the molecules of oxygen:—

$$CO + OO + OH_2 = CO_2 + O_2H_2$$

and that the peroxide so formed is again reduced to steam by more carbonic oxide:—

$$CO + O_2H_2 = CO_2 + OH_2.$$

CAP. IX.—THE EXPLOSION OF HYDROGEN AND CHLORINE.

The coincidences and the divergences found between the calculated and observed rates of explosion, as described above, made it important to examine a reaction in which no steam should be formed, and, if possible, one in which no alteration in volume should be produced. The simplest case fulfilling these conditions is the explosion of hydrogen and chlorine; a reaction which has the further advantage of forming, as the product of explosion, a gas which approximates to a perfect gas. So far as I know the rate of explosion of hydrogen and chlorine had never been measured.

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[*] 'Ber. Deut. Chem. Ges.,' vol. 18, p. 1890. (See 'Chem. Soc. Journ.,' 1886, vol. 1, p. 94.)

Before devising apparatus to carry out the experiments, it was necessary to determine whether the action between hydrogen and chlorine was a direct one, or whether it was influenced by the presence of other gases, *e.g.*, steam.[*] Electrolytic hydrogen and chlorine were accordingly sealed up in glass vessels with anhydrous phosphoric acid, and were kept in the dark for several weeks. The dried gases were found to be far less sensitive to the action of *light* than when a trace of aqueous vapour was present, a result in accordance with PRINGSHEIM's[†] observations. But on the passage of an induction spark by means of platinum-iridium wires the dried gases exploded violently.

Since these experiments left it doubtful whether the presence of steam influenced the rate of explosion of hydrogen and chlorine, a preliminary series of measurements were made with the dry and moist gases.[‡] The explosion-vessel consisted of two long glass tubes[§] (34 and 30 feet) cemented together. The ends were provided with steel flanges so that they could be rapidly joined to two smaller tubes which carried the silver bridges. The explosion tube was hung to the wall of a long, darkened corridor.

The gas passed directly from the electrolytic cell through washing tubes into the long explosion tube—driving out the air before it: at the further end it passed through an "analyser," which served to determine the composition of the mixture issuing from the tube. When the long tube was completely filled with hydrogen and chlorine, the two smaller tubes carrying the bridges were fastened to its two ends by the steel flanges. These smaller tubes had previously been filled with hydrogen and oxygen. The electric connections were then made to the chronograph, and the gases fired by the passage of a spark through the hydrogen and oxygen in one of the smaller tubes. The flame travelling down this tube, which was about 4 feet long, broke the silver bridge at the end of it, and communicated the explosion to the hydrogen and chlorine. On reaching the end of the glass tube, the flame broke the silver bridge carried by the second smaller tube. In the few seconds that elapsed between the joining on of the small tubes and the firing of the mixture but little diffusion of the gases was possible; and the silver bridges, being coated with paraffin, were hardly acted on by the chlorine. By this arrangement the electrolytic hydrogen and chlorine were kept from contact with metals or india-rubber, and from all liquids but water, and, in the "dry" experiments, oil of vitriol saturated with chlorine. The drying apparatus consisted of a "Winkler" worm, a series of bulbs, and a large U-tube packed with pumice, all filled with boiled oil of vitriol.

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[*] "On the Combination of Hydrogen and Chlorine, alone, and in Presence of other Gases." By H. B. DIXON and J. A. HARKER. 'Manch. Memoirs' [IV.], vol. 3, p. 118.

[†] 'Annalen der Physik,' N.F., vol. 32, p. 421, 1887.

[‡] 'Manch. Memoirs' [IV.], vol. 4, p. 3. (The numbers given in this paper were afterwards found to be affected by a constant error.)

[§] These glass tubes were made for me by Messrs. MOLINEAUX and WEBB, of Manchester. They were unannealed, but stood the shock of many explosions.

The following table gives the mean of each pair of experiments with the dry and with the moist gases saturated at 14°.

TABLE XXXVI.—Rate of Explosion of Hydrogen and Chlorine.

Dry.	Moist.
1748 1745 1737 1748 1750	1724 1707 1730
Mean 1745	Mean 1720

It appeared, therefore, that the explosion, once started, travelled at a slightly faster rate in the dry than in the wet gases, the steam appearing to act merely as a diluent.

These experiments, together with the analysis of the mixture which issued from the end of the explosion tube, showed that it was possible to fill a long tube directly from the electrolytic cell with hydrogen and chlorine in equal volumes, and that the mixture *detonated* in a glass tube of 10 mm. diameter. To prove the constancy of the rate of explosion, and at the same time to have a longer interval of time to measure, it was necessary to obtain a longer explosion tube. The difficulty of joining up long pieces of unannealed glass-tubing led me to try the glass-lined wrought-iron tubing manufactured by Messrs. RYLANDS, at Barnsley. Having found that this tubing answered the purpose, I made an explosion tube 40 yards long, and $\frac{1}{2}$-inch diameter, by joining together a number of six-feet lengths. The junctions were made by fitting an asbestos washer on to the ends of each tube, and screwing them together with a nut. The tube was supported at intervals by wooden brackets fastened to the wall of a long corridor. The explosion tube so put together was perfectly gas-tight; it was unaffected by the chlorine passed into it during the experiments, and it stood the shock of many explosions without leaking. This tube was filled, as before, direct from the electrolytic cell of aqueous hydrochloric acid. The cell and drying vessels were all of glass with ground glass joints. Two short tubes charged with hydrogen and oxygen were used, as before, to carry the silver bridges. They were joined to the long tube immediately before the explosion. The mean rate of explosion of dry hydrogen and chlorine in this tube was found to be 1723 metres per second—as the result of three pairs of experiments :—

TABLE XXXVII.—Rate of Explosion of Hydrogen and Chlorine. 2nd Series.

	1718
	1727
	1725
Mean	1723

If the preliminary series of experiments with the dry gases is regarded as of equal value with *one* of these pairs, the mean rate of explosion is found to be 1729 metres per second.

The heat of combination of hydrogen and chlorine was found by FAVRE and SILBERMANN to be 23,780 calories; by J. THOMSEN it was found to be 22,000 calories. BERTHELOT gives one determination 22,100 calories. The mean of these numbers is 22,630 calories. Taking THOMSEN's number for the heat of combination, the mean rate of translation of the products of combustion according to BERTHELOT's formula (corrected to 13° C.) is found to be 1551 metres per second; taking the mean number 22,630 calories for the heat of combination, BERTHELOT's formula gives 1571 metres for the mean rate. The rate of explosion of hydrogen and chlorine under ordinary conditions is, therefore, considerably faster than the rate given by BERTHELOT's formula. The velocity of the sound-wave (Σ) calculated by my formula is 1805 or 1830, according as the lower or higher heat of combination is taken. The rate of the explosion thus falls appreciably below the calculated velocity of the sound-wave. Since BERTHELOT contends that his formula gives the maximum velocity of explosion, his hypothesis fails in the case of the explosion of hydrogen and chlorine. On the other hand, two questions arise: (1) Does dissociation occur in the explosion? (2) On diluting the mixture will the rate of explosion approach the velocity of the sound-wave?

When the hydrochloric acid gas, produced by the explosion in the long tube, was driven out by a stream of hydrogen through a washing apparatus containing potassium iodide, a quantity of iodide was liberated, showing the presence of free chlorine in the products of combustion. And when the gases were driven out by a stream of carbonic acid through a washing apparatus containing caustic soda, free hydrogen was found to be present also. Even after passing the current for many hours the gases coming from the electrolytic cell, and issuing from the further end of the explosion tube, were always found to contain a slight excess (amounting to between ·2 and ·5 per cent.) of hydrogen. The presence of free chlorine in the vessel after the explosion was thus a proof of dissociation occurring in the explosion, and the determination of this free chlorine by the liberation of iodine afforded a ready means of measuring the volume of gases left unburnt. About 1 per cent. of the combustible gases were found uncombined.

On diluting the electrolytic gas with hydrogen the rate of explosion was found to increase, and to approximate to the calculated velocity of the sound-wave. The mixtures were made by bringing the electrolytic gas from the cell, and the hydrogen from a holder, into a sulphuric acid wash-bottle through two tubes of the same size dipping side by side beneath the liquid. By regulating the flow of hydrogen the bubbles of each gas could be kept passing at an equal rate. In this way a mixture containing three volumes of hydrogen to one volume of chlorine was passed into the explosion tube. The correctness of the mixture was checked by collecting and analysing a sample of the gas escaping from the end of the tube. By passing the hydrogen at half the rate a mixture containing approximately two volumes of hydrogen to one of chlorine was obtained. While the addition of diluent hydrogen to the electrolytic gas makes little difference in the calculated rates, the observed velocities of explosion increased appreciably. On the assumption that the heat of combination of hydrogen and chlorine is 22,630 calories, the calculated and observed velocities are as follow :—

TABLE XXXVIII.—Rate of Explosion of Hydrogen and Chlorine with addition of Hydrogen.

Mixture.	H_2+Cl_2	H_4+Cl_2	H_6+Cl_2
BERTHELOT'S θ	1571	1581	1589
Rate of explosion . . .	1729	1849	1855
Σ	1830	1832	1832

These experiments show that in the explosion of hydrogen and chlorine as in the explosion of hydrogen and oxygen, in equivalent proportions, the rate falls below the calculated velocity of the sound-wave, but on diluting the gases the calculated and observed velocities come together. This divergence at the highest temperature may be explained either by the dissociation of the hydrochloric acid or by a rise in its specific heat.* As I have shown, we have direct evidence of *some* dissociation. On the other hand, the physical properties of hydrochloric acid more nearly resemble those of carbonic oxide and the elementary gases than those of steam or carbonic acid; we should expect, therefore, that the specific heat of hydrochloric acid should remain as constant as that of carbonic oxide at high temperatures.

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* MM. MALLARD and LE CHATELIER conclude from their experiments that some dissociation occurs when equal volumes of hydrogen and chlorine are exploded. When excess of hydrogen is added they find no dissociation.

The fact that the rate of explosion of hydrogen and chlorine exceeds the calculated velocity of the sound-wave, as calculated from THOMSEN's Heat of Combination (22,000 cals.), is possibly due to the behaviour of the unburnt chlorine in the wave-front. The rates agree fairly with the sound-wave calculated from the *mean* heat of combination, which I have given in the table.

CAP. X.—THE INSTANTANEOUS PRESSURES PRODUCED IN THE EXPLOSION-WAVE.

There is, lastly, one important phenomenon in the propagation of explosions in gases which I have not touched on—the pressure produced in the reacting gases. BUNSEN was the first who attempted to measure this pressure. In his well-known work* he describes the explosion of mixtures of hydrogen and of carbonic oxide with oxygen in a strong vessel closed with a loaded lid. When the pressure produced in the vessel exceeded the pressure on the lid, the latter was raised in the manner of a safety valve, and some of the heated gas escaped; when the pressure was less than that on the lid, no gas escaped and but little noise was heard.

By successive trials he thus found the pressure produced by the explosion of carbonic oxide and oxygen to be about 10 atmospheres, and the pressure produced by the explosion of hydrogen and oxygen to be about 9·5 atmospheres. The comparison of these pressures with the numbers calculated from the heat of combination and the specific heats of the products led BUNSEN to the conclusion that, at the highest temperature reached in the explosion, only one-third of the gases entered into combination. BERTHELOT and VIEILLE, working in a manner essentially the same, only replacing BUNSEN's loaded lid by a light piston moving against a spring, found the pressures produced in the explosion of carbonic oxide and hydrogen to be 10·1 and 9·8 atmospheres respectively. BERTHELOT rightly calls these pressures "*effective pressures.*"† On the other hand, MALLARD and LE CHATELIER, using the more delicate indicator of DEPREZ, found pressures produced in the explosion of hydrogen and oxygen, of far greater amount, but lasting an exceedingly short time. These fugitive pressures of great intensity were only obtained with mixtures which burnt very rapidly:—‡

"Cette pression très fugitive est d'autant plus grande que la vitesse de propagation est plus considérable. Elle peut devenir énorme avec l'onde explosive. C'est à ces pressions, si instantanées qu'elles soient, que sont dus les surprenants effets des substances explosives telles que la nitroglycérine. Ce sont précisément ces pressions passagères, sans relation directe avec le phénomène que nous voulions mesurer, qu'enregistrait notre appareil."

On substituting for the DEPREZ indicator a less sensitive Bourdon manometer,

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* 'Gasometrische Methoden,' 1857.

† 'Ann. Chim. et Phys.' [vi.] 4, p. 14, 1885.

‡ 'Combustions des Mélanges gazeux,' p. 123.

LE CHATELIER found "effective pressures" in agreement with those obtained by BUNSEN and by BERTHELOT, *i.e.*, about 10 atmospheres.

It is evident from these experiments that sudden pressures are produced in the explosion-wave greater than those shown by the experiments of BUNSEN and of BERTHELOT. If the explosion is propagated by compression after the manner of a sound-wave enormously high temperatures must be produced in the wave, as I have already shown, and correspondingly high pressures must also exist. This aspect of the matter has been well stated by MALLARD and LE CHATELIER :—

"A gas, increasing in volume by combustion, exerts a certain pressure on the unburnt gas next to it : it is possible to conceive that this pressure may be raised sufficiently high to bring the neighbouring layer to the temperature of ignition. The possibility of the propagation of the inflammation by the propagation of the pressure can thus be understood.

"Take for example the explosive mixture of hydrogen and oxygen. According to our experiments it inflames at 555°, and the pressure exerted after the combustion in a closed vessel is 10 atmospheres. An infinitely thin layer inflamed by heating, reaches this pressure at the greatest, and cannot exert any higher pressure on the uninflamed layer next to it.

"A compression of 10 atmospheres evolves a quantity of heat which can readily be calculated by thermo-dynamics : it is not sufficient to produce an elevation of temperature equal to that inflammation. To obtain this temperature a compression of 30 atmospheres would be needed. It would appear, then, that the explosion-wave could not be initiated under these conditions. . . . .

"Suppose, however, that the inflammation of the first layer is not brought about by heating, but by a suitable pressure, *i.e.*, one of not less than thirty atmospheres, which may readily be produced by the detonation of a little fulminate. After the combustion the pressure of the inflamed layer will be multiplied by ten, or at least will be far higher than the initial pressure of thirty atmospheres, and will be able, consequently, to exert on the next layer a pressure sufficient to inflame it. This will continue then to propagate itself from layer to layer by the same mechanism."—('Combustion des Mélanges Gazeux,' p. 88.)

The sudden state of pressure produced in each layer of burning gas ceases as suddenly as it is caused, by the expansion of the gas which compresses the unburnt layer in front; the high temperature of the burnt layer, therefore, rapidly falls by the transformation of heat into work. MALLARD and LE CHATELIER have shown by photographs of the flame that, in the explosion-wave, the luminosity of each layer does not last longer than one thousandth part of a second ; while in the *régime* of ordinary combustion each layer remains luminous more than ten times as long. I can fully confirm these statements. By photographing the explosion-wave of hydrogen and oxygen on a fixed and on a rapidly moving plate at the same time, I have found that the interval during which the gas remains bright enough to effect a sensitive isochromatic plate cannot be more than *one five-thousandth of a second.*

The problem of measuring the pressure in a thin layer of gas, lasting for so small a fraction of a second, is one of the greatest difficulty. I believe the method tried by MALLARD and LE CHATELIER for explosions of carbon bisulphide and oxygen is probably the best that has yet been devised. This depends on the principle that if a pressure is produced in a glass vessel greater than the glass can stand, the vessel will be broken, although the pressure may endure for a very small interval of time. If the vessel is broken by the explosion, I believe that there must have been a pressure produced in the layer of gas next the glass *at least* equal to that which would break the vessel when gradually applied. By compressing gas in the vessel by means of a pump, it can be shown that the vessel is able to withstand a certain pressure. The vessel can then be connected with a long tube, the whole filled with an explosive mixture, and the explosion-wave set up in the tube. If the vessel is fractured, a pressure was exerted on its walls greater than the pressure applied by the pump.

To test this method, I have carried out a few preliminary experiments. Out of a piece of strong combustion tubing I made twelve short tubes, carefully sealed at one end, and slightly opened at the other. These tubes were then firmly attached to a metal piece by a collar and nut working against india-rubber rings—care being taken that the glass did not come anywhere into contact with the metal, but only with the rubber rings. The metal piece could be attached either to the pump or to a leaden firing tube. The twelve tubes were first tested with compressed air, which was slowly forced into the tube while an observer read the manometer. One tube (it was rather unevenly sealed) broke down between nineteen and twenty atmospheres. A second broke at twenty-three atmospheres. The other ten stood a pressure of twenty-five atmospheres for two minutes. These tubes were then in turn attached to one end of a firing-tube about twenty feet long. On filling the whole with a mixture of carbonic oxide and oxygen, and passing a spark through the gases at the other end of the tube, the explosion-wave was propagated through the leaden pipe into the glass vessel. All ten tubes were blown into small fragments by the explosion. Four stronger green glass tubes were tested by compressed air at fifty atmospheres. All these tubes stood the explosion of carbonic oxide and oxygen, and hydrogen and oxygen. Two of these green tubes were blown to pieces by the explosion of equal volumes of cyanogen and oxygen, a third was fractured by compressed air at seventy-eight atmospheres. These experiments confirm LE CHATELIER's results with the DEPREZ indicator, showing that very high pressures are produced for the moment in the explosion-wave; they do not, of course, *measure* the pressures, they only indicate a lower limit.

The quiet burning of carbonic oxide and oxygen, when this mixture is lighted at the open end at a short tube, such as a test-tube, is in striking contrast to the violence of the explosion when the "wave" is set up in the same gases. As a lecture experiment a test-tube full of the mixture may be ignited by a taper, when the quiet passage of the blue flame down to the tube can be followed by the eye; the tube is then refilled and screwed on to the end of a few feet of leaden pipe filled with the mixture. The test-tube is surrounded by metal gauze and a thick glass cylinder. On applying a flame to the open end of the pipe, or passing a spark near the extremity, a loud report is heard, and the test-tube is reduced to powder.* The explosion-wave in a glass tube produces a brilliant flash of orange light.

Professor A. SCHUSTER† has drawn my attention to a memoir, by RIEMANN, on the propagation of intense disturbances in gases, and has suggested a mode of calculating the momentary pressures produced in the explosion-wave from its known velocity of propagation, and the density of the unburnt gas. On the assumption that RIEMANN'S theory holds for an explosion, I have calculated, from the observed rates of the wave and the densities of the unburnt gas, the pressures existing in the wave in the explosion of different gaseous mixtures. In parallel columns I have given the "effective" pressures as registered in the apparatus used by BERTHELOT for the same mixtures. The pressures calculated for the wave are rather more than four times larger than these effective pressures. In the third column I have given the calculated pressures divided by 4·2.

The pressure in the wave is given by the equation

$$\pi = \frac{\nu^2 \rho}{100}$$

where π is the pressure in atmospheres, ν the rate in metres per second, and ρ is the weight in grammes of 1 c.c. of the unburnt gas.

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* Professor DEWAR has noticed the great violence with which carbonic oxide and oxygen explode under certain conditions. He says: "Sometimes the explosions were so violent as to break the plate closing the end of the tube, though this had resisted the explosions of the hydrogen mixture; while at the other times the wave of explosion passed slowly along the tube." 'Roy. Soc. Proc.,' April 3, 1884. Probably owing to differences in the nature of the electric sparks, the "explosion-wave" was sometimes set up in his tube and sometimes not.

† See note at end of cap.

TABLE XXXIX.—Pressures in the Explosion-wave calculated from RIEMANN'S Theory.

Explosive Mixture.	Calculated Pressure. Π	$\frac{\Pi}{4\cdot2}$	Effective Pressure measured by BERTHELOT.
$C_2N_2+O_2$	135·2	32·2	25·11
C_2N_2+2NO	123·0	29·3	23·34
$C_2N_2+2N_2O$	121·6	28·9	26·02
$C_2N_2+2O_2$	90·1	21·5	20·96
$C_2N_2+O_2+2N_2$	71·0	16·9	15·25
$C_2H_4+O_2$	81·5		
$C_2H_4+O_3$	94·3		
$C_2H_4+O_4$	88·4		
$C_2H_4+O_6$	75·2	17·9	16·13
CH_4+O_4	62·2	14·8	16·34
$C_2H_2+O_2$	110·0		
$C_2H_2+O_3$	94·5		
$C_2H_2+O_3+N_{12}$	45·3		
H_8+O	28·8	6·9	7·05
H_2+O	41·3	9·8	9·8
H_2+O+N_5	31·6	7·5	7·41
H_2+N_2O	52·9	12·6	13·6
$CO+O$	35·7	8·5	10·1
$H+Cl$	46·9	11·2	By LE CHATELIER. 10·0

It is to be noticed that these calculated pressures are in substantial agreement with my experiments on the fracturing of glass tubes by the explosion of carbonic oxide and of cyanogen with oxygen; and also that the minimum pressures calculated for the diluted electrolytic gas are close to 30 atmospheres—the pressure required to fire electrolytic gas by the heat of compression.

NOTE BY PROFESSOR ARTHUR SCHUSTER, F.R.S.

In 1859 RIEMANN* deduced an equation for the propagation of abrupt variations in the density and pressure of a gas, on the supposition that such an abrupt variation could be propagated without change of type. Lord RAYLEIGH† criticising this investigation, draws attention to the fact that a steady wave is only possible for a particular relation between the pressure and density of the gas, which is different from the one actually holding. In the case of the explosion-waves it seems possible, however, that the temperature, pressure, and density of the gas should so adjust themselves as to make RIEMANN's equations applicable. In fact, they must do so if the front of the wave keeps its type, which it probably does when the velocity has become constant.

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* 'Göttingen Abhandlungen,' vol. 8 (1860).

† RAYLEIGH, 'Sound,' vol. 2, p. 41.

If ρ_0 and ρ are the densities of the gas in the undisturbed state, and at some point in the wave respectively, then, referring the motion to a system of coordinates moving forward with the velocity V of the wave, we have the condition of steady motion

$$\rho u = \rho_0 V$$

where u is the velocity at the point at which the density is ρ;
also

$$\frac{dp}{dx} = -\rho u \cdot \frac{du}{dx}.$$

As ρu is a constant quantity we may integrate the latter equation and obtain

$$p - p_0 = -\rho u (u - u_0) = \frac{\rho_0}{\rho} V^2 (\rho - \rho_0),$$

which gives for the velocity of the wave propagation

$$V = \sqrt{\frac{(p - p_0)}{(\rho - \rho_0)} \cdot \frac{\rho}{\rho_0}}.$$

This equation gives a relation between the density ρ at any point of the wave and the pressure p at that point, the pressure and density in the initial state being p_0 and ρ_0. The equation can be simplified if ρ is large compared to ρ_0 for

$$\frac{\rho}{\rho_0 (\rho - \rho_0)} = \frac{1}{\rho_0} / (1 - \rho_0/\rho)$$

so that we may write

$$V = \sqrt{\frac{p - p_0}{\rho_0}} \quad \sqrt{\frac{1}{1 - \rho_0/\rho}}$$

Putting, for the sake of argument, $\rho = 10\rho_0$, the second square root in the above expression becomes 1·05, so that as a first approximation it may be put equal to one. Writing P for the *excess* of pressure over the atmospheric pressure, the equation now takes the simple form

$$V = \sqrt{\frac{P}{\rho_0}}.$$

In this expression the value of P has to be written in dynes per square centimetre, and the velocity would be given in centimetres. For more convenient numerical calculation we may adopt provisionally the suggestion of GUILLAUME that a pressure of 75 cms. of mercury under the normal conditions should be called an *atmosphere*. The atmosphere then would be exactly one megadyne per square centimetre, and if P is given in atmospheres, the equation becomes

$$V = 1000 \sqrt{\frac{P}{\rho_0}},$$

or if V is expressed in metres per second,

$$V = 10 \sqrt{\frac{P}{\rho_0}}.$$

The excess of pressure in the waves, therefore, will be, as far as the equations hold,

$$P = \frac{V^2 \rho_0}{100}.$$

If the factor

$$\sqrt{\frac{\rho}{\rho - \rho_0}}$$

be retained, the calculated pressure P would be rather smaller than that given; thus if the maximum density of the wave is only double that of the undisturbed state, the calculated value of P would have to be reduced in the ratio of 1·4 to 1.

In order to test the equation

$$V = \sqrt{\frac{P}{\rho_0}}$$

I have applied it to compare the rates of explosions in the mixtures $H_7 + O$, $H_2 + O + N_5$, $H_2 + O_6$. In these three cases the explosive mixture $H_2 + O$ is diluted with the same volume of inert gases, and according to the equation the rates should be inversely as the square roots of the densities of unburnt gases, for the pressure P may be expected to be the same in all three cases. The rates found by Professor DIXON are 3532; 1822; 1707. Assuming 3532 to be right, I calculate for the other two combinations 1806 and 1711, which so far confirms the equation. But I am doubtful whether this means much, as the same result could be deduced from arguments relating merely to the dimensions of the quantities involved.

I have not discussed the question whether a steady wave is really possible, but assuming it to be steady I believe that the equation

$$V = \sqrt{\frac{p - p_0}{\rho - \rho_0} \cdot \frac{\rho}{\rho_0}}$$

holds. In the strict sense of the word I do not think the explosion-wave can be steady, because if the motion is, as assumed, linear, compression must precede the explosion, and Lord RAYLEIGH'S objection would hold for the front part of the wave in which no combination takes place.

But it seems possible to me that the motion may not strictly be a linear one, and that yet taking the average velocities over a cross-section of the tube the ordinary equations would apply. It seems probable that jets of hot gases are projected bodily forward from that part of the wave in which the combination takes place, and that these jets, which would correspond to the spray of a breaking wave, really fire the mixture.

CAP. XI.—SOME DIFFICULTIES IN THE "WAVE" THEORY.

§ 1. In calculating a theoretical velocity for the rate of the explosion-wave in gases, I am aware that I have passed over many difficulties which demand consideration. For some of the problems raised here we have, I believe, no data which might enable us to solve them. RIEMANN'S theory of the propagation of an intense disturbance in a gas requires, in order to calculate the velocity of propagation, a knowledge of the highest pressure reached in the wave. But concerning the pressures produced for extremely short periods in the explosion-wave, we have less exact knowledge than of the velocities, and possibly the rates of explosion may be used to calculate the pressures produced.

The question, indeed, may be raised whether the propagation of the ignition is due to a forward movement *en masse* of the burning molecules, or to the advance of a comparatively few molecules with extreme rapidity. In the latter case the explosion-wave might be propagated with a velocity far higher than the *average* velocity of the molecules of burning gas. I think the available evidence precludes the idea that the ignition can be propagated by a few molecules moving with velocities far above the mean. For, if this were the case, the presence of a diluent gas ought not to hinder the advance of the wave in the manner it is found to do. The heat evolved on the formation of a compound molecule is the same whether inert molecules are in the neighbourhood or not, and for the moment the temperature of the compound is the same. If, therefore, the ignition could be propagated by the advance of individual molecules, we might expect the explosion to be propagated at the same, or nearly the same, rate in the presence of an inert gas. My experiments have, however, conclusively shown that an inert gas retards the wave by an amount which depends upon its volume and density. It would seem to follow that each layer is heated as a whole and that the advance of the wave depends upon the average motion of the molecules. Two other facts appear to support this view. It is well known that small electric sparks may be sent through explosive mixtures of gases, *e.g.*, through electrolytic gas, without causing explosion. Yet the temperature in the spark must be extremely high, and combination occurs in the path of the spark. It would seem that the number of molecules heated by each spark was not sufficient to communicate ignition to the rest. Again, when a sound-wave is started in a mixture of gases, its rate of propagation depends upon the mean density of the gaseous mixture; there is no selective propagation of the wave by one of the constituents of the mixture.[*] This fact, of which I could find no experimental demonstration for gases of widely different density, has been verified, at my request, by Mr. F. J. SMITH in the MILLARD Laboratory, at Trinity College, Oxford. According to Mr. SMITH's careful experiments, sound is propagated at the same rate in ammonia gas (density 8·5) as in a mixture of equal volumes of hydrogen and oxygen of the same mean density as ammonia.

On the other hand, it may be that the wave does not advance perfectly regularly, but *breaks*, like a water-wave is broken by the quicker movement of the crest. In breaking, jets of heated gas may be projected in front of the wave into the unburnt gas beyond—such jets becoming the centres of fresh disturbances which coalesce to re-form a wave. If the jets were produced in sufficient number their average effect would be uniform.

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[*] FARADAY thought that gases of very different density would not mix perfectly. "As the atmosphere is now constituted, there exists a permanency of sonorous pitch; any tone being once generated remains the same tone until it dies away. . . . If, however, the atmosphere had been composed of two gases, each having widely different specific gravities, there would have been a difference. No permanency of tone could then have been depended upon—the pitch of every original note would have been continually varying as its transmission might be propagated by the vibrations of ever-varying mixtures of the two gases of dissimilar specific gravities." FARADAY's Lectures, SCOFFERN, 1853.

§ 2. *The Specific Heat of Gases at High Temperatures.*

In calculating the theoretical velocity of the explosion-wave, I have assumed that the specific heats of the simple gases do not rise with increase of temperature. On the other hand, I have assumed that the specific heats of gases formed from their elements with a condensation of volume—such as steam, nitrous oxide, ethylene, etc.—are equal at a high temperature to the sum of the specific heats of their elements. Thus, I have taken the specific heat of steam at constant volume to be 7·2 instead of 6·5, as given by determinations at low temperatures. In making these assumptions, I have followed the example of BERTHELOT in his investigations on the explosion-wave. But in his later work on the specific heat of the gaseous elements at very high temperatures, BERTHELOT comes to the conclusion that these specific heats rise so rapidly with an increase of temperature that they are doubled between 1600° and 4400° C.[*] He arrives at this result by exploding cyanogen to carbonic oxide in a bomb, and observing the pressures registered by a piston moving against a spring when the cyanogen is burnt by oxygen alone, and when nitrogen is added or oxides of nitrogen are used instead of oxygen.

TABLE XL.—BERTHELOT'S Experiments on the Specific Heats of Simple Gases.

Mixture.	Pressure developed.	Temperature.	Specific heat of N_2 or CO.
	At	°	
$C_2N_2 + O_2$	25·11	4394	9·60
$C_2N_2 + O_2 + 1\frac{1}{2}N_2$.	20·67	4024	8·39
$C_2N_2 + O_2 + 2N_2$ . .	15·26	3191	7·93
$C_2N_2 + O_2 + \frac{19}{21}N_2$.	11·78	2810	6·67
$C_2N_2 + 2NO$ . . .	23·34	4309	9·85
$C_2N_2 + 2N_2O$ . . .	26·02	3993	8·43

These values of the specific heats depend upon the accuracy with which the maximum pressure developed in the explosion is registered by the moving piston. Now the more rapidly the combustion takes place, the more nearly does the pressure exerted resemble a blow. I have found that a short column of mercury in a pressure-gauge suffers a far greater movement under the influence of a slow explosion than it does under the influence of a rapid explosion, although the quantity of heat evolved and the specific heats of the products were nearly the same in the two cases. No doubt this gauge was less sensitive to sudden pressures than the instrument used by M. BERTHELOT; but the experiment warns us that the pressures registered by a moving piston may fall below the true value, and the more rapid the explosion the greater this error may be. That the pressures registered, and, therefore, the specific heats deduced from them, vary with the instruments employed appears from a comparison of the numbers obtained by M. BERTHELOT, and those of MM. MALLARD and LE CHATELIER in the explosion of cyanogen. According to BERTHELOT's experiments, the specific heat of nitrogen and carbonic oxide is 6·2 at 2500°; according to MM. MALLARD and LE CHATELIER, it is 7·2 at this temperature. Again the latter have determined the specific heat of the simple gases by adding measured volumes of hydrogen, oxygen, and nitrogen to electrolytic gas, and measuring the pressures produced on explosion. From the seven experiments[*] made at temperatures near 2000° (calculated), they obtain, as a mean, the number 5·6 for the specific heat of the elementary gases at this temperature; in a former paper they gave the value 7·5 as the specific heat at this temperature. Neither of these values agrees with the 7·2 deduced from their cyanogen experiment at 2500°. The great difficulties attending determinations of this nature, in spite of the skill and patience with which the French chemists have attacked the problem, still leave it an open question whether the specific heats of the simple gases increase with the temperature.

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[*] 'Ann. Chim. et Phys.' (VI.), vol. 4, p. 66. 1885.

§ 3. The observations I have made on the effect of changes of initial pressure and temperature on the rate of explosion in gases are not such as might have been expected from the theory of ordinary sound-waves. There appears to be for each mixture a *crucial pressure*, above which variations of pressure do not affect the rate of explosion. Below this point the rates fall with diminution of pressure. In the case of electrolytic gas an increase of pressure from one to two atmospheres increases the rate of explosion from 2821 to 2872 metres per second. When the electrolytic gas is largely diluted, an increase in pressure from one to two atmospheres has less effect. The following rates were found at one and two atmospheres for electrolytic gas diluted with oxygen, nitrogen, and hydrogen respectively:—

TABLE XLI.

(1.) MIXTURE $H_2 + O_6$.

Pressure	1 At.	2 At.
Rate of explosion . . .	1696	1718

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[*] 'Combustion des Mélanges Gazeux,' p. 259.

(2.) MIXTURE $H_2 + O + N_5$.

Pressure	1 At.	2 At.
Rate of explosion . . .	1815	1830

(3.) MIXTURE $H_8 + O$.

Pressure	1 At.	2 At.
Rate of explosion . . .	3532	3538

No doubt the numbers obtained point to a slight increase in the rate on doubling the pressure, but the differences are so small as to be confounded with the experimental error. I have not, therefore, given the higher rates as the true rates of the explosion, but have taken the mean of the two in each case.

The difference in the effect of pressure on electrolytic gas when pure, and when diluted, may be due to one of two causes. In the case of the pure gas increase of pressure may check dissociation, and thus increase the rate of explosion; whereas the diluted gas does not dissociate, and is consequently unaffected by changes of pressure. Or, more probably, since the diluted gas suffers less contraction in the chemical change, its crucial pressure is lower than that of the pure gas, and the change from one to two atmospheres being wholly, or in large part, *above* the crucial pressure, the effect on the rate is nil or very slight. In all the mixtures I have tried the rate of the explosion-wave is diminished by a sufficient reduction of pressure, even when the products of combustion are carbonic oxide and nitrogen alone. Changes of pressure appear, therefore, to affect the rate apart from any dissociation; and consequently may influence the rate of explosion of electrolytic gas independently of any effect on dissociation. Why changes of pressure affect the rate of explosion has yet received no satisfactory explanation.

As the initial pressure of a gaseous mixture is reduced, it is found that the flame travels further from the ignition-point before its rate becomes constant; it requires a longer run to attain its maximum velocity. By sufficiently reducing the pressure it is possible to stop the explosion of gases altogether; but the flame is still propagated slowly at pressures below that at which the explosion-wave dies out. For instance, with electrolytic gas, the explosion-wave is propagated at 200 mm. pressure, but not at 150 mm. At 200 mm. pressure the flame travels about 2 metres before attaining its maximum rate. According to my experiments[*] the flame is still propagated in electrolytic gas at 75 mm., but not at 70 mm., when an induction spark is passed through the mixture. LOTHAR MEYER and SEUBERT* found that a spark from a RUHMKORFF'S coil caused partial ignition of electrolytic gas at 70 mm., and complete combustion at 72 mm., the exact pressure under which the gases unite varying with the strength of the spark.

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[*] 'Phil. Trans.,' 1884, p. 634.

It is conceivable, on the "breaking-wave" hypothesis, that the projected jets of heated gas do not under reduced pressure initiate their own explosion-waves so rapidly as under higher pressure.

Similarly with regard to changes of initial temperature, I do not know why a rise of temperature should diminish the rate of explosion. The limited range in which it is possible to conduct the experiments, and the small differences found within this range, do not permit of a curve being drawn which might indicate the general effect of changes of temperature upon the rate. In the case of electrolytic gas the effect of heating was found to lower the rate both at low and at high pressures. Experiments made at atmospheric pressure with ethylene, burning to carbonic oxide and steam, at 10° and 100°, showed a lowering of the rate rather greater than that of electrolytic gas; while similar experiments made with cyanogen, burning to carbonic oxide and nitrogen, showed a smaller difference—hardly, perhaps, to be distinguished from the experimental error.

§ 3. *The Rate of Explosion of Hydrocarbons with insufficient Oxygen.*

The explosion of hydrocarbons with insufficient oxygen has been studied by DALTON, J. DAVY, KERSTEN, and E. VON MEYER. In the 'New System of Chemical Philosophy,' Part II., p. 442 (1810), DALTON thus describes the explosion of equal volumes of ethylene and oxygen:—

"If 100 measures of oxygen be put to 100 of olefiant gas, and electrified, an explosion ensues, not very violent; but instead of a diminution, as usual, there is a great increase of gas; instead of 200 measures, there will be found about 360; some traces of carbonic acid are commonly observed, which disappear on passing two or three times through lime water; there will then remain perhaps 350 measures of a permanent gas, which is all combustible, yielding by an additional dose of oxygen, carbonic acid, and water, the same as if entirely burnt in the first instance. What, therefore, is this new gas in the intermediate state? The answer is clear. It is carbonic oxide and hydrogen mixed together, an equal number of atoms of each."

DALTON found that marsh gas could not be exploded with less than its own volume of oxygen, in which case half the hydrogen formed water and the other half was liberated:—

"If 100 measures of carburetted hydrogen be mixed with 100 measures of oxygen (the least that can be used with effect), and a spark passed through the mixture, there is an explosion without any material change of volume: after passing a few times through lime water, it is reduced a little, manifesting signs of carbonic acid. This residue is found to possess the characters of a mixture of equal volumes of carbonic oxide and hydrogen."

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* 'Chem. Soc. Journ.,' 1884, vol. 1. p. 587.

John Davy,[*] in 1832, described some experiments on the explosion of ethylene with insufficient oxygen, in which the volume was nearly doubled after detonation.

In 1861 Kersten[†] opposed the view, which he stated was common at that time, that in the combustion of hydrocarbons the hydrogen burns before the carbon. He describes experiments on the explosion of ethylene and of marsh gas with electrolytic gas, and with air. His conclusion is "that before a portion of hydrogen is burnt, all the carbon is burnt to carbonic oxide, and that then the excess of oxygen divides itself between the carbonic oxide and hydrogen."

In 1874, Ernst von Meyer[‡] (who does not refer to the previous work on the subject) showed that equal volumes of ethylene and oxygen formed carbonic oxide and hydrogen when exploded in an eudiometer. Acetylene, diluted with hydrogen to diminish the force of the explosion, behaved in the same way.

On the other hand, in many of the text-books of the present day the doctrine is taught that in the combustion of hydrocarbons the hydrogen burns before the carbon.[§] Berthelot comes to the same conclusion from observations of the rate of explosion of ethylene mixed with hydrogen, and that of other hydrocarbons rich in hydrogen. Measuring the rate of explosion from the point of ignition through a space some 130 mm. long, Berthelot found that a mixture of hydrogen and ethylene burnt at a rate which was quicker than the mean of the rates of hydrogen and of ethylene when exploded with oxygen in the same apparatus:

Mixture.	Time taken by flame to travel 130 mm.	
$C_2H_4 + O_6$	2·86 thousandths of second	1·95 mean.
$H_2 + O$	1·04 ,, ,,	
$C_2H_4 + H_2 + O_7$ . .	1·37 ,, ,,	

He concludes from these experiments that the hydrogen burns before the ethylene in the mixture. It must again be pointed out that these velocities refer to the irregular period of vibration, and that everything depends on the quickness with which the explosion-wave is developed. Again, BERTHELOT compares the rate of explosion of hydrocarbons with that of hydrogen in the same apparatus :—

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[*] 'Edin. Journal of Science,' New Ser., vol. 6, p. 50.

[†] 'Journ. Prakt. Chem.,' vol. 84, p. 303.

[‡] 'Journ. Prakt. Chem.' [II.], vol. 10, p. 308.

[§] Faraday, lecturing at the Royal Institution, said:—"The volatile matter raised by combustion from the tallow of a candle is a vapour composed of carbon and hydrogen; and our experiment teaches us that the forces which hold these elements together are so nicely balanced that the hydrogen is made to combine first, the carbon afterwards" ('Lectures at Royal Institution,' p. 280, J. Scoffern, 1853).

For a lucid discussion of this question, as far as it affects the liberation of carbon in hydrocarbon flames, see Professor A. Smithells "On the Structure of Luminous Flames" ('Journ. Chem. Soc.,' 1892, vol. 1, p. 217).

Gas.	Time.
Acetylene	1·94
Ethylene	2·86
Ethane	·83
Marsh Gas	1·24
Hydrogen	1·04

"The rate of combustion of gases rich in hydrogen is very close to that of hydrogen. This seems to indicate that the hydrogen burns before the carbon, even in total combustions."

I do not think this conclusion is justified by the figures ; nor do I see how the second argument can be reconciled with the first.

I have made experiments on the rate of the explosion-wave in ethylene and in acetylene mixed with their own volume of oxygen. In both cases the main reaction is the formation of carbonic oxide and hydrogen ; but the rates found are not so near the calculated velocities as I have found in other cases. The explosions being made in a gas-tight tube surrounded with water, the increase in pressure gave the expansion due to the chemical changes. With ethylene and oxygen the volume was almost doubled ; with acetylene and oxygen the volume was increased by one-half. In the following table I have given the observed rates of explosion for ethylene and acetylene, with an equal volume of oxygen, compared with the rate calculated by BERTHELOT's formula and with the sound-wave Σ :—

TABLE XLII.—Rates of Explosion of Ethylene and Acetylene with an equal volume of Oxygen.

1. $C_2H_4 + O_2 = 2CO + 2H_2$ (Q = 67,000 cals.)
2. $C_2H_2 + O_2 = 2CO + H_2$ (Q = 112,700 cals.)

Mixture.	$C_2H_4+O_2$	$C_2H_2+O_2$
BERTHELOT'S θ	2134	2733
Rate of explosion	2507	2961
Σ	2330	3084

The rate of explosion of ethylene and an equal volume of oxygen was found to be

2505 metres per second as the mean of four double determinations. The rate of the sound-wave "Σ," calculated on the hypothesis that the mixture burns directly to carbonic oxide and hydrogen, is 2330. The calculated velocity is, therefore, about 7 per cent. too low. This is the only instance I have yet found where the observed rate is largely in excess of the calculated sound-wave. On the other hand, the rate of explosion of acetylene and an equal volume of oxygen was found to be 2961 metres per second as the mean of four double determinations. The rate of the sound-wave, calculated on the hypothesis that the mixture burns directly to carbonic oxide and hydrogen, is 3084. The calculated velocity is, therefore, about 4 per cent. too high. This difference between the explosion of ethylene and acetylene, with an equal volume of oxygen, is somewhat striking; it seems to point to some different mode of decomposition in the two cases.

As stated by DALTON, a mixture of marsh gas with half its volume of oxygen will not explode; but a mixture of equal volumes of marsh gas and oxygen explodes. I have determined the rate of explosion of this mixture, and find the velocity is $2\frac{1}{2}$ per cent. faster than the sound-wave calculated for the reaction—

$$CH_4 + O_2 = CO + H_2 + H_2O.$$

TABLE XLIII.—Rate of Explosion of Marsh Gas with an equal volume of Oxygen.

Mixture.	$CH_4 + O_2$.
BERTHELOT'S θ	2184
Rate of explosion . .	2528
Σ	2466

At the temperature produced in this reaction the steam must be partly dissociated —lowering the rate. The observed velocity, therefore, is undoubtedly too high to reconcile with the calculated sound-wave. Marsh gas must, therefore, be classed with ethylene as giving exceptionally high rates.

At this point a difficulty must be pointed out which I have hitherto avoided. I have endeavoured to show that the explosion-wave is in part propagated by the movements of the yet unburnt molecules. The question arises—Are these unburnt molecules ever broken up by impact before they are burnt? In the case of an endothermic compound, such as cyanogen or acetylene, is the propagation of the flame partly due to the explosive decomposition of the compound, previous to the combustion of its constituents? If so, the unburnt gas would be diminished in density, and one might expect to find the explosion of endothermic compounds travelling at velocities greater than those calculated for sound in the undecomposed gas. But this is exactly the reverse of what is observed. The formula, which gives the correct velocity for explosions of cyanogen, gives numbers which are rather too high for acetylene (also a strongly endothermic compound), and too low for ethylene and for marsh gas (an exothermic compound). It would appear as if ethylene and marsh gas were unstable bodies, and acetylene and cyanogen were stable bodies, in the particular condition of the explosion-wave. Let me recall the words used by Professor OSTWALD in his address last year at Halle, to the sections of Physics and Chemistry, when speaking on a cognate subject—the spectra of gases at high temperatures :—

"It is held as an undoubted dogma that at the highest temperatures, as, for example, in the electric arc, all compounds must be dissociated into their elements. This view is certainly not justified. What we do know about the stability of compounds is, on the contrary, that all compounds which are formed with absorption of heat become *more stable* with rising temperature, and the reverse. Because the majority of the compounds known to us are formed from the elements with evolution of heat, and correspondingly become more unstable with rising temperature, the conclusion has been drawn that this is in general the case. But if we reflect that cyanogen and acetylene, two compounds formed with great absorption of energy, are readily formed in quantity, at the highest temperatures, in the blast furnace and in the arc light, we become conscious that spectra occurring at high temperatures may belong to compounds which have a fleeting existence confined to those temperatures only."*

We know, then, that cyanogen and acetylene are formed from their elements at the highest temperature; but BERTHELOT has shown that these gases are decomposed by the shock of a fulminate detonator.† Is it possible that there is an essential difference between the two acts—that molecules of acetylene, charged by highly heated molecules, do not break up, but that they are decomposed when they are dashed against cold molecules, or the cold walls of the containing vessel? I have found that the decomposition of acetylene by a fulminate is confined to the neighbourhood of the detonator. When a charge of fulminate is fired in a steel bomb filled with acetylene, the decomposition of the acetylene is not propagated along a tube attached to the bomb and filled with the gas. The flame of the detonator is "carried on" for some little distance, but *no explosion-wave is set up in the acetylene*. Similar results were found with carbon bisulphide vapour, though in this case the decomposition is propagated to a greater distance from the detonator. These experiments show that "the decomposition by shock of acetylene and carbon bisulphide is not propagated like the explosion-wave at a constant velocity as far as the gas extends, but the decomposition set up by the fulminate dies out at a distance from the detonator, depending on the nature of the gas, and also probably on the intensity of the initial shock and the cooling power of the walls."[*]

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* See 'Nature,' April 21, 1892.

† 'La Force des Matières Explosives,' vol. 1, p. 109. See also THORPE, 'Journ. Chem. Soc.,' 1889, p. 220.

On the other hand, there is some evidence of the breaking up of ethylene in the explosion-wave. When ethylene is exploded with an equal volume of oxygen some carbon is deposited, and an appreciable quantity of marsh gas is found in the products of combustion. It is difficult to account for this formation of marsh gas except by the decomposition of the ethylene, either directly or by partial oxidation :—

$$\text{(i.)}\ C_2H_4 \ . \quad . = C + CH_4.$$
$$\text{(ii.)}\ C_2H_4 + O = CO + CH_4.$$

When ethylene is exploded with less than its own volume of oxygen, the proportion of marsh gas found in the residue increases as the volume of oxygen is diminished.[†] This points to the formation of marsh gas as an intermediate product in the explosion of ethylene.

Again, it is a matter of frequent observation in the analysis of coal gas, when the residue of marsh gas, hydrogen, and carbonic oxide is exploded in an eudiometer, with a large excess of oxygen, that carbon is deposited towards the bottom of the tube. When, however, the mixture is sufficiently diluted with air, the marsh gas is completely burnt. This fact is easily explained on the assumption that in the explosion-wave (which is set up in marsh gas with oxygen but not with air) the marsh gas is more or less decomposed in the wave-front, and some of the liberated carbon escapes combustion by cooling.[‡]

I do not forget that the question raised concerning the breaking up of yet unburnt compounds in the explosion-wave, may also be asked concerning the elementary molecules. What happens, for instance, to the molecules of oxygen in the wave? When a molecule of oxygen reacts with a molecule of hydrogen, is an oxygen atom liberated, and, if so, how far does its existence influence the rate? It was with the object of obtaining some answer to these questions that experiments were made on the rate of explosion of hydrogen with nitrous oxide, in which reaction it is possible that no free atoms are liberated :—

$$H_2 + N_2O = H_2O + N_2.$$

So far as the experiments go, I cannot trace the influence of liberated atoms of oxygen.

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[*] DIXON and HARKER, "On the Decomposition by Shock of Endothermic Compounds." 'Manchester Memoirs' [IV.], vol. 5, 1892.

[†] LEAN and BONE, "The Behaviour of Ethylene on Explosion with less than its own Volume of Oxygen." 'Chem. Soc. Journ.,' 1892.

[‡] Results which might be explained on the same hypothesis—viz., the decomposition of unburnt compounds in the wave-front, are afforded by the explosion of ammonia with nitrous oxide, and of chlorine with hydrogen. In both cases the rate is somewhat higher than might be expected.

§ 4. *Conclusion.*

The difficulties I have touched on show how far we are from a complete theory of explosions.

The tentative formula which I have proposed must be taken for nothing more than a "working hypothesis." It has been found so far *parallel to the truth* that it may be useful to suggest further experiments on the nature of explosions. As BERTHELOT'S formula helped me to interpret the reactions of gaseous carbon, and suggested experiments which led me to modify the hypothesis I was using; so I will venture to hope that this modified hypothesis may in turn lead other workers to a truer knowledge of explosions.

In concluding, I wish to thank three former pupils of mine, who have rendered me devoted service in the conduct of these experiments—Mr. J. E. MARSH, of Balliol College, Oxford, and Messrs. H. W. SMITH and J. A. HARKER, of Owens College, Manchester.

GENERAL CONCLUSIONS.

1. BERTHELOT'S measurements of the rates of explosion of a number of gaseous mixtures have been confirmed. The rate of the explosion-wave for each mixture is constant. It is independent of the diameter of the tube above a certain limit.

2. The rate is not absolutely independent of the initial temperature and pressure of the gases. With rise of temperature the rate falls; with rise of pressure the rate increases; but above a certain *crucial pressure* variations in pressure appear to have no effect.

3. In the explosion of carbonic oxide and oxygen in a long tube the presence of steam has a marked influence on the rate. From measurements of the rate of explosion with different quantities of steam the conclusion is drawn that at the high temperature of the explosion-wave, as well as in ordinary combustion, the oxidation of the carbonic oxide is effected by the interaction of the steam.

4. Inert gases are found to retard the explosion-wave according to their volume and density. Within wide limits an excess of one of the combustible gases has the same retarding effect as an inert gas (of the same volume and density) which can take no part in the reaction.

5. Measurements of the rate of explosion can be employed for determining the course of some chemical changes.

In the explosion of a volatile carbon-compound with oxygen, the gaseous carbon burns first to carbonic oxide, and afterwards, if oxygen is present in excess, the carbonic oxide first formed burns to carbonic acid.

6. The theory proposed by BERTHELOT—that in the explosion-wave the flame travels at the mean velocity of the products of combustion—although in agreement with the rates observed in a certain number of cases, does not account for the velocities found in other gaseous mixtures.

7. It seems probable that in the explosion-wave—
   (1) The gases are heated at *constant volume,* and not at *constant pressure;*
   (2) Each layer of gas is raised in temperature *before* being burnt;
   (3) The wave is propagated not only by the movements of the burnt molecules, but also by those of the heated but yet unburnt molecules;
   (4) When the permanent volume of the gases is changed in the chemical reaction an alteration of temperature is thereby caused which affects the velocity of the wave.

8. In a gas, of the mean density and temperature calculated on these assumptions, a sound-wave would travel at a velocity which nearly agrees with the observed rate of explosion in those cases where the products of combustion are perfect gases.

9. With mixtures in which steam is formed, the rate of explosion falls below the calculated rate of the sound-wave. But when such mixtures are largely diluted with an inert gas, the calculated and found velocities coincide. It seems reasonable to suppose that at the higher temperatures the lowering of the rate of explosion is brought about by the dissociation of the steam, or by an increase in its specific heat, or by both these causes.

10. The propagation of the explosion-wave in gases must be accompanied by a very high pressure lasting for a very short time. The experiments of MM. MALLARD and LE CHATELIER, as well as my own, show the presence of these fugitive pressures. It is possible that data for calculating the pressures produced may be derived from a knowledge of the densities of the unburnt gases and of their rates of explosion.

APPENDIX.

Received July 22, 1892.

Part I.—*The Preparation of the Gases.*

Hydrogen.—The hydrogen used in most of the experiments was prepared by the action of pure dilute sulphuric acid on re-distilled zinc, free from arsenic. The gas was washed with potash and silver nitrate solutions, and was then passed through a scrubbing tower and a series of bulbs filled with a strong solution of alkaline permanganate. The gas so prepared had only the faintest odour.

For a few experiments the gas was prepared by the electrolysis of pure dilute sulphuric acid with platinum electrodes.

Marsh Gas.—The marsh gas was prepared by GLADSTONE and TRIBE'S method,[*] in which pure methyl iodide is decomposed by the zinc-copper couple in presence of absolute alcohol. The "scrubber," packed with the zinc-copper couple, was kept moistened by alcohol dropping from a tap-funnel, and the gas was further washed with strong alcoholic potash and finally with oil of vitriol. The generating flask (of 2 litres capacity) was kept at 30° C.

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[*] 'Chem. Soc. Journ.,' 1884, p. 1.

An analysis of the gas by explosion with air in McLeod's gas apparatus showed 99·3 per cent. of marsh gas.

Ethylene.—The ethylene was prepared by the method given by Erlenmeyer and Bunte. 25 grms. of absolute alcohol are mixed with 150 grms. of pure sulphuric acid, and the mixture is heated in a large flask, on a sand bath, to 160°–170° C. A mixture of one part alcohol and two parts sulphuric acid is then slowly added by a tap-funnel. Great difficulty was experienced in getting rid of all acid gases from the ethylene. The gas was passed (1) through two wash-bottles containing oil of vitriol, (2) through a tower packed with pieces of solid potash, (3) through two wash-bottles containing strong potash solution, (4) through a second tower packed with pieces of solid potash. The large quantity of air contained in the flask and washing apparatus rendered it difficult to obtain the gas quite free from air. The gas was analysed and found to be free from sulphurous acid, but it contained small quantities of nitrogen. Allowance was made for the presence of this diluent gas in making up the mixtures of ethylene with oxygen, &c. The percentage impurity in the ethylene could be reduced to 1 per cent.; no other gas used in these experiments presented the same difficulty in making pure.

Acetylene.—The acetylene was prepared by the action of strong alcoholic potash or ethylene dibromide. The potash was dissolved in amyl alcohol. The gas evolved from the boiling solution was passed through a second flask containing boiling potash solution, and then into an ammoniacal solution of cuprous chloride. The precipitated cuprous acetylide was washed with dilute ammonia solution and kept till required under this solution. On warming the red precipitate with strong hydrochloric acid to 50° C. a regular stream of acetylene was given off. It was washed with water, dried with caustic potash, and collected over mercury. Analysis showed that the gas was pure.

Ammonia.—The ammonia was distilled from a strong aqueous solution, was dried by caustic potash, and was collected over mercury.

Carbonic Oxide.—This gas was prepared by gently warming formic acid with pure sulphuric acid diluted with an equal volume of water. It was washed with potash solution.

Cyanogen.—Re-crystallised mercuric cyanide was powdered and carefully heated to drive off moisture. The dry powder was introduced into a steel retort with a long neck. After a considerable quantity of cyanogen had been allowed to escape, the gas was collected directly in a mercury holder. The retort was raised to dull redness in a fireclay furnace, heated by gas.

Oxygen.—The oxygen was prepared by heating powdered chlorate of potash in a hard glass flask. It was washed by a solution of caustic potash.

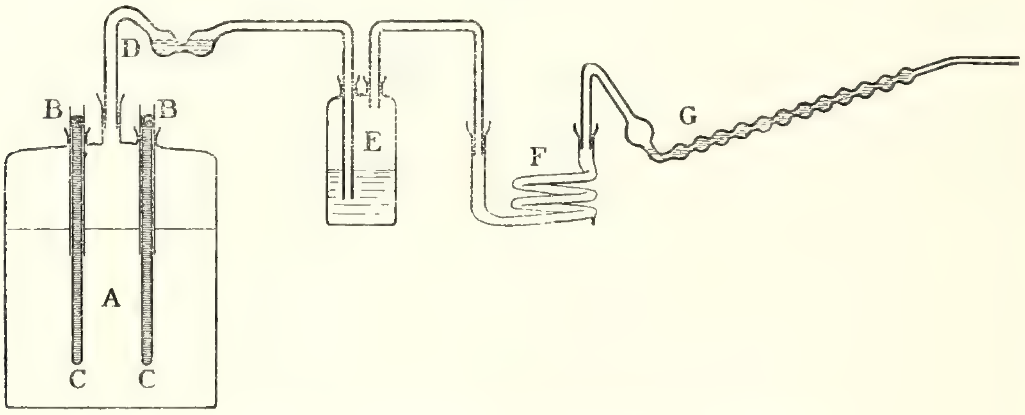
Nitrous Oxide.—This gas was prepared by heating pure ammonium nitrate in a hard glass flask. It was washed with a solution of ferrous sulphate, dried by sulphuric acid, and collected over mercury.

Nitric Oxide.—The nitric oxide was made by mixing eight parts of crystallised ferrous sulphate with one part of crystallised nitre, and pouring on to this mixture sulphuric acid diluted with three volumes of water. If the acid is warm the action starts at once, afterwards the flask is gently warmed.

Nitrogen.—The nitrogen was prepared by VERNON HARCOURT's method, in which a mixture of air and ammonia is passed through a heated tube containing copper and copper oxide. The nitrogen issuing from the tube was washed with sulphuric acid.

Chlorine.—The chlorine employed in determining the rates of explosion of hydrogen and chlorine was prepared by the electrolysis of aqueous hydrochloric acid, and was passed, mixed with hydrogen, into the explosion-tube without standing in a holder. The gas was washed first with water, and then with sulphuric acid, and was dried in a glass worm and a series of bulbs filled with sulphuric acid. The electrolytic cell was made entirely of glass, and the washing bottles and drying vessels were all fitted together with ground glass joints with cups.

Fig. 1.



Apparatus for preparing chlorine and hydrogen.

The electrolytic vessel A was made with three tubulures, into which ground glass tubes were fitted. Two of these, BB, passed into the vessel, and dipped about an inch below the level of the acid. Through these tubes passed the gas-carbon electrodes CC, the small annular space between the rods and tubes being made tight with paraffin. A little mercury poured on top of the carbon rods served to connect them with the battery wires. Into the third tubulure was ground a gas delivery tube D, containing washing bulbs filled with water, and passing through a tubulure in the sulphuric acid wash-bottle E. From E the gas passed through the worm F and through the bulbs G to a long flexible glass tube leading into the explosion-tube.

For mixtures containing an excess of hydrogen, the wash-bottle E was replaced by a vessel fitted with three tubulures similar in shape to A. Two of the tubulures were fitted with tubes of the same size dipping beneath the acid, the third tubulure serving for the exit tube. The electrolytic gas entered through one tube, while hydrogen was driven from a holder through the other.

At the further end of the explosion tube the gases passed through (1) a wash bottle; (2) a long graduated tube fitted with taps at either end; (3) a thin glass bulb; (4) a tower filled with lime and powdered charcoal. The thin bulb could be readily detached and submitted to a "magnesium flash." The graduated tube served to analyse the gas. It was connected with a movable reservoir from which a solution of caustic potash could be introduced into the tube. The volume of chlorine present could thus be readily estimated, the residual hydrogen being measured under atmospheric pressure by bringing the level of the liquid in the tube and reservoir to the same height.

The current for the electrolysis was given by 7 secondary cells (Elwell and Parker), E.M.F. 13–15 volts; current 4–6 amperes. The preliminary saturation occupied many hours. It was found best to leave a small current (about ½ ampere) passing through the cell for two days before making an experiment. When the liquids were all saturated it took about four hours to fill the explosion tube. An analysis of the issuing gas at the end of that time gave—

Chlorine	49·82
Hydrogen	50·18

In another experiment an analysis at the end of 4½ hours gave—

Chlorine	49·90
Hydrogen	50·10

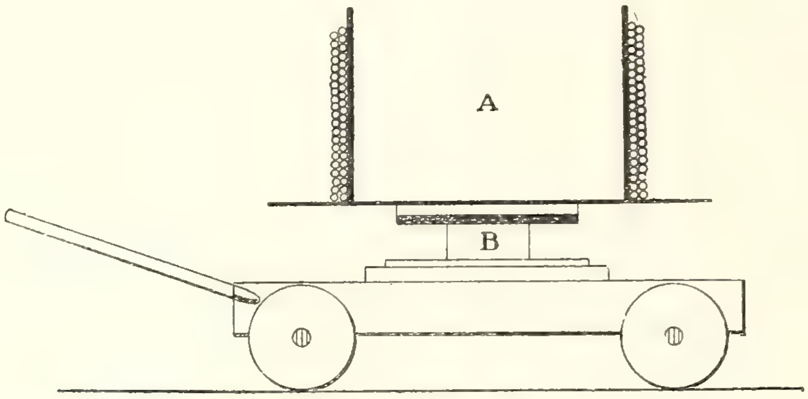
Part II.—*Arrangement of the Firing Tubes.*

The leaden pipes used for the explosion tubes were 5, 6½, 9, and 13 mm. in internal diameter. The pipe was wound on a galvanised iron drum about 2 feet in diameter, provided with a flange at the bottom.

In order to wind the pipe it was first laid straight along a corridor. The drum A was attached to a turntable B fixed to a low trolley, which was pushed forward as the pipe was wound on the revolving drum.

The drum with the pipe wound on it was placed in an iron water-bath, which could be heated by a large ring burner. The ends of the pipe passed out through two openings near the top of the bath. To each end of the explosion tube was soldered a short steel tube, ending in a cap with a polished steel face 30 mm. in diameter, perforated in the centre by a hole 10 mm. in diameter. The steel face of each cap is pressed by a clamp against a similar steel face, as in Regnault's gas apparatus, from which these joints are taken. One end of the explosion tube is thus joined to the "firing piece," the other end to a shorter tube, called the "end piece," carrying the second "bridge."

Fig. 2.

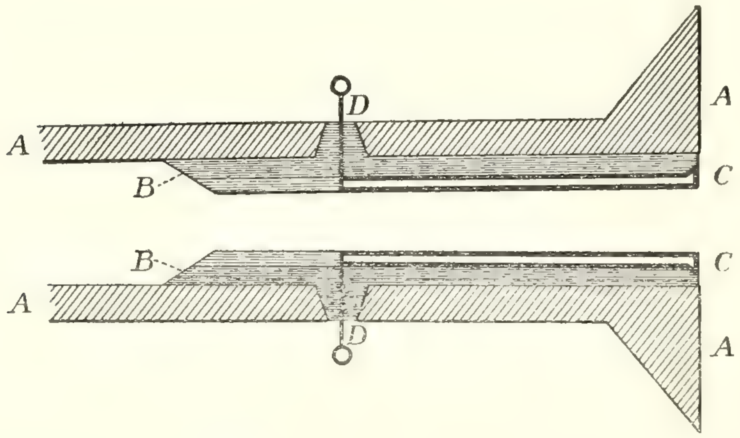


Section of drum mounted on turntable, and trolley for winding the explosion tube.

The caps which carried the bridges to be broken by the flame were constructed in the following way:—

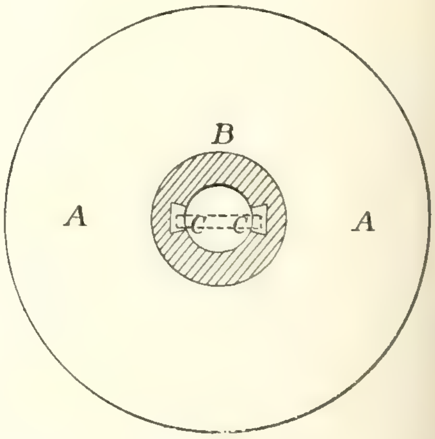
The steel tube AA was pierced at a distance of 30 mm. from the face by two conical holes opposite each other. The interior of the tube was lined with a cylinder of vulcanite, which also filled the conical holes. Two thick platinum wires, DD, passed through the holes, and were soldered to two pieces of silver, CC, embedded in the vulcanite, and thus insulated from the steel tube. The silver pieces were broadened out at the face end into two small plates, on to which the narrow strip of silver foil forming the bridge was soldered. The silver plates were brought not quite flush with the polished steel face, so that there was no danger of breaking the bridge when clamping up the two caps.

Fig. 3.



Section of bridge-piece.

Fig. 4.

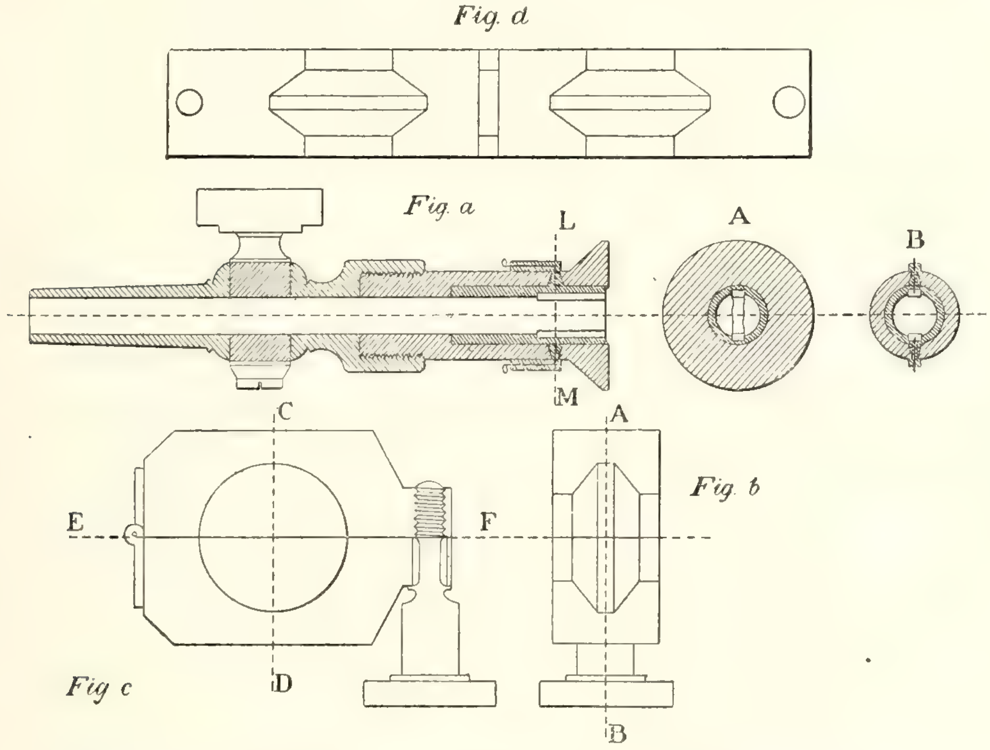


Front view.

For the experiments with chlorine and hydrogen a glass tube was fixed inside the vulcanite so as to protect it from the action of the chlorine. The silver bridge, after being soldered to the silver plates, was warmed and coated with a layer of paraffin as a protection from the chlorine.

The apparatus being arranged as shown in fig. 7, the explosive mixture from the holder was driven through the washing vessel A, partly filled with boiled oil of vitriol, and through the two drying towers B, B, packed with pieces of asbestos saturated with boiled oil of vitriol. The gas then passed through a short length of black rubber tubing to the end piece C carrying the second bridge. From the end piece the mixture passed into the explosion tube coiled on a drum standing in the large iron water-bath D. A second clamp united the explosion tube to the firing piece E, carrying the first bridge at one end, and the firing wires *f*, *f*, sealed through a stout glass tube, and a tap G at the other. The taps were made of hardened steel. After leaving the firing piece the gas passed by a glass tube M to a three-way tap H, by which it could be sent either to the testing tube I or by the pipe K to the window. In many of the experiments a sulphuric acid wash-bottle L was joined to the tube at M, and connected with a holder containing electrolytic gas. Before starting the experiment the air was displaced from the vessel L by a stream of electrolytic gas. When the explosion tube and firing piece were full of the explosive mixture, the tap H was shut, and a few bubbles of electrolytic gas were driven into the firing piece so as to fill about one-third of its length. This insured the explosion-wave being set up in the firing piece before the flame reached the first bridge, and was a convenient mode of firing mixtures—such as those containing cyanogen—which do not readily ignite.

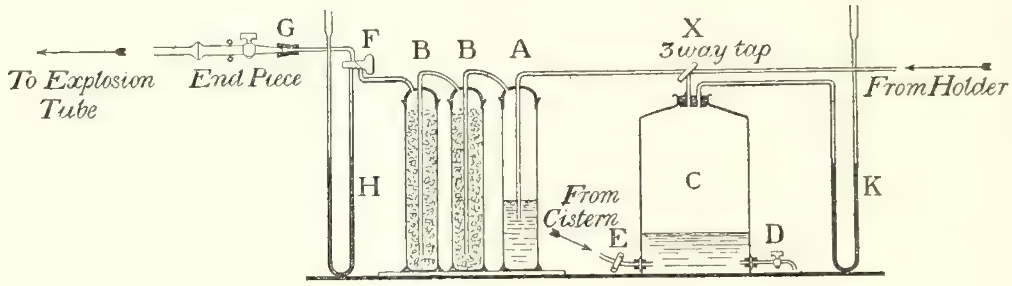
Fig. 5.



Shows the construction of a second pair of bridge pieces and clamps used during the last two years.

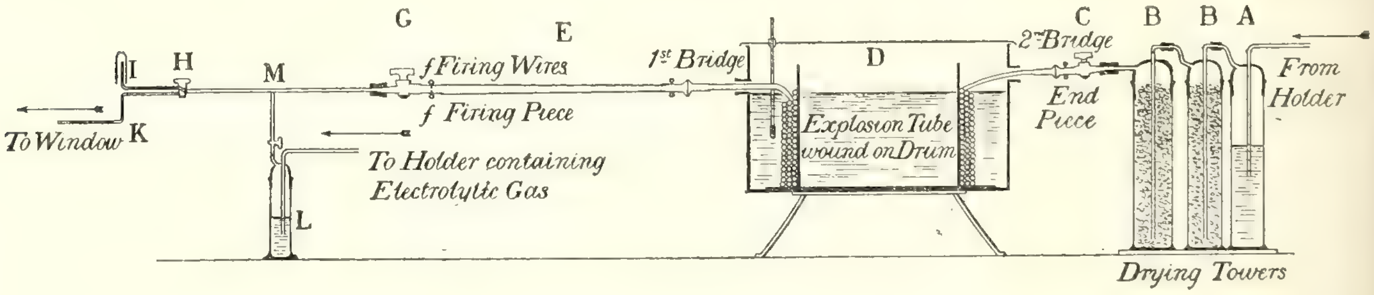
For experiments under increased pressure the arrangement shown in fig. 6 was adopted. The strong glass cylinder C, filled with water from a high cistern through the tube E, or (when necessary) with mercury from a second movable cylinder, was filled with the mixture from the holder by allowing the liquid to run out at the tap D. After the cylinder was filled, the three-way tap X was turned so as to send the gas from the holder through the drying vessels A, B, B, and through the three-way tap F, into the explosion tube. When the explosion tube was full, the tap X was turned so as to put it in communication with the cylinder C. The tap at the further end of the explosion tube being closed, water or mercury was admitted to C, forcing the gas into the explosion tube through the drying vessels. When the manometer K showed the desired pressure had been reached, the taps were closed and the connections made with the chronograph and firing wires. Just before firing, the taps were opened to the manometer K and the pressure in the explosion tube was read off. The taps were then shut and the mixture fired.

Fig. 6.



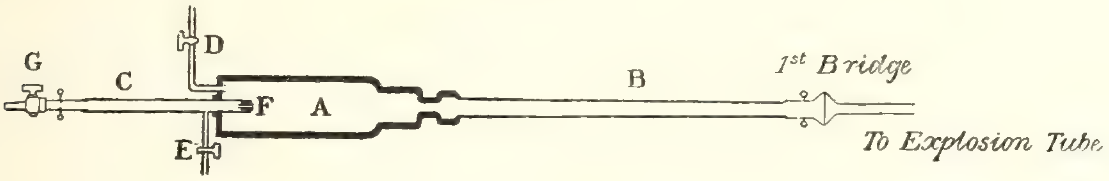
For experiments under reduced pressure the explosion tube was filled in the usual way. The gas was then sucked from the apparatus by a pump attached to the tube K (Fig. 7), a manometer sealed in at M (in place of the wash bottle L) giving the pressure in the explosion tube.

Fig. 7.



For firing the mixture of cyanogen with nitric oxide the following arrangement was adopted. The firing piece B was firmly screwed to a stout steel cylinder A. Through the other end of this cylinder a tube C was screwed so as to project 2 inches into the cylinder. Into the end of this tube a copper detonator holding fulminate of mercury was attached, by a caoutchouc stopper with a large hole, the open end of the detonator pointing inwards. The explosive mixture passed from the explosion tube through the firing piece and cylinder, escaping at the tube D which was also screwed into the base of the cylinder. The tube C was then filled with electrolytic gas through the tap G, the air escaping at E. On firing the gas in C by the firing wires the explosion travelled to F, and there set off the detonator in the middle of the cyanogen mixture. The explosion thus started was propagated through the tube B, into the explosion tube which it traversed to the end.

Fig. 8.



Part III.—*Chronographic Apparatus.*

The arrangement of the chronograph and electrical connections are shown in fig. 9. The chronograph, made by Messrs. Elliot, consisted of a heavy pendulum carrying a smoked glass plate. The electro-magnetic styli used were made by Pfeil, of Berlin, and by Mr. F. J. Smith,* of Trinity College, Oxford, to whom I am greatly indebted for several of these instruments. In most of the experiments recorded in this paper, one of Pfeil's instruments and one of Mr. Smith's were employed. These two were chosen out of several, for the great constancy of their latent period, and for the sharpness of their action. The styli were attached to an upright bar on a heavy iron stand, with three pointed feet. The bar could be tilted back on two feet and exactly replaced.

At the lowest point of its swing the pendulum acted on two breaks, X and X′, which by a rack and pinion could be brought exactly in a line. By bringing the pendulum slowly against the uprights the position of the two breaks could be adjusted so that they were broken at the same instant.

In the preliminary experiment the wires were arranged as shown in the figure, except that the wires to the firing battery were detached at C′C′, and the wires from *bb*, *b′b′* to the two bridges were not connected. The stylus S was put in circuit through a resistance RR, and the break X; the stylus S′ was put in circuit through a resistance R′R′ and the break X′. On letting the pendulum fall the two styli were released, and registered their marks on the moving plate. The resistances RR and R′R′ were now removed, the firing battery and coil put into circuit at C′C′, the wires from *bb* and *b′b′* attached to the two bridges, and the wires from the coil attached to the firing piece. The pendulum, being again let fall, broke the primary circuit of the coil on striking X′, causing a spark to pass between the platinum wires in the firing piece. The flame travelling down the tube broke both bridges in turn, and the styli connected with them registered their marks on the moving plate. The connections were immediately adjusted as in the preliminary experiment, and the pendulum was again let fall. If the "confirmatory" marks passed directly over the "preliminary" it was a proof that nothing had been displaced during the experiment.*

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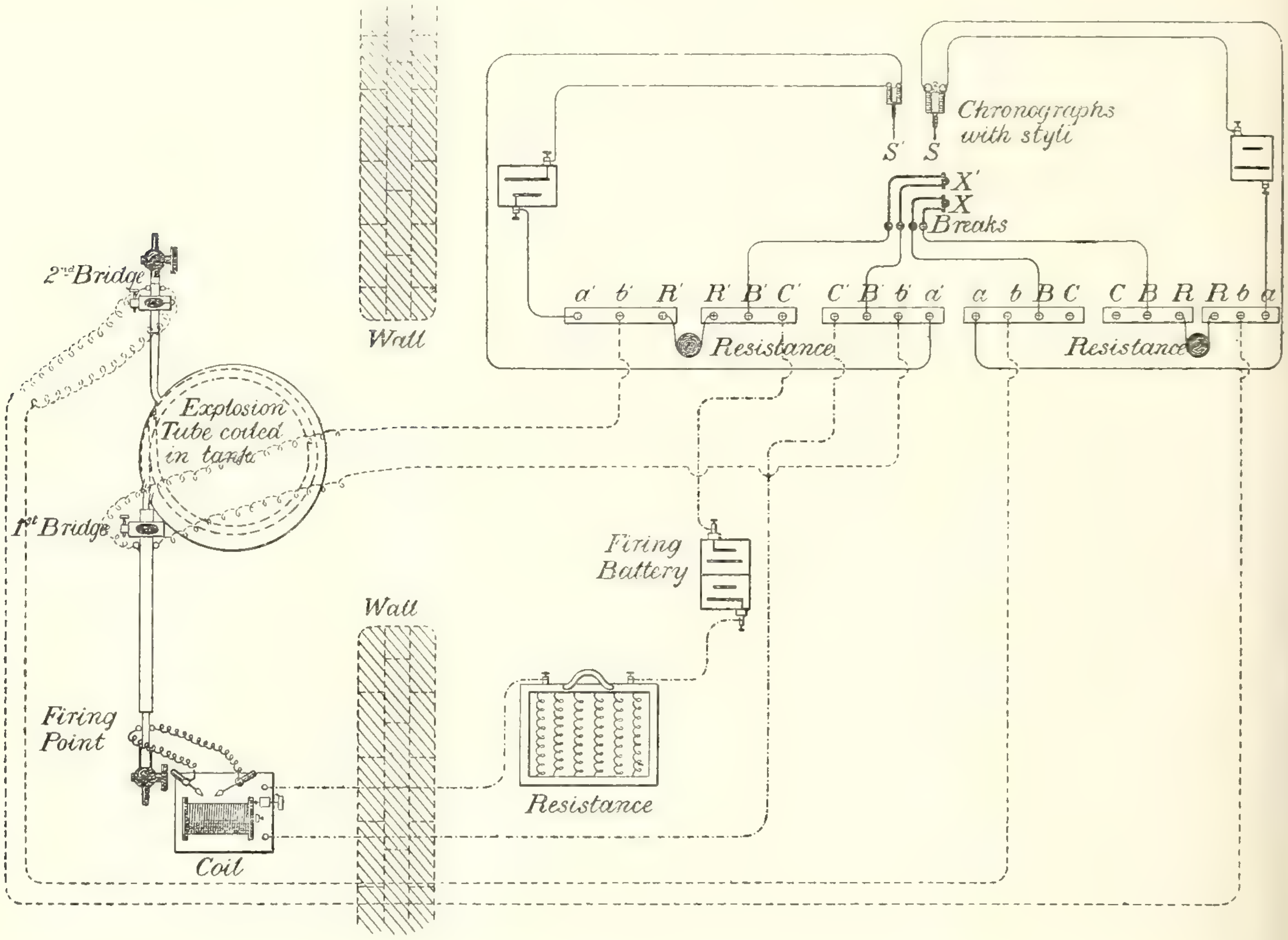
* 'Phil. Mag.,' 1890, p. 377.

Fig. 9.



In the companion experiment, which was made as soon as possible after the first, the wires from the break X were removed from BB to B'B', and the wires from X' from B'B' to BB. X was now the firing break. Any difference between the two breaks which might affect the moment of firing was thus eliminated by taking the mean of the pair of experiments.

The breaks were constructed as shown in fig. 10. The contact was made between a platinum plate A and a platinum point B joined to a copper strip. The platinum plate was fastened to a piece of ebonite turning on a horizontal axis; a spiral spring pressed the plate against the platinum point until the arm C of the pendulum struck the copper plate D attached to the ebonite. The whole piece could be moved to the right or left by a pinion working in the rack E.

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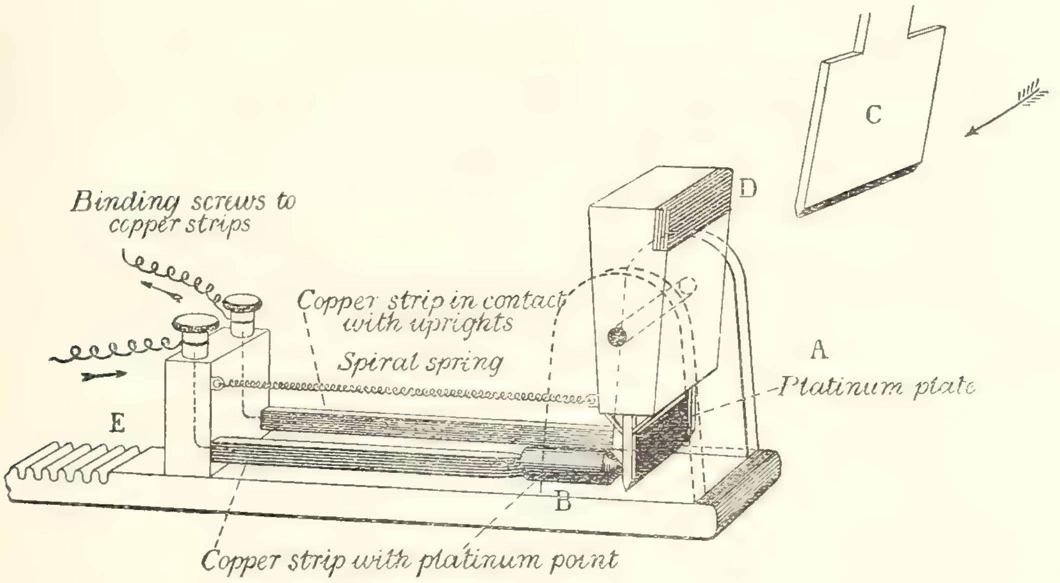
* Mr. F. J. Smith has pointed out that the difference in the latencies of two electro-magnetic styli may be eliminated by making two experiments, and reversing the connections for the second experiment. I think my plan of making a "preliminary experiment" (without firing the gases) is quicker. In both cases it is assumed that the latencies remain constant during the experiment: the quicker the operation the more likely is this condition to be attained.

Fig. 10.


C
D
Binding screws to copper strips
Copper strip in contact with uprights
Spiral spring
A
Platinum plate
E
B
Copper strip with platinum point

Side view of one of the breaks.

The tuning fork was by Koenig. It gave 100 complete vibrations per second at 15° C. Its temperature correction was determined by Mr. R. H. M. Bosanquet in the Laboratory of St. John's College, Oxford. By measuring the length of the wave traced by the tuning fork on the plate at different heights, a table of wave-lengths was constructed, giving the value of $\frac{1}{100}$ second at different heights on the plate, and for different lengths of swing. After measuring the distance between two marks of a stylus and the height of the tracing on the plate, a reference to the table gave the value of the distance in time. But, to make sure that the pendulum was moving at the same rate as when the table was constructed, a tuning fork trace was taken after each experiment—everything being in the same state except that the tuning fork was substituted for the electro-magnetic styli. From the length of the observed trace, compared with that given in the table, a correction could readily be applied to the table readings. A small correction was also made, when necessary, for the effect of temperature on the tuning fork.

The distance between the marks of the styli was measured by cutting the rising trace 2 mm. above the datum line with a pointed style, and drawing a perpendicular line through the points of intersection, as shown in fig. 11. At *a* the upper stylus began to rise in the preliminary experiment, and the lower stylus at *b*. In the explosion the lower stylus was released by the breaking of the first bridge, and began to rise at *b*′; the upper stylus was released by the breaking of the second bridge, and began to rise at *a*′. The distances between the lines AA′ and between the lines BB′ being measured, as well as the position of *aa*′ and *bb*′ on the plate, the value of BB′

(corrected for position) is subtracted from AA′, the remainder giving the time of explosion.

I take one example from my note-book of the actual measurements obtained in a pair of experiments :—

mm.

A.—Distance between "firing" and "second bridge" marks = 70·5
Distance between "firing" and "first bridge" marks . = 4·7

Distance between first and second bridge marks . . . = 65·8

At this point on the plate and for this length of swing the distance, which is equal to $\frac{1}{100}$ second, is 24·62 mm. Length of tube between bridges, 75·35 metres.

Rate of explosion . . . . = 2820 metres per second.

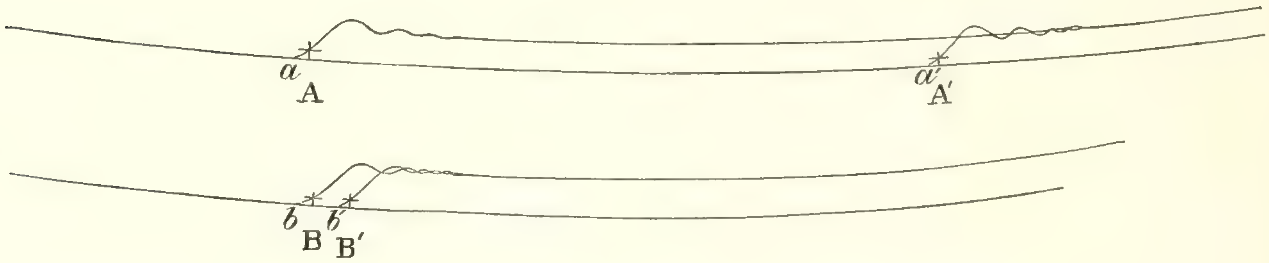
B. (with connections reversed)— mm.
Distance between "firing" and "second bridge" marks = 68·6
Distance between "firing" and "first bridge" marks . = 2·4

Distance between first and second bridge marks . . . = 66·2

At this point on the plate and for this length of swing, the distance, which is equal to $\frac{1}{100}$ second, is 24·70 mm.

Rate of explosion . . . . = 2811 metres per second.
Mean rate . . . . = 2815 ,, ,,

Fig. 11.



Traces made by electro-magnetic styli on the moving plate.

Part IV.—*The Rates of Explosion as Measured.*

(1.) HYDROGEN and Oxygen under Ordinary Temperature and Pressure.

The rate of explosion of the mixture $H_2 + O$ was measured in a tube 9 mm. in diameter and 55 metres long.

Rates Found.

SERIES A.
Metres per sec. 2804 2812 2835
Mean 2817

In a tube of same diameter and 100 metres long :—

SERIES B.
2822 2847 2795
2821

In a tube 13 mm. diameter and 100 metres long :—

SERIES C.
2805 2840 2831 2800
2819

In 1888–90 these experiments were repeated. The numbers which follow are all the means of pairs of measurements with the connections of the chronograph reversed :—

Series D.
2818 2820 2831 2828
2824

Series E.	Series F.	Series G.
In 9 mm. tube.	In $6\frac{1}{2}$ mm. tube.	In $6\frac{1}{2}$ mm. tube.
2852	2795	2827
2814	2781	2840
2825	2814	2803
2825	2807	2836
2830		
2828	2799	2826

Most of the experiments given in these three series, E, F, and G, were *control* experiments interpolated in other series to check the working of the whole apparatus.

In the following series the mixture was fired alternately in a straight tube and in the same tube coiled on a drum. Each number is the mean of a pair of measurements with connections reversed :—

Series H.	
9 mm. tube 100 metres long.	
2828	straight
2829	coiled
2820	straight
2812	coiled
2818	straight
2821	

(2.) Hydrogen and Oxygen under Varying Pressure.

1. At 10° C.				
200 mm.	300 mm.	500 mm.	1100 mm.	1500 mm.
2636	2716	2784	2869	2869
2619	2693	2764	2850	2871
2626	2680	2777	2864	2873
	2713		2844	2883
	2723		2852	2861
			2845	2873
2627	2705	2775	2856	2872

2. At 100° C.				
390 mm.	500 mm.	760 mm.	1000 mm.	1450 mm.
2704	2730	2784	2833	2831
2689	2746	2793	2825	2837
		2799	2824	2849
		2796	2832	2849
		2789		
		2800		
		2782		
		2774		
2697	2738	2790	2828	2842

(3.) Hydrogen and Oxygen in Different Proportions, under Ordinary Temperature and Pressure.

In 13 mm. tube.	In 9 mm. tube.				In 13 mm. tube.
H_8+O.	H_6+O.	H_4+O.	H_2+O_2.	H_2+O_4.	H_2+O_6.
3547	3517	3265	2315	1945	1703
3518	3526	3281	2331	1905	1728
3537	3538	3261	2344	1943	1708
3526	3527	3266	2322	1914	1691
3532	3527	3268	2328	1927	1707

(4.) HYDROGEN and Oxygen in Different Proportions under 1500 mm.

$H_8 + O$	$H_2 + O_6$
3534	1723
3542	1713
3538	1718

(5.) HYDROGEN and Oxygen Diluted with Nitrogen.

At 760 mm. and 10°.		
In 9 mm. tube.		In 13 mm. tube.
H_2+O+N	H_2+O+N_3	H_2+O+N_5
2429	2065	1822
2408	2045	1805
2443	2065	1809
2425	2063	1824
	2035	
2426	2055	1815

Under 1500 mm. and 10°.
H_2+O+N_5
1821
1838
1830

MARSH Gas and Oxygen in Varying Proportions.

At 760 mm. and 10°.				
CH_4+O_2	CH_4+O_3	CH_4+O_4	CH_4+O_6	CH_4+O_8
2531 2510 2552 2520	2479 2469 2463	2322 2318 2326	2147 2145	1954 1972
2528	2470	2322	2146	1963

MARSH Gas and Oxygen diluted with Nitrogen.

At 760 mm. and 10°.		
CH_4+O_3+N	$CH_4+O_3+N_3$	$CH_4+O_3+N_5$
2349 2364 2348 2334	2146 2162 2155	1872 1888
2349	2154	1880

MARSH Gas and Oxygen under Varying Pressures.

CH_4+O_3		CH_4+O_4	
500 mm.	1000 mm.	500 mm.	1000 mm.
2414 2422	2480 2496	2273 2287	2324 2314
2413	2488	2280	2319

ETHYLENE and Oxygen in Different Proportions.

At 760 mm. and 10° C.						
$C_2H_4+O_2$	$C_2H_4+2O_2$	$C_2H_4+3O_2$	$C_2H_4+4O_2$	$C_2H_4+6O_2$	$C_2H_4+8O_2$	$C_2H_4+10O_2$
2516 2527 2472 2515	2606 2578 2562 2571 2586	2364 2359 2380 2369	2239 2255	2131 2105	1974 1986	1852 1860
2507	2581	2368	2247	2118	1980	1856

ETHYLENE and Oxygen Diluted with Nitrogen.

$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+4N_2$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
2430 2401 2408	2202 2220	2009 2039	1881 1874	1737 1731
2413	2211	2024	1878	1734

ETHYLENE and Oxygen.

At 760 mm. and 100° C.
$C_2H_4+2O_2$
2541 2532 2532 2548
2538

ACETYLENE and Oxygen in Different Proportions.

At 760 mm. and 10° C.		
$C_2H_2+O_2$	$C_2H_2+O_3$	$C_2H_2+O_5$
2961 2950 2971	2698 2747 2691 2703 2722 2733	2389 2399 2382
2961	2716	2391

ACETYLENE and Oxygen Diluted with Nitrogen.

$C_2H_2+O_3+N_2$	$C_2H_2+O_3+N_6$	$C_2H_2+O_3+N_8$	$C_2H_2+O_3+N_{10}$	$C_2H_2+O_3+N_{12}$
2410 2418 2415	2197 2221	2110 2122	2028 2009	1903 1913
2414	2209	2116	2019	1908

ACETYLENE and Oxygen under Varying Pressures.

At 500 mm.
2685 2693
2689

CYANOGEN and Oxygen in Varying Proportions.

At 760 mm. and 10°.		
$C_2N_2+O_2$	$C_2N_2+2O_2$	$C_2N_2+3O_2$
2714 2742 2727	2319 2318 2326	2108 2112
2728	2321	2110

CYANOGEN and Oxygen Diluted with Nitrogen.

$C_2N_2+O_2+N_2$	$C_2N_2+O_2+2N_2$
2394 2401	2156 2175
2398	2166

CYANOGEN and Oxygen under Varying Pressures.

In 5 mm. tube. 10° C.		
$C_2N_2+O_2$		
At 500 mm.	At 760 mm.	At 1000 mm.
2533 2539	2663 2686	2667 2675
2536	2677	2671

At 760 mm. and 100° C.
In 9 mm. tube.
2697 2725
2711

CYANOGEN and Nitrous Oxide.

In 9 mm. tube.		
$C_2N_2+2N_2O$	$C_2N_2+2N_2O+N_2$	$C_2N_2+2N_2O+2N_2$
2451 2458	2273 2292	2101 2095
2454	2283	2098

CYANOGEN and Nitric Oxide.

In 9 mm. tube. (Fired with fulminate detonator.)
C_2N_2+2NO
2766 2740 2773
2760

HYDROGEN and Nitrous Oxide in Varying Proportions.

At 760 mm. and 10° C.			
In 13 mm. tube.	In 9 mm. tube.		
H_7+N_2O	H_6+N_2O	H_4+N_2O	H_2+N_2O
2728 2737	2700 2711	2552 2538	2307 2302
2732	2705	2545	2305

HYDROGEN and Nitrous Oxide Diluted with Nitrogen.

In 9 mm. tube.	In 13 mm. tube.	
$H_2+N_2O+N_2$	$H_2+N_2O+N_3$	$H_2+N_2O+N_4$
2001 1980	1886 1875	1650 1416 went out ditto
1991	1880	irregular

HYDROGEN with Nitrous Oxide under Varying Pressures.

H_2+N_2O	
At 500 mm.	At 1000 mm.
2089 2098	2306 2298
2094	2302

AMMONIA and Oxygen.

At 760 mm. and 10° C.
$4NH_3 + 3O_2$.
2381 2399
2390

AMMONIA and Nitrous Oxide.

$2NH_3 + 3N_2O$.
2199 2202
2200

CARBONIC Oxide and Oxygen with Varying Proportions of Steam.

CO + O at 760 mm.				
Well dried.	Dried.	Saturated at 10°.	Saturated at 20°.	Saturated at 28°.
1267	1289	1670	1710	1711
1262	1297	1673	1704	1726
	1320	1679	1685	1702
	1316	1674	1718	
		1687	1700	
		1675	1701	
1264	1305	1676	1703	1713

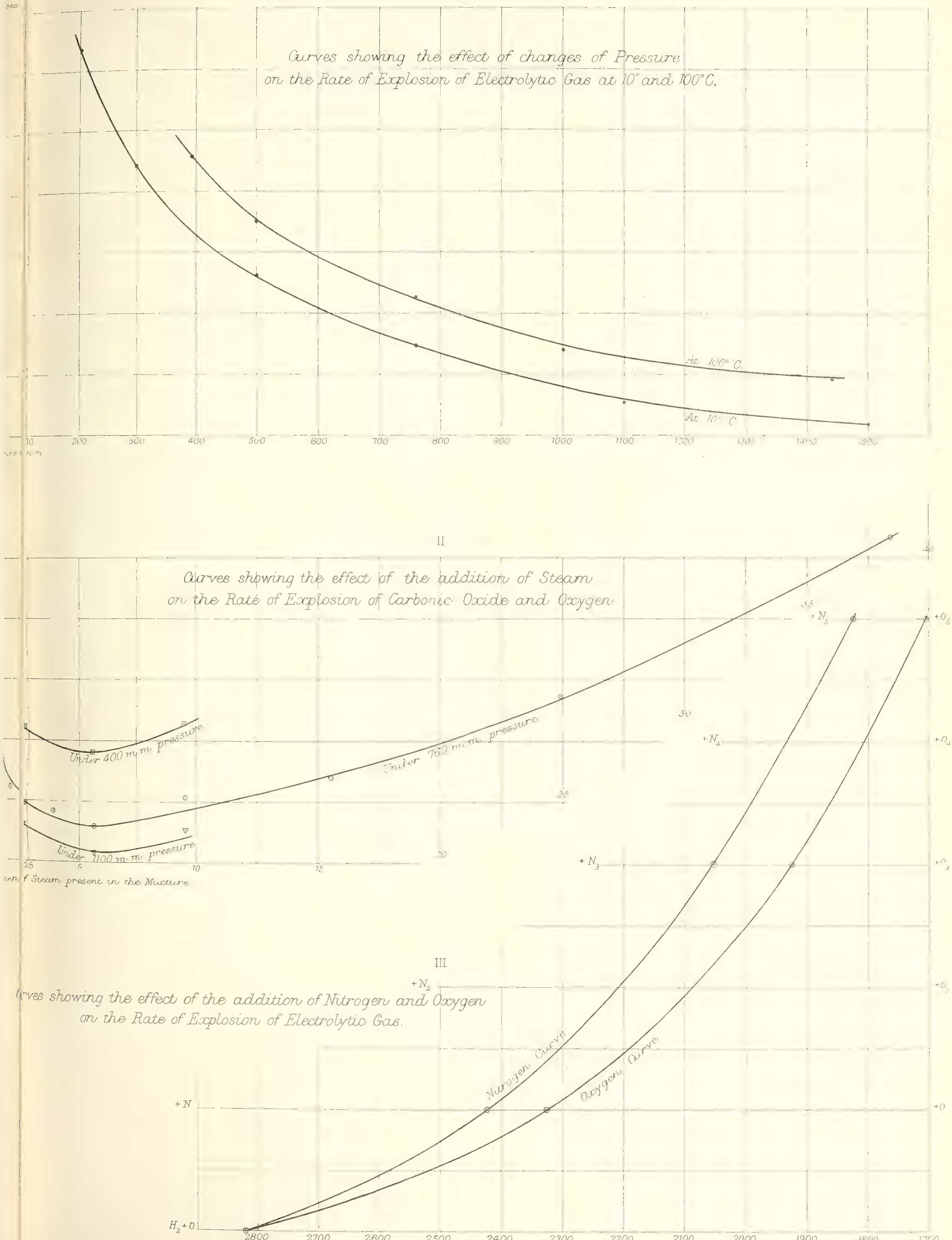
CO + O at 760 mm.				
Saturated at 35°.	Saturated at 45°.	Saturated at 55°.	Saturated at 65°.	Saturated at 75°.
1741	1700	1673	1519	1260
1731	1681	1650	1534	1272
1726	1697	1674		
1750				
1742				
1738	1693	1666	1526	1266

At 400 mm.		
Saturated at 10°.	Saturated at 25°.	Saturated at 33°.
1573	1607	1580
1582	1617	1568
1574	1625	1562
1576	1616	1570

At 1100 mm.		
Saturated at 26°.	Saturated at 43°.	Saturated at 53°.
1747	1767	1730
1737	1799	1751
1728	1779	1745
1737	1782	1742

HYDROGEN and Oxygen Diluted with Carbonic Oxide [K = 1 vol. CO.].

$H_2 + O + K$.	$H_2 + O + K_3$.	$H_2 + O_2 + N_2$.	$H_2 + O_2 + K_2$.
2467	2086	1993	2144
2446	2070	2014	2142
2458	2084		
2448			
2455	2080	2003	2143



I

Curves showing the effect of changes of Pressure on the Rate of Explosion of Electrolytic Gas at 10° and 100° C.

II

Curves showing the effect of the addition of Steam on the Rate of Explosion of Carbonic Oxide and Oxygen

III

Curves showing the effect of the addition of Nitrogen and Oxygen on the Rate of Explosion of Electrolytic Gas.

West, Newman, lith.

NATURAL HISTORY

IV. *On the Residues of Powers of Numbers for any Composite Modulus, Real or Complex.*

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Communicated by Professor Cayley, *F.R.S.*

Received April 8,—Read May 5, 1892.

The present work consists of two parts, with an appendix to the second. Part I. deals with real numbers, Part II. with complex.

In the simple cases when the modulus is a real number, which is an odd prime, a power of an odd prime, or double the power of an odd prime, we know that there exist primitive roots of the modulus ; that is, that there are numbers whose successive powers have for their residues the complete set of numbers less than and prime to the modulus. A primitive root may be said to generate by its successive powers the complete set of residues. It is also known that, in general, when the modulus is any composite number, though primitive roots do not exist, there may be laid down a set of numbers which will here be called *generators,* the products of powers of which give the complete set of residues prime to the modulus.

The principal object of Part I. is to investigate the relations which must subsist among any such set of generators ; to determine the most general form that they can take ; to show how to form any such set of generators, and conversely to furnish tests for the efficiency, as generators, of any given set of numbers. Other results which are obtained as instrumental in effecting these objects, such as the determination of the number of numbers that belong to any exponent, may also possess independent interest.

The object of Part II. is to make, for complex numbers, an investigation which shall be as nearly as possible parallel to that of Part I. for real numbers. Much of the work of Part I. may be applied immediately to complex numbers ; of the rest some will need slight modification, and some will need replacing by propositions leading to corresponding results. Of those cases which thus call for independent treatment, the most noticeable is that of the modulus $(1 + i)^\lambda$, which is the complex analogue of the real modulus 2^λ.

The work is put in the form of a series of propositions, and is started almost from first principles. The early part is consequently elementary, but the advantages of completeness and ease of reference may be more than sufficient to compensate for this. A large number of illustrative examples are given. These will sometimes, perhaps, assist in elucidating the symbolical proofs which they follow; in all cases they will help to maintain clearly the actual arithmetical meaning of the results arrived at, a meaning which may easily seem obscure if it be noticed only in its symbolical and generalised form.

The Appendix contains tables of indices for complex numbers for all moduli whose norms do not exceed 100.

PART I.—ON THE RESIDUES OF POWERS OF NUMBERS FOR ANY COMPOSITE REAL MODULUS.

In what follows (except when the contrary is explicitly stated) we shall be treating of the residues of powers of numbers which are *prime* to the modulus with regard to which those residues are taken; and the modulus will be taken to be any composite number whatever. In this first part, moreover, all the numbers dealt with are real.

(1.) The residues (modulus m) of the successive powers of a number a prime to m form a recurring series of periods of terms, the first period beginning with the first term.

Consider the series of numbers

$$a,\ a^2,\ a^3 \dots$$

Since a is prime to m, therefore any power of a is prime to m, and therefore the residue of a^s for modulus m is prime to m.

Hence each term of the series of residues,

$$a,\ a^2,\ a^3 \dots (\text{mod } m),$$

is one of the numbers less than m and prime to it.

There are $\phi(m)$ numbers less than m and prime to it.

Hence in the above infinite series there are only $\phi(m)$ *different* terms.

Suppose that the first term which occurs for the second time is a^{t+s} (mod m), and suppose that this is congruent to a^s.

Then

$$a^{t+s} \equiv a^s \ (\text{mod } m),$$

where s and t are both to have as small values as possible.

$$a^s(a^t - 1) \equiv 0 \ (\text{mod } m),$$

and since a^s is prime to m,

$$a^t - 1 \equiv 0 \ (\text{mod } m),$$

and so

$$a^{t+\sigma} \equiv a^\sigma \ (\text{mod } m),$$

for *every* value of σ

Hence the term which first appears a second time is the first term a; which appears next as a^{t+1}, t being the least number for which

$$a^t \equiv 1 \pmod{m}.$$

Definition.—The smallest number t which makes $a^t \equiv 1 \pmod{m}$, where a is prime to m, is called the *exponent* of a for modulus m. CAUCHY uses the word "*indicator*" in the same sense. He also uses "*maximum indicator*" to denote what will be called the *highest exponent.*

Thus the infinite series of residues of a, a^2, a^3, . . . consists of a repetition of the period of t terms beginning from the first term.

(2.) If t be the exponent of a and $a^s \equiv 1 \pmod{m}$ then t divides s.

Let

$$s = qt + r \text{ where } r < t.$$

Then

$$a^s = a^{qt+r} \equiv 1 \pmod{m},$$
$$(a^t)^q \,.\, a^r \equiv 1 \pmod{m},$$
$$a^r \equiv 1 \pmod{m},$$

whereas t is the smallest value (not zero) which makes $a^t \equiv 1$. Therefore

$$r = 0,$$

therefore

$$t \text{ divides } s.$$

(3.) FERMAT'S *Theorem.*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Let $a_1, a_2, a_3, \ldots a_{\phi(m)}$ be the $\phi(m)$ numbers less than m and prime to it.

Take any one of them a.

Then since $aa_1, aa_2, aa_3, \ldots aa_{\phi(m)} \pmod{m}$ are all prime to m, and no two congruent, they must be the same set of numbers as $a_1, a_2, \ldots a_{\phi(m)}$; therefore

$$a_1 a_2 a_3 \ldots a_{\phi(m)} a^{\phi(m)} \equiv a_1 a_2 \ldots a_{\phi(m)} \pmod{m},$$

and, therefore,

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Corollary.—It follows from this proposition and proposition (1) that the exponent of any number (modulus m) is always a divisor of $\phi(m)$. The propositions which follow will determine that divisor.

(4.) If t be the exponent of a then the exponent of a^s is τ: where $t = \kappa\tau$ and κ is the G.C.M. of s and t.

Let T be the exponent of a^s.

We have

$$a^t \equiv 1 \pmod{m},$$

therefore

$$a^{\kappa\tau} \equiv 1 \pmod{m},$$

$$a^{\kappa\sigma\tau} \equiv 1 \pmod{m},$$

$$(a^s)^\tau \equiv 1 \pmod{m},$$

therefore

$$\text{T divides } \tau. \quad \text{(Prop. 2.)}$$

Again,

$$(a^s)^{\text{T}} \equiv 1 \pmod{m},$$

therefore

$$a^{s\text{T}} \equiv 1 \pmod{m},$$

therefore

$$t \text{ divides } s\text{T},$$

therefore

$$\kappa\tau \text{ divides } \kappa\sigma\text{T},$$

therefore

$$\tau \text{ divides } \sigma\text{T},$$

therefore

$$\tau \text{ divides T},$$

therefore

$$\text{T} = \tau.$$

Example.—The exponent of 3 for modulus 308 is 30: the residues of its successive powers are given in the following table:—

Number . . .	3	9	27	81	243	113	31	93	279	221
Power of 3 . .	1	2	3	4	5	6	7	8	9	10
Exponent . .	30	15	10	15	6	5	30	15	10	3

Number . . .	47	141	115	37	111	25	75	225	59	177
Power of 3 . .	11	12	13	14	15	16	17	18	19	20
Exponent . .	30	5	30	15	2	15	30	5	30	3

Number . . .	223	53	159	169	199	289	251	137	103	1
Power of 3 . .	21	22	23	24	25	26	27	28	29	30
Exponent .	10	15	30	5	6	15	10	15	30	1

where the exponents are all immediately deducible from the proposition.

(5.) The exponent of a is t and of a' is t' and t and t' are co-prime; then the exponent of aa' is tt'.

Let T be the exponent of aa'.

Then,

$$a^t \equiv 1 \pmod{m} \qquad a'^{t'} \equiv 1 \pmod{m},$$

therefore

$$a^{tt'} \equiv 1 \pmod{m} \qquad a'^{tt'} \equiv 1 \pmod{m},$$

therefore

$$(aa')^{tt'} \equiv 1 \pmod{m},$$

therefore

$$\text{T divides } tt'. \quad \text{(Prop. 2.)}$$

Again,

$$(aa')^{\mathrm{T}} \equiv 1 \pmod{m},$$
$$(aa')^{\mathrm{T}t} \equiv 1 \pmod{m},$$
$$a'^{\mathrm{T}t} \equiv 1 \pmod{m},$$

therefore

$$t' \text{ divides T}t \text{ (Prop. 2)},$$

therefore

$$t' \text{ divides T}.$$

Similarly,

$$t \text{ divides T},$$

and therefore (since t and t' are co-prime),

$$tt' \text{ divides T},$$

therefore

$$\mathrm{T} = tt'.$$

Corollary.—It follows that if the numbers $a, a', a'', \ldots$ have exponents (for modulus m) $t, t' t'', \ldots$ these exponents being all co-prime, then the exponent of $aa'a'' \ldots$ is $tt't'' \ldots$

Example.—The exponent of

$$23 \bmod 308 \text{ is } 6; \ (23.\ 221.\ 155.\ 177.\ 67.\ 1),$$

and the exponent of

$$113 \bmod 308 \text{ is } 5; \ (113.\ 141.\ 225.\ 169.\ 1).$$

Since 5 and 6 are co-prime, it follows that the exponent of

$$135 \equiv 23.\ 113 \text{ is } 30.$$

Corollary. Example.—The exponent of

$$\left.\begin{array}{l}111 \text{ is } 2\\ 221 \text{ is } 3\\ 113 \text{ is } 5\end{array}\right\}\text{therefore the exponent of } 111.\ 221.\ 113 \equiv 3 \text{ is } 30.$$

(6.) Let a have exponent t and a' exponent t' and suppose that t and t' are not co-prime. Then, if t and t' contain no prime factor raised to the same power in both, the exponent of aa' is the L.C.M. of t and t'.

Let

$$\left.\begin{array}{r}t = \kappa\tau\\ t' = \kappa\tau'\end{array}\right\}\text{where } \tau \text{ and } \tau' \text{ are co-prime,}$$

then

$$a^{\kappa} \text{ has exponent } \tau \text{ (Prop. 4),}$$

$$a'^{\kappa} \text{ has exponent } \tau' \text{ (Prop. 4),}$$

therefore,

$$(aa')^{\kappa} \text{ has exponent } \tau\tau' \text{ (Prop. 5).}$$

So if aa' has exponent T, then

$$\frac{\mathrm{T}}{\kappa} = \frac{\tau\tau'}{z}$$

where $\tau\tau'$ divides T, z divides κ and z is prime to $\tau\tau'$. (Prop. 4.)

Now since t and t' contain no prime raised to the same power in each, therefore $\tau\tau'$ contains every prime factor which occurs in t and t' and therefore contains every prime factor which occurs in κ. Hence z cannot divide κ and be prime to $\tau\tau'$ unless it be unity.

Therefore,

$$\mathrm{T} = \kappa\tau\tau' = \text{L.C.M. of } t \text{ and } t'.$$

(7.) If a has exponent t, a' has exponent t', a'' — t'', &c., for modulus m, and if, of the $tt't''\ldots$ numbers $a^r a'^{r'} a''^{r''}\ldots$ (mod m) formed by giving to r all values modulus t, to r' all values modulus t' . . ., no two are congruent; and if

$$a^s a'^{s'} a''^{s''} \ldots \equiv 1 \pmod{m};$$

then we must have

$$a^s \equiv 1, \quad a'^{s'} \equiv 1, \quad a''^{s''} \equiv 1, \ldots \pmod{m}.$$

For suppose that at least one of these congruences is violated.

Say

$$a^s \not\equiv 1 \pmod{m},$$

and, therefore,

$$s \not\equiv 0 \pmod{t}.$$

Then, because

$$a^s a'^{s'} a''^{s''} \dots \equiv 1 \pmod{m},$$

therefore (multiplying by a^{t-s}),

$$a^t a'^{s'} a''^{s''} \dots \equiv a^{t-s} \pmod{m},$$

or

$$a^0 a'^{s'} a''^{s''} \dots \equiv a^{t-s} \pmod{m},$$

or

$$a^0 a'^{s'} a''^{s''} \dots \equiv a^{t-s} a'^0 a''^0 \dots \pmod{m},$$

which is contrary to the supposition that no two numbers of the form $a^r a'^{r'} \dots$ (mod m) are congruent; for the last congruence obtained shows that if we make

$$\left.\begin{array}{lll} r = 0 & r' = s' & r'' = s'' \dots \\ r = t - s & r' = 0 & r'' = 0 \dots \end{array}\right\}$$

where

$$t - s \not\equiv 0 \pmod{t},$$

the two numbers are congruent.

Hence, we must have

$$s \equiv 0 \pmod{t},$$

and, therefore,

$$a^s \equiv 1 \pmod{t}.$$

Similarly,

$$a'^{s'} \equiv 1 \pmod{t'}, \text{ \&c.}$$

Definition.—If a, a', a'' . . . have exponents t, t', t'', . . . and if no two of the $tt't''$ . . . numbers that can be formed by products of their powers are congruent modulus m (as in the last proposition), then these numbers, a, a', a'', . . . which generate the $tt't''$ . . . incongruent numbers, will be called *independent* generators.

The last proposition may then be stated thus :—If a product of powers of a set of independent generators be congruent to unity, then each of those powers is itself congruent to unity.

(8.) If a, a', a'', . . . independent generators, have exponents t, t', t'', . . . then the exponent of $aa'a''$ . . . is the L.C.M. of t, t', t'' . . .

Let T be the L.C.M. of t, t', t'', . . . and τ the required exponent of $aa'a''$ . . .

$$a^t \equiv 1 \pmod{m},$$

therefore,

$$a^{\mathrm{T}} \equiv 1 \pmod{m}.$$

Similarly,

$$a'^{\mathrm{T}} \equiv 1 \pmod{m}, \text{ \&c.},$$

therefore,

$$(aa'a'' \ldots)^{\mathrm{T}} \equiv 1 \pmod{m},$$

therefore,

$$\tau \text{ divides T.} \quad \text{(Prop. 2.)}$$

Again,

$$(aa'a'' \ldots)^{\tau} \equiv 1 \pmod{m},$$

therefore,

$$a^{\tau}a'^{\tau}a''^{\tau} \ldots \equiv 1 \pmod{m},$$

therefore,

$$\left.\begin{array}{l}\tau \equiv 0 \pmod{t} \\ \tau \equiv 0 \pmod{t'} \\ \tau \equiv 0 \pmod{t''}\end{array}\right\} \text{(Prop. 7)},$$

therefore,

$$\text{T divides } \tau,$$

therefore,

$$\mathrm{T} = \tau.$$

Corollary.—If the exponents t, t', t'', be all of them powers of the same prime, and a, a', a'', . . . are independent generators, then the exponent of the product $aa'a'' \ldots$ is equal to the greatest of the exponents t, t', $t'' \ldots$

(9.) If the exponent of a for modulus m is t, and for modulus n is t', and if m and n are co-prime, then the exponent of a for modulus mn is the L.C.M. of t and t'.

Let $t'' = t\tau = t'\tau'$ where τ and τ' are co-prime, so that t'' is the L.C.M. of t and t'.

Let the exponent of a for modulus mn be T.

Thus we have

$$a^{t} \equiv 1 \pmod{m},$$

and, therefore, raising to the power τ,

$$a^{t''} \equiv 1 \pmod{m}.$$

Similarly,

$$a^{t''} \equiv 1 \pmod{n},$$

and, therefore (since m and n are co-prime),

$$a^{t''} \equiv 1 \pmod{mn},$$

therefore

$$\text{T divides } t''. \quad \text{(Prop. 2.)}$$

Again,

$$a^{\mathrm{T}} \equiv 1 \pmod{mn},$$

therefore

$$a^{\mathrm{T}} \equiv 1 \pmod{m},$$

therefore

$$t \text{ divides T.}$$

Similarly,

$$t' \text{ divides } \mathrm{T},$$

therefore

$$t'' \text{ divides } \mathrm{T}.$$

Therefore

$$\mathrm{T} = t''.$$

Corollary.—If the exponents of a for moduli m, m', m'', . . . are respectively t, t', t'', . . . and the moduli are co-prime, then the exponent of a for modulus $mm'm''$ . . . is the L.C.M. of t, t', t'' . . .

Examples.—The exponent of

$$\left.\begin{array}{l} 3 \text{ for mod } \ 4 \text{ is } 2 \\ 3 \text{ for mod } \ 7 \text{ is } 6 \\ 3 \text{ for mod } 11 \text{ is } 5 \end{array}\right\} \qquad \begin{array}{l} (3.\ 1.) \\ (3.\ 2.\ 6.\ 4.\ 5.\ 1.) \\ (3.\ 9.\ 5.\ 4.\ 1.) \end{array}$$

Therefore the exponent of

$$3 \text{ for mod } 4.\ 7.\ 11 = 308 \text{ is } 30.$$

The exponent of

$$\begin{array}{ll} 5 \text{ for mod } \ 4 \text{ is } 1. & \\ 5 \text{ for mod } \ 7 \text{ is } 6 & (5.\ 4.\ 6.\ 2.\ 3.\ 1.) \\ 5 \text{ for mod } 11 \text{ is } 5 & (5.\ 3.\ 4.\ 9.\ 1.) \end{array}$$

and, therefore, the exponent of

$$5 \text{ for mod } 308 \text{ is } 30.$$

The exponent of

$$\begin{array}{ll} 9 \text{ for mod } \ 4 \text{ is } 1. & \\ 9 \text{ for mod } \ 7 \text{ is } 3 & (2.\ 4.\ 1.) \\ 9 \text{ for mod } 11 \text{ is } 5 & (9.\ 4.\ 3.\ 5.\ 1.) \end{array}$$

and, therefore, the exponent of

$$9. \text{ mod } 308 \text{ is } 15.$$

(10.) If the exponent of a is t, and $t = pqr$ . . . where p, q, r are co-prime factors of t, to express a as a product of numbers whose exponents are p, q, r . . .

Let

$$\left.\begin{array}{l} \mathrm{P} \equiv 0 \ (\text{mod } qr \ldots) \\ \quad \equiv 1 \ (\text{mod } p) \end{array}\right\} \quad \left.\begin{array}{l} \mathrm{Q} \equiv 0 \ (\text{mod } pr \ldots) \\ \quad \equiv 1 \ (\text{mod } q) \end{array}\right\} \ \text{\&c.}$$

These congruences determine one value each and one only (mod t) for P, Q, . . .

$$\begin{array}{l} a^{\mathrm{P}} \text{ has exp } p \text{ (Prop. 4)}, \\ a^{\mathrm{Q}} \quad ,, \quad ,, \quad q \\ \quad \&\text{c.} \end{array}$$

From the above congruences we obtain

$$\begin{array}{l} \mathrm{P} \equiv 1 \pmod{p}, \\ \mathrm{Q} \equiv 0 \pmod{p}, \\ \mathrm{R} \equiv 0 \pmod{p}, \\ \quad \&\text{c.} \end{array}$$

therefore, by addition,

$$\mathrm{P} + \mathrm{Q} + \mathrm{R} + \ldots \equiv 1 \pmod{p}.$$

Similarly,

$$\mathrm{P} + \mathrm{Q} + \mathrm{R} + \ldots \equiv 1 \pmod{q},$$

$$\&\text{c.},$$

and, therefore, since p, q, r, . . . are co-prime,

$$\mathrm{P} + \mathrm{Q} + \mathrm{R} + \ldots \equiv 1 \pmod{t},$$

therefore

$$a^{\mathrm{P}}.\, a^{\mathrm{Q}}.\, a^{\mathrm{R}} \ldots \equiv a \pmod{m}.$$

And so a is expressible (in one way only) as the product of numbers a^{P}, a^{Q} . . . whose exponents are p, q . . .

Example.—3 has exponent 30 mod 308.

To express it as a product of 3 numbers with exponents 2. 3. 5,

$$\begin{array}{lll} \mathrm{P} \equiv 0 \pmod{15} & \mathrm{Q} \equiv 0 \pmod{10} & \mathrm{R} \equiv 0 \pmod{6} \\ \phantom{\mathrm{P}} \equiv 1 \pmod{2} & \phantom{\mathrm{Q}} \equiv 1 \pmod{3} & \phantom{\mathrm{R}} \equiv 1 \pmod{5}, \end{array}$$

therefore

$$\mathrm{P} = 15 \qquad \mathrm{Q} = 10 \qquad \mathrm{R} = 6,$$

and, therefore,

$$3 \equiv 3^{15}.\, 3^{10}.\, 3^{6} \equiv 111.\ 221.\ 113 \pmod{308}$$

where

$$\left.\begin{array}{l} 111 \text{ has exp } 2 \\ 221 \quad ,, \quad ,, \quad 3 \\ 113 \quad ,, \quad ,, \quad 5 \end{array}\right\}$$

Similarly 79 has exponent 30 (mod 308), and

$$79 \equiv 79^{15}.\, 79^{10}\ 79^{6} \equiv 43.\ 177.\ 141 \pmod{308}$$

where

$$\left.\begin{array}{rl} 43 & \text{has exp } 2 \\ 177 & \text{,, \quad ,, } 3 \\ 141 & \text{,, \quad ,, } 5 \end{array}\right\}$$

(11.) The number of numbers that belong to any exponent t when the modulus is a prime p is $\phi(t)$.

If there are any numbers which have exponent t, mod p, let α be one.

Then because

$$\alpha^t \equiv 1 \pmod{p},$$

therefore each of the t incongruent numbers

$$\alpha, \alpha^2, \alpha^3, \ldots \alpha^t$$

when raised the t^{th} power is $\equiv 1$.

Hence each is a root of $x^t \equiv 1 \pmod{p}$ which has only t incongruent roots.

Therefore every number β which has exponent t, and which is therefore such that $\beta^t \equiv 1$ is included in the above set. Hence every number with exponent t is to be found in the above set. Now, of the powers of α there are $\phi(t)$ which have their index prime to t, and which, therefore, have exponent t (Prop. 4).

Therefore if there is *one* number with exponent t there are $\phi(t)$, and no more.

Now if $t_1, t_2, \ldots$, are all the divisors of $p-1$, and, therefore, all possible exponents (Prop. 3, corollary),

$$\phi(t_1) + \phi(t_2) + \ldots = \phi(p).$$

Corresponding to each value t there are either $\phi(t)$ numbers or none with t as exponent. The number of numbers altogether is $\phi(p)$. Hence in no case can there fail to be $\phi(t)$ numbers with exponent t.

Corollary.—In particular there are $\phi(p-1)$ numbers with exponent $p-1$, modulus p, *i.e.*, any odd prime p has $\phi(p-1)$ primitive roots.

(12.) The exponents to which a number a belongs for successive powers of a prime p as moduli.

We suppose that a is prime to p and that $p \neq 2$.

Let the exponents to which a belongs for the moduli $p, p^2, p^3, \ldots p^\lambda \ldots$ be $t_1, t_2, t_3, \ldots$ respectively.

Then because

$$a^{t_{\lambda+1}} \equiv 1 \pmod{p^{\lambda+1}},$$

therefore

$$a^{t_{\lambda+1}} \equiv 1 \pmod{p^\lambda},$$

therefore

$$t_\lambda \text{ divides } t_{\lambda+1}. \quad \text{(Prop. 2.)}$$

Again

$$a^{t_\lambda} \equiv 1 \pmod{p^\lambda},$$

therefore (raising to the p^{th} power $a^{t_\lambda} = 1 + \kappa p^\lambda$)

$$a^{pt_\lambda} \equiv 1 \pmod{p^{\lambda+1}},$$

therefore

$$t_{\lambda+1} \text{ divides } pt_\lambda. \quad \text{(Prop. 2.)}$$

It follows from these two results that either $t_{\lambda+1} = t_\lambda$ or $t_{\lambda+1} = pt_\lambda$.

Each exponent in the series $t_1, t_2, \ldots t_\lambda, \ldots$ is either equal to that immediately preceding it or is p times that value.

We can show, however, that after the first set of equal exponents $t_1 = t_2 = \ldots$ comes to an end, that each exponent is p times that which immediately precedes it.

For suppose that, if possible, after

$$t_{\lambda+1} = pt_\lambda$$

we can have

$$t_{\lambda+2} = t_{\lambda+1}.$$

We thus have

$$t_{\lambda+2} = t_{\lambda+1} = pt_\lambda.$$

Now

$$a^{t_\lambda} \equiv 1 \pmod{p^\lambda} = 1 + xp^\lambda + yp^{\lambda+1},$$

say, where $x < p$.

Also,

$$a^{pt_\lambda} = a^{t_{\lambda+2}} \equiv 1 \pmod{p^{\lambda+2}},$$

therefore

$$(1 + xp^\lambda + yp^{\lambda+1})^p \equiv 1 \pmod{p^{\lambda+2}}$$

$$1 + xp^{\lambda+1} \equiv 1 \pmod{p^{\lambda+2}}$$

therefore

$$x = 0,$$

and, therefore,

$$a^{t_\lambda} = 1 + yp^{\lambda+1}$$

$$\equiv 1 \pmod{p^{\lambda+1}}.$$

Therefore

$$t_{\lambda+1} \text{ divides } t_\lambda,$$

which is not so. Therefore we cannot have

$$t_{\lambda+2} = t_{\lambda+1}$$

if

$$t_{\lambda+1} = pt_\lambda,$$

and therefore we must have

$$t_{\lambda+2} = pt_{\lambda+1}.$$

If, then, the exponent of a for modulus p be $t_1 = t$ and $a^t - 1$ contain p^s as the highest power of p, we have

$$a^t \equiv 1 \pmod{p^s},$$

and

$$t_1 = t_2 = \ldots = t_s = t.$$

And after these

$$t_{s+1} = pt_s,$$
$$t_{s+2} = pt_{s+1},$$
$$\&c.$$

Hence the exponent of a for modulus p^λ is

$$t \text{ if } \lambda \leqq s,$$
$$tp^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for modulus p, and p^s is the greatest power of p that will divide $a^t - 1$.

Corollary.—The greatest value that t can have is $p - 1$. This is so when a is congruent to a primitive root of p (Proposition 11, corollary). The greatest value that $p^{\lambda - s}$ can have is got by making s as small as possible, viz., by making $s = 1$, *i.e.*, by taking a so that $a^{p-1} - 1$ (though necessarily divisible by p) is not divisible by p^2.

Therefore the greatest possible exponent that a number can have for modulus p^λ is $(p - 1) \, . \, p^{\lambda - 1}$, and as this is equal to $\phi\,(p^\lambda)$ it follows that primitive roots exist for a modulus a power of a prime.

Examples.—Exponent of 3 for mod 5^6.

The exp of 3 for mod 5 is 4 (3. 4. 2. 1).

$3^4 - 1 = 80$ is divisible by 5^1.

Therefore

exp of 3 is $4.\ 5^5 \pmod{5^6}$.

Exponent of 24 for mod 5^6.

The exp of 24 for mod 5 is 2 ($t = 2$).

$24^2 - 1 = 575$ which is divisible by 5^2. ($s = 2$).

Therefore

exp of 24 mod $5^6 = 2.\ 5^4$.

(13.) The exponent to which a number a belongs for a power of 2 as modulus.

The number a is now to be considered odd.

Let $t_1, t_2, t_3, \ldots t_\lambda \ldots$ be the exponents of a for moduli $2, 2^2, \ldots 2^\lambda \ldots$

We have

$$a^{t_{\lambda+1}} \equiv 1 \pmod{2^{\lambda+1}}.$$

Therefore

$$a^{t_{\lambda+1}} \equiv 1 \pmod{2^{\lambda}};$$

therefore

$$t_\lambda \text{ divides } t_{\lambda+1}. \quad \text{(Prop. 2.)}$$

Again

$$a^{t_\lambda} \equiv 1 \pmod{2^\lambda} = 1 + x.2^\lambda + y.2^{\lambda+1} \text{ (where } x = 1 \text{ or } 0).$$

Therefore

$$a^{2t_\lambda} \equiv 1 \pmod{2^{\lambda+1}},$$

and therefore

$$t_{\lambda+1} \text{ divides } 2t_\lambda.$$

Therefore either

$$\left.\begin{array}{l} t_{\lambda+1} = t_\lambda \\ \text{or} \quad t_{\lambda+1} = 2t_\lambda \end{array}\right\}.$$

Suppose now that after

$$t_{\lambda+1} = 2t_\lambda$$

we can have

$$t_{\lambda+2} = t_{\lambda+1}.$$

Then

$$a^{t_\lambda} \equiv 1 \pmod{2^\lambda} = 1 + x.2^\lambda + y.2^{\lambda+1}.$$

Therefore

$$a^{2t_\lambda} \equiv 1 + x.2^{\lambda+1} + x^2 2^{2\lambda} \pmod{2^{\lambda+2}}.$$

Now if $\lambda \geqq 2$, then

$$2\lambda \geqq \lambda + 2,$$

and therefore

$$a^{2t_\lambda} \equiv 1 + x.2^{\lambda+1} \pmod{2^{\lambda+2}}.$$

Now

$$a^{2t_\lambda} = a^{t_{\lambda+2}} \equiv 1 \pmod{2^{\lambda+2}},$$

therefore

$$x = 0.$$

Therefore

$$a^{t_\lambda} = 1 + x.2^\lambda + y2^{\lambda+1} \equiv 1 \pmod{2^{\lambda+1}},$$

therefore

$$t_{\lambda+1} \text{ divides } t_\lambda,$$

which is not so.

Hence we have the result, when $\lambda \geqq 2$ and $t_{\lambda+1} = 2t_\lambda$, then $t_{\lambda+2} = 2t_{\lambda+1}$.

The first exponent t_1, of the series, is equal to unity.

If this be followed by a set of 1's (at least one), then by what has been proved they will be followed by the series 2, 2^2, 2^3. . . .

If the second exponent be a 2, the third may be also, and so on; the series then continues with $2^2, 2^3, 2^4, \ldots$ Of the 2's there are at least two; for otherwise the first three exponents would be $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, making 4 an exponent for mod $2^3 = 8$, which is impossible.

Hence the series of exponents run either thus—

$$1.\ 1.\ 1.\ \ldots\ 1.\ 2.\ 2^2.\ 2^3.\ 2^4\ \ldots$$

or thus

$$1.\ 2.\ 2.\ \ldots\ 2.\ 2.\ 2^2.\ 2^3.\ 2^4\ \ldots$$

These results can be expressed thus—

Let the highest power of 2 which divides $a^2 - 1$ be 2^{s+1}. (Since a is an odd number, s is at least equal to 2.)

Then the exponent of a is

$$\text{if } \lambda > s,\ 2^{\lambda - s}$$

$$\text{if } \lambda \overset{=}{<} s, \begin{cases} 2 \text{ if } a^2 \equiv 1 \pmod{2^\lambda} \text{ and } a \not\equiv 1 \pmod{2^\lambda}. \\ 1 \text{ if } a \equiv 1 \pmod{2^\lambda}. \end{cases}$$

Corollary.—The greatest exponent possible for mod 2^λ is $2^{\lambda-2}$; and as $\phi(2^\lambda) = 2^{\lambda-1}$, primitive roots do not exist.

Examples.—Exponent of 3 for mod 2^6.

Exp of 3 for mod 2 is 1,
,, ,, 2^2 is 2,
,, ,, 2^3 is 2,
,, ,, 2^4 is 4,

and therefore

exp of 3 for mod 2^6 is 16

Exponent of 35 for mod $64 = 2^6$ ($\lambda = 6$).

$$35^2 - 1 = 1224 = 153.2^3,$$

therefore

$$s = 2,$$

therefore

exp of 35 is 2^4.

Exp of 41 for mod $128 = 2^7$ ($\lambda = 7$).

$$41^2 - 1 = 1680 = 105.2^4,$$

therefore

$$s = 3,$$

therefore

$$\text{exp of } 41 = 2^{7-3} = 2^4.$$

(14.) The numbers which belong to exponent 2 for modulus 2^κ ($\kappa \geqq 3$).

Let a have exponent 2, modulus 2^κ.

Then

$$a^2 \equiv 1 \pmod{2^\kappa}$$

and

$$a \not\equiv 1 \pmod{2^\kappa}.$$

The congruence

$$a^2 \equiv 1 \pmod{2^\kappa}$$

gives

$$(a - 1)(a + 1) \equiv 0 \pmod{2^\kappa}.$$

Since a is odd, if 2^s is the highest power of 2 that divides $a - 1$ and $s > 1$, then the highest power of 2 that divides $a + 1$ is 2, and *vice versâ.*

Hence either

$$a + 1 \equiv 0 \pmod{2^{\kappa-1}},$$

or

$$a - 1 \equiv 0 \pmod{2^{\kappa-1}}.$$

Therefore (excluding $a \equiv 1$ (modulus 2^κ)) we have three numbers whose exponents are 2 for modulus 2^κ, viz.,

$$2^{\kappa-1} + 1, \quad 2^\kappa - 1, \quad 2^{\kappa-1} - 1. \quad \pmod{2^\kappa}.$$

The product of each pair of these is congruent to the third.

$$\left.\begin{array}{l} (2^{\kappa-1} + 1)(2^\kappa - 1) \equiv 2^{\kappa-1} - 1 \\ (2^\kappa - 1)(2^{\kappa-1} - 1) \equiv - 2^{\kappa-1} + 1 \equiv 2^{\kappa-1} + 1 \\ (2^{\kappa-1} - 1)(2^{\kappa-1} + 1) \equiv 2^\kappa - 1 \end{array}\right\} \pmod{2^\kappa}.$$

(15.) The numbers which have exponent 2^s for modulus 2^κ.

We have already seen that $s \not> \kappa - 2$, and we have already treated (Prop. 14) of the case when $s = 1$.

Let a have exponent 2^s.

The exponents to which it belongs for successive powers of 2 as moduli are given either by

$$\left.\begin{array}{llllllll} \text{exp} & 1. \ 1. \ 1. & \ldots & 1 & 2. & 2^2. & \ldots & 2^s \\ \text{mod} & 2. \ 2^2. \ 2^3. & \ldots & 2^{\kappa-s}. & 2^{\kappa-s+1}. & 2^{\kappa-s+2}. & \ldots & 2^\kappa \end{array}\right\},$$

or by

$$\left.\begin{array}{l}\text{Exp } 1.\ 2.\ 2.\ \ldots\ \ 2.\ \ \ 2^2.\ \ldots\ 2^s \\ \text{Mod } 2.\ 2^2.\ 2^3.\ \ldots\ 2^{\kappa-s+1}.\ 2^{\kappa-s}.\ \ldots\ 2^\kappa\end{array}\right\}$$

In either case

$$a^2 \equiv 1 \pmod{2^{\kappa-s+1}}$$
$$a^2 \not\equiv 1 \pmod{2^{\kappa-s+2}},$$

therefore

$$a^2 \equiv 1 + 2^{\kappa-s+1} \pmod{2^{\kappa-s+2}},$$

and therefore

$$a \equiv \pm 1 + 2^{\kappa-s} \pmod{2^{\kappa-s+1}}.$$

Thus the numbers (modulus 2^κ) which have exponent 2^s are the 2^s numbers given by

$$\pm 1 + 2^{\kappa-s} \pmod{2^{\kappa-s+1}}.$$

In particular, the numbers with the greatest exponent $2^{\kappa-2}$ are the $2^{\kappa-2}$ numbers

$$\pm 1 + 2^2 \pmod{2^3}.$$

We have seen that there is one number (viz., unity) with exponent 1.

there are	3	numbers with exponent			2.
,, ,,	2^2	,,	,,	,,	2^2.
	⋮				
,, ,,	$2^{\kappa-2}$	,,	,,	,,	$2^{\kappa-2}$.

In all, $1 + 3 + 2^2 + 2^3 + \ldots + 2^{\kappa-2} = 2^{\kappa-1} = \phi(2^\kappa)$, the number of odd numbers less than 2^κ, as it should be.

(16.) If we take any number g which has exponent $2^{\kappa-2}$ for modulus 2^κ, the successive powers of g give $2^{\kappa-2}$ incongruent numbers, one half of the complete set of odd numbers less than 2^κ.

Of these, one and only one, has exponent 2, viz., $g^{2^{\kappa-3}}$ (Prop. 4).

Now

$$g \equiv \pm 1 + 2^2 \quad \pmod{2^3}$$

squaring,

$$g^2 \equiv \quad 1 + 2^3 \quad \pmod{2^4}$$

and successively squaring

$$g^{2^2} \equiv \quad 1 + 2^4 \quad \pmod{2^5}$$
$$\vdots$$
$$g^{2^{\kappa-3}} \equiv \quad 1 + 2^{\kappa-1} \pmod{2^\kappa}.$$

So of the three numbers $2^\kappa - 1$, $2^{\kappa-1} + 1$, $2^{\kappa-1} - 1$, it is always $2^{\kappa-1} + 1$ to which the power $2^{\kappa-3}$ of any number with exponent $2^{\kappa-2}$ is congruent.

Let f be either of the two numbers $2^\kappa - 1$, $2^{\kappa-1} - 1$; so that $g^{2^{\kappa-3}} \not\equiv f \pmod{2^\kappa}$.

Consider the $2^{\kappa-1}$ numbers

$$\left.\begin{matrix} g\,, g^2\,, g^3 & \ldots & g^{2^{\kappa-2}} \\ fg, fg^2, fg^3 & \ldots & fg^{2^{\kappa-2}} \end{matrix}\right\} \pmod{2^\kappa}.$$

Clearly no two in the first row are congruent, nor in the second.

The supposition

$$fg^r \equiv g^{r+t} \pmod{2^\kappa}$$

leads to

$$g^r (g^t - f) \equiv 0 \pmod{2^\kappa}$$
$$g^t \equiv f \pmod{2^\kappa},$$

which is contrary to the supposition that $g^{2^{\kappa-3}}$, and therefore *no* power of g is congruent to f.

Therefore no two of the above $2^{\kappa-1}$ odd numbers are congruent mod 2^κ, and hence they are the $2^{\kappa-1}$ numbers (mod 2^κ).

If for f we take $2^\kappa - 1 \equiv -1 \pmod{2^\kappa}$, the $2^{\kappa-1}$ numbers may be written

$$\pm g, \pm g^2, \pm g^3, \ldots \pm g^{2^{\kappa-2}} \pmod{2^\kappa}.$$

Whichever number be taken for f, any number (mod 2^κ) is expressible in the form

$$a \equiv f^i g^j \pmod{2^\kappa},$$

where

$$\left.\begin{matrix} i \text{ is referred to mod } 2, \\ j \text{ is referred to mod } 2^{\kappa-2}. \end{matrix}\right\}$$

Note.—In the last propositions relating to mod 2^κ, κ is supposed to be $\geqq 3$.

In the case $\kappa = 2$, when the modulus is $2^2 = 4$, there are two odd numbers, less than the modulus, viz., 1 and 3, and 3 (having exponent 2) is a primitive root.

In the case $\kappa = 1$ when the modulus is 2, the only odd number is unity.

We see that the case when the modulus is a power of 2, differs very much from the case when the modulus is a power of an odd prime.

When the mod is 2^κ ($\kappa \geqq 3$).

(i.) The highest exponent is not $\phi(2^\kappa) = 2^{\kappa-1}$, but $2^{\kappa-2}$, and hence there are no primitive roots.

(ii.) The form of the numbers which belong to any exponent is known, and the numbers can be at once written down when κ is known. In particular the numbers $\pm 1 + 2^2$ (mod 8) always belong to the highest exponent.

When the modulus is a power of an odd prime.

(i.) Primitive roots always exist.

(ii.) The determination of primitive roots depends on a knowledge of those of the prime in question.

Examples.—For modulus $2^5 = 32$ the numbers which belong to the different exponents are

Exp 1. 1.
Exp 2. 15. 17. 31.
Exp 4. 7. 23. 9. 25. . . . . . ($\pm 1 + 8$ (mod 16)).
Exp 8. 3. 11. 19. 27. 5. 13. 21. 29 ($\pm 1 + 4$ (mod 8)).

The residues of powers of 3 are

3. 9. 27. 17. 19. 25. 11. 1.

Multiplying each by $2^5 - 1 = 31$ we get

29. 23. 5. 15. 13. 7. 21. 31,

and, multiplying by $2^4 - 1 = 15$, we get

13. 7. 21. 31. 29. 23. 5. 15,

the same set.

THE Table of Indices for Generators 3 and 15 is—

..	0	1	2	3	4	5	6	7	(Index of 3.)
0	1	3	9	27	17	19	25	11	
1	15	13	7	21	31	29	23	5	

(Index of 15.)

(17). From propositions (9), (12) and (13) the exponent to which any number a belongs for modulus $m = 2^\kappa p^\lambda q^\mu \ldots$ is readily determined.

For by (9) the exponent is the L.C.M. of the separate exponents of a for moduli $2^\kappa, p^\lambda, q^\mu, \ldots$ separately.

These exponents are separately determined by propositions (12) and (13).

The greatest possible separate exponents are

$$\left.\begin{array}{ll} 2^{\kappa-2} & \text{if } \kappa \geqq 3 \\ 2 & \text{if } \kappa = 2 \\ 1 & \text{if } \kappa = 1 \end{array}\right\}\text{for mod } 2^\kappa,\ (p-1)p^{\lambda-1} \text{ for mod } p^\lambda, \text{ \&c.,}$$

and, hence, the greatest exponent possible for modulus m is the L.C.M. of these.

The value of

$$\phi(m) \text{ is } 2^{\kappa-1}(p-1)\,p^{\lambda-1}(q-1)\,q^{\mu-1}\ldots$$

and, since $p-1$, $q-1$, . . . are all even, the L.C.M. can only equal $\phi(m)$ when

(i.) there is the only one odd prime p present; and

(ii.) $\kappa = 1$ or 0.

Hence the only moduli which admit of primitive roots (which exist only when the highest exponent is equal to $\phi(m)$), are powers of odd primes, and double the powers of odd primes.

Examples.—What is the exponent of 3 for mod $1000000 = 2^6 . 5^6$?

The exp of 3 for mod 2^6 is 2^4.
,, ,, 5^6 is $4 . 5^5$.

Therefore

exp of 3 mod $10^6 = 2^4 . 5^5 = 50000$.

How many decimal places are there in the period of the product of $\cdot\dot{0}\dot{1}$ and $\cdot\dot{0}\dot{1}$, *i.e.*, what is the exponent of 10 for mod $99 \times 99 = 3^4 . 11^2$?

Exp of 10 for mod $3 = 1$.
,, ,, $3^2 = 1$.
,, ,, $3^3 = 3$.
,, ,, $3^4 = 9$.
,, ,, $11 = 2$.
,, ,, $11^2 = 22$.

Therefore

exp of 10 for mod $3^4 . 11^2 = 9 \times 22 = 198$.

Hence there are 198 figures in the period of $(\cdot\dot{0}\dot{1})^2$.

How many decimals are there in the period of $(\cdot\dot{0}0\dot{1})^n$, *i.e.*, what is the exponent of 10 for mod $(999)^n = 37^n . 3^{3n}$?

The exp of 10 mod 3 is 1,

and

$10 - 1$ is divisible by 3^2,

therefore

exp of 10 mod 3^{3n} is 3^{3n-2}.

$$\text{Exp of } 10 \bmod 37 = 3,$$

and

$$37^1 \text{ divides } 10^3 - 1,$$

therefore

$$\text{exp of } 10 \bmod 37^n = 3 \,.\, 37^{n-1},$$

therefore

$$\text{exp of } 10 \bmod 37^n \,.\, 3^{3n} = 3^{3n-2} \,.\, 37^{n-1},$$

the number of figures in the period.

(18.) We will next briefly consider the residues of successive powers of a number not prime to the modulus.

Let a be the number, m the modulus.

Let $m = p\mathrm{P}$: where p consists of powers of those primes which occur as factors of a, and P is prime to a.

Consider the series of residues

$$a \,.\, a^2 \,.\, a^3 \ldots (\text{mod } m).$$

Suppose that the first term which is repeated is a^r and suppose that

$$a^{r+t} \equiv a^r \,(\text{mod } m),$$

for which we seek the smallest values of r and t.

Then after the first $r - 1$ terms we shall have a period of t terms constantly repeated.

We have

$$a^r (a^t - 1) \equiv 0 \,(\text{mod } \mathrm{P}p),$$

and P is prime to a. Therefore

$$a^t - 1 \equiv 0 \,(\text{mod } \mathrm{P}) \quad \ldots\ldots\ldots\ldots \quad (\text{i.}).$$

Each prime factor of p is a factor of a, therefore $a^t - 1$ is prime to p. Therefore

$$a^r \equiv 0 \,(\text{mod } p) \quad \ldots\ldots\ldots\ldots \quad (\text{ii.}).$$

(i.) Shows that t is the exponent of a for modulus P.

(ii.) Shows that r is the least number that makes a^r divisible by p.

Corollaries.—(i.) When $a \equiv 1$ (mod P) $t = 1$ and the period consists of one term only. (ii.) When a is divisible by p the period starts from the first term. (iii.) When both these hold good, then every power of a is $\equiv a$ (mod m).

Examples.—Residues of powers of 2 mod 100.

$$100 = 5^2 \,.\, 2^2.$$

$$\text{Exp of } 2 \bmod 5 = 4$$

$$\text{,,} \qquad \text{,,} \qquad 5^2 = 20$$

therefore

$$t = 20,$$

and

$$2^2 \text{ is divided by } 2^2$$

therefore

$$r = 2.$$

$$2.\ \dot{4}.\ 8.\ 16.\ 32.\ 64.\ 28.\ 56.\ 12.\ 24.\ 48.\ 96.\ 92.\ 84.\ 68.\ 36.\ 72.\ 44.\ 88.\ 76.\ 5\dot{2},$$

the points indicating the period.

Powers of 5 mod $1000 = 2^3 \,.\, 5^3$.

$$5 \text{ has exp } 2 \text{ mod } 2^3,$$

therefore

$$t = 2,$$

$$5^3 \text{ is divided by } 5^3,$$

therefore

$$r = 3.$$

$$5.\ 25.\ 1\dot{2}5.\ 6\dot{2}5.$$

Powers of 5 mod $1000000 = 2^6 \,.\, 5^6$.

$$5 \text{ has exp } 2^4 = 16 \text{ mod } 2^6,$$

therefore

$$t = 16,$$

and

$$5^6 \text{ is divisible by } 5^6,$$

therefore

$$r = 6.$$

$$5.\ 25.\ 125.\ 625.\ 3125.\quad \begin{array}{r} \left.\begin{array}{r} 15625 \\ 78125 \\ 390625 \\ 953125 \\ 765625 \\ 828125 \\ 140625 \\ 703125 \\ 5\dot{1}5625 \\ 578125 \\ 890625 \\ 453125 \\ 265625 \\ 328125 \\ 640625 \\ 203125 \end{array}\right\} \\ \hline 15625 \end{array}$$

Powers of $57 \cdot 5^6 = 890625$ mod 1000000.

$$\text{Exp of } 57 \cdot 5^6 \text{ mod } 2^6 \text{ is } 1 \; (890625 \equiv 1 \pmod{2^6}),$$

and

$$57 \cdot 5^6 \text{ is divisible by } 5^6,$$

therefore

$$t = 1$$

and

$$r = 1,$$

and so all powers of 890625 are $\equiv 890625 \pmod{1000000}$.

$$\begin{array}{l} 890625 \\ 890625 \\ \hline 453125 \\ 81250 \\ 3750 \\ 25 \\ \hline 890625 \end{array}$$

(19.) Let $m = p_1 p_2 p_3 \ldots$, where $p_1, p_2, p_3, \ldots$ are co-prime.

Take any number a prime to m.

Suppose that

$$\begin{aligned} a &\equiv \alpha_1 \pmod{p_1}, \\ &\equiv \alpha_2 \pmod{p_2}, \\ &\equiv \alpha_3 \pmod{p_3}, \\ &\&c. \end{aligned}$$

Since a is prime to m and therefore to p_1, it follows that α_1 is prime to p_1. So α_2 is prime to p_2, α_3 to p_3, &c.

Suppose now that, conversely, $\alpha_1, \alpha_2, \alpha_3, \ldots$ prime to $p_1, p_2, p_3, \ldots$ are given, and we wish to find a, such that it is congruent to $\alpha_1 \pmod{p_1}$, $\alpha_2 \pmod{p_2}$, &c.

Let x_1 be determined from

$$x_1 p_2 p_3 \ldots \equiv 1 \pmod{p_1},$$

which can be done, and in one way only, since $p_2 p_3 \ldots$ is prime to p_1; and when x_1 is found let $x_1 p_2 p_3 \ldots \equiv \xi_1$. Determine similarly $\xi_2, \xi_3 \ldots$ Then the number a is given by

$$a \equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 + \ldots \pmod{m}.$$

For

$$\xi_2,\ \xi_3,\ \ldots \text{ are all congruent to zero mod } p_1,$$

and

$$\xi_1 \text{ is congruent to unity mod } p_1.$$

Therefore

$$a \equiv \alpha_1 \ (\text{mod } p_1),$$

and similarly

$$\equiv \alpha_2 \ (\text{mod } p_2),$$

&c.,

and therefore

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \ldots \ (\text{mod } m),$$

a formula which gives the value of a corresponding to given values of the α's, the coefficients ξ being independent of the α's and depending only on the moduli $p_1, p_2 \ldots$

Note.—The principal use to be made of this proposition and the next will be for the special case when $p_1, p_2, \ldots$ are the powers of primes of which the modulus m is the product.

(20.) Let us take two numbers expressed in this form,

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \ (\text{mod } m),$$
$$b \equiv \beta_1\xi_1 + \beta_2\xi_2 + \ldots \ (\text{mod } m),$$

in which

$$\left.\begin{aligned} \xi_1 &\equiv 1 \ (\text{mod } p_1) \\ &\equiv 0 \left(\text{mod } \frac{m}{p_1}\right) \end{aligned}\right\}$$

with similar relations for $\xi_2,\ \xi_3,\ \ldots$

Let us form the product of a and b.

By Proposition (18), Corollary (iii.), all powers of ξ_1 are congruent to ξ_1, mod m, therefore

$$\alpha_1\beta_1\xi_1^2 \equiv \alpha_1\beta_1\xi_1 \ (\text{mod } m).$$

Taking any cross-term such as $\alpha_1\beta_2\xi_1\xi_2$, since

$$\xi_1 \equiv 0 \ (\text{mod } p_2p_3 \ldots),$$
$$\xi_2 \equiv 0 \ (\text{mod } p_1p_3 \ldots),$$

therefore

$$\xi_1\xi_2 \equiv 0 \ (\text{mod } m).$$

Hence

$$ab \equiv \alpha_1\beta_1\xi_1 + \alpha_2\beta_2\xi_2 + \alpha_3\beta_3\xi_3 + \ldots \ (\text{mod } m),$$

i.e., the multiplication of numbers expressed thus is simply effected by forming the products of the coefficients of $\xi_1, \xi_2 \ldots$

Corollary (i.). If

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots + (\text{mod } m),$$

then

$$a^s \equiv \alpha_1^s\xi_1 + \alpha_2^s\xi_2 + \ldots (\text{mod } m).$$

Corollary (ii.). If

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots (\text{mod } m),$$

then a is congruent to the product of the numbers

$$\begin{cases} a_1 \equiv \alpha_1\xi_1 + \xi_2 + \ldots (\text{mod } m), \\ a_2 \equiv \xi_1 + \alpha_2\xi_2 + \ldots (\text{mod } m), \\ \text{\&c.} \end{cases}$$

Or, since $\xi_1 + \xi_2 + \ldots \equiv 1 \ (\text{mod } m)$ (*cf.* Prop. 10),

$$\begin{cases} a_1 \equiv (a_1 - 1)\,\xi_1 + 1 \ (\text{mod } m), \\ a_2 \equiv (\alpha_2 - 1)\,\xi_2 + 1 \ (\text{mod } m), \\ \ldots\ldots\ldots\ldots \end{cases}$$

Examples.—To find a, modulus 308, so that

$$\begin{aligned} a &\equiv \alpha_1 \ (\text{mod } 4) \\ &\equiv \alpha_2 \ (\text{mod } 7) \\ &\equiv \alpha_3 \ (\text{mod } 11) \end{aligned}$$

$$\begin{array}{rrr} 7.\ 11.\ x_1 \equiv 1 \ (\text{mod } 4) & 4.\ 11.\ x_2 \equiv 1 \ (\text{mod } 7) & 4.\ 7.\ x_3 \equiv 1 \ (\text{mod } 11) \\ 3.\ \ 3.\ x_1 \equiv 1 \ (\text{mod } 4) & 2.\ x_2 \equiv 1 \ (\text{mod } 7) & 6.\ x_3 \equiv 1 \ (\text{mod } 11) \\ x_1 \equiv 1 \ (\text{mod } 4) & x_2 \equiv 4 \ (\text{mod } 7) & x_3 \equiv 2 \ (\text{mod } 11) \\ \xi_1 \equiv 77 \ (\text{mod } 308) & \xi_2 \equiv 176 \ (\text{mod } 308) & \xi_3 \equiv 56 \ (\text{mod } 308) \end{array}$$

and so

$$a \equiv 77\,\alpha_1 + 176\,\alpha_2 + 56\,\alpha_3 \ (\text{mod } 308),$$

e.g.,

$$\begin{aligned} a &\equiv 3 \ (\text{mod } 4) \\ &\equiv 6 \ (\text{mod } 7) \\ &\equiv 1 \ (\text{mod } 11) \end{aligned}$$

then

$$\begin{aligned} a &\equiv 77.\,3 + 176.\,6 + 56.\,1 \ (\text{mod } 308) \\ &\equiv 111 \ (\text{mod. } 308), \end{aligned}$$

and

$$\begin{aligned} 111 &\equiv (77.\ 3 + 176 + 56)\,(77 + 176.\ 6 + 56)\,(77 + 176 + 56) \\ &\equiv 155.\ 265 \pmod{308}. \end{aligned}$$

(21.) The number of numbers which belong to a given exponent when the modulus is a power of a prime.

I. Let the prime be an odd prime, and p^λ the modulus.

In Proposition (12), Corollary, we saw that for mod p^λ primitive roots exist.

Let g be a primitive root.

The numbers $g, g^2, g^3, \ldots g^{\phi(p^\lambda)} \pmod{p^\lambda}$ are congruent to the complete set of numbers less than p^λ and prime to it.

The exponent to which any one of these numbers, g^s, belongs is t, where

and
$$\left.\begin{aligned} \phi(p^\lambda) &= \kappa t \\ s &= \kappa\sigma \end{aligned}\right\} \text{ and } t \text{ and } \sigma \text{ are co-prime (Prop. 4).}$$

For any given value of t the value of $\kappa = \phi(p^\lambda)/t$ is given.

σ may then have any value prime to t such that

$$s \ngtr \phi(p^\lambda),$$

i.e.,

$$\kappa\sigma \ngtr \kappa t$$

$$\sigma \ngtr t.$$

Hence σ may have each of the $\phi(t)$ values of the numbers less than t and prime to it.

Therefore there are $\phi(t)$ numbers having t as their exponent (mod m).

II. Let the modulus be 2^κ. ($\kappa \geqq 3$).

In this case we have seen (Proposition 15) that there are 2^s numbers with exponent 2^s (if $s > 1$): and 3 numbers with exponent 2; and 1 number with exponent unity.

When the modulus is 2^2 there is one number with exponent 2 and one with unity.

When the modulus is 2 there is one number (unity) with exponent unity.

Definition.—When a number m is expressed in the form $m = 2^\kappa P_1^{\lambda_1} P_2^{\lambda_2} \ldots$ where $P_1, P_2, \ldots$ are different odd primes, it will be convenient to speak of $2^\kappa, P_1^{\lambda_1}, P_2^{\lambda_2} \ldots$ as the *principal factors* of m.

(22.) The number of numbers, each of which has, as exponent, some power of a prime p, for modulus m, p being a divisor of $\phi(m)$.

Let

$$m = 2^\kappa P_1^{\lambda_1} P_2^{\lambda_2}, \ldots$$

$$\phi(P_1^{\lambda_1}) = 2^{\kappa_1} p^{l_1} q^{m_1}, \ldots$$

$$\phi(P_2^{\lambda_2}) = 2^{\kappa_2} p^{l_2} q^{m_2}, \ldots$$

&c.

Any number has for its exponent, modulus m, the L.C.M. of its separate exponents for moduli 2^κ, $P_1^{\lambda_1}$, $P_2^{\lambda_2}$, . . . the principal factors of m. (Proposition (9), Corollary.)

Hence, when the exponent, modulus m, is a power of a prime p, the exponent for each of the principal factors of m as moduli, must each be either unity or some power of p.

Conversely, if we take a set of numbers α_0, α_1, α_2, . . . one for each of the moduli 2^κ, $P_1^{\lambda_1}$, . . . such that the exponent of each, for its own modulus, is unity or some power of p, then the number $a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \ldots \pmod{m}$, will have a power of p as exponent, modulus m.

The numbers ξ are given by

$$\begin{array}{ll} \xi_0 \equiv 1 \pmod{2^\kappa}, & \xi_1 \equiv 1 \pmod{P_1^{\lambda_1}}, \\ \quad \equiv 0 \left(\text{mod } \dfrac{m}{2^\kappa}\right), & \quad \equiv 0 \left(\text{mod } \dfrac{m}{P^{\lambda_1}}\right), \quad \text{\&c.} \end{array}$$

Thus, by giving to the α's all possible sets of values consistently with each having unity or a power of p as exponent, we shall obtain all the numbers (mod m) which have (unity or) a power of p as exponent.

There are

$$\left\{\begin{array}{lllll} \phi(p^{l_1}) & \text{numbers} & \text{which have} & \text{exp } p^{l_1}, & \text{mod } P_1^{\lambda_1}. \quad \text{(Prop. 21.)} \\ \phi(p^{l_1-1}) & ,, & ,, & ,, \ \ p^{l_1-1} & ,, \\ \quad \text{\&c.} & & & & \\ \quad 1 & ,, & ,, & ,, \ \ 1 & ,, \end{array}\right.$$

Hence there are $\phi(p^{l_1}) + \phi(p^{l_1-1}) + \ldots + 1 = p^{l_1}$ numbers, mod $P_1^{\lambda_1}$, which have unity or a power of p as exponent.

Similarly, there are p^{l_2} for mod $P_2^{\lambda_2}$, and so on.

Hence, in

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m},$$

$$\left.\begin{array}{l} \text{we can give 1 value (viz., unity) to } \alpha_0 \\ \text{any one of } p^{l_1} \text{ values to } \alpha_1 \\ \quad ,, \qquad p^{l_2} \qquad ,, \qquad \alpha_2 \\ \qquad \text{\&c.,} \qquad\qquad \text{\&c.} \end{array}\right\}$$

and then a has unity or a power of p as exponent, mod m; moreover, in this way, all such numbers are obtained.

We thus obtain $p^{l_1+l_2+\cdots} = p^{\Sigma l}$ incongruent numbers, each of which has a power of p (or unity) for exponent.

In the case when $p = 2$

for mod 2^κ, all $2^{\kappa-1}$ numbers have powers of 2 as exponents;
,, $P_1^{\lambda_1}$, $\phi(2^{\kappa_1}) + \phi(2^{\kappa_1-1}) + \ldots + 1 = 2^{\kappa_1}$ have powers of 2 as exponents,
,, $P_2^{\lambda_2}$, $\phi(2^{\kappa_2}) + \phi(2^{\kappa_2-1}) + \ldots + 1 = 2^{\kappa_2}$,, ,, ,,
&c. &c. &c.

Hence, in all, there are $2^{\kappa-1+\kappa_1+\kappa_2+\cdots}$ numbers, which have unity or a power of 2 as exponent.

Note.—Any number a, mod m, with any exponent, must be congruent to a product of one number from each set of $2^{\kappa-1+\kappa_1+\kappa_2+\cdots}$ numbers, with powers of 2 as exponent, $p^{\Sigma l}$ with powers of p as exponent, . . . &c. (Proposition 10.)

Hence in all we get from these

$$
\begin{aligned}
& 2^{\kappa-1+\kappa_1+\cdots} p^{\Sigma l} q^{\Sigma m} \ldots \text{ numbers,} \\
&= \phi(2^\kappa)\,\phi(P_1^{\lambda_1}) \ldots \qquad \text{,,} \\
&= \phi(m) \text{ numbers, } i.e., \text{ the complete set of numbers prime to } m.
\end{aligned}
$$

(23.) The number of numbers having exponent p^s, mod m, p^s being a divisor of the greatest exponent.

Let a be such a number and let

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m},$$

then the exponent of

$$\left.\begin{array}{l} \alpha_0 \text{ mod } 2^\kappa \\ \alpha_1 \text{ mod } P_1^{\lambda_1} \\ \alpha_2 \text{ mod } P_2^{\lambda_2} \\ \quad \&c. \end{array}\right\}$$

must each be unity or some power of p, and the greatest power of p must be p (Proposition (9), Corollary); p being an odd prime α_0 is necessarily unity.

There are

$\phi(p^{l_1})$ numbers, mod $P_1^{\lambda_1}$ with exp p^{l_1},
$\phi(p^{l_1-1})$,, ,, ,, ,, p^{l_1-1},
&c , &c.,

and therefore there are

p^{l_1} numbers with exp a power of $p \overset{=}{<} p^{l_1}$,
p^{l_1-1} ,, ,, ,, ,, $\overset{=}{<} p^{l_1-1}$;
&c., &c.

Hence if

$$\left.\begin{array}{l} l_1 < s \text{ there are } p^{l_1} \text{ numbers, mod } \mathrm{P}_1^{\lambda_1}\text{, with exp a power of } p \not> p^s \\ l_1 \overset{=}{>} s \text{ there are } p^{s} \text{ numbers, mod } \mathrm{P}_1^{\lambda_1}\text{, with exp a power of } p \not> p^s \end{array}\right\}$$

and if

Suppose that $(l_1)_s$ stands for s if $l_1 \overset{=}{>} s$ and for l_1 if $l_1 < s$, then in either case there are $p^{(l_1)_s}$ numbers, mod $\mathrm{P}_1^{\lambda_1}$, whose exponents are powers of $p \not> p^s$.

Similarly there are $p^{(l_2)_s}$ numbers mod $\mathrm{P}_2^{\lambda_2}$, whose exponents are powers of $p \not> p^s$ and so for $\mathrm{P}_2^{\lambda_2}$, &c.

Giving any one of these $p^{(l_1)_s}$ values to α_1, $p^{(l_2)_s}$ values to α_2, &c., we obtain

$$p^{(l_1)_s + (l_2)_s + \dots} = p^{(\Sigma l)_s} \text{ numbers, mod } m,$$

whose exponents are powers of $p \not> p^s$; where in $p^{(\Sigma l)_s}$ each number l in $l_1 + l_2 + \dots$ is to be replaced by s if it exceeds s.

Similarly, there are $p^{(\Sigma l)_{s-1}}$ numbers, mod m, whose exponents are powers of $p \not> p^{s-1}$ &c., &c., $p^{(\Sigma l)_1}$ numbers with exponents p or unity, and 1 number with exponent 1.

Hence the number of numbers whose exponent is p^s, mod m, is

$$p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}, \text{ which when } s = 1 \text{ is } p^{(\Sigma l)_1} - 1.$$

Corollary.—If p^s be the highest power of p which can be an exponent for mod m, *i.e.*, if p^s is the highest power of p that divides the greatest exponent, *i.e.*, if s is the greatest of the numbers l_1, l_2, . . . then

$$1 + (p^{(\Sigma l)_1} - 1) + (p^{(\Sigma l)_2} - p^{(\Sigma l)_1}) + \dots + (p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}) = p^{(\Sigma l)_s}$$

is the number of numbers having as exponent (mod m) a power of p (or unity) as exponent.

Now since s is equal to the greatest of the quantities l_1, l_2, . . . (or rather is not less than any) $(\Sigma l)_s = \Sigma l$ and therefore $p^{(\Sigma l)_s} = p^{\Sigma l}$: the result which was obtained in the last proposition.

(23A.) To find the number of numbers having exponent 2^σ, mod m.

(It will be convenient to write now $\kappa = \kappa' + 2$, so that $m = 2^{\kappa' + 2}\mathrm{P}_1^{\lambda_1}\mathrm{P}_2^{\lambda_2} \dots$).

If a be any such number, and

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \dots (\text{mod } m),$$

the exponents of

$$\left.\begin{array}{l} \alpha_0 \text{ mod } 2^\kappa \\ \alpha_1 \text{ mod } \mathrm{P}_1^{\lambda_1} \\ \alpha_2 \text{ mod } \mathrm{P}_2^{\lambda_2} \\ \quad\&\text{c.} \end{array}\right\}$$

must each be unity or a power of 2, and the greatest power of 2 must be 2^σ.

First suppose that $\kappa \geqq 3$, and so $\kappa' \geqq 1$.

Then there are, mod $2^{\kappa'+2}$,

$$\left.\begin{array}{lllll} 2^{\kappa'} & \text{numbers with exponent} & & & 2^{\kappa'} \\ 2^{\kappa'-1} & ,, & ,, & ,, & 2^{\kappa'-1} \\ \vdots & & & & \\ 2^2 & ,, & ,, & ,, & 2^2 \\ 3 & ,, & ,, & ,, & 2 \\ 1 & \text{number} & ,, & ,, & 1 \end{array}\right\}$$

Hence there are

$$\begin{array}{llllll} 2^{\kappa'+1} & \text{numbers with exponent a power of } 2 & & & & \ngtr 2^{\kappa'} \\ 2^{\kappa'} & ,, & ,, & ,, & ,, & \ngtr 2^{\kappa'-1} \\ \vdots & & & & & \\ 2^2 & ,, & ,, & ,, & ,, & \ngtr 2 \\ 1 & \text{number} & ,, & ,, & ,, & \ngtr 2^0 \end{array}$$

If

$\sigma > \kappa'$ then there are $2^{\kappa'+1}$ numbers with exponent a power of $2 \ngtr 2^{\sigma}$, mod $2^{\kappa'+2}$,

and if

$\sigma \leqq \kappa'$ then ,, ,, $2^{\sigma+1}$,, ,, ,, ,, ,, ,,

(this holds unless $\sigma = 0$, and then there is one number (unity) with exponent 1).

In either case, then, there are $2^{(\kappa')_\sigma+1}$ numbers with powers of $2 \ngtr 2^{\sigma}$ as exponents, where $(\kappa')_\sigma$ is to be replaced by

$$\left.\begin{array}{l} \sigma \text{ if } \kappa' > \sigma \\ \kappa' \text{ ,, } \kappa' \leqq \sigma \end{array}\right\}.$$

Next for mod $P_1^{\lambda_1}$ there are

$$\begin{array}{llllll} 2^{\kappa_1-1} & \text{numbers with exponent a power of } 2 & & & & \ngtr 2^{\kappa_1-1} \\ 2^{\kappa_1-2} & ,, & ,, & ,, & ,, & \ngtr 2^{\kappa_1-2} \\ \&\text{c.} & & & & & \end{array}$$

Hence (using the same notation) there are $2^{(\kappa_1)_\sigma}$ numbers (mod $P_1^{\lambda_1}$) having exponent a power of $2 \ngtr 2^{\sigma}$.

Thus α_0 may have each of $2^{(\kappa')_\sigma}$ values, α_1 each of $2^{(\kappa_1)_\sigma}$ values, &c., and the corresponding value of α has exponent a power of 2 (mod m), $\ngtr 2^{\sigma}$.

Hence the number of these is $2^{(\Sigma\kappa)_\sigma+1}$, where in $\Sigma\kappa = \kappa' + \kappa_1 + \kappa_2 + \ldots$ each number κ is to be replaced by σ if it exceeds σ.

Similarly the number of numbers whose exponent, mod m, is a power of $2 \not> 2^{\sigma-1}$ is

$$2^{(\Sigma\kappa)_{\sigma-1}+1}.$$

Hence the number of numbers with exponent 2^σ (mod m) is

$$2^{(\Sigma\kappa)_\sigma+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1}.$$

This holds for $\sigma \geqq 2$. When $\sigma = 1$ the number of numbers with exponent 2 is

$$2^{(\Sigma\kappa)_1+1} - 1.$$

Secondly suppose that $\kappa = 2$, and so $\kappa' = 0$.

Then, if σ is > 1, α_0 may be either 1 or 3, two values.

Hence the number of numbers with exponent 2^σ is

$$2^{(\Sigma\kappa)_\sigma+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1},$$

which agrees with the above when κ' is omitted.

If $\sigma = 1$, the number of numbers with exponent 1 or 2, mod 2^2, is 2.

Therefore the number of numbers with exponent 2, mod m, is

$$2^{(\Sigma\kappa)_1+1} - 1,$$

which agrees with the above when κ' is put $= 0$.

Thirdly, let $\kappa = 1$. Then α_0 must be unity and the number of numbers with exponent 2^σ is $2^{(\Sigma\kappa)_\sigma} - 2^{(\Sigma\kappa)_{\sigma-1}}$ where $\Sigma\kappa = \kappa_1 + \kappa_2 + \dots$

Fourthly, when ($\kappa = 0$) m is odd, then again the number required is $2^{(\Sigma\kappa)_\sigma} - 2^{(\Sigma\kappa)_{\sigma-1}}$.

To collect the results :—

When

$$\left.\begin{array}{l} m = 2^{\kappa'+2}P_1^{\lambda_1}P_2^{\lambda_2}\dots \\ \text{or} \\ \quad\;\; 2^2P_1^{\lambda_1}P_2^{\lambda_2}\dots \end{array}\right\} \begin{array}{l} \text{there are } 2^{(\Sigma\kappa)_1+1} - 1 \text{ numbers with exp } 2, \\ \\ \qquad\qquad 2^{(\Sigma\kappa)_\sigma+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1} \text{ numbers with exp } 2^\sigma. \end{array}$$

And when

$$\left.\begin{array}{l} m = 2P_1^{\lambda_1}P_2^{\lambda_2}\dots \\ \text{or} \\ \quad\;\; P_1^{\lambda_1}P_2^{\lambda_2}\dots \end{array}\right\} \begin{array}{l} \text{there are } 2^{(\Sigma\kappa)_1} - 1 \text{ numbers with exp } 2, \\ \\ \qquad\qquad 2^{(\Sigma\kappa)_\sigma} - 2^{(\Sigma\kappa)_{\sigma-1}} \text{ numbers with exp } 2^\sigma. \end{array}$$

(23B.) The last two propositions give us the number of numbers belonging to any exponent $t = 2^a p^b q^c \dots$

For the complete set of these numbers is formed by taking all possible products of a number with exponent 2^a, one with exponent p^b, &c.

Hence the number of numbers with exponent t is the product of the number of numbers that belong to each of its principal factors as exponent.

Examples.—The number of numbers that belong to any exponent for mod $m = 2^4 . 13 . 17 . 19$.

The greatest exponent is the L.C.M. of

$$\left.\begin{array}{l} \phi(2^4) = 2^3 \\ \phi(13) = 2^2 . 3 \\ \phi(17) = 2^4 \\ \phi(19) = 2 . 3^2 \end{array}\right\} = 2^4 . 3^2. \qquad \phi(m) = 2^{10} . 3^3.$$

For the 2-power exponent numbers we have for the κ's

$$\kappa' = 2 \qquad \kappa_1 = 2 \qquad \kappa_2 = 4 \qquad \kappa_3 = 1.$$

$$\left.\begin{array}{l} (\Sigma\kappa)_1 = 4 \\ (\Sigma\kappa)_2 = 7 \\ (\Sigma\kappa)_3 = 8 \\ (\Sigma\kappa)_4 = 9 \end{array}\right\}.$$

Therefore

$$\left.\begin{array}{llll} \text{belonging to exp } 2^4 & \text{there are } 2^{10} & - 2^9 & \text{numbers} \\ \quad ,, \qquad ,, \quad 2^3 & \text{there are } 2^9 & - 2^8 & \text{numbers} \\ \quad ,, \qquad ,, \quad 2^2 & \text{there are } 2^8 & - 2^5 & \text{numbers} \\ \quad ,, \qquad ,, \quad 2 & \text{there are } 2^5 & - 1 & \text{numbers} \end{array}\right\}.$$

For the 3-power exponent numbers we may take $l_1 = 1, \quad l_2 = 2$,

$$(\Sigma l)_1 = 2,$$
$$(\Sigma l)_2 = 3.$$

$$\left.\begin{array}{lll} \text{Belonging to exp } 3^2 & \text{there are } 3^3 - 3^2 & \text{numbers} \\ \quad ,, \qquad ,, \quad 3 & \text{there are } 3^2 - 1 & \text{numbers} \end{array}\right\}.$$

The number of numbers belonging to any other exponent is at once got from these by multiplying, *e.g.*, the number of numbers with the greatest exponent $2^4 . 3^2$ is $(2^{10} - 2^9)(3^3 - 3^2)$.

(24.) We shall now establish a particular set of independent generators which generate the $\phi(m)$ numbers (mod m) prime to m.

Any number a, mod m, is expressible in the form

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m}.$$

Let

and let

$$\left.\begin{array}{l} g_1 \text{ be a primitive root of } P_1^{\lambda_1} \\ g_2 \qquad ,, \qquad\qquad ,, \qquad P_2^{\lambda_2} \\ \qquad \&c. \qquad\qquad \&c., \\ \\ g_0 \text{ have exp } 2^{\kappa-2} \bmod 2^{\kappa} \\ f \text{ be} \equiv 2^{\kappa} - 1 \text{ or } 2^{\kappa-1} - 1 \bmod 2^{\kappa} \end{array}\right\}$$

(In the case when $\kappa = 2$, f does not occur, when $\kappa = 1$ or 0, neither f nor g_0.)
Then

and

$$\left.\begin{array}{l} \alpha_1 \equiv g_1^{i_1} \pmod{P_1^{\lambda_1}} \\ \alpha_2 \equiv g_2^{i_2} \pmod{P_2^{\lambda_2}} \\ \qquad \&c., \\ \\ \alpha_0 \equiv g_0^{i_0} f^j \pmod{2^{\kappa}} \end{array}\right\} \text{say.}$$

Therefore

$$\begin{aligned} a &\equiv g_0^{i_0} f^j \xi_0 + g_1^{i_1} \xi_1 + \ldots \pmod{m} \\ &\equiv (f\xi_0 + \xi_1 + \ldots)^j (g_0\xi_0 + \xi_1 + \ldots)^{i_0} (\xi_0 + g_1\xi_1 + \ldots)^{i_1} \ldots \pmod{m} \\ &\equiv [(f-1)\xi_0 + 1]^j [(g_0 - 1)\xi_0 + 1]^{i_0} [(g_1 - 1)\xi_1 + 1]^{i_1} \ldots \pmod{m}, \end{aligned}$$

where

$$\begin{array}{l} j \text{ is referred to mod } 2, \\ i_0 \quad ,, \qquad ,, \quad \text{mod } 2^{\kappa-2}, \\ i_1 \quad ,, \qquad ,, \quad \text{mod } \phi(P_1^{\lambda_1}), \\ \qquad\qquad \&c., \end{array}$$

and so the set of indices $j, i_0, i_1 \ldots$ correspond uniquely to the number a: and the numbers

$$(f-1)\xi_0 + 1, \quad (g_0 - 1)\xi_0 + 1, \quad (g_1 - 1)\xi_1 + 1, \ldots$$

are a set of independent generators, with exponents

$$2, \quad 2^{\kappa-2}, \quad \phi(P_1^{\lambda_1}), \ldots$$

which generate completely the $\phi(m)$ numbers, mod m, which are prime to m.

Example.—Take the modulus

$$m = 112 = 2^4 . 7. \qquad \phi(m) = (2^3)(2 . 3) = 2^4 . 3 = 48.$$

If

$$a \equiv \alpha_0 \pmod{2^4},$$

and

$$a \equiv \alpha_1 \pmod{7},$$
$$a \equiv \alpha_0 \xi_0 + \alpha_1 \xi_1 \pmod{112}.$$

$$\begin{array}{ll} 7x_0 \equiv 1 \pmod{2^4} & 2^4 x_1 \equiv 1 \pmod{7} \\ x_0 \equiv 7 \pmod{2^4} & 2x_1 \equiv 1 \pmod{7} \\ \xi_0 \equiv 49 \pmod{112} & x_1 \equiv 4 \pmod{7} \\ & \xi_1 \equiv 64 \pmod{112}, \end{array}$$

$$a \equiv 49\alpha_0 + 64\alpha_1 \pmod{112}.$$

We will take

$$f = 7 \ (= 2^3 - 1)$$
$$g_0 = 3 \text{ (having exp } 2^2),$$

and

$$g_1 = 3 \text{ (a primitive root of 7)}.$$

Then

$$(f - 1)\,\xi_0 + 1 = 6.\,49 + 1 = 295 \equiv 71 \pmod{112},$$
$$(g_0 - 1)\,\xi_0 + 1 = 2.\,49 + 1 = 99 \pmod{112},$$
$$(g_1 - 1)\,\xi_1 + 1 = 2.\,64 + 1 = 129 \equiv 17 \pmod{112};$$

and

$$\left.\begin{array}{l} 71 \text{ with exp } 2 \\ 99 \text{ with exp } 4 \\ 17 \text{ with exp } 6 \end{array}\right\} \text{generate the 48 numbers prime to 112.}$$

The following table gives the indices corresponding to any number :—

Numbers	1.	3.	5.	9.	11.	13.	15.	17.	19.	23.	25.	27.	29.	31.	33.	37.	39.	41.	43.	45.	47.	51.	53.	55.	57.	59.
Ind. of 17	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1
Ind. of 99	0	1	1	2	3	3	2	0	1	0	2	3	3	2	0	1	0	2	3	3	2	1	1	0	2	3
Ind. of 71	0	0	1	0	0	1	1	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0

Numbers	61.	65.	67.	69.	71.	73.	75.	79.	81.	83.	85.	87.	89.	93.	95.	97.	99.	101.	103.	107.	109.	111
Ind. of 17	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3
Ind. of 99	3	0	1	1	0	2	3	2	0	1	1	0	2	3	2	0	1	1	0	3	3	2
Ind. of 71	1	0	0	1	1	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	1	1

We proceed now to the point we have been approaching from the beginning, viz., the investigation of the mode of formation of and the relations among the most general set of numbers capable of generating the $\phi(m)$ numbers (modulus m) which are prime to the modulus m.

(25.) Suppose $G_1, G_2, G_3, \ldots$ with exponents $t_1, t_2, \ldots$ are capable of acting as complete generators of the $\phi(m)$ numbers.

Then

$$\text{(i.)}\ t_1 t_2 t_3 \ldots = \phi(m)$$

and

(ii.) the generators must be independent.

Since the numbers G generate the complete set of $\phi(m)$ numbers they can generate in particular the numbers which have a power of a prime p (which is a factor of $\phi(m)$) as exponent.

Suppose the highest powers of p which occur in $t_1, t_2, \ldots$ are $p^{s_1}, p^{s_2}, \ldots$ respectively.

Say

$$\left.\begin{array}{l} t_1 = p^{s_1} t'_1, \\ t_2 = p^{s_2} t'_2, \\ \text{\&c.} \end{array}\right\}$$

Then we can express the numbers G thus (Proposition 10) :—

$$\begin{array}{l} G_1 \equiv g_1 h_1 \pmod{m}, \\ G_2 \equiv g_2 h_2 \pmod{m}, \\ \text{\&c.,} \end{array}$$

when g_1 has exponent p^{s_1} and h_1 has exponent t'_1, &c.

Suppose now that

$G_1^{i_1} G_2^{i_2} G_3^{i_3} \ldots$ is congruent to a number with a power of p as exponent,

and so

$(g_1^{i_1} g_2^{i_2} \ldots)(h_1^{i_1} h_2^{i_2} \ldots)$,, ,, ,, ,, ,,

g_1, and therefore $g_1^{i_1}$, has a power of p as exponent (Proposition 4), and so for $g_2^{i_2}$, &c.

Therefore

$g_1^{i_1} g_2^{i_2} \ldots$ has a power of p as exponent.

The exponent of h_1, and therefore of $h_1^{i_1}$, is prime to p, and so for $h_2^{i_2}, \ldots$ &c.

Therefore the exponent of $h_1^{i_1} h_2^{i_2} \ldots$ which divides the L.C.M. of the exponents of $h_1^{i_1}, h_2^{i_2}, \ldots$ is prime to p.

Therefore the exponent of $(g_1^{i_1} g_2^{i_2} \ldots)(h_1^{i_1} h_2^{i_2} \ldots)$ is the L.C.M. of a power of p and of a number (the exponent of $h_1^{i_1} h_2^{i_2} \ldots$) prime to p. Now the exponent is to be a power of p. Hence the exponent of $h_1^{i_1} h_2^{i_2} \ldots$ must be unity.

Therefore

$$h_1^{i_1} h_2^{i_2} \ldots \equiv 1 \pmod{m}.$$

Now we can show that this cannot be unless

$$h_1^{i_1} \equiv 1,\ h_2^{i_2} \equiv 1, \ldots \pmod{m}.$$

For if not suppose that at least $i_1 \not\equiv 0$ (modulus t'_1).
Then

$$h_1^{i_1} h_2^{i_2} \ldots \equiv h_1^{t'_1} \pmod{m},$$
$$h_2^{i_2} h_3^{i_3} \ldots \equiv h_1^{t'_1 - i_1} \pmod{m}.$$

Let

$I_2 \equiv i_2 \pmod{t'_2}$ and the same for I_3, &c.: and $I_1 \equiv - i_1 \pmod{t'_1}$
$\equiv 0 \pmod{p^{s_2}}$, $\equiv 0 \pmod{p^{s_1}}$,
so that $I_2 < t_2$ so that $I_1 < t_1$

and $\left.\begin{matrix} g_2^{I_2} \equiv 1 \\ h_2^{I_2} \equiv h_2^{i_2} \end{matrix}\right\} \pmod{m}$ $\begin{matrix} g_1^{I_1} \equiv 1 \pmod{m} \\ h_1^{I_1} \equiv h_1^{t'_1 - i_1} \pmod{m}, \end{matrix}$

then

$$G_2^{I_2} G_3^{I_3} \ldots \equiv G_1^{I_1} \pmod{m},$$

where

$$I_1 \not\equiv 0 \pmod{t_1},$$

which is contrary to the supposition that the generators G are *independent*, and that therefore no two numbers of the $t_1 t_2 \ldots$ that they generate shall be congruent.

Therefore we must have

$$i_1 \equiv 0 \pmod{t'_1},$$
$$i_2 \equiv 0 \pmod{t'_2},$$
&c.

and so

$$h_1^{i_1} \equiv 1 \pmod{m},$$
$$h_2^{i_2} \equiv 1 \pmod{m},$$
&c.

and so

$$G_1^{i_1} \equiv g_1^{i_1} \pmod{m},$$
&c.

We have shown, then, that when a product of powers of the generators G, $G_1^{i_1} G_2^{i_2} \ldots$ is congruent to a number with a power of a prime p as exponent, then each factor, $G_1^{i_1}$, $G_2^{i_2}$, ... must have a power of p for its exponent.

Thus, of the factors of the product to which G_1 is congruent, each factor having a principal factor of t_1 as exponent (Proposition 10), only one factor g_1 (viz., that which has, as exponent, a power of p) is effective in any power of G_1 which can be used to form a number with exponent a power of p, the product of the remaining factors h_1 being necessarily raised to such a power i_1 that $h_1^{i_1} \equiv 1 \pmod{m}$.

(It will be convenient to call the numbers which have a power of p as exponent, the "p-power-exponent numbers").

Suppose then that we take the generators G, and express each as a product in the manner of Proposition 10. Then take from each the factor (if any) which has a power of the prime p as exponent. Then, since the numbers G can generate all the numbers, modulus m, which have a power of p as exponent, and, since in the number G the factor g is alone effective in so doing, it follows that the set of numbers g generate completely the p-power-exponent numbers.

If we take from each generator G the factor which has a power of any other prime q as exponent, we obtain a set of numbers g', which generate the q-power-exponent numbers, and so on for each prime which divides $\phi(m)$.

We see now, that any set of complete independent generators, G, must be formed from these special sets of generators; the formation of each G being effected by taking one number (which may be unity) from each set and forming their product.

In order to obtain the most general set of generators, G, we have now only to obtain the most general method of producing each of these subsidiary sets of generators. We may then combine them as products (one from each set) in any manner we please.

(26.) Suppose that $\Gamma_1, \Gamma_2, \ldots$ independent generators, generate completely the p-power-exponent numbers.

The exponent of each number, Γ, must be some power of p.

Let them be $p^{n_1}, p^{n_2}, \ldots$ respectively.

One condition that the numbers, Γ, must satisfy is that the number of numbers they generate, which belong to any power of p, p^s as exponent, should agree with the number already found. (Proposition 23.)

Consider any number generated

$$a \equiv \Gamma_1^{i_1}\Gamma_2^{i_1} \ldots \pmod{m}.$$

Since $\Gamma_1, \Gamma_2, \ldots$ are *independent* generators, therefore the exponent of a is the greatest of the separate exponents of $\Gamma_1^{i_1}, \Gamma_2^{i_2}, \ldots$ (Proposition 8. Corollary).

Now Γ_1 has exponent p^{n_1}, therefore of the numbers of the form $\Gamma_1^{i_1}$, there are

p^{n_1} with exponent a power of $p \ngtr p^{n_1}$,
p^{n_1-1} ,, ,, ,, $\ngtr p^{n_1-1}$,
&c.,

and similarly for each of the others, $\Gamma_2, \Gamma_3. \ldots$

Hence the number of numbers of the form a which have, as exponent, a power of $p \ngtr p^s$, is $p^{(\Sigma n)_s}$ (using the notation of Proposition 23).

Hence the number of numbers, with exponent p^s generated by the numbers Γ, is

$$p^{(\Sigma n)_s} - p^{(\Sigma n)_{s-1}},$$

and, therefore, we must have

$$p^{(\Sigma n)_s} - p^{(\Sigma n)_{s-1}} = p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}},$$

for all values of s from 1 up to the greatest of the values of the numbers l.

Hence we get

$$\begin{aligned}(\Sigma n)_1 &= (\Sigma l)_1,\\ (\Sigma n)_2 &= (\Sigma l)_2,\\ (\Sigma n)_3 &= (\Sigma l)_3,\\ &\text{\&c.}\end{aligned}$$

The first of these equations shows that the number of numbers l is the same as that of the numbers n.

The second then shows that, of each set, the number of numbers which exceed 1 is the same.

The third then shows that, of each set, the number of numbers which exceed 2 is the same.

And so on.

Hence the two sets of numbers l_1, l_2, . . . and n_1, n_2, . . . *are identical* (in some order) *term for term.*

We have, therefore, shown that the most general set of p-power-exponent generators must have as exponents the powers of p, p^{l_1}, p^{l_2}, . . . , which occur, one each, as a principal factor of $\phi(P_1^{\lambda_1})$, $\phi(P_2^{\lambda_2})$ . . .

When $p = 2$ we have the special case of the 2-power-exponent generators.

First suppose $\kappa' \geqq 1$, then

$$2^{(\Sigma\kappa)_\sigma + 1} - 2^{(\Sigma\kappa)_{\sigma-1} + 1} = 2^{(\Sigma n)_\sigma} - 2^{(\Sigma n)_{\sigma-1}},$$

and so

$$\begin{aligned}(\Sigma\kappa)_1 + 1 &= (\Sigma n)_1,\\ (\Sigma\kappa)_2 + 1 &= (\Sigma n)_2,\\ &\text{\&c.}\end{aligned}$$

Hence the set of numbers n are identical with the set κ *together with unity:* and so the exponents of the 2-power-exponent generators are

$$2.\ 2^{\kappa'}.\ 2^{\kappa_1}.\ 2^{\kappa_2}.\ .\ .\ .$$

Secondly, suppose $\kappa = 2$, then $\kappa' = 0$, and the exponents of the generators are

$$2.\ 2^{\kappa_1}.\ 2^{\kappa_2}.\ .\ .\ .$$

Lastly, when $\kappa = 0$ or 1, we have

$$2^{(\Sigma\kappa)_\sigma} - 2^{(\Sigma\kappa)_\sigma - 1} = 2^{(\Sigma n)_\sigma} - 2^{(\Sigma n)_\sigma - 1},$$

and therefore

$$\begin{aligned} (\Sigma\kappa)_1 &= (\Sigma n)_1, \\ (\Sigma\kappa)_2 &= (\Sigma n)_2, \\ &\text{\&c.}, \end{aligned}$$

and so the numbers n are identical with the numbers κ.

Hence the exponents of the 2-power-exponent generators are

$$2^{\kappa_1},\ 2^{\kappa_2},\ 2^{\kappa_3} \ldots$$

We can now see the least possible number of numbers G that can generate the complete set of $\phi(m)$ numbers.

Since each number G contains not more than one generator from each of the subsidiary sets of generators as a factor, it follows that there cannot be less generators G than the number of generators in that subsidiary set which contains most.

Now, since $\phi(\mathrm{P}_1{}^{\lambda_1}) = \mathrm{P}^{\lambda_1 - 1}(\mathrm{P}_1 - 1)$, and P_1 is odd, therefore $\mathrm{P}_1 - 1$ is even. Hence $\kappa_1 = 1$ at least. Therefore of the generators of the 2-power-exponent numbers there are at least as many as there are odd principal factors $\mathrm{P}_1{}^{\lambda_1}, \mathrm{P}_2{}^{\lambda_2}, \ldots$, in m: and so the number of 2-power-exponent generators is never less than the number of generators in any other subsidiary set.

Therefore when

$$\left.\begin{aligned} m = \mathrm{P}_1{}^{\lambda_1}, \ldots \\ \text{or} \qquad 2.\mathrm{P}_1{}^{\lambda_1}, \ldots \end{aligned}\right\}$$

the least number of generators G is the number of primes P_1, when

$$m = 2^2.\mathrm{P}_1{}^{\lambda_1}, \ldots$$

the least number of generators G is the number of primes P_1, $+ 1$, and when

$$m = 2^{\kappa' + 2}\mathrm{P}_1{}^{\lambda_1}, \ldots$$

the least number of generators G is the number of primes P_1, $+ 2$.

(27.) We shall next form a set of p-power-exponent generators of a particular kind, similar to the complete generators of Proposition (24).

Let a be any p-power-exponent number

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m}.$$

Then we must have

$\alpha_0 \equiv 1 \pmod{2^\kappa}$
α_1 with exp a power of p (or unity) mod $P_1^{\lambda_1}$
α_2 ,, ,, p ( ) mod $P_2^{\lambda_2}$
&c.

Let

γ_1 be a number with exp p^{l_1} mod $P_1^{\lambda_1}$
γ_2 ,, ,, p^{l_2} mod $P_2^{\lambda_2}$,
&c.,

(if in any case p is not a factor of $\phi(P^\lambda)$ γ is $\equiv 1$).

Then we may put

$\alpha_1 \equiv \gamma_1^{i_1} \pmod{P_1^{\lambda_1}}$
$\alpha_2 \equiv \gamma_2^{i_2} \pmod{P_2^{\lambda_2}}$,
&c.,

and therefore

$$\begin{aligned} a &\equiv \xi_0 + \gamma_1^{i_1}\xi_1 + \gamma_2^{i_2}\xi_2 + \ldots \pmod{m} \\ a &\equiv (\xi_0 + \xi_1 + \ldots)(\xi_0 + \gamma_1\xi_1 + \ldots)^{i_1}(\xi_0 + \xi_1 + \gamma_2\xi_2 + \ldots)^{i_2} \ldots \pmod{m} \\ &\equiv g_1^{i_1} g_2^{i_2} \ldots \pmod{m}. \end{aligned}$$

The first factor and each of the following in which the γ is $= 1$ is $\equiv 1 \pmod{m}$.

The number of factors remaining is the number of the principal factors P^λ which have a power of p as a factor of $\phi(P^\lambda)$.

These factors generate the p-power-exponent numbers: corresponding to each set of indices i is one of the numbers a and *vice versâ*.

These generators are only a very special kind of p-power-exponent generators inasmuch as each is congruent to unity for each but one of the principal factors of m as modulus. They will be called unitary generators and will be useful for the discussion of the more general type.

In the case when $p = 2$, let

f be $\equiv 2^\kappa - 1$ or $2^{\kappa-1} - 1 \pmod{2^\kappa}$
γ_0 have exp $2^{\kappa-2} \pmod{2^\kappa}$

(if $\kappa = 2$, f does not occur; if $\kappa = 0$ or 1, neither g nor f),

γ_1 have exp $2^{\kappa_1} \pmod{P_1^{\lambda_1}}$
γ_2 ,, $2^{\kappa_2} \pmod{P_2^{\lambda_2}}$,
&c.,

and thus

$$a \equiv (f\xi_0 + \xi_1 + \ldots)^{j}(\gamma_0\xi_0 + \xi_1 + \ldots)^{i_0}(\xi_0 + \gamma_1\xi_1 + \ldots)^{i_1}\ldots (\text{mod } m),$$

and all the factors in this product (omitting those whose γ is $= 1$) form a complete set of 2-power-exponent generators.

Thus, in either case, when p is an odd prime or is equal to 2 we can form a set of p-power-exponent generators (having the exponents found to be necessary in Proposition 26), such that each is congruent to unity for all but one of the principal factors of m.

Example.—Let

$$m = 308 = 2^2.\ 7.\ 11.$$

$$\phi(m) = (2).\ (2.\ 3)\ (2.\ 5).$$

The highest exponent is 30.

There are

and $$\left.\begin{array}{l} 7 \text{ numbers with exp } 2 \\ 1 \qquad ,, \qquad ,, \qquad 1 \end{array}\right\}\text{ (Example Proposition 26.)}$$

We will form unitary generators of these eight numbers. Since 2 enters only in the *square* into m, f is not needed.

We must take

$$\begin{aligned} \gamma_0 &= 3 \text{ (with exp 2 mod } 2^2), \\ \gamma_1 &= 6 \text{ (with exp 2 mod 7)}, \\ \gamma_2 &= 10 \text{ (with exp 2 mod 11)}. \end{aligned}$$

$$\xi_0 = 77, \qquad \xi_1 = 176, \qquad \xi_2 = 56.$$

Thus

$$\left.\begin{aligned} g_0 &\equiv (3 - 1)\ 77 + 1 = 155 \text{ (mod 308)} \\ g_1 &\equiv (6 - 1)\ 176 + 1 = 881 \equiv 265 \text{ (mod 308)} \\ g_2 &= (10 - 1)\ 56 + 1 = 505 \equiv 197 \text{ (mod 308)} \end{aligned}\right\}.$$

are a (in this case *the*) set of 2-power-exponent unitary generators.

The numbers with exponent 2 are given as products of powers of these by the following table of indices.

Numbers.	197.	265.	155.	111.	43.	153.	307.
Index of 155 . . .	0	0	1	1	1	0	1
Index of 265 . . .	0	1	0	1	0	1	1
Index of 197 . . .	1	0	0	0	1	1	1

(28.) In what follows we shall need the following lemma.

Consider

$$\begin{aligned} a_1x + b_1y + c_1z + \ldots &\equiv k_1 \pmod{p^a} \\ a_2x + b_2y + c_2z + \ldots &\equiv k_2 \pmod{p^a} \\ a_3x + b_3y + c_3z + \ldots &\equiv k_3 \pmod{p^a} \\ \ldots\ldots\ldots\ldots\ldots & \end{aligned}$$

as many congruences as unknown quantities.

What is necessary in order that the congruences may have one solution, for any assigned set of values of the k's, and *one only?*

Multiply the equations in order by the minors of the elements of the 1st column of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \text{ and add.}$$

We get

$$x \begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \equiv \begin{vmatrix} k_1 & b_1 & c_1 & . \\ k_2 & b_2 & c_2 & . \\ k_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \pmod{p^a}.$$

In order that this may have one solution, and one only, the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix}$$

must be prime to p^a, and, therefore, prime to p.

This being so, then to each set of values of the k's corresponds a single set of values of $x, y, z. \ldots$

(29.) We have now to find the most general type of a set of p-power-exponent generators. (The work is in no respect different in the case when $p = 2$.)

Let the exponents necessary for a set of generators be $p^{l_1}, p^{l_2}, \ldots p^{l_\mu}$ as determined by Proposition (26).

Let

$$\left.\begin{array}{lll} g_1 \text{ with exp.} & & p^{l_1} \\ g_2 \quad ,, & ,, & p^{l_2} \\ \vdots & & \\ g_\mu \quad ,, & ,, & p^{l_\mu} \end{array}\right\} \text{be a set of } \textit{unitary} \text{ generators. (Proposition 27.)}$$

Suppose that $\Gamma_1, \Gamma_2, \ldots \Gamma_\mu$ are the most general set of p-power-exponent generators. We know already (Proposition 27) that their exponents are $p^{l_1}, p^{l_2}, \ldots$ and we may suppose them to be so in this order.

Since each number Γ is itself a p-power-exponent number, each is expressible as a product of powers of the g's.

Let

$$\left.\begin{array}{l} \Gamma_1 \equiv g_1^{i_{11}} g_2^{i_{21}} g_3^{i_{31}} \ldots g_\mu^{i_{\mu 1}} \pmod{m} \\ \Gamma_2 \equiv g_1^{i_{12}} g_2^{i_{22}} g_3^{i_{32}} \ldots g_\mu^{i_{\mu 2}} \pmod{m} \\ \vdots \\ \Gamma_\mu \equiv g_1^{i_{1\mu}} g_2^{i_{2\mu}} g_3^{i_{3\mu}} \ldots g_\mu^{i_{\mu\mu}} \pmod{m} \end{array}\right\}$$

(*Note.*—$i_{r,\,s,}$ is the index of g_r in the value of Γ_s.)

Since the exponent of Γ_1 is p^{l_1}, it follows that p^{l_1} is the least multiplier that makes

$$i_{21} p^{l_1} \equiv 0 \pmod{p^{l_2}},$$
$$i_{31} p^{l_1} \equiv 0 \pmod{p^{l_2}},$$
$$\&c.,$$

(Proposition 8, Corollary), and similarly for p^{l_2}, &c.

We may insert here a lemma, which will be useful presently.

Lemma.—If $l_r > l_s$, then i_{rs} is divisible by p.

$$i_{rs} \text{ is the index of } g_r \text{ in the product } \Gamma_s,$$
$$\Gamma_s \equiv \ldots g_r^{i_{rs}} \ldots \pmod{m}.$$

Hence (Proposition 8, Corollary) the exponent of Γ_s is $\nless$ exponent of $g_r^{i_{rs}}$.

Suppose, if possible, that i_{rs} can be prime to p.

Then the exponent of $g_r^{i_{rs}}$ is p^{l_r} (Proposition 4).

Now the exponent of Γ_s is p^{l_s},

therefore

$$p^{l_s} \nless p^{l_r},$$

therefore

$$l_s \nless l_r,$$

which is contrary to the supposition $l_s < l_r$, and therefore i_{rs} cannot be prime to p.

Hence, if $l_r > l_s$, then i_{rs} is divisible by p.

Take any one whatever of the p-power-exponent numbers, $g_1^{I_1} g_2^{I_2} \ldots g_\mu^{I_\mu}$ (modulus m). Then, since the numbers Γ are also generators of these numbers (the p-power-exponent numbers), the numbers $g_1^{I_1} g_2^{I_2} \ldots g_\mu^{I_\mu}$ (modulus m) must be expressible in the form $\Gamma_1^{x_1} \Gamma_2^{x_2} \ldots \Gamma_\mu^{x_\mu}$ (modulus m) in one way, and one way only.

Therefore

$$g_1^{I_1} g_2^{I_2} \ldots g_\mu^{I_\mu} \equiv \Gamma_1^{x_1} \Gamma_2^{x_2} \ldots \Gamma_\mu^{x_\mu} \pmod{m}$$

must lead to

$$\begin{aligned} &\text{one value of } x_1 \pmod{p^{l_1}}, \\ &\quad ,, \qquad ,, \quad x_2 \pmod{p^{l_2}}, \\ &\&\text{c.} \end{aligned}$$

Substituting the values of the Γ's in terms of the g's,

$$(g_1^{i_{11}} g_2^{i_{21}} \ldots g_\mu^{i_{\mu 1}})^{x_1} (g_1^{i_{12}} g_2^{i_{22}} \ldots g_\mu^{i_{\mu 2}})^{x_2} \ldots (g_1^{i_{1\mu}} g_2^{i_{2\mu}} \ldots g_\mu^{i_{\mu\mu}})^{x_\mu} \equiv g_1^{\mathrm{I}_1} g_2^{\mathrm{I}_2} \ldots g_\mu^{\mathrm{I}_\mu} \pmod{m},$$

$$g_1^{i_{11}x_1 + i_{12}x_2 + \ldots + i_{1\mu}x_\mu} \,.\, g_2^{i_{21}x_1 + i_{22}x_2 + \ldots i_{2\mu}x_\mu} \ldots g_\mu^{i_{\mu 1}x_1 + i_{\mu 2}x_2 + \ldots i_{\mu\mu}x_\mu} \equiv g_1^{\mathrm{I}_1} g_2^{\mathrm{I}_2} \ldots g_\mu^{\mathrm{I}_\mu} \pmod{m}.$$

Therefore (Proposition 7) it follows that

$$\begin{aligned} i_{11}x_1 + i_{12}x_2 + \ldots + i_{1\mu}x_\mu &\equiv \mathrm{I}_1 \pmod{p^{l_1}}, \\ i_{21}x_1 + i_{22}x_2 + \ldots + i_{2\mu}x_\mu &\equiv \mathrm{I}_2 \pmod{p^{l_2}}, \\ &\vdots \\ i_{\mu 1}x_1 + i_{\mu 2}x_2 + \ldots + i_{\mu\mu}x_\mu &\equiv \mathrm{I}_\mu \pmod{p^{l_\mu}}, \end{aligned}$$

and we need that these shall have *one* set of values of x_1 (modulus p^{l_1}), x_2 (modulus p^{l_2}), &c., . . . and only one.

This being so the numbers Γ will generate completely the p-power-exponent numbers.

We may suppose that the moduli $p^{l_1}, p^{l_2} \ldots$ are in ascending order of magnitude, so that $l_s \overset{=}{<} l_{s+1}$.

Suppose that

$$l_1 = l_2 = l_3 = \ldots = l_a < l_{a+1} = l_{a+2} = \ldots = l_b < l_{b+1} = l_{b+2} = \ldots = l_c < \&\text{c.} \ldots = l_\mu.$$

Any one of the unknowns x_s is to be found with regard to mod p^{l_s}.

We can write x_s in the form

$$x_s \equiv \xi_s + \xi_{sa} p^{l_a} + \xi_{sb} p^{l_b} + \xi_{sc} p^{l_c} + \ldots \pmod{p^{l_s}},$$

where

$$\left.\begin{aligned} \xi_s &< p^{l_a} \\ \xi_{sa} &< p^{l_b - l_a} \\ \xi_{sb} &< p^{l_c - l_b} \end{aligned}\right\}$$
$$\&\text{c.}$$

This substitution gives in particular,

$$\begin{aligned} x_1 &\equiv \xi_1 \pmod{p^{l_a}}. \\ x_2 &\equiv \xi_2 \pmod{p^{l_a}}. \\ &\vdots \\ x_a &\equiv \xi_a \pmod{p^{l_a}}. \end{aligned}$$

$$
\begin{aligned}
x_{a+1} &\equiv \xi_{a+1} + \xi_{a+1.a}\, p^{l_a} \pmod{p^{l_b}}.\\
x_{a+2} &\equiv \xi_{a+2} + \xi_{a+2.a}\, p^{l_a} \pmod{p^{l_b}}.\\
&\vdots\\
x_b &\equiv \xi_b + \xi_{ba}\, p^{l_a} \pmod{p^{l_b}}.\\[1em]
x_{b+1} &\equiv \xi_{b+1} + \xi_{b+1.a}\, p^{l_a} + \xi_{b+1.b}\, p^{l_b} \pmod{p^{l_c}}.\\
x_{b+2} &\equiv \xi_{b+2} + \xi_{b+2.a}\, p^{l_a} + \xi_{b+2.b}\, p^{l_b} \pmod{p^{l_c}}.\\
&\vdots\\
x_c &\equiv \xi_c + \xi_{ca}\, p^{l_a} + \xi_{cb}\, p^{l_b} \pmod{p^{l_c}}.\\
&\qquad \&c. \qquad\qquad \&c.
\end{aligned}
$$

First replace each modulus which exceeds p^{l_a} by p^{l_a}, and substitute the assumed values of the x's.

We get the μ congruences

$$
\begin{aligned}
i_{11}\xi_1 + i_{12}\xi_2 + \ldots + i_{1\mu}\xi_\mu &\equiv \mathrm{I}_1 \pmod{p^{l_a}},\\
i_{21}\xi_1 + i_{22}\xi_2 + \ldots + i_{2\mu}\xi_\mu &\equiv \mathrm{I}_2 \pmod{p^{l_a}},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\
i_{\mu 1}\xi_1 + i_{\mu 2}\xi_2 + \ldots + i_{\mu\mu}\xi_\mu &\equiv \mathrm{I}_\mu \pmod{p^{l_a}},
\end{aligned}
$$

and we need that these shall determine a single set of values for $\xi_1, \xi_2, \ldots \xi_\mu$ (mod p^{l_a}).

The necessary and sufficient condition is that the determinant

$$(i_{11}, i_{22}, i_{33}, \ldots i_{\mu\mu})$$

should be prime to p (Proposition 28).

Suppose that this is so, and that the numbers $\xi_1, \xi_2, \ldots \xi_\mu$ are determined.

When these values are substituted above, suppose that the values of the left-hand sides are $\mathrm{I}_1 - p^{l_a}\mathrm{I}'_1$, $\mathrm{I}_2 - p^{l_a}\mathrm{I}'_2$, &c. . . . (where the negative signs are written for convenience in what follows, and $\mathrm{I}'_1, \mathrm{I}'_2, \ldots$ are therefore negative).

Next replace each modulus which exceeds p^{l_b} by p^{l_b}, and substitute the assumed values of the x's.

The first a congruences are already satisfied.

The rest give

$$
\begin{aligned}
\mathrm{I}_{a+1} - p^{l_a}\mathrm{I}'_{a+1} + p^{l_a}\,(i_{a+1.a+1}\,\xi_{a+1.a} + i_{a+1.a+2}\,\xi_{a+2.a} + \ldots + i_{a+1.\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{a+1} \pmod{p^{l_b}}.\\
\mathrm{I}_{a+2} - p^{l_a}\mathrm{I}'_{a+2} + p^{l_a}\,(i_{a+2.a+1}\,\xi_{a+1.a} + i_{a+2.a+2}\,\xi_{a+2.a} + \ldots + i_{a+2.\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{a+2} \pmod{p^{l_b}}.\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\
\mathrm{I}_{\mu} - p^{l_a}\mathrm{I}'_{\mu} + p^{l_a}\,(i_{\mu.a+1}\,\xi_{a+1.a} + i_{\mu.a+2}\,\xi_{a+2.a} + \ldots\ i_{\mu\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{\mu} \pmod{p^{l_b}}.
\end{aligned}
$$

Therefore

$$\begin{array}{l} i_{a+1.a+1}\,\xi_{a+1.a} + i_{a+1.a+2}\,\xi_{a+2.a} + \ldots + i_{a+1.\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{a+1} \pmod{p^{l_b-l_a}} \\ i_{a+2.a+1}\cdot\xi_{a+1.a} + i_{a+2.a+2}\,\xi_{a+2.a} + \ldots + i_{a+2.\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{a+2} \pmod{p^{l_b-l_a}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ i_{\mu.a+1}\,\xi_{a+1.a} + i_{\mu.a+2}\cdot\xi_{a+2.a} + \ldots \quad i_{\mu\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{\mu} \pmod{p^{l_b-l_a}}, \end{array}$$

and we need that these shall determine a single set of values of

$$\xi_{a+1.a},\ \xi_{a+2.a},\ \ldots\ \xi_{\mu a} \pmod{p^{l_b-l_a}}.$$

The necessary and sufficient condition is that the determinant

$$(i_{a+1,\,a+1},\ i_{a+2,\,a+2},\ \ldots\ i_{\mu\mu})$$

should be prime to p.

Suppose that this is so, and the numbers $\xi_{a+1,\,a}$, $\xi_{a+2.a}$, $\ldots$ $\xi_{\mu a}$ are determined: and suppose that when substituted above the values of the left-hand sides become $\mathrm{I}'_{a+1} - p^{l_b-l_a}\mathrm{I}''_{a+1}$, &c.

Next replace each modulus that exceeds p^{l_c} by p^{l_c}, and substitute again the values of x.

The first b congruences are now satisfied.

The rest give

$$\begin{array}{l} (\mathrm{I}_{b+1} - \mathrm{I}'_{b+1}p^{l_a}) + (\mathrm{I}'_{b+1} - \mathrm{I}''_{b+1}p^{l_b-l_a})\,p^{l_a} + p^{l_b}\,(i_{b+1.b+1}\,\xi_{b+1.b} + \ldots\ i_{b+1.\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{b+1} \pmod{p^{l_c}} \\ (\mathrm{I}_{b+2} - \mathrm{I}'_{b+2}p^{l_a}) + (\mathrm{I}'_{b+2} - \mathrm{I}''_{b+2}p^{l_b-l_a})\,p^{l_a} + p^{l_b}\,(i_{b+2.b+1}\,\xi_{b+1.b} + \ldots\ i_{b+2.\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{b+2} \pmod{p^{l_c}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\mathrm{I}_{\mu} - \mathrm{I}'_{\mu}p^{l_a}) \quad + (\mathrm{I}'_{\mu} - \mathrm{I}''_{\mu}p^{l_b-l_a}) \quad + p^{l_b}(i_{\mu.b+1}\,\xi_{b+1.b} \quad + \ldots\ i_{\mu\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{\mu} \pmod{p^{l_c}}. \end{array}$$

Therefore

$$\begin{array}{l} i_{b+1.b+1}\,\xi_{b+1.b} + i_{b+1.b+2}\,\xi_{b+2.b} + \ldots + i_{b+1.\mu}\,\xi_{\mu b} \equiv \mathrm{I}''_{b+1} \pmod{p^{l_c-l_b}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ i_{\mu.b+1}\,\xi_{b+1.b} + i_{\mu.b+2}\,\xi_{b+2.b} + \ldots\ i_{\mu\mu}\,\xi_{\mu b} \qquad \equiv \mathrm{I}''_{\mu} \pmod{p^{l_c-l_b}}, \end{array}$$

and we need that these shall determine a single set of values of

$$\xi_{b+1.b},\ \xi_{b+2,\,b},\ \ldots\ \xi_{\mu b} \pmod{p^{l_c-l_b}}.$$

The necessary and sufficient condition for this is that the determinant

$$(i_{b+1.b+1},\ i_{b+2,\,b+2},\ \ldots\ i_{\mu\mu})$$

should be prime to p.

These successive steps are to be continued until all the quantities ξ are determined. They, and therefore the numbers x will be determined uniquely, provided that the conditions that

$$\left.\begin{array}{l} (\dot{i}_{11} \ldots \dot{i}_{\mu\mu}) \\ (\dot{i}_{a+1}\, \dot{i}_{a+1}, \ldots \dot{i}_{\mu\mu}) \\ (\dot{i}_{b+1,\, b+1}, \ldots \dot{i}_{\mu\mu}) \\ \quad \text{\&c.,} \end{array}\right\} \text{ shall all be prime to } p \text{, are satisfied.}$$

When these conditions are satisfied the generators Γ generate the complete set of p-power-exponent numbers.

(30.) We have not yet seen whether the conditions just found are independent or not. We shall find that the first condition $(\dot{i}_{11}, \dot{i}_{22}, \ldots \dot{i}_{\mu\mu})$ includes all the others; it may, however, be replaced by others of a similar kind, but practically simpler.

Let us write down the complete determinant and divide it into squares and rectangles thus:—

$$\begin{array}{cccc|cc|c}
\dot{i}_{11} & \dot{i}_{12} & \ldots\ \dot{i}_{1a-1} & \dot{i}_{1a} & \dot{i}_{1a+1} & \dot{i}_{1b} & \\
\dot{i}_{21} & \dot{i}_{22} & \ldots\ \dot{i}_{2.\,a-1} & \dot{i}_{2a} & \vdots & & \\
\vdots & & & & \vdots & & \text{\&c.} \\
\dot{i}_{a-1.\,1} & \dot{i}_{a-1.\,2} & \ldots\ \dot{i}_{a-1.\,a-1} & \dot{i}_{a-1.\,a} & \vdots & & \\
\dot{i}_{a1} & \dot{i}_{a2} & \ldots\ \dot{i}_{a.\,a-1} & \dot{i}_{aa} & \dot{i}_{aa+1} & \dot{i}_{ab} & \\
\hline
\dot{i}_{a+1.\,1} & & \ldots & \dot{i}_{a+1.\,a} & \dot{i}_{a+1.\,a+1} & \dot{i}_{a+1.\,b} & \\
\vdots & & & & \vdots & & \\
\dot{i}_{b1} & & \ldots & \dot{i}_{ba} & \dot{i}_{ba+1} & \dot{i}_{bb} & \\
\hline
& & \text{\&c.} & & & &
\end{array}$$

For reference, we may name these squares and rectangles, thus:—

$$\begin{array}{c|c|c}
(aa) & (ab) & (ac) \\
\hline
(ba) & (bb) & (bc) \quad \text{\&c.} \\
\hline
(ca) & (cb) & (cc)
\end{array}$$

$$\text{\&c.}$$

The lemma proved in the course of the last proposition (that if $l_r > l_s$, then i_{rs} is divisible by p), shows that every element i, which is found to the left and below the determinants (aa), (bb), (cc) . . . is divisible by p, *i.e.*, every element in

(ba)			
(ca)	(cb)		
(da)	(db)	(dc)	

is divisible by p. &c.,

The complete determinant $(i_{11}, \dots i_{\mu\mu})$ can be expressed as the sum of products of pairs of complementary determinants, one of these in each such product being the determinant formed by any a rows taken from (aa), (ba), (ca) . . .

Now, any row taken from (ba), (ca), . . . has every element divisible by p. Hence, the only determinant formed by a rows taken from (aa), (ba), (ca), . . . which is not immediately divisible by p, is the determinant $(aa) = (i_{11}, \dots i_{aa})$.

Hence

$$(i_{11}, \dots i_{\mu\mu}) \equiv (i_{11}, \dots i_{aa})\,(i_{a+1.\,a+1} \dots i_{\mu\mu}) \pmod p.$$

Therefore, if $(i_{11} \dots i_{\mu\mu})$ is prime to p, then $(i_{11}, \dots i_{aa})$ and $(i_{a+1.\,a+1}, \dots i_{\mu\mu})$ are both prime to p (and *vice versâ*).

Again, take the determinant $(i_{a+1.\,a+1}, \dots i_{\mu\mu})$,

(bb)	(bc)	
(cb)	(cc)	&c.

&c.

This can be expressed as a sum of products of determinants in the same way: one determinant in each product being formed by $(b - a)$ rows taken from (bb), (cb) . . . Of these, only (bb) is not immediately divisible by p.

Therefore

$$(i_{a+1.\,a+1}, \dots i_{\mu\mu}) \equiv (i_{a+1.\,a+1}, \dots i_{bb})\,(i_{b+1.\,b+1}, \dots i_{\mu\mu}) \pmod{p},$$

and, since

$$(i_{a+1.\,a+1}, \dots i_{\mu\mu}) \text{ is prime to } p,$$

therefore

$$(i_{a+1.\,a+1}, \dots i_{bb}) \text{ and } (i_{b+1.\,b+1}, \dots i_{\mu\mu}) \text{ are both prime to } p,$$

and so on.

Hence, as stated above, the single condition $(i_{11}, \dots i_{\mu\mu})$ prime to p, includes all the other conditions found necessary in Proposition (29), and is therefore a necessary and sufficient condition that the numbers Γ (having the proper set of exponents) may completely generate the p-power-exponent numbers.

Also, this single condition may be replaced (with much advantage practically) by the equivalent set of conditions—

$$\left.\begin{array}{c} (i_{11}, \dots i_{aa}) \\ (i_{a+1}, i_{a+1}, \dots i_{bb}) \\ (i_{b+1}, i_{b+1}, \dots i_{cc}) \\ \&\text{c.} \end{array}\right\} \text{all prime to } p.$$

(31.) We have now a complete series of methods for ascertaining whether a given set of numbers G can act as generators of the complete system of $\phi(m)$ numbers (modulus m).

(1.) Find the exponent to which each belongs. (Proposition 17.)

(2.) The principal factors of these exponents must be the numbers found in Proposition (26).

(3.) Express each number G as a product of numbers with principal factors of exponent of G as exponents. (Proposition 10.)

(4.) Express each of these last as a product of powers of unitary generators, and thus get for each a set of indices.

(5.) Apply the series of conditions of Proposition (30).

If (2) and (5) are both satisfied then the numbers G are complete generators.

Conversely to form any set of generators—

(1.) For each prime p in $\phi(m)$ form a set of unitary generators.

(2.) Form from these any set of p-power-exponent generators satisfying the conditions of Proposition 30.

(3.) Multiply together one generator from each of the sets just formed.

Examples.

Let us form a set of generators for mod $308 = 2^2.\ 7.\ 11.$

$$m = 2^2.\ 7.\ 11.$$

$$\phi(7) = 2.\ 3.$$

$$\phi(11) = 2.\ 5.$$

The principal factors of the exponents of the generators are

2. 2. 2. 3. 5.

We form first the numbers with exponent 2 to generate the 2-exponent numbers.

These numbers expressed in terms of the unitary generators, 155, 265, 197, are given by

No.	Ind. of 155.	Ind. of 265.	Ind. of 197.
197.	0.	0.	1.
265.	0.	1.	0.
155.	1.	0.	0.
111.	1.	1.	0.
43.	1.	0.	1.
153.	0.	1.	1.
307.	1.	1.	1.

Of these (by Proposition 30) we can take any three such that the determinant formed by their indices is prime to 2, *i.e.*, odd.

For example,

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1,$$

and so 43, 153, 307 may be taken as generators.

Secondly we need a number with exponent 3.

Of these there are two, 177 and 221. Let us take 177.

Lastly we need a number with exponent 5.

Of these there are four, 113, 141, 169, 225. Let us take 113.

We have now 43, 153, 307, independent generators with exponent 2,

177 with exp 3,
113 ,, ,, 5.

These we may combine in any manner we please as products, taking only one from each set in each product.

Let us take 43. 177. 113 $\equiv$ 107 (mod 308) with exponent 30.

We have thus obtained three generators

$$\left.\begin{array}{lll} 107 & \text{with exp} & 30 \\ 153 & ,, \quad ,, & 2 \\ 307 & ,, \quad ,, & 2 \end{array}\right\}$$

which generate the complete set of $\phi(308)$ numbers.

Conversely, we can verify that 107. 153. 307 are complete generators.

First we find that

$$\left.\begin{array}{llll} 107 & \text{has} & \text{exp} & 30 \\ 153 & ,, & ,, & 2 \\ 307 & ,, & ,, & 2 \end{array}\right\}$$

and so the exponents have the necessary principal factors.

$$\begin{aligned} 107 &\equiv 107^{15} . \, 107^{10} . \, 107^{6} \pmod{308}, \\ &\equiv 43 . \, 177 . \, 113 \pmod{308}, \end{aligned}$$

where

$$\left.\begin{array}{llll} 43 & \text{has} & \text{exp} & 2 \\ 177 & ,, & ,, & 3 \\ 113 & ,, & ,, & 5 \end{array}\right\}$$

We only need to show that 43, 153, 307 are independent; and since the indices of these when referred to the unitary generators are

$$\begin{array}{rccc} 43, & 1 & 0 & 1 \\ 153, & 0 & 1 & 1 \\ 307, & 1 & 1 & 1 \end{array}$$

and the determinant is prime to 2, we see that the generators 43. 153. 307 are independent.

Let us form generators for modulus $999 = 3^3 . \, 37$.

$$\begin{aligned} m &= 3^3 . \, 37, \\ \phi(m) &= (3^2 . \, 2)(3^2 . \, 2^2). \end{aligned}$$

The principal factors of the exponents of the generators are

$$2 . \, 2^2 . \, 3^2 . \, 3^2 .$$

If

$$\begin{aligned} a &\equiv \alpha_1 \pmod{3^3}, \\ &\equiv \alpha_2 \pmod{37}, \\ a &\equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 \pmod{3^3 . \, 37}. \end{aligned}$$

$$\begin{aligned} 37 x_1 &\equiv 1 \pmod{3^3} \\ 10 x_1 &\equiv 1 \pmod{27} \\ x_1 &\equiv 19 \pmod{3^3} \\ \xi_1 &\equiv 37 . \, 19 \equiv 703 \pmod{999} \end{aligned}$$

$$\begin{aligned} 27 x_2 &\equiv 1 \pmod{37} \\ x_2 &\equiv 11 \pmod{37} \\ \xi_2 &\equiv 297 \pmod{999} \end{aligned}$$

and, therefore,

$$a \equiv 703 \alpha_1 + 297 \alpha_2 \pmod{999}.$$

We want first any number with exponent 2.

Suppose we take $\alpha_1 = 26$ and $\alpha_2 = 36$ each with exponent 2 (for moduli 3^3 and 37 respectively).

Then we get

$$\begin{aligned} \alpha &\equiv 703.\ 26 + 297.\ 36 \pmod{999} \\ &\equiv 998 \pmod{999} \text{ with exp } 2. \end{aligned}$$

We want next a number with exponent 4.

Suppose we take

$$\alpha_1 = 1$$

and

$$\alpha_2 = 6 \text{ (with exp 4, mod 37)}.$$

Then we get

$$\begin{aligned} a &\equiv 703 + 297.\ 6 \pmod{999} \\ &\equiv 487 \pmod{999} \text{ with exp } 4. \end{aligned}$$

Lastly, we want two independent generators with exponent 3^2.

We first form unitary generators with exponents 3^2

$$\alpha_1 = 4 \quad \alpha_2 = 1 \text{ gives } 112 \text{ with exp } 3^2,$$

and

$$\alpha_1 = 1 \quad \alpha_2 = 12 \text{ gives } 271 \text{ with exp } 3^2.$$

We may take for our 3-power-exponent generators any two numbers of the form $112^{i_1}\ 271^{i_2}$, $112^{i'_1}\ 271^{i'_2}$, provided that

$$\begin{vmatrix} i_1 & i_2 \\ i'_1 & i'_2 \end{vmatrix} \text{ is prime to } 3.$$

Suppose we take it

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

and then the generators are

$$112.\ 271 \equiv 382 \pmod{999}$$

and

$$112^2.\ 271 \equiv 826 \pmod{999}.$$

Thus we have

998 with exp 2,
487 with exp 4,

and

$$382 \text{ and } 826 \text{ with exp } 3^2.$$

Hence we may form, say,

$$998.\ 826 \equiv 173 \pmod{999} \text{ with exp } 2.\ 3^2.$$

and

$$487.\ 382 \equiv 220 \pmod{999} \text{ with exp } 2^2.\ 3^2.$$

Thus

$$\left.\begin{array}{l} 173 \text{ with exp } 18 \\ 220 \text{ with exp } 36 \end{array}\right\}$$

generate completely the $2^3 . 3^4$ numbers prime to 999.

INDICES AND TABLES OF INDICES.

(32.) Let $G_1\, G_2 \ldots G_\kappa$ be a complete set of independent generators for modulus m. Let their exponents be $t_1, t_2, \ldots t_\kappa$.

Then any number a prime to m is expressible in the form

$$a \equiv G_1^{i_1}\, G_2^{i_2} \ldots G_\kappa^{i_\kappa} \pmod{m},$$

where

$$i_1 < t_1,\ i_2 < t_2, \text{ \&c.}$$

We may thus make a table giving the set of indices $i_1, i_2, \ldots i_\kappa$ which correspond to any number a, and conversely the number a which corresponds to any set of indices.

We may conveniently write

$$a \equiv (i_1, i_2, \ldots i_\kappa),$$

and then if

$$a' \equiv (i'_1, i'_2, \ldots i'_\kappa)$$

we shall have

$$aa' \equiv (i_1 + i'_1, i_2 + i'_2, \ldots i_\kappa + i'_\kappa)$$

and

$$a^s \equiv (i_1 s, i_2 s, \ldots i_\kappa s).$$

With tables of this kind for every modulus we can at once solve any congruence of the form

$$ax^n \equiv b \pmod{m}$$

whatever m may be, if a and b are both prime to m. For suppose that

$$a \equiv (i_1\, i_2 \ldots)$$
$$b \equiv (i'_1\, i'_2 \ldots)$$
$$x \equiv (I_1, I_2, \ldots)$$

Then we must have

$$i_1 + nI_1 \equiv i'_1 \pmod{t_1}$$
$$i_1 + nI_2 \equiv i'_2 \pmod{t_2},$$
&c.

The G.C.M. of n and t_1 must divide $i'_1 - i_1$, &c., in order that the congruences may be soluble.

These being so we find $I_1, I_2, \ldots$ by reference to tables with moduli $t_1, t_2, \ldots$ which are (or may be) composite numbers.

If

ν_1 is the G.C.M. of n and t_1, I_1 will have ν_1 values mod t_1,
ν_2 ,, ,, n and t_2, I_2 ,, ν_2 ,, t_2,
&c.

Hence, by reference again to the table for modulus m, we obtain at once the $\nu_1, \nu_2, \nu_3 \ldots$ numbers which satisfy the congruence $ax^n \equiv b \pmod{m}$.

Let us now solve the same congruence with the help only of tables of indices for powers of primes as moduli.

If

α_0 is a solution of the congruence $ax^n \equiv b \pmod{2^\kappa}$,
α_1 ,, ,, ,, $ax^n \equiv b \pmod{P_1^{\lambda_1}}$,
&c.,

then

$$x \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 \ldots \pmod{m}$$

is a solution of the congruence

$$ax^n \equiv b \pmod{m}.$$

Hence the solution is conducted as follows :—

By means of the tables of indices find

all the values α_0 which satisfy $ax^n \equiv b \pmod{2^\kappa}$,
,, ,, α_1 ,, ,, $\pmod{P_1^{\lambda_1}}$,
&c.

Next find the values of the ξ's, Proposition (19), each being found by the tables.

Lastly the values of x which satisfy $ax^n \equiv b \pmod{m}$ must each be calculated separately by giving to each of $\alpha_0, \alpha_1, \ldots$ one of its possible values.

In the case of a multiple solution the labour involved in this last step may be very great; and consequently the saving of labour effected by the use of tables of indices for composite moduli proportionately so.

As an example of the use of tables of indices for composite moduli let us use the table given in the example to Proposition (24).

Solve

$$9x^5 \equiv 53 \pmod{112}.$$
$$53 \equiv (4.\ 1.\ 1.)$$

(written in this order the indices are referred to moduli 6. 4. 2),

$$9 \equiv (2.\ 2.\ 0),$$

and, therefore,

$$x^5 \equiv (2.\ 3.\ 1).$$

If, then,

$$x \equiv (a.\ b.\ c)$$
$$x^5 \equiv (5a.\ 5b.\ 5c),$$

and, therefore,

$$\begin{aligned} 5a &\equiv 2 \pmod{6}, \text{ and, therefore, } a = 4, \\ 5b &\equiv 3 \pmod{4}, \text{ and, therefore, } b = 3, \\ 5c &\equiv 1 \pmod{2}, \text{ and, therefore, } c = 1, \end{aligned}$$

therefore

$$x \equiv (4.\ 3.\ 1),$$
$$x \equiv 109 \pmod{112}.$$

Solve

$$47x^2 \equiv 55 \pmod{112}.$$
$$55 \equiv (3.\ 0.\ 1),$$
$$47 \equiv (5.\ 2.\ 1),$$

therefore

$$x^2 \equiv (4.\ 2.\ 0),$$

therefore, if

$$x \equiv (a.\ b.\ c)$$

$$\begin{array}{lll} 2a \equiv 4 \pmod{6}, & 2b \equiv 2 \pmod{4}, & 2c \equiv 0 \pmod{2}, \\ a \equiv 2 \pmod{3}, & b \equiv 1 \pmod{2}, & c = 0 \text{ or } 1 \\ a = 2 \text{ or } 5. & b = 1 \text{ or } 3. & \end{array}$$

Hence there are eight solutions

$$\begin{array}{ll} (2\quad 1\quad 0) \equiv 51 & (5\quad 1\quad 0) \equiv 19 \\ (2\quad 1\quad 1) \equiv 37 & (5\quad 1\quad 1) \equiv 5 \\ (2\quad 3\quad 0) \equiv 107 & (5\quad 3\quad 0) \equiv 75 \\ (2\quad 3\quad 1) \equiv 93 & (5\quad 3\quad 1) \equiv 61. \end{array}$$

(33.) The last proposition shows how, by means of tables of indices for composite moduli, to solve (when possible) the congruence

$$ax^n \equiv b \pmod{m},$$

where m is any composite modulus and a and b are both prime to m.

To complete the question of the solution of the congruence for all cases we have now to show how it is solved when a and b are not both prime to m.

I. First, if a is not prime to m, the congruence can be at once reduced to the case in which it is so; for if a and m have G.C.M. κ, then κ must divide b, and we have

$$\left(\frac{a}{\kappa}\right) x^n \equiv \frac{b}{\kappa}\left(\bmod \frac{m}{\kappa}\right), \text{ where now } \frac{a}{\kappa} \text{ is prime to } \frac{m}{\kappa}\cdot$$

Any one solution x of this gives κ solutions of the original congruence $ax^n \equiv b \pmod{m}$, viz.,

$$x,\ x + \frac{m}{\kappa},\ x + \frac{2m}{\kappa}, \ldots x + \overline{\kappa - 1}\,\frac{m}{\kappa}\cdot$$

II. We have now, therefore, only to deal with the congruence of the form $ax^n \equiv b \pmod{m}$, in which a is prime to m and b is not prime to m. Let κ be the G.C.M. of b and m, and express κ as the product of its principal factors $\kappa = p^r p'^{r'} \ldots$

Since κ divides b and m, it divides ax^n.

Now a is prime to m, and therefore to κ; therefore κ divides x^n, *i.e.*, $p^r p'^{r'} \ldots$ divides x^n. Therefore x is divisible by $p^\rho p'^{\rho'} \ldots$ where

$$\begin{gathered} n\rho \geqq r, \\ n\rho' \geqq r', \\ \&\text{c.} \end{gathered}$$

Take ρ to be equal to r/n if r/n is an integer. If r/n is not an integer take ρ equal to the integer next greater.

Put

$$x = \xi p^\rho p'^{\rho'} \ldots,$$

then the congruence

$$a\,\frac{x^n}{\kappa} \equiv \frac{b}{\kappa}\left(\bmod \frac{m}{\kappa}\right) \quad \ldots\ldots\ldots\ldots \quad \text{(i.)}$$

gives

$$a\,(p^{n\rho - r} p'^{n\rho' - r'} \ldots)\,\xi^n \equiv \frac{b}{\kappa}\left(\bmod \frac{m}{\kappa}\right) \quad \ldots\ldots\ldots \quad \text{(ii).}$$

If $n\rho - r \neq 0$ (*i.e.*, if r/n is not an integer), then the coefficient of ξ^n in this congruence is divisible by a power of p, and therefore, if the congruence is possible, m/κ is not (for then b/κ would be also, whereas m/κ and b/κ are co-prime).

Thus

$$p^{n\rho - r} p'^{n\rho' - r'} \ldots \text{ is prime to } m/\kappa,$$

and, therefore, since a and b/κ are also prime to m/κ, the congruence (ii.) is soluble (if possible) by means of the tables giving the indices of numbers prime to the modulus.

Each solution ξ of the congruence (ii.) gives a single solution $x \equiv \xi p^{\rho} p'^{\rho'} \ldots$ (mod m) of the congruence $ax^n \equiv b \pmod{m}$.

Example.—Solve

$$22x^5 \equiv 54 \pmod{672}.$$

The congruence is

$$2.\ 11.\ x^5 \equiv 2.\ 3^3 \pmod{2^5.\ 3.\ 7} \quad \ldots\ldots\ldots\ldots \quad \text{(i.)}$$

Dividing by 2 we have

$$11.\ x^5 \equiv 3^3 \pmod{2^4.\ 3.\ 7} \quad \ldots\ldots\ldots\ldots \quad \text{(ii.)}$$

and any solution x of this gives the two solutions x, $x + 336$ of the congruence (i.).

From (ii.) we get, dividing by 3,

$$11\frac{x^5}{3} \equiv 9 \pmod{2^4.\ 7}.$$

In congruence (ii.) let $x = 3\xi$.

Then

$$11.\ 3^4.\ \xi^5 \equiv 9 \pmod{2^4.\ 7}$$

$$11.\ 3^2.\ \xi^5 \equiv 1 \pmod{2^4.\ 7}.$$

Now (see table, p. 222)

$$11 \equiv (4.\ 3.\ 0),$$

$$9 \equiv (2.\ 2.\ 0),$$

therefore

$$9.\ 11 \equiv (0.\ 1.\ 0),$$

therefore

$$\xi^5 \equiv (0.\ 3.\ 0),$$

therefore

$$\xi \equiv (0.\ 3.\ 0)$$

$$\xi \equiv 43 \pmod{2^4.\ 7}.$$

Hence (ii.) has the solution

$$x \equiv 129 \pmod{336},$$

and therefore the solutions of (i.) are

$$x \equiv 129,\ 465 \pmod{672}.$$

PART II.—ON THE RESIDUES OF POWERS OF NUMBERS FOR ANY MODULUS, COMPOSITE AND COMPLEX.

In order to maintain as far as possible the parallelism of Parts I. and II. a number of facts and relations peculiar to complex numbers are noted (most without proof) in the following Preface to Part II.

PREFACE TO PART II.

Complex primes.—i.e., numbers either real or complex which have no real or complex factors.

These are of two kinds

(i.) Real primes of the form $4k + 3$; *e.g.* 3, 7, 11, 19, . . .

(ii.) The factors of real primes of the form $4k + 1$, which are expressible as the sum of two squares, *e.g.*, $1 + 2i$, $2 + i$, $3 + 2i$, $2 + 3i$ . . . Among the last we include the factors of the real prime 2, viz., $1 + i$.

These two kinds of prime we shall call respectively *pure* and *mixed*, speaking of either as a *complex* prime. By a real prime we shall mean the primes ordinarily so called, real numbers which have no real factors.

To express any number as a product of its prime factors.

Let $a + bi$ be the number. Let d be the G.C.M. of a and b, and suppose that

$$a + bi = d\,(\alpha + \beta i).$$

d is a real number and can be expressed as a product of real primes.

Of these, those that are not complex primes can be separated each into its two factors.

All the factors of $\alpha^2 + \beta^2$ are themselves expressible as the sum of two squares.

Say

$$\alpha^2 + \beta^2 = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)\ldots$$

Then the prime factors of $\alpha + \beta i$ are

$$\alpha_1 \pm i\beta_1,$$
$$\alpha_2 \pm i\beta_2, \ldots$$

where, in each pair, the sign must be so chosen that

$$\alpha + \beta i = (\alpha_1 \pm i\beta_1)(\alpha_2 \pm i\beta_2)\ldots$$

Example.

$$60 + 105i = 15\,(4 + 7i)$$
$$15 = 3.\,5 = 3\,(2 + i)(2 - i)$$
$$4^2 + 7^2 = 65 = 5.\,13 = (2^2 + 1^2)(3^2 + 2^2)$$

and

$$4 + 7i = (2 + i)(3 + 2i)$$

therefore

$$60 + 105i = 3(2 - i)(2 + i)^2(3 + 2i).$$

The number of incongruent residues for any modulus m.

The number of incongruent residues (including zero) is the *norm* of the modulus. Thus if $m = \alpha + \beta i$, the number of incongruent residues modulus m is $\alpha^2 + \beta^2$. When the modulus is a pure number, $m = p$, then the number of incongruent residues is p^2.

Set of numbers all incongruent for modulus m.

Let $m = d(\alpha + \beta i)$ where α and β are co-prime.

Then the $N(m) = d^2(\alpha^2 + \beta^2)$ numbers $x + iy$, where x has any one of the values $0, 1, 2, \ldots d(\alpha^2 + \beta^2) - 1$, and y any one of the values $0, 1, 2, 3, \ldots \overline{d - 1}$, are a complete set of $N(m)$ incongruent residues for mod m.

We have two special cases:—

(i.) If $m = \alpha + \beta i$, the residues are $0, 1, 2, \ldots \alpha^2 + \beta^2 - 1$.

(ii.) If $m = d$, the residues are the numbers $x + iy$, where x and y have each any one of the values $0, 1, 2, \ldots \overline{d - 1}$.

For most purposes the above set of residues are most convenient. It is sometimes, however, a saving of arithmetical labour (in examples) to make use of the "absolutely least residues," *i.e.*, the set of residues whose norms are not greater than half the norm of the modulus. We may also make use of "least positive residues," *i.e.*, the set of numbers whose norms are as small as possible consistently with each number having both its parts positive.

In what follows we shall always use the above set of residues and we shall need an expeditious method for finding to which of them any given number is congruent.

To reduce a number to its residue.

Let $X + iY$ be the number whose residue we wish to find for mod $d(\alpha + \beta i)$, where α and β are co-prime.

We have to find x and y so that

$$X + iY \equiv x + iy \pmod{d\,\overline{\alpha + \beta i}},$$

where

$$\left.\begin{array}{l} x < d(\alpha^2 + \beta^2) \\ y < d \end{array}\right\}\text{and both are positive numbers.}$$

Let

$$X + iY = (x + iy) + (\xi + i\eta)(\alpha + \beta i)\,d.$$

Then

$$\begin{aligned} \mathrm{X} &= x + d\,(\alpha\xi - \beta\eta) \\ \mathrm{Y} &= y + d\,(\beta\xi + \alpha\eta). \end{aligned}$$

The second equation gives $y \equiv \mathrm{Y} \pmod{d}$, which determines y.

Then

$$\beta\xi + \alpha\eta = \frac{\mathrm{Y} - y}{d},$$

α and β are co-prime, and therefore we can find α' and β' so that $\alpha'\beta - \alpha\beta' = 1$, and then

$$\left.\begin{aligned} \xi &= \alpha'\frac{\mathrm{Y}-y}{d} + \lambda\alpha \\ \eta &= -\beta'\frac{\mathrm{Y}-y}{d} - \lambda\beta \end{aligned}\right\}\text{where } \lambda \text{ is some integer.}$$

Therefore,

$$\begin{aligned} \mathrm{X} &= x + d\left[\frac{\mathrm{Y}-y}{d}(\alpha\alpha' + \beta\beta') + \lambda\,(\alpha^2 + \beta^2)\right] \\ &= x + (\mathrm{Y} - y)\,(\alpha\alpha' + \beta\beta') + \lambda\,.\,d\,(\alpha^2 + \beta^2), \end{aligned}$$

and therefore

$$x \equiv \mathrm{X} - (\mathrm{Y} - y)\,(\alpha\alpha' + \beta\beta') \;(\mathrm{mod}\ d\,.\,\overline{\alpha^2 + \beta^2}),$$

which determines x.

So x and y are given by

$$\begin{aligned} & y \equiv \mathrm{Y} \pmod{d} \\ & x \equiv \mathrm{X} - (\mathrm{Y} - y)\,(\alpha\alpha' + \beta\beta') \;(\mathrm{mod}\ d\,.\,\alpha^2 + \beta^2) \end{aligned}$$

where $\alpha\alpha' + \beta\beta'$ is a constant depending on the modulus.

Example.

Mod 3 (3 + 2i)

$$d = 3. \qquad \left.\begin{matrix} \alpha = 3 & \beta = 2 \\ \alpha' = -1 & \beta' = -1 \end{matrix}\right\} \;\alpha\alpha' + \beta\beta' = -5.$$

The reducing formulæ are

$$\begin{aligned} & y \equiv \mathrm{Y} \pmod{3} \\ & x \equiv \mathrm{X} + 5\,(\mathrm{Y} - y) \pmod{39}. \end{aligned}$$

Thus to find the residues of the successive powers of $1 + i$:—

$$\begin{aligned}
 & 1 + i \\
 & 2i \\
-2 + 2i & \equiv 37 + 2i \\
35 + 39i & \equiv 35 \\
35 + 35i & \equiv 5 + 2i \\
3 + 7i & \equiv 33 + i \\
32 + 34i & \equiv 2 + i \\
1 + 3i & \equiv 16 \\
16 + 16i & \equiv 13 + i \\
12 + 4i & \equiv 33 + 2i \\
31 + 35i & \equiv 1 + 2i \\
-1 + 3i & \equiv 14 \\
14 + 14i & \equiv 35 + 2i \qquad (\text{mod } 9 + 6i) \\
33 + 37i & \equiv 18 + i \\
17 + 19i & \equiv 29 + i \\
28 + 30i & \equiv 22 \\
22 + 22i & \equiv 10 + i \\
9 + 11i & \equiv 15 + 2i \\
13 + 17i & \equiv 10 + 2i \\
8 + 12i & \equiv 29 \\
29 + 29i & \equiv 8 + 2i \\
6 + 10i & \equiv 12 + i \\
11 + 13i & \equiv 32 + i \\
31 + 33i & \equiv 1
\end{aligned}$$

In the two special cases

(i.) $m = \alpha + \beta i$

$$\mathrm{X} + i\mathrm{Y} \equiv x \ (\text{mod } \alpha + \beta i) \quad \text{and} \quad x$$

is determined by

$$x \equiv \mathrm{X} - \mathrm{Y}\ (\alpha\alpha' + \beta\beta')\ (\text{mod } \alpha^2 + \beta^2).$$

(ii.) $m = d$

$$\mathrm{X} + i\mathrm{Y} \equiv x + iy \ (\text{mod } d) \quad \text{and} \quad x \quad \text{and} \quad y$$

are given by

$$x \equiv \mathrm{X} \ (\text{mod } d),$$

$$y \equiv \mathrm{Y} \ (\text{mod } d).$$

Number of residues prime to any modulus.

Let the modulus be expressed as a product of powers of its prime factors, $m = p^{\alpha}q^{\beta} \ldots$ then of the N (m) residues,

$$\Phi(m) = [\mathrm{N}(p^{\alpha}) - \mathrm{N}(p^{\alpha-1})][\mathrm{N}(q^{\beta}) - \mathrm{N}(q^{\beta-1})] \ldots$$

are prime to m.

In particular for a pure prime p,

$$\Phi(p) = p^2 - 1 \quad \text{and} \quad \Phi(p^{\alpha}) = p^{2\alpha} - p^{2\alpha-2};$$

and for a mixed prime, $a + bi$,

$$\Phi(a + bi) = a^2 + b^2 - 1 \quad \text{and} \quad \Phi(a + bi)^{\alpha} = (a^2 + b^2)^{\alpha} - (a^2 + b^2)^{\alpha-1}.$$

This value Φ, when we are dealing with complex numbers, must be distinguished from the value ϕ when we are concerned only with real numbers.

To a mixed prime ϕ is inapplicable, and to a real it has the relations $\phi(p) = p - 1$, $\phi(p^{\alpha}) = p^{\alpha} - p^{\alpha-1}$.

Any modulus may be multiplied by -1, i, *or* $-i$, for if the modulus is a factor of any number it remains a factor on being multiplied by -1, i, or $-i$. This enables us to change any modulus into one entirely positive :—

For

$$\left.\begin{aligned} -\alpha - \beta i &= -1(\alpha + \beta i) \\ -\beta + \alpha i &= i(\alpha + \beta i) \\ \beta - \alpha i &= -i(\alpha + \beta i) \end{aligned}\right\}.$$

PART II.—RESIDUES OF POWERS OF NUMBERS FOR ANY MODULUS, COMPOSITE AND COMPLEX.

(i.) a being a number prime to m each term of the series of residues

$$a, a^2, a^3, \ldots (\mathrm{mod}\ m)$$

is one of the $\Phi(m)$ residues which are prime to m. Hence, as in (Proposition 1), we see that, if t is the smallest integer for which $a^t \equiv 1 \ (\mathrm{mod}\ m)$, the infinite series of residues consists of a repetition of t terms beginning from the first term. The number t is called the exponent of a for the modulus m.

Example.—The series of residues of the first twenty-four powers of the number $2 + i$ for the modulus 9 are

$2 + i$
$3 + 4i$
$2 + 2i$
$2 + 6i$
$7 + 5i$
$8i$
$1 + 7i$
$4 + 6i$
$2 + 7i$
$6 + 7i$
$5 + 2i$
8
$7 + 8i$
$6 + 5i$
$7 + 7i$
$7 + 3i$
$2 + 4i$
i
$8 + 2i$
$5 + 3i$
$7 + 2i$
$3 + 2i$
$4 + 7i$
1

after which the set of twenty-four terms constantly repeats itself.

(ii.) The proof is identical with 2, that if a has exponent t mod m and $a^s \equiv 1$, then t divides s.

(iii.) The proof of FERMAT'S theorem $a^{\Phi(m)} \equiv 1$ (mod m) is identical with that in (3) if for "the $\phi(m)$ numbers less than m and prime to it," we substitute "the $\Phi(m)$ residues of m that are prime to it."

Hence the corollary, that the exponent of any number, modulus m, is a divisor of $\Phi(m)$.

Example.—Mod $9 = 3^2$. $\Phi(3^2) = 3^4 - 3^2 = 72$. In Proposition (i.) we saw that the exponent of $2 + i$, mod 9, is 24, a divisor of 72.

(iv.) The proof is identical with that of (4). If a has exponent t, mod m, then a^s has exponent τ: where $t = \kappa\tau$ and κ is the G.C.M. of s and t.

Example.—Taking again the successive powers of $2 + i$, mod 9, we find the exponent of each of the twenty-four numbers.

No.	s.	τ = exp.
$2 + i$	1	24
$3 + 4i$	2	12
$2 + 2i$	3	8
$2 + 6i$	4	6
$7 + 5i$	5	24
$8i$	6	4
$1 + 7i$	7	24
$4 + 6i$	8	3
$2 + 7i$	9	8
$6 + 7i$	10	12
$5 + 2i$	11	24
8	12	2
$7 + 8i$	13	24
$6 + 5i$	14	12
$7 + 7i$	15	8
$7 + 3i$	16	3
$2 + 4i$	17	24
i	18	4
$8 + 2i$	19	24
$5 + 3i$	20	6
$7 + 2i$	21	8
$3 + 2i$	22	12
$4 + 7i$	23	24
1	24	1

(v.) The proof is the same as that of (5).

If the exponent of a is t, and of a' is t', and t and t' are co-prime, then the exponent of aa' is tt'.

And hence the corollary, that the same is true for any number of numbers: if $a, a', a'', \ldots$ have exponents $t, t', t'', \ldots$ co-prime, then $aa'a'' \ldots$ has exponent $tt't'' \ldots$

Example.—The exponent of 3 for mod $1 + 5i$ is 3.

Its successive powers have residues 3. 9. 1.

The exponent of 5 for mod $1 + 5i$ is 4.

Its successive powers have residues 5. 25. 21. 1.

Hence the exponent of $5 \times 3 = 15$ is $4 \times 3 = 12$.

The successive powers of 15 have residues 15. 17. 21. 3. 19. 25. 11. 9. 5. 23. 7. 1.

Example.—The exponent of 2, mod 7, is 3. The residues are 2. 4. 1.

The exponent of $5 + 2i$ is 8. The residues are $5 + 2i$, $6i$, $2 + 2i$, 6, $2 + 5i$, i, $5 + 5i$, 1.

Hence the exponent of $10 + 4i \equiv 3 + 4i$ is $3 \times 8 = 24$.

The successive residues are

$$\begin{array}{cccccc}
3 + 4i & 3i & 2 + 2i & 5 & 1 + 6i & i. \\
3 + 3i & 4 & 5 + 2i & 5i & 1 + i & 6. \\
4 + 3i & 4i & 5 + 5i & 2 & 6 + i & 6i. \\
4 + 4i & 3 & 2 + 5i & 2i & 6 + 6i & 1.
\end{array}$$

(vi.) The proof is the same as for (6).

Suppose that a has exponent t, and a' has exponent t', and that t and t' are not co-prime. Then, if t and t' contain no prime factor raised to the same power in both, the exponent of aa' is the L.C.M. of t and t'.

Example.—The exponent of 35 for mod $9 + 6i$ is 6.

The residues of its powers are 35. 16. 14. 22. 29. 1.

The exponent of $3 + 7i$ for the same modulus is 4.

The residues of its powers are $3 + 7i$. 14. $15 + 2i$. 1.

Hence the exponent of $35\,(3 + 7i)$ is the L.C.M. of 6 and $4 = 12$.

$$35\,(3 + 7i) = 105 + 245i \equiv 33 + 2i \pmod{9 + 6i}.$$

The residues of its powers are

$$\begin{array}{cccccc}
33 + 2i & 29 & 33 + i & 22 & 2i & 14. \\
12 + i & 16 & 15 + 2i & 35 & 18 + i & 1.
\end{array}$$

(vii.) The proof is the same as for (7).

If a has exponent t, a' has exponent t', a'' has exponent t'', &c. for modulus m, and if of the $tt't''$ . . . numbers $a^r a'^{r'} a''^{r''}$ . . . (modulus m), formed by giving to r all values modulus t, to r' all values modulus t', &c., no two are congruent; then if

$$a^s a'^{s'} \ldots \equiv 1 \pmod{m},$$

we must have

$$a^s \equiv 1,\ a'^{s'} \equiv 1,\ \ldots$$

in other words, if a product of powers of numbers that are independent generators be congruent to unity, then each of these powers is itself congruent to unity.

(viii.) Proof identical with (8).

The exponent of the product of a number of independent generators is the L.C.M. of their exponents.

In particular if the separate exponents are all powers of the same prime, then the exponent is equal to the greatest of them.

(ix.) Proof the same as for (9).

If the exponent of a for modulus m is t, and for modulus n is t', and m and n are co-prime, then the exponent of a for modulus mn is the L.C.M. of t and t'.

Also the corollary, if the exponents of a for moduli $m, m', m'', \ldots$ are respectively $t, t', t'', \ldots$ $m, m', m'' \ldots$ being co-prime, then the exponent of a for modulus $mm'm'' \ldots$ is the L.C.M. of the numbers $t, t', t' \ldots$

Example.

$$1 + i \equiv 4 \ (\text{mod } 2 + i),$$

and the residues of its powers are 4. 1. Hence the exponent of $1 + i$ for modulus $2 + i$ is 2.

$$1 + 1 \equiv 6 \ (\text{mod } 3 + 2i),$$

and the residues of its powers are 6. 10. 8. 9. 2. 12. 7. 3. 5. 4. 11. 1. Hence the exponent of $1 + i$ for modulus $3 + 2i$ is 12.

The moduli $2 + i$, $3 + 2i$ are co-prime, therefore the exponent of $1 + i$ for modulus $(2 + i)(3 + 2i)$ is 12.

$$1 + i \equiv 19 \ (\text{mod } 4 + 7i),$$

and the residues of its powers are

19. 36. 34. 61. 54. 51. 59. 16. 44. 56. 24. 1.

Example.

$$1 + 2i \text{ has exp } 8, \text{ mod } 3\ ;$$

the residues of its powers are

$$1 + 2i.\ \ i.\ \ 1 + i.\ \ 2.\ \ 2 + i.\ \ 2i.\ \ 2 + 2i.\ \ 1.$$

$$1 + 2i \equiv 11 \ (\text{mod } 3 + 2i), \text{ and has exp } 12\ ;$$

the residues of its powers are

11. 4. 5. 3. 7. 12. 2. 9. 8. 10. 6. 1.

$$1 + 2i \equiv 26 \ (\text{mod } 6 + 1), \text{ and has exp } 3,$$

the residues of its powers are

26. 10. 1.

Hence, since $3, 3 + 2i, 6 + i$ are co-prime, $1 + 2i$ has for modulus $3(3 + 2i)(6 + i)$ exponent = L.C.M. of 8, 12, 3, *i.e.*, $1 + 2i$ has exponent 24 for modulus $48 + 45i$. The residues of its powers are

$$1+2i.\ 90+i.\ 1339+i.\ 692.\ 275+i.\ 1419+2i.\ 1370+2i.$$
$$1231.\ 1012+2i.\ 291+i.\ 1015+i.\ 482.$$

$$32+i.\ 540+2i.\ 827+2i.\ 211.\ 244+2i.\ 969+i.\ 439+i.$$
$$269.\ 1043+i.\ 741+2i.\ 503+2i.\ \ 1.$$

(x.) Proof identical with (10).

If the exponent of a is t, and $t = pqr \ldots$ where $p, q, r \ldots$ are co-prime factors of t, then a can be expressed as a product of numbers whose exponents are $p, q, r \ldots$

Example.—$2 + i$ has exponent 24 for modulus 9. (See example Proposition iv.)

$$24 = 3.\,8 \qquad \left.\begin{array}{r} 16 \equiv 1 \pmod 3 \\ \equiv 0 \pmod 8 \end{array}\right\} \quad \begin{array}{r} 9 \equiv 1 \pmod 8. \\ \equiv 0 \pmod 3. \end{array}$$

Hence

$$\begin{aligned} 2 + i &\equiv (2+i)^{16}.\,(2+i)^9 \pmod 9, \\ &\equiv (7+3i)\,(2+7i) \pmod 9, \end{aligned}$$

where

and

$$\left.\begin{array}{l} 7 + 3i \text{ has exp } 3 \\ 2 + 7i \text{ has exp } 8 \end{array}\right\} \quad \text{(See example Proposition iv.)}$$

Example.—$33 + 2i$ has exponent 12 for modulus $9 + 6i$. (See example Proposition vi.)

$$12 = 3.\,4 \qquad \begin{array}{r} 4 \equiv 1 \pmod 3 \\ \equiv 0 \pmod 4 \end{array} \qquad \begin{array}{r} 9 \equiv 1 \pmod 4 \\ \equiv 0 \pmod 3. \end{array}$$

Therefore

$$\begin{aligned} 33 + 2i &\equiv (33+2i)^4\,(33+2i)^9 \pmod{9+6i} \\ &\equiv 22\,(15+2i) \pmod{9+6i}, \end{aligned}$$

where 22 has exponent 3, and $15 + 2i$ has exponent 4.

(xi.) The number of numbers with exponent t, when the modulus is a prime (pure or mixed), is $\phi(t)$.

Any exponent t is a divisor of $\Phi(p)$, p being the prime modulus (Corollary, Proposition iii.). Exactly as in (11) we see that if there be *one* number with exponent t there are $\phi(t)$ and no more.

Now $t_1, t_2, \ldots$ being all the divisors of any real number $\Phi(p)$,

$$\phi(t_1) + \phi(t_2) + \ldots = \Phi(p).$$

Corresponding to each value t there are $\phi(t)$ numbers, or none with t as exponent.

The number of residues altogether (prime to the modulus p) is $\Phi(p)$. Hence, in no case can there fail to be $\phi(t)$ numbers with exponent t, t being any divisor of $\Phi(p)$.

Corollary.—In particular, any prime modulus p has numbers with exponents $\Phi(p)$, *i.e.*, *has primitive roots.*

The number of these primitive roots is $\phi[\Phi(p)]$.

If p is a pure prime this is $\phi(p^2 - 1)$.

If p is a mixed prime, $= \alpha + \beta i$, the number is $\phi(\alpha^2 + \beta^2 - 1)$.

Example.—Modulus $5 + 2i$. 10 is a primitive root with exponent

$$\Phi(5 + 2i) = (5^2 + 2^2 - 1) = 28.$$

The residues of its powers are given in the table.

Number	10	13	14	24	8	22	17	25	18	6	2	20	26	28
Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number	19	16	15	5	21	7	12	4	11	23	27	9	3	1
Index	15	16	17	18	19	20	21	22	23	24	25	26	27	28 or 0

The $\phi(28) = 12$ primitive roots are 10 14 8 18 2 26 19 15 21 11 27 3.

Example.—Modulus 7. Primitive roots have exponent $\Phi(7) = 7^2 - 1 = 48$. $2 + i$ is a primitive root, and the residues of its powers are given in the table.

Number	$2 + i$	$3 + 4i$	$2 + 4i$	$3i$	$4 + 6i$	$2 + 2i$	$2 + 6i$	5
Index	1	2	3	4	5	6	7	8
Number	$3 + 5i$	$1 + 6i$	$3 + 6i$	i	$6 + 2i$	$3 + 3i$	$3 + 2i$	4
Index	9	10	11	12	13	14	15	16
Number	$1 + 4i$	$5 + 2i$	$1 + 2i$	$5i$	$2 + 3i$	$1 + i$	$1 + 3i$	6
Index	17	18	19	20	21	22	23	24
Number	$5 + 6i$	$4 + 3i$	$5 + 3i$	$4i$	$3 + i$	$5 + 5i$	$5 + i$	2
Index	25	26	27	28	29	30	31	32
Number	$4 + 2i$	$6 + i$	$4 + i$	$6i$	$1 + 5i$	$4 + 4i$	$4 + 5i$	3
Index	33	34	35	36	37	38	39	40
Number	$6 + 3i$	$2 + 5i$	$6 + 5i$	$2i$	$5 + 4i$	$6 + 6i$	$6 + 4i$	1
Index	41	42	43	44	45	46	47	0

The $\phi(48) = 16$ primitive roots are

$2 + i$	$4 + 6i$	$2 + 6i$	$3 + 6i$
$6 + 2i$	$1 + 4i$	$1 + 2i$	$1 + 3i$
$5 + 6i$	$3 + i$	$5 + i$	$4 + i$
$1 + 5i$	$6 + 3i$	$6 + 5i$	$6 + 4i$

(xii.) The exponents to which a number a belongs for successive powers of a prime as moduli.

If we make one slight alteration the proof of (12) holds good throughout in the case of complex numbers for powers of a *pure* prime p as moduli. [In place of "$x < p$" read "$x =$ one of the $p^2 - 1$ residues, mod p."]

The exponent of a for mod p^λ is

$$t \text{ if } \lambda \leqq s,$$
$$tp^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for mod p, and p^s is the greatest power of p that divides $a^t - 1$.

Corollary.—The greatest value of t is $p^2 - 1$. (Proposition xi.)

The greatest value that $p^{\lambda - s}$ can have is got by making $s = 1$, *i.e.*, by taking a so that $a^{p^2-1} - 1$ (though necessarily divisible by p) is not divisible by p^2.

Hence the greatest exponent that a number can have for mod p^λ is

$$(p^2 - 1)p^{\lambda - 1}.$$

Now $\Phi(p^\lambda) = p^{2\lambda} - p^{2(\lambda-1)} = (p^2 - 1)p^{2(\lambda - 1)}$, which is $p^{\lambda-1}$ times the highest exponent.

Hence for a *power* of a pure prime as modulus primitive roots do not exist.

Consider next the case of a mixed prime (not $1 + i$).

Suppose $\alpha + \beta i$ is the prime.

Then, if $(\alpha + \beta i)^\lambda = A + Bi$, A and B must be co-prime, for otherwise $A + Bi$ would be divisible by some number other than $\alpha + \beta i$.

Hence the $N(\alpha + \beta i)^\lambda = (\alpha^2 + \beta^2)^\lambda$ residues can be taken to be the pure numbers

$$0, 1, 2, \ldots (\alpha^2 + \beta^2)^\lambda - 1.$$

Suppose that a is any one of these numbers.

Then, if we have any congruence of the form

$$a^r \equiv b \,[\text{mod } (\alpha + \beta i)^\lambda],$$
$$a^r - b \equiv 0 \,[\text{mod } (\alpha + \beta i)^\lambda],$$

it follows, since $a^r - b$ is a pure number, that

$$a^r - b \equiv 0 \left[\bmod (\alpha - \beta i)^\lambda\right],$$

therefore

$$a^r - b \equiv 0 \left[\bmod (\alpha^2 + \beta^2)^\lambda\right],$$
$$a^r \equiv b \left[\bmod (\alpha^2 + \beta^2)^\lambda\right].$$

Hence for the modulus $(\alpha + \beta i)^\lambda$ the residue of any power of a number is the same as for the modulus $(\alpha^2 + \beta^2)^\lambda$.

Now $\alpha^2 + \beta^2$ is a *real prime.*

Therefore the exponent to which a belongs for modulus $(\alpha + \beta i)^\lambda$ is

$$t \text{ if } \lambda \leqq s,$$
$$t\,(\alpha^2 + \beta^2)^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for mod $\alpha^2 + \beta^2$, and $(\alpha^2 + \beta^2)^s$ is the highest power of $\alpha^2 + \beta^2$ that divides $a^t - 1$.

The greatest exponent is

$$(\alpha^2 + \beta^2 - 1)\,(\alpha^2 + \beta^2)^{\lambda - 1},$$

and

$$\Phi\,(\alpha + \beta i)^\lambda = (\alpha^2 + \beta^2)^\lambda - (\alpha^2 + \beta^2)^{\lambda - 1} = (\alpha^2 + \beta^2 - 1)\,(\alpha^2 + \beta^2)^{\lambda - 1}.$$

Thus for powers of a *mixed prime* primitive roots *do* exist.

(xii*a*.) We see from the last proposition that for a power of a mixed prime as modulus primitive roots exist, and any one of them generates by its powers all the residues prime to the modulus. But in the case of a power of a pure prime p^λ, the highest exponent is $p^{\lambda-1}\,(p^2 - 1)$, whereas $\Phi\,(p^\lambda) = p^{2(\lambda-1)}\,(p^2 - 1)$.

We wish now to find how all the $p^{2(\lambda-1)}\,(p^2 - 1)$ numbers can be generated.

$$\left.\begin{array}{l} \text{Take a number } g \text{ with exp } p^{\lambda-1}\,(p^2 - 1) \\ \text{and ,, } \quad \text{,, } \quad f \quad \text{,, } \quad \text{,, } \quad p^{\lambda-1} \end{array}\right\}$$

g can be expressed as a product

$$g \equiv f'h \;(\bmod p^\lambda),$$

where

$$\left.\begin{array}{l} f' \text{ has exp } p^{\lambda-1} \\ h \quad \text{,, } \quad \text{,, } \quad p^{2}-1 \end{array}\right\}, \text{ Proposition x.}$$

$$\left.\begin{array}{r} f \text{ and } f' \text{ are each} \equiv 1 \;(\bmod p) \\ \not\equiv 1 \;(\bmod p^2) \end{array}\right\}, \text{ Proposition xii.}$$

Suppose

$$f \equiv 1 + \alpha p \pmod{p^2} \qquad \alpha \not\equiv 0 \pmod{p};$$
$$f' \equiv 1 + \alpha' p \pmod{p^2} \qquad \alpha' \not\equiv 0 \pmod{p}.$$

The $p^{\lambda-1} \times p^{\lambda-1}(p^2-1)$ numbers generated by products of powers of f and g will be the $p^{2(\lambda-1)}(p^2-1)$ residues prime to the modulus p^λ, provided that no two of them are congruent. We shall now show that no two can be congruent provided that k is a *complex* and not a *real* number, k being determined by the congruence $k\alpha' \equiv \alpha \pmod{p}$.

Suppose that two of the numbers generated are congruent, say

$$g^i f^j \equiv g^{i'} f^{j'} \pmod{p^\lambda},$$

where

$$i \not\equiv i' \pmod{p^{\lambda-1} . p^2 - 1},$$
$$j \not\equiv j' \pmod{p^{\lambda-1}}.$$

This is equivalent to

$$g^{i''} \equiv f^{j''} \pmod{p^\lambda},$$

where

$$i'' \equiv i - i' \not\equiv 0 \pmod{p^{\lambda-1} . p^2 - 1},$$
$$j'' \equiv j - j' \not\equiv 0 \pmod{p^{\lambda-1}},$$

j'' may be divisible by a power of p.

Suppose

$$j'' = \beta p^{\lambda-s},$$

where β is prime to p, and s may be any one of the numbers $2, 3, \ldots \lambda$.

Then

$$f^{j''} \text{ has exp } p^{s-1}. \quad \text{(Proposition iv.)},$$

therefore

$$g^{i''} \text{ has exp } p^{s-1},$$

therefore

$$i'' = \beta'(p^2-1)p^{\lambda-s},$$

where β' is prime to p. (Proposition iv.) Therefore

$$g^{\beta'(p^2-1)p^{\lambda-s}} \equiv f^{\beta p^{\lambda-s}} \pmod{p^\lambda},$$

and, therefore, since

$$h^{p^2-1} \equiv 1 \pmod{p^\lambda},$$
$$f'^{\beta'(p^2-1)p^{\lambda-s}} \equiv f^{\beta p^{\lambda-s}} \pmod{p^\lambda}.$$

Raise both sides to the power p^{s-2}, then

$$f'^{\beta'(p^2-1)p^{\lambda-2}} \equiv f^{\beta p^{\lambda-2}} \pmod{p^\lambda}.$$

Let

$$\beta\xi \equiv 1 \pmod{p}$$

determine ξ, and suppose

$$\beta'(p^2 - 1)\,\xi \equiv \gamma \pmod{p}.$$

Raise both sides to the power ξ, then

$$f'^{\gamma p^{\lambda-2}} \equiv f^{p^{\lambda-2}} \pmod{p^\lambda}.$$

Now

$$f \equiv 1 + \alpha p \pmod{p^2},$$

therefore

$$f^{p^{\lambda-2}} \equiv 1 + \alpha p^{\lambda-1} \pmod{p^\lambda},$$

and

$$f' \equiv 1 + \alpha' p \pmod{p^2},$$

therefore

$$f'^{\gamma p^{\lambda-2}} \equiv 1 + \alpha'\gamma p^{\lambda-1} \pmod{p^\lambda},$$

therefore

$$1 + \alpha'\gamma p^{\lambda-1} \equiv 1 + \alpha p^{\lambda-1} \pmod{p^\lambda},$$

therefore

$$\gamma\alpha' \equiv \alpha \pmod{p},$$

therefore

$$\gamma \equiv k \pmod{p}.$$

Now i'', j'', β, β', and, therefore, ξ, and finally γ, are *real* numbers, whereas, by supposition, k is complex, therefore on this supposition g and f are complete generators.

Example.—Modulus 3^2.

$$\Phi(3^2) = 3^4 - 3^2 = 72.$$

The highest exponent is

$$3(3^2 - 1) = 24.$$

We shall find that $2 + i$ and $4 + 3i$ can be taken for generators.

$$\left.\begin{array}{l} g = 2 + i \text{ has exp } 24 \bmod 3^2 \\ f = 4 + 3i \text{ has exp } \ \ 3 \bmod 3^2 \end{array}\right\}$$

$$\begin{aligned} 2 + i &\equiv (2 + i)^{16}(2 + i)^9 \pmod{9} \\ &\equiv (7 + 3i)(2 + 7i) \pmod{9}, \end{aligned}$$

and

$$f' = 7 + 3i \text{ has exp } 3.$$

Now

$$f \equiv 1 + (1 + i)\,.\,3 \pmod{3^2}$$

$$f' \equiv 1 + (2 + i)\,.\,3 \pmod{3^2}$$

and

$$k(2+i) \equiv (1+i) \pmod 3$$

gives

$$k \equiv 2i,$$

which is not a real number.

The following table gives the indices of the 72 numbers.

	0	1	2
1	$2+i$	$5+i$	$8+i$
2	$3+4i$	$7i$	$6+i$
3	$2+2i$	$2+5i$	$2+8i$
4	$2+6i$	$8+3i$	5
5	$7+5i$	$4+5i$	$1+5i$
6	$8i$	$3+5i$	$6+2i$
7	$1+7i$	$1+4i$	$1+i$
8	$4+6i$	7	$1+3i$
9	$2+7i$	$5+7i$	$8+7i$
10	$6+7i$	$3+i$	$4i$
11	$5+2i$	$5+5i$	$5+8i$
12	8	$5+6i$	$2+3i$
13	$7+8i$	$4+8i$	$1+8i$
14	$6+5i$	$2i$	$3+8i$
15	$7+7i$	$7+4i$	$7+i$
16	$7+3i$	$1+6i$	4
17	$2+4i$	$5+4i$	$8+4i$
18	i	$6+4i$	$3+7i$
19	$8+2i$	$8+5i$	$8+8i$
20	$5+3i$	2	$8+6i$
21	$7+2i$	$4+2i$	$1+2i$
22	$3+2i$	$6+8i$	$5i$
23	$4+7i$	$4+4i$	$4+i$
0	1	$4+3i$	$7+6i$

[The pair of indices for any number are found at the end of the row and at the head of the column in which the number is situated.

Thus

$$1+8i \equiv (2+i)^{13}(4+3i)^2 \pmod{9}.]$$

(xiii.–xvi.) Propositions (13) to (16) are concerned with the moduli 2^λ. In Propositions xiii.–xvi. we shall investigate the moduli analogous to these in the case of complex numbers, viz. $(1+i)^\lambda$. [Propositions xiii.–xvi. will not be made to correspond individually to Propositions (13)–(16): they are intended to cover the same ground.]

(xiii.) In the case of the smaller values of λ, from 1 to 7, we shall find results, some of which, though they are really particular cases of the general results for any value of λ, are not conveniently included under them.

We shall start by a separate treatment of each of the first seven moduli, finding—

(i.) the numbers which belong to each exponent;
(ii.) what numbers must be taken in order to generate the complete set of residues.

Mod $1+i$. There is only one residue, viz., 1, whose exponent is 1.

Mod $(1+i)^2$. $\Phi(1+i)^2 = 2$. There are two numbers, viz.,

$$\text{and} \qquad \left.\begin{array}{r} i \text{ with exp } 2 \\ 1 \text{ with exp } 1 \end{array}\right\}.$$

Thus, for this modulus, i is a primitive root.

Mod $(1+i)^3$. $\Phi(1+i)^3 = 4$. The four residues are—

$$\begin{array}{r} i,\ 2+i \text{ with exp } 4, \\ 3 \text{ with exp } 2, \\ 1 \text{ with exp } 1. \end{array}$$

i and $2+i$ are both primitive roots.

Mod $(1+i)^4$. $\Phi(1+i)^4 = 8$. The 8 numbers with their exponents are

Exp 4.	i,	$3i$,	$2+i$,	$2+3i$.
Exp 2.	3,	$1+2i$,	$3+2i$.	
Exp 1.	1.			

The square of each of the four numbers with exponent 4 is congruent to the same number with exponent 2, viz. 3.

Hence, to generate the 8 numbers, we must take any one of the four numbers with exponent 4, and either of the two numbers with exponent 2, viz. $1+2i$, $3+2i$.

Mod $(1+i)^5$. $\Phi(1+i)^5 = 16$. The 16 numbers, with their exponents are

Exp 4.	i	$3i$	$2+i$	$2+3i$	$4+i$	$6+i$	$4+3i$	$6+3i$,
Exp 2.	3	5	7	$1+2i$	$3+2i$	$5+2i$	$7+2i$	
Exp 1.	1.							

The square of each of the 8 numbers with exponent 4 gives the same number with exponent 2, viz. 7.

Also $7 \equiv 3 \,.\, 5 \equiv (1 + 2i)\,(3 + 2i) \equiv (5 + 2i)\,(7 + 2i)\,.\,[\text{mod }(1 + i)^5]$.

Hence, firstly, we must take as generators

$$\left.\begin{array}{ll} & \text{one number with exponent } 4 \\ \text{and} & \\ & \text{two numbers with exponent } 2 \end{array}\right\} ;$$

and, secondly, the two numbers with exponent 2 must neither of them be 7, and they must not be a pair of the numbers as arranged above whose product is congruent to 7.

Mod $(1 + i)^6$. $\Phi\,(1 + i)^6 = 32$. The 32 numbers with their exponents are arranged below.

Exp 1.	Exp 2.	Exp 4.							
1	$5+4i$	$1+2i$	$1+6i$	$3+2i$	$3+6i$	$5+2i$	$5+6i$	$7+2i$	$7+6i$
	$3+4i$	$2+i$	$2+3i$	$2+5i$	$2+7i$	$6+i$	$6+3i$	$6+5i$	$6+7i$
	7	$4+i$	$4+3i$	$4+5i$	$4+7i$	i	$3i$	$5i$	$7i$
	3								
	5								
	$1+4i$								
	$7+4i$								

The 24 numbers with exponent 4 are written in three rows, and the number with exponent 2, to which the square of any one of them is congruent, is to be found written at the end of the row.

Moreover, the three numbers, with exponent 2 (viz. $5 + 4i$, $3 + 4i$ and 7), which thus appear as the squares of numbers with exponent 4, are such that the product of any two is congruent to the third; for

$$7\,(5 + 4i)\,(3 + 4i) \equiv 1\ [\text{mod }(1 + i)^6].$$

Hence, to generate the 32 numbers, we must take

two numbers, with exponent 4, not in the same row

and

one number, with exponent 2, selected from 3, 5, $1 + 4i$, $7 + 4i$.

Mod $(1+i)^7$. $\Phi(1+i)^7 = 64$. The 64 numbers, with their exponents, are arranged below.

Exp 1.	*Exp* 2.	*Exp* 4.							
1	7	$4+i$	$12+i$	$3i$	$8+3i$	$5i$	$8+5i$	$4+7i$	$12+7i$
	9	3	5	11	13	$1+4i$	$7+4i$	$9+4i$	$15+4i$
	15	i	$8+i$	$4+3i$	$12+3i$	$4+5i$	$12+5i$	$7i$	$8+7i$
	$3+4i$	$2+i$	$10+i$	$2+3i$	$10+3i$	$6+5i$	$14+5i$	$6+7i$	$14+7i$
	$5+4i$	$5+2i$	$7+2i$	$13+2i$	$15+2i$	$1+6i$	$3+6i$	$9+6i$	$11+6i$
	$11+4i$	$6+i$	$14+i$	$6+3i$	$14+3i$	$2+5i$	$10+5i$	$2+7i$	$10+7i$
	$13+4i$	$1+2i$	$3+2i$	$9+2i$	$11+2i$	$5+6i$	$7+6i$	$13+6i$	$15+6i$

The 56 numbers with exponent 4 are arranged in rows collinearly with the numbers, with exponent 2, to which their squares are congruent.

The 7 numbers with exponent 2 can be arranged in 7 sets of 3, so that of each 3 the product of any two is congruent to the third (or the product of the three congruent to unity). Thus:

$$\left.\begin{aligned} 7.\ 9.\ 15 &\equiv 1 \\ 7\,(13+4i)\,(3+4i) &\equiv 1 \\ 7\,(5+4i)\,(11+4i) &\equiv 1 \\ 9\,(3+4i)\,(11+4i) &\equiv 1 \\ 9\,(5+4i)\,(13+4i) &\equiv 1 \\ 15\,(3+4i)\,(5+4i) &\equiv 1 \\ 15\,(11+4i)\,(13+4i) &\equiv 1 \end{aligned}\right\}\ (\text{mod } 1+i)^7.$$

To generate the 64 numbers, we must take three numbers each with exponent 4. The product of powers of three such numbers can only be congruent to unity without each being separately congruent to unity, if the product of their squares be so.

Hence, the three generators must be chosen, one each from three of the above rows, and the numbers, with exponent 2 found in these rows, must not have their product congruent to unity.

(xiv.) *The numbers which have as exponents* 2, 4, *and* 8 *for mod* $(1+i)^\lambda$.

We shall use the results of the three following Lemmas:

Lemma (1).—If

$$a^2 \equiv 1\ [\text{mod } (1+i)^\kappa]$$

$$(a-1)\,(a+1) \equiv 0\ [\text{mod } (1+i)^\kappa].$$

Now, if

$$a \equiv 1\ [\text{mod } (1+i)^3]$$

$$a+1 \equiv 2\ [\text{mod } (1+i)^3]$$

$$\equiv 0\ [\text{mod } (1+i)^2],$$

i.e., if $a - 1$ is divisible by $(1 + i)^3$ or any higher power, then $a + 1$ is divisible by $(1 + i)^2$ as the highest power.

Similarly, we can show that if a higher power than $(1 + i)^2$ divides $a + 1$, then no higher power than $(1 + i)^2$ divides $a - 1$.

Therefore, if $\kappa > 4$, either

$$a \equiv 1 \,[\text{mod } (1 + i)^{\kappa-2}]$$

or

$$a \equiv - 1 \,[\text{mod } (1 + i)^{\kappa-2}],$$

therefore, κ being > 4, the solutions of

$$a^2 \equiv 1 \,[\text{mod } (1 + i)^{\kappa}]$$

are given by

$$a \equiv \pm 1 \,[\text{mod } (1 + i)^{\kappa-2}].$$

Lemma (2).—If

$$a^2 \equiv - 1 \,[\text{mod } (1 + i)^{\kappa}]$$

$$(a + i)\,(a - i) \equiv 0 \,[\text{mod } (1 + i)^{\kappa}]$$

Now, if

$$a \equiv i \,[\text{mod } (1 + i)^3]$$

$$a + i \equiv 2i \,[\text{mod } (1 + i)^3]$$

$$\equiv 0 \,[\text{mod } (1 + i)^2].$$

Thus as above, if either of $a + i$, $a - i$ is divisible by a power of $(1 + i)$ above the second, then $(1 + i)^2$, is the highest power that divides the other.

Therefore, if $\kappa > 4$ the solutions of

$$a^2 \equiv - 1 \,[\text{mod } (1 + i)^{\kappa}]$$

are given by

$$a \equiv \pm i \,[\text{mod } (1 + i)^{\kappa-2}].$$

Lemma (3).

$$a^2 \equiv \pm i \,[\text{mod } (1 + i)^{\kappa}]$$

is impossible for any value of $\kappa > 1$.

For if so,

$$a^2 \equiv \pm i \,[\text{mod } (1 + i)^2],$$

or

$$a^2 \equiv i \,[\text{mod } (1 + i)^2],$$

since

$$- i \equiv i \,[\text{mod } (1 + i)^2],$$

which is impossible, for

$$1^2 \not\equiv i \,[\text{mod } (1 + i)^2],$$

and

$$\iota^2 \not\equiv i \,[\text{mod } (1 + i)^2].$$

Numbers with exponent 2 *mod* $(1+i)^\lambda$. $(\lambda > 4.)$

All these numbers, together with 1, which has exponent 1, satisfy the congruence

$$a^2 \equiv 1\,[\mathrm{mod}\ (1+i)^\lambda]\,;$$

therefore

$$a \equiv \pm 1\,[\mathrm{mod}\ (1+i)^{\lambda-2}]$$

gives (together with unity) the numbers which have exponent 2, mod $(1+i)^\lambda$.

We get thus 7 numbers with exponent 2, mod $(1+i)^\lambda$. They are congruent to

$$\left.\begin{array}{l} -1 \\ \pm 1 + (1+i)^{\lambda-2} \\ \pm 1 + (1+i)^{\lambda-1} \\ \pm 1 + (1+i)^{\lambda-2} + (1+i)^{\lambda-1} \end{array}\right\}[\mathrm{mod}\ (1+i)^\lambda].$$

The product of any two of these is congruent to a third. If we name the 7 numbers thus—

$$\begin{aligned} \alpha &\equiv -1, \\ \beta &\equiv +1 + (1+i)^{\lambda-2}, \\ \gamma &\equiv -1 + (1+i)^{\lambda-2}, \\ \delta &\equiv +1 + (1+i)^{\lambda-1}, \\ \epsilon &\equiv -1 + (1+i)^{\lambda-1}, \\ \eta &\equiv +1 + (1+i)^{\lambda-2} + (1+i)^{\lambda-1}, \\ \theta &\equiv -1 + (1+i)^{\lambda-2} + (1+i)^{\lambda-1}, \end{aligned}$$

then these relations may be written thus—

$$\begin{aligned} \alpha\beta\gamma &\equiv 1, \\ \alpha\delta\epsilon &\equiv 1, \\ \alpha\eta\theta &\equiv 1, \\ \beta\delta\eta &\equiv 1 \\ \beta\epsilon\theta &\equiv 1, \\ \gamma\delta\theta &\equiv 1, \\ \gamma\epsilon\eta &\equiv 1, \end{aligned}$$

[which includes $\alpha\beta \equiv \gamma$, $\alpha\gamma \equiv \beta$, $\beta\gamma \equiv \alpha$, because $\alpha^2 \equiv 1$, $\beta^2 \equiv 1$, $\gamma^2 \equiv 1$].

Numbers with exponent 4, *mod* $(1+i)^\lambda$. $(\lambda > 6.)$

The numbers which satisfy

$$a^4 \equiv 1\,[\mathrm{mod}\,(1+i)^\lambda],$$

and which do not satisfy

$$a^2 \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

are the numbers with exponent 4.

Hence Lemma (1) gives us

$$a^2 \equiv \pm 1 \,[\text{mod}\,(1+i)^{\lambda-2}],$$

and then Lemma (2) gives

$$a \equiv \pm 1, \ \pm i \,[\text{mod}\,(1+i)^{\lambda-4}].$$

If of these numbers we exclude

$$a \equiv \pm 1 \,[\text{mod}\,(1+i)^{\lambda-2}],$$

we have the numbers with exponent 4.

Hence the numbers with exponent 4 are

$$\equiv \pm i \,[\text{mod}\,(1+i)^{\lambda-4}],$$

and

$$\left.\begin{aligned} &\equiv \pm 1 + (1+i)^{\lambda-4} \\ &\quad \pm 1 + (1+i)^{\lambda-3} \\ &\quad \pm 1 + (1+i)^{\lambda-4} + (1+i)^{\lambda-3} \end{aligned}\right\} \text{mod}\,(1+i)^{\lambda-2}.$$

The number of these is

$$\begin{aligned} &2\text{N}\,[(1+i)^4] + 6\text{N}\,[(1+i)^2] \\ &= 2 \,.\, 2^4 + 2 \,.\, 3 \,.\, 2^2 \\ &= 2^5 + 3 \,.\, 2^3 \\ &= 2^5 + 2^4 + 2^3 = 56. \end{aligned}$$

Numbers with exponent 8 *mod* $(1+i)^\lambda$. $\lambda > 8$.

The numbers are those that satisfy

$$a^8 \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

excluding those that satisfy

$$a^4 \equiv 1 \,[\text{mod}\,(1+i)^\lambda].$$

Now

$$a^8 \equiv 1 \,[\text{mod}\,(1+i)^\lambda]$$

gives

$$a^4 \equiv \pm 1 \,[\text{mod}\,(1+i)^{\lambda-2}]. \quad \text{Lemma (1).}$$

$$a^2 \equiv \pm 1 \,[\text{mod}\,(1+i)^{\lambda-4}]. \quad \text{Lemmas (1), (2), and (3)}$$

$$a \equiv \pm 1, \pm i \,[\text{mod}\,(1+i)^{\lambda-6}]. \quad \text{Lemmas (1) and (2)}$$

Of these we have to exclude

$$a \equiv \pm 1 \ \ \pm i, \,[\text{mod}\,(1+i)^{\lambda-4}].$$

Hence the numbers with exponent 8 are

$$\left.\begin{array}{l}\pm 1, \pm i, +(1+i)^{\lambda-6} \\ \pm 1, \pm i, +(1+i)^{\lambda-5} \\ \pm 1, \pm i, +(1+i)^{\lambda-6}+(1+i)^{\lambda-5}\end{array}\right\} \text{mod } (1+i)^{\lambda-4}.$$

In number they are

$$\begin{aligned} & 4 \times 3 \times \mathrm{N}\,(1+i)^4 \\ = {} & 2^2 . 3 . 2^4 \\ = {} & 2^6 + 2^7. \end{aligned}$$

We have not specially examined the modulus $(1+i)^8$. The general results already obtained give us the numbers with exponents 2 and 4. We will find the numbers with exponent 8.

Of the solutions of

$$a^8 \equiv 1 \, [\text{mod } (1+i)^8]$$

we must exclude those of

$$a^4 \equiv 1 \, [\text{mod } (1+i)^8].$$

$$a^8 \equiv 1 \, [\text{mod } (1+i)^8]$$

gives

$$a^4 \equiv \pm 1 \, [\text{mod } (1+i)^6]$$

and

$$a^2 \equiv \pm 1 \, [\text{mod } (1+i)^4],$$

and therefore

$$a \equiv i,\ 3i,\ 2+i,\ 2+3i,\ 3,\ 1+2i,\ 3+2i,\ 1 \, [\text{mod } (1+i)^4]. \quad [\text{See mod } (1+i)^4.]$$

The numbers to be excluded are

$$\begin{aligned} a^4 &\equiv 1 \, [\text{mod } (1+i)^8] \\ a^2 &\equiv \pm 1 \, [\text{mod } (1+i)^6] \\ a &\equiv \pm 1, \pm i \, [\text{mod } (1+i)^4], \end{aligned}$$

i.e.,

$$a \equiv 1,\ 3,\ i,\ 3i \, [\text{mod } (1+i)^4].$$

Hence the numbers $\equiv 2+i,\ 1+2i,\ 2+3i,\ 3+2i \, [\text{mod } (1+i)^4]$ have exponent 8. The number of them is $4 \times 2^4 = 2^6$.

(xv.) *The numbers which have exponent* 2^s *for mod* $(1 + i)^\lambda$. [$\lambda > 8$ and $s > 3$.]

The numbers are those which satisfy

$$a^{2^s} \equiv 1 \; [\text{mod } (1 + i)^\lambda],$$

and which do not satisfy

$$a^{2^{s-1}} \equiv 1 \; [\text{mod } (1 + i)^\lambda].$$

The first gives

$$a \equiv \pm 1, \; \pm i \; [\text{mod } (1 + i)^{\lambda - 2s}] \text{ if } \lambda - 2s > 2,$$

and the second

$$a \equiv \pm 1, \pm i, \; [\text{mod } (1 + i)^{\lambda - 2s + 2}].$$

Hence the numbers with exponent 2^s for mod $(1 + i)^\lambda$, $(\lambda - 2s > 2)$, are

$$\left.\begin{array}{l} \pm 1, \; \pm i, \; + (1 + i)^{\lambda - 2s} \\ \pm 1, \; \pm i, \; + (1 + i)^{\lambda - 2s + 1} \\ \pm 1, \; \pm i, \; + (1 + i)^{\lambda - 2s} + (1 + i)^{\lambda - 2s + 1} \end{array}\right\} \text{mod } (1 + i)^{\lambda - 2s + 2}.$$

The number of them is

$$\begin{aligned} & 12 \times \text{N } (1 + i)^{2s - 2} \\ & = 2^2 . \; 3 . \; 2^{2s - 2} \\ & = 2^{2s} + 2^{2s + 1}. \end{aligned}$$

If λ is odd, $= 2\mu + 1$ the greatest value of s is $\mu - 1$, and then $\lambda - 2s > 2$. So there are $2^{2\mu - 2} + 2^{2\mu - 1}$ numbers with exponent $2^{\mu - 1}$ for mod $(1 + i)^{2\mu + 1}$, viz.,

$$\left.\begin{array}{l} \pm 1, \pm i, + (1 + i)^3 \\ \pm 1, \pm i, + (1 + i)^4 \\ \pm 1, \pm i, + (1 + i)^3 + (1 + i)^4 \end{array}\right\} \text{mod } (1 + i)^5,$$

$$\equiv \left.\begin{array}{cccc} 7 + 2i, & 5 + 2i, & 6 + 3i, & 6 + i \\ 5 & 3 & 4 + i & 3i \\ 3 + 2i, & 1 + 2i, & 2 + 3i, & 2 + i \end{array}\right\} \text{mod } (1 + i)^5.$$

So for mod $(1 + i)^{2\mu + 1}$ there are

$$\left.\begin{array}{llllll} 2^{2\mu - 1} + 2^{2\mu - 2} & \text{numbers} & \text{with} & \text{exp} & 2^{\mu - 1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 2^{2s + 1} + 2^{2s} & ,, & ,, & ,, & 2^s \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 2^7 + 2^6 & ,, & ,, & ,, & 2^3 \\ 2^5 + 2^4 + 2^3 & ,, & ,, & ,, & 2^2 \\ 2^2 + 2 + 1 & ,, & ,, & ,, & 2 \\ 1 & ,, & ,, & ,, & 1 \end{array}\right\}$$

making, in all, $2^{2\mu} = \Phi \; (1 + i)^{2\mu + 1}$ numbers, as it should.

If λ is even and $= 2\mu$, then we may have $s = \mu + 1$, and in this case $\lambda - 2s = 2$. We wish, therefore, to find the numbers with exponent $2^{\mu-1}$ for mod $(1 + i)^{2\mu}$. We have to take the solutions of

$$a^{2^{\mu-1}} \equiv 1 \; [\text{mod } (1 + i)^{2\mu}]$$

and exclude those of

$$a^{2^{\mu-2}} \equiv 1 \; [\text{mod } (1 + i)^{2\mu}].$$

The first leads to

$$a^2 \equiv \pm 1 \; [\text{mod } (1 + i)^4],$$

and the second to

$$a \equiv \pm 1, \; \pm i \; [\text{mod } (1 + i)^4],$$

therefore

$$a \equiv 1 + 2i, \; 2 + i, \; 3 + 2i, \; 2 + 3i \; [\text{mod } (1 + i)^4].$$

Hence the numbers with exponent $2^{\mu-1}$ for mod $(1 + i)^{2\mu}$ are

$$\equiv 1 + 2i, \; 2 + i, \; 3 + 2i, \; 2 + 3i \; [\text{mod } (1 + i)^4].$$

The number of them is $4 \times \text{N} \, (1 + i)^{2\mu-4}$

$$\begin{aligned} &= 2^2 . \, 2^{2\mu-4} \\ &= 2^{2\mu-2}. \end{aligned}$$

For mod $(1 + i)^{2\mu}$ there are

$$\left.\begin{array}{llllr} 2^{2\mu-2} & \text{numbers with exp} & & & 2^{\mu-1} \\ 2^{2\mu-3} + 2^{2\mu-4} & ,, & ,, & ,, & 2^{\mu-2} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot \\ 2^{2s+1} + 2^{2s} & ,, & ,, & ,, & 2^s \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot \\ 2^7 + 2^6 & ,, & ,, & ,, & 2^3 \\ 2^5 + 2^4 + 2^3 & ,, & ,, & ,, & 2^2 \\ 2^2 + 2 + 1 & ,, & ,, & ,, & 2 \\ 1 & ,, & ,, & ,, & 1 \end{array}\right\}$$

making, in all, $2^{2\mu-1} = \Phi \, (1 + i)^{2\mu}$ numbers as it should.

(xvi.) *Generators for the modulus* $(1 + i)^\lambda$.

If $\lambda = 2\mu + 1$, it is easy to see that in order to generate $2^{2\mu}$ numbers, of which for each exponent there shall be the right number of numbers (as just found), we must take three generators :—

$$\left.\begin{array}{l} \text{two generators with exp } 2^{\mu-1} \\ \text{and} \qquad \text{one generator with exp } 2^2 \end{array}\right\}.$$

If $\lambda = 2\mu$, we can similarly see that we must take three generators

$$\left.\begin{array}{lll}\text{one generator with exp} & & 2^{\mu-1} \\ \quad ,, & ,, & 2^{\mu-2} \\ \quad ,, & ,, & 2^{2}\end{array}\right\} .$$

We have now to show how these may be selected, so that all the numbers generated may be incongruent.

Any number, with exponent > 2 modulus $(1+i)^\lambda$, has one of its powers, and one only, which has exponent 2.

Suppose all the numbers modulus $(1+i)^\lambda$, arranged in 7 groups, so that all the numbers in a group may have the same number, with exponent 2, as a power.

We shall now prove four Lemmas with regard to these groups.

Lemma (i.). A power of a number belonging to any group belongs to the same group.

Let a have exponent 2^s, and belong to the group α, *i.e.*,

$$a^{2^{s-1}} \equiv \alpha \, [\bmod\,(1+i)^\lambda].$$

Take any power of a, say $a^{2^\sigma i}$ where i is odd.

Its exponent is $2^{s-\sigma}$. (Proposition iv.)

Therefore

$$(a^{2^\sigma i})^{2^{s-\sigma-1}} \text{ has exp } 2,$$

i.e.,

$$a^{2^{s-1}i} \text{ has exp } 2,$$

and it is $\equiv \alpha^i$, and, therefore, $\equiv \alpha$, because i is odd.

Therefore, any power of a belongs to the same group.

Lemma (ii.). If a and a' have exponent 2^s, and belong to the same group α, then

$$(aa')^{2^{s-1}} \equiv 1 \, [\bmod\,(1+i)^\lambda].$$

For

$$a^{2^{s-1}} \equiv \alpha.$$

$$a'^{2^{s-1}} \equiv \alpha.$$

Therefore

$$(aa')^{2^{s-1}} \equiv \alpha^2 \equiv 1 \, [\bmod\,(1+i)^\lambda].$$

Lemma (iii). If

a has exp 2^s and belongs to group α,

and

b ,, 2^s ,, ,, β,

then

ab ,, 2^s ,, ,, γ,

where

$$\alpha\beta\gamma \equiv 1.$$

For

$$a^{2^{s-1}} \equiv \alpha,$$
$$b^{2^{s-1}} \equiv \beta.$$

Therefore

$$(ab)^{2^{s-1}} \equiv \alpha\beta \equiv \gamma \; [\text{mod } (1+i)^{\lambda}],$$

and, therefore,

ab has exp 2^s, and belongs to group γ.

Lemma (iv.). If

a has exp $2^{s+\sigma}$ and belongs to group α,

and

b ,, 2^s ,, ,, β,

then

ab ,, $2^{s+\sigma}$,, ,, α.

For

$$a^{2^{s+\sigma-1}} \equiv \alpha,$$
$$b^{2^{s-1}} \equiv \beta,$$

and, therefore,

$$b^{2^{s+\sigma-1}} \equiv 1.$$

Therefore

$$(ab)^{2^{s+\sigma-1}} \equiv \alpha.$$

Therefore

ab has exp $2^{s+\sigma}$, and belongs to group α.

Now, suppose that we take these generators with the necessary exponents. They must be chosen so that the numbers they generate are all incongruent.

Let g_1, g_2, g_3 be the generators.

Then, $g_1^{i_1}, g_2^{i_2}, g_3^{i_3} \equiv g_1^{i_1'}, g_2^{i_2'}, g_3^{i_3'}$ [mod $(1+i)^{\lambda}$] must be impossible unless

$$i_1 \equiv i_1' \; (\text{mod exp } g_1),$$
$$i_2 \equiv i_2' \; (\text{mod exp } g_2),$$
$$i_3 \equiv i_3' \; (\text{mod exp } g_3),$$

i.e., $g_1^{j_1}, g_2^{j_2}, g_3^{j_3} \equiv 1$ [mod $(1+i)^{\lambda}$] must be possible only when

$$j_1 \equiv 0 \; (\text{mod exp } g_1), \quad j_2 \equiv 0 \; (\text{mod exp } g_2), \quad j_3 \equiv 0 \; (\text{mod exp } g_3).$$

We shall now show that to render this so, the three generators must belong to three different groups, and that those three groups must not be any one of the seven sets of three, such as α, β, γ, for which $\alpha\beta\gamma \equiv 1$.

Let us denote the exponents of g_1, g_2, g_3 by 2^{s_1}, 2^{s_2}, 2^{s_3}. [When λ is odd these are $2^{\mu-1}$, $2^{\mu-1}$, 2^2, and when λ is even, $2^{\mu-1}$, $2^{\mu-2}$, 2^2.]

First suppose that any two of the generators belong to the *same* group.

Say g_1 and g_2 both belong to the same group; s_1 and s_2 may be equal or unequal.

If $s_1 = s_2$, Lemma (ii.) shows that the exponent of g_1g_2 divides 2^{s_1-1}, therefore

$$(g_1g_2)^{2^{s_1-1}} \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

$$g_1^{2^{s_1-1}} g_2^{2^{s_2-1}} \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

where

$$2^{s_1-1} \not\equiv 0 \pmod{2^{s_1}},$$

$$2^{s_2-1} \not\equiv 0 \pmod{2^{s_2}},$$

therefore g_1, g_2, cannot belong to the same group.

If $s_1 = s_2 + \sigma$, then $g_2^{2^\sigma}$ has exponent s_1, and Lemma (ii.) shows that

$$g_1^{2^{s_1-1}} g_2^{2^{\sigma+s_1-1}} \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

where

$$2^{s_1-1} \equiv 0 \,[\text{mod}\,(1+i)^\lambda],$$

therefore g_1 and g_2 cannot belong to the same group.

Hence the three generators must belong to three different groups.

Next suppose that the three generators belong to three groups, such as α, β, γ. Then

$$g_1^{2^{s_1-1}} g_2^{2^{s_2-1}} g_3^{2^{s_3-1}} \equiv \alpha\beta\gamma \equiv 1 \,[\text{mod}\,(1+i)^\lambda],$$

where

$$2^{s_1-1} \not\equiv 0 \pmod{2^{s_1}},$$

$$2^{s_2-1} \not\equiv 0 \pmod{2^{s_2}},$$

$$2^{s_3-1} \not\equiv 0 \pmod{2^{s_3}},$$

and therefore the three generators must not belong to three such groups.

Finally, we can see that if the three generators are taken from three different groups, excluding the seven sets of three groups, then

$$g_1^{j_1} g_2^{j_2} g_3^{j_3} \equiv 1 \,[\text{mod}\,(1+i)^\lambda]$$

is not possible unless

$$j_1 \equiv 0 \pmod{2^{s_1}},$$

$$j_2 \equiv 0 \pmod{2^{s_2}},$$

$$j_3 \equiv 0 \pmod{2^{s_3}}.$$

Lemma (i.) shows that $g_1^{j_1}$, $g_2^{j_2}$, $g_3^{j_3}$ belong to the same groups as do g_1, g_2, g_3. Suppose that these are α, β, and δ.

The Lemmas (iii.) and (iv.) show that the group to which $g_1^{j_1} g_2^{j_2} g_3^{j}$ belongs is one of

$$\alpha,\ \beta,\ \delta,\quad \alpha\beta,\ \alpha\delta,\ \beta\delta,\quad \text{or}\quad \alpha\beta\delta,$$

i.e.,

$$\alpha,\ \beta,\ \delta,\quad \gamma,\ \epsilon,\ \eta,\quad \text{or}\quad \theta,$$

and in no case is the product congruent to unity, save when each factor is so separately.

The result, therefore, is that for modulus $(1 + i)^\lambda$ we have to take three generators with the exponents determined in the last proposition, and such that the product of the three numbers with exponent 2 that they separately produce shall not be congruent to unity.

Example.—Mod $(1 + i)^8$. $\Phi (1 + i)^8 = 2^7$.

The exponents of the generators are

$$2^3,\ 2^2,\ 2^2.$$

The 2^6 numbers with exp 2^3 are

$$\left.\begin{matrix} 1 + 2i, & 3 + 2i \\ 2 + i, & 2 + 3i \end{matrix}\right\} \text{(mod 4)}.$$

The $2^5 + 2^4 + 2^3$ numbers with exp 2^2 are

$$\left.\begin{matrix} i \\ 3i \end{matrix}\right\} \text{(mod 4) and } 3,\ 5,\ 3 + 4i,\ 5 + 4i,\ 1 + 4i,\ 7 + 4i \text{ (mod 8)}.$$

The $2^2 + 2 + 1$ numbers with exp 2 are

$$15,\ 9,\ 7,\ 1 + 8i,\ 15 + 8i,\ 9 + 8i,\ 7 + 8i.$$

If of these we take

$$2 + i \text{ with exp } 8,$$
$$3 \text{ with exp } 4,$$

and

$$i \text{ with exp } 4,$$

we have

$$(2 + i)^4 \equiv 9 + 8i\ [\text{mod } (1 + i)^8],$$
$$3^2 \equiv 9\ [\text{mod } (1 + i)^8],$$

and

$$i^2 \equiv 15\ [\text{mod } (1 + i)^8],$$

and

$$9 + 8i \,.\, 9 \,.\, 15 \equiv 15 + 8i \not\equiv 1\ [\text{mod } (1 + i)^8],$$

therefore, $2 + i$, 3, and i generate the 2^7 numbers.

Example.—Mod $(1+i)^9$. $\Phi(1+i)^9 = 2^8$. The three generators have exponents $2^3, 2^3, 2^2$.

The $2^7 + 2^6$ numbers with exp 2^3 are (see Proposition xv.)—

$$\equiv 7+2i,\ 5+2i,\ 6+3i,\ 6+i,\ 5,\ 3,\ 4+i,\ 3i,\ 3+2i,\ 1+2i,\ 2+3i,\ 2+i\,[\text{mod}\,(1+i)^5].$$

The $2^5 + 2^4 + 2^3$ numbers with exp 2^2 are

$$\equiv \pm i\,[\text{mod}\,(1+i)^5] \text{ and } 7,\ 9,\ 3+4i,\ 5+4i,\ 11+4i,\ 13+4i\,[\text{mod}\,(1+i)^7].$$

The $2^2 + 2 + 1$ numbers with exp 2, are

$$15,\ 17,\ 31,\ 7+8i,\ 23+8i,\ 25+8i,\ 9+8i.$$

If of these we take

$$2+i \text{ with exp } 8,$$
$$3 \text{ with exp } 8,$$

and

$$i \text{ with exp } 4,$$

we have

$$\left.\begin{array}{r}(2+i)^4 \equiv 9+8i\\ 3^4 \equiv 17\\ i^2 \equiv 15\end{array}\right\} \text{ and } 15\,.\,17\,.\,(9+8i) \equiv 7+8i \not\equiv 1.$$

Therefore $2+i$, 3, and i generate the 2^8 numbers.

(xvii.) From Propositions ix., xii., xiii., and xiv. the exponent of a number for any modulus (to which it is prime) is readily determined.

Let the modulus be expressed as a product of powers of its prime factors,

$$m = (1+i)^{\kappa} \mathrm{P}_1^{\lambda_1} \mathrm{P}_2^{\lambda_2} \ldots \mathrm{Q}_1^{\mu_1} \mathrm{Q}_2^{\mu_2} \ldots,$$

where $\mathrm{P}_1, \mathrm{P}_2, \ldots$ are pure primes, and $\mathrm{Q}_1, \mathrm{Q}_2, \ldots$ are mixed primes, and equal to $\alpha_1 + \beta_1 i,\ \alpha_2 + \beta_2 i \ldots$

Then, by Proposition ix., the exponent of any number a is the L.C.M. of its separate exponents for the moduli

$$(1+i)^{\kappa},\ \mathrm{P}_1^{\lambda_1},\ \mathrm{P}_2^{\lambda_2}, \ldots \mathrm{Q}_1^{\mu_1},\ \mathrm{Q}_2^{\mu_2} \ldots$$

The greatest exponent possible.—The greatest exponent for mod $(1+i)^{\kappa}$ is

$$1 \text{ if } \kappa = 1,$$
$$2 \text{ if } \kappa = 2,$$
$$4 \text{ if } 2 < \kappa < 8,$$
$$\ldots\ldots\ldots\ldots$$
$$2^{\kappa'-1} \text{ if } 8 \leqq \kappa = 2\kappa' \text{ or } 2\kappa' + 1.$$

The greatest exponent for mod $P_1^{\lambda_1}$ is

$$P_1^2 - 1 \text{ if } \lambda_1 = 1,$$
$$P_1^{\lambda_1 - 1}(P_1^2 - 1) \text{ if } \lambda_1 > 2.$$

The greatest exponent for mod $Q_1^{\mu_1}$ is

$$\alpha_1^2 + \beta_1^2 - 1 \text{ if } \mu_1 = 1,$$
$$(\alpha_1^2 + \beta_1^2)^{\mu_1 - 1}(\alpha_1^2 + \beta_1^2 - 1) \text{ if } \mu_1 > 1,$$

and the greatest exponent for mod m is the L.C.M. of these separate greatest exponents.

Primitive roots exist when the greatest exponent is equal to $\Phi(m)$, and

$$\Phi(m) = \Phi(1+i)^{\kappa} \, . \, \Phi(P_1^{\lambda_1}) \, . \, \Phi(P_2^{\lambda_2}) \, . \, . \, . \, \Phi(Q_1^{\mu_1}) \, \Phi(Q_2^{\mu_2}) \, . \, . \, .$$

and

$$\begin{aligned} \Phi(1+i)^{\kappa} &= 1 \text{ if } \kappa = 1 \\ &= 2^{\kappa - 1} \text{ if } \kappa > 1, \\ \Phi(P_1^{\lambda_1}) &= P_1^2 - 1 \text{ if } \lambda_1 = 1 \\ &= P_1^{2(\lambda_1 - 1)}(P_1^2 - 1) \text{ if } \lambda_1 > 2, \\ \Phi(Q_1^{\mu_1}) &= \alpha_1^2 + \beta_1^2 - 1 \text{ if } \mu_1 = 1 \\ &= (\alpha_1^2 + \beta_1^2)^{\mu_1 - 1}(\alpha_1^2 + \beta_1^2 - 1). \end{aligned}$$

The only cases in which the greatest exponent can be equal to $\Phi(m)$ are those in which the separate exponents are each equal to the Φ of the modulus they refer to and are also co-prime.

Hence we have the following moduli possessed of primitive roots :—

(1) The moduli $1 + i$, $(1 + i)^2$, $(1 + i)^3$;

(2) Any pure prime and $(1 + i) \times$ any pure prime ;

(3) A power of a mixed prime and $(1 + i) \times$ a power of a mixed prime.

Example.—To find the exponent of $3 + 2i$ for mod 1000.

$$1000 = 2^3 \, . \, 5^3 = -(1+i)^6 (2+i)^3 (1+2i)^3.$$

The exp of $3 + 2i$ for mod $(1 + i)^6$ is 4.

$$3 + 2i \equiv 1 \pmod{2 + i},$$

and therefore has exponent 1.

$$(3 + 2i)^1 \not\equiv 1 \, [\text{mod } (2 + i)^2]$$

therefore

$$3+2i \text{ has exp } 5^2 \text{ for mod } (2+i)^3.$$

$$3+2i \equiv 2 \;(\text{mod } 1+2i),$$

and therefore has exp 4.

$$(3+2i)^4 \equiv 21 \;[\text{mod } (1+2i)^2]$$

$$\not\equiv 1,$$

therefore

$$3+2i \text{ has exp } 4\,.\,5^2 \text{ for mod } (1+2i)^3.$$

The exponent required is the L.C.M. of 4, 5^2 and $4\,.\,5^2 = 100$.

Therefore the exponent of $3+2i$ for mod 1000 is 100.

(xviii.) Proof identical with (18).

If a be not prime to m and $m = p\mathrm{P}$ where p consists of powers of those primes which occur in a, and P is prime to a, then the series of residues

$$a, a^2, a^3, \ldots a^r, \ldots a^{r+t}, \ldots (\text{mod } m)$$

consists of periods of t terms, the first period beginning at the r^{th} term : where t is the exponent of a for mod P, and r is the least number that makes a^r divisible by p.

Corollary.—When $a \equiv 1$ (mod P) $t = 1$ and the period consists of one term.

When a is divisible by p the first period starts from the first term.

When both these conditions hold good then every power of a is $\equiv a$ (mod m).

Example.—Residues of powers of $2+i$ for mod 10.

$$10 = -(1+i)^2(2+i)(1+2i),$$

$$p = 2+i,$$

$$\mathrm{P} = (1+i)^2(1+2i),$$

therefore $r = 1$, and t is the exponent of $2+i$ for mod $(1+i)^2(1+2i)$.

Now

$$\left.\begin{array}{l} 2+i \equiv i\,[\text{mod } (1+i)^2] \text{ and therefore has exp } 2 \\ \text{and} \\ 2+i \equiv 4 \;(\text{mod } 1+2i) \qquad ,, \qquad ,, \qquad ,, \qquad 2 \end{array}\right\}$$

therefore

$$t = 2.$$

The period consists of two terms and the first period begins at the first term

$$2+i,$$

$$(2+i)^2 \equiv 3+4i,$$

$$(2+i)^3 \equiv 2+i,$$

&c.

Example.—Residues of powers of $(1 + i)$ for mod $100 = (1 + i)^4 (2 + i)^2 (1 + 2i)^2$,

$$p = (1 + i)^4,$$

and therefore

$$r = 4,$$
$$\mathrm{P} = (2 + i)^2 (1 + 2i)^2.$$

$$1 + i \text{ has exp } 4 \text{ mod } 2 + i \text{ and } (1 + i)^4 \not\equiv 1 \,[\text{mod } (2 + 1)^2],$$

therefore

$$1 + i \text{ has exp } 20 \text{ mod } (2 + i)^2.$$

Similarly

$$1 + i \text{ has exp } 4 \text{ mod } 1 + 2i \text{ and } (1 + i)^4 \not\equiv 1 \,[\text{mod } (1 + 2i)^2],$$

therefore

$$1 + i \text{ has exp } 20 \text{ mod } (1 + 2i)^2,$$

and therefore

$$t = 20.$$

The residues are

$$\begin{array}{llllllllll}
1 + i & 2i & 98 + 2i & 96 & 96 + 96i & 92i & 8 + 92i & 16 & 16 + 16i & 32i \\
 & & 68 + 32i & 36 & 36 + 36i & 72i & 28 + 72i & 56 & 56 + 56i & 12i \\
 & & 88 + 12i & 76 & 76 + 76i & 52i & 48 + 52i. & & &
\end{array}$$

Example.—The residues of the powers of $688 + 784i$ for mod 1000.

$$1000 = -(1 + i)^6 (2 + i)^3 (1 + 2i)^3.$$

$$\begin{aligned}
688 + 784i &\equiv 0 \,[\text{mod } (1 + i)^6]. \\
688 + 784i &\equiv 688 - 57\,(784)\,[\text{mod } (2 + i)^3]. \quad \text{(See preface.)} \\
&\equiv 0 \qquad\qquad\qquad\quad [\text{mod } (2 + i)^3].
\end{aligned}$$

and

$$\begin{aligned}
688 + 784i &\equiv 688 + 57\,(784)\,[\text{mod } (1 + 2i)^3]. \\
&\equiv 1 \qquad\qquad\qquad\quad [\text{mod } (1 + 2i)^3].
\end{aligned}$$

Hence, $r = 1$ and $t = 1$, and so all powers of the number $688 + 784i$ are congruent to itself for mod 1000.

$$\begin{array}{lcl@{\qquad\qquad}l}
688 & \qquad & 688 & 784 \\
688 & & 784 & 784 \\
\cline{1-1}\cline{3-4}
504 & & 752 & 136 \\
04 & & 04 & 72 \\
8 & & 6 & 8 \\
\cline{1-1}\cline{3-4}
344 & & 392 & 656 \\
 & & 2 & \\
\cline{3-3}
 & & 784 &
\end{array}$$

therefore

$$(688 + 784i)^2 \equiv (344 - 656 + 784i) \pmod{1000}$$
$$\equiv 688 + 784i \pmod{1000}.$$

(xix. and xx.) The proofs and results are the same as in (19) and (20).

Example.—Mod $10 = 2.\ 5 = -(1+i)^2(2+i)(1+2i)$.

To find a so that

$$a \equiv \alpha_1 \ [\text{mod}\ (1+i)^2]$$
$$\equiv \alpha_2 \ [\text{mod}\ (2+i)]$$
$$\equiv \alpha_3 \ [\text{mod}\ (1+2i)].$$

We shall have

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 \pmod{10},$$

where ξ_1, ξ_2, ξ_3 are found thus:—

$$\xi_1 = (2+i)(1+2i)\,x_1 \equiv 1 \ [\text{mod}\ (1+i)^2],$$
$$5i\,x_1 \equiv 1 \ [\text{mod}\ (1+i)^2],$$
$$i\,x_1 \equiv 1 \ [\text{mod}\ (1+i)^2],$$
$$x_1 \equiv i,$$

and therefore

$$\xi_1 \equiv 5 \pmod{10}.$$
$$\xi_2 = (1+i)^2(1+2i)\,x_2 \equiv 1 \ (\text{mod}\ 2+i),$$
$$(2i-4)\,x_2 \equiv 1 \ (\text{mod}\ 2+i),$$
$$-8x_2 \equiv 1 \ (\text{mod}\ 2+i),$$
$$x_2 \equiv 3 \ (\text{mod}\ 2+i),$$

and therefore

$$\xi_2 \equiv 8 + 6i \pmod{10}.$$
$$\xi_3 = (1+i)^2(2+i)\,x_3 \equiv 1 \ (\text{mod}\ 1+2i),$$
$$(4i-2)\,x_3 \equiv 1 \ (\text{mod}\ 1+2i),$$
$$-4x_3 \equiv 1 \ (\text{mod}\ 1+2i),$$
$$x_3 \equiv 1 \ (\text{mod}\ 1+2i),$$

and therefore

$$\xi_3 \equiv 8 + 4i \pmod{10}.$$

Hence

$$a \equiv 5\alpha_1 + (8+6i)\,\alpha_2 + (8+4i)\,\alpha_3 \pmod{10},$$

e.g., if

$$\left.\begin{array}{l} \alpha_1 \equiv 1 \ [\text{mod}\ (1+i)^2] \\ \alpha_2 \equiv 4 \ (\text{mod}\ 2+i) \\ \alpha_3 \equiv 3 \ (\text{mod}\ 1+2i) \end{array}\right\}$$

then

$$\begin{aligned} a &\equiv 5 + (8 + 6i)\,4 + (8 + 4i)\,.\,3\,.\ (\mathrm{mod}\ 10), \\ &\equiv 5 + 2 + 4i + 4 + 2i\ (\mathrm{mod}\ 10), \\ &\equiv 1 + 6i\ (\mathrm{mod}\ 10). \end{aligned}$$

(xxi.) The number of numbers which belong to a given exponent when the modulus is a power of a prime.

We have three cases to consider

(1) When the modulus is a power of a pure prime.
(2) When the modulus is a power of a mixed prime.
(3) When the modulus is a power of $1 + i$.

(1.) We saw in Proposition xii*a*, that for the modulus p^λ we can generate the $\Phi(p^\lambda) = p^{2(\lambda-1)}(p^2 - 1)$ residues prime to the modulus by three generators

$$\text{and} \qquad \left.\begin{array}{l} f \text{ and } f' \text{ each with exp } p^{\lambda-1} \\ h \text{ with exp } p^2 - 1. \end{array}\right\}$$

How many of the residues have exponent $p^s t$, where $s \not> \lambda - 1$ and t divides $p^2 - 1$?

Of the $p^2 - 1$ powers of h, $\phi(t)$ have exponent t.

Of the $p^{\lambda-1}$ powers of f, p^s have exponent a power of $p \not> p^s$.

Of the $p^{\lambda-1}$ powers of f', p^s have exponent a power of $p \not> p^s$.

Therefore f and f' generate p^{2s} numbers with exponent $\not> p^s$, and similarly p^{2s-2} numbers with exponent $\not> p^{s-1}$.

Therefore f and f' generate $p^{2s} - p^{2s-2}$ numbers with exponent p^s.

Therefore f, f', and g generate $(p^{2s} - p^{2s-2})\,\phi(t)$ numbers with exponent $p^s t$, *i.e.*, the number of numbers with exponent $p^s t$ mod p^λ is $(p^{2s} - p^{2s-2})\,\phi(t)$.

In particular the number of numbers with exponent p^s is $p^{2s} - p^{2s-2}$, and the number of numbers with exponent t is $\phi(t)$ if t is prime to p.

(2.) Any power of a mixed prime, $(\alpha + \beta i)^\lambda$, has primitive roots, and hence, as in (21), the number of numbers with exponent t [any divisor of $(\alpha^2 + \beta^2)^\lambda - (\alpha^2 + \beta^2)^{\lambda-1}$] is $\phi(t)$.

(3.) The number of numbers with exponent a given power of 2 for modulus $(1 + i)^\nu$ was found completely in Propositions xiv. and xv.

(xxi*a*.) It will be convenient for the succeeding propositions to express the number of numbers which have a given exponent for modulus $(1 + i)^\lambda$, in terms of the exponents of the generators. By so doing we shall avoid the detailed discussion of the cases arising from different values of λ, which was necessary in Part I.

Suppose 2^κ, $2^{\kappa'}$, $2^{\kappa''}$ are the exponents of the generators.

The exponent of the product of any powers of the generators is equal to the highest exponent of the three (Proposition viii.).

The generator with exponent 2^κ generates

$$2^s \text{ numbers with exp } \ngtr 2^s \text{ if } \kappa > s,$$
$$2^\kappa \text{ numbers with exp } \ngtr 2^s \text{ if } \kappa \leqq s.$$

Suppose by $(\kappa)_s$ we denote that κ is to be replaced by s if κ exceeds s. Then in either case the generator with exponent 2^κ generates

$$2^{(\kappa)_s} \text{ numbers with exp } \ngtr 2^s.$$

Similarly the two other generators generate respectively

$$2^{(\kappa')_s} \text{ numbers with exp } \ngtr 2^s,$$

and

$$2^{(\kappa'')_s} \text{ numbers with exp } \ngtr 2^s.$$

Therefore the three generators generate

$$2^{(\kappa + \kappa' + \kappa'')_s} \text{ numbers with exp } \ngtr 2^s,$$

i.e., there are, for modulus $(1 + i)^\lambda$,

$$2^{(\kappa + \kappa' + \kappa'')_s} \text{ numbers with exp } \ngtr 2^s.$$

Similarly there are

$$2^{(\kappa + \kappa' + \kappa'')_{s-1}} \text{ numbers with exp } \ngtr 2^{s-1}.$$

Hence there are

$$2^{(\kappa + \kappa' + \kappa'')_s} - 2^{(\kappa + \kappa' + \kappa'')_{s-1}}$$

numbers with exponent 2^s for modulus $(1 + i)^\lambda$.

This result clearly holds good when one or two of the κ's is absent, as is the case when λ is less than 5.

(xxii.) The number of numbers, for modulus m, each of which has as exponent some power of a prime p, p being a divisor of $\Phi(m)$.

Let

$$m = (1 + i)^\kappa P_1^{\lambda_1} P_2^{\lambda_2} \ldots Q_1^{\mu_1} Q_2^{\mu_2} \ldots$$

where $P_1, P_2, \ldots$ are pure primes and $Q_1, Q_2, \ldots$ are mixed primes.

$$\Phi(m) = \Phi(1 + i)^\kappa \,.\, \Phi(P_1^{\lambda_1}) \,.\, \Phi(P_2^{\lambda_2}) \ldots \Phi(Q_1^{\mu_1})\, \Phi(Q_2^{\mu_2}) \ldots$$
$$\Phi(1 + i)^\kappa = 2^{\kappa_0} 2^{\kappa_0'} 2^{\kappa_0''}$$

where 2^{κ_0}, $2^{\kappa_0'}$, $2^{\kappa_0''}$, are the exponents of the generators for mod $(1 + i)^\kappa$, and suppose that

$$\Phi(P_1^{\lambda_1}) = P_1^{2(\lambda_1 - 1)} (P_1^2 - 1) = 2^{\kappa_1} p^{l_1} q^{m_1} \ldots$$
$$\Phi(P_2^{\lambda_2}) = P_2^{2(\lambda_2 - 1)} (P_2^2 - 1) = 2^{\kappa_2} p^{l_2} q^{m_2} \ldots$$
$$\&c. \qquad\qquad \&c.$$

With regard to these we shall follow a convention which will be useful in simplifying the next proposition.

Firstly, we write $2^{\kappa_0}2^{\kappa_0'}2^{\kappa_0''}$, and not $2^{\overline{\kappa_0+\kappa_0'+\kappa_0''}}$. Thus the value of $(\Sigma\kappa)_s$ will be $(\kappa_0+\kappa_0'+\kappa_0''+\kappa_1+\kappa_2+\ldots)_s$, and not $\overline{(\kappa_0+\kappa_0'+\kappa_0''+\kappa_1+\kappa_2+\ldots)_s}$.

Secondly, in $2^{\kappa_1}p^{l_1}q^{m_1}\ldots$ occurs the prime P_1 raised to the power $2(\lambda_1-1)$.

We shall suppose it written $P_1^{\lambda_1-1}\,.\,P_1^{\lambda_1-1}$, and not $P_1^{2(\lambda_1-1)}$.

For the rest of the principal factors no such arrangement is necessary.

Let

$$\begin{array}{l} \Phi(Q_1^{\mu_1}) = 2^{\kappa_1'}p_1^{l_1'}q_1^{m_1'}\ldots \\ \Phi(Q_2^{\mu_2}) = 2^{\kappa_2'}p_1^{l_2'}q_1^{m_2'}\ldots \\ \quad\&c. \qquad\qquad \&c. \end{array}$$

As in (22) the number of numbers which have some power of p as exponent is the product of the number of such numbers for each separate modulus $(1+i)^\kappa$, $P_1^{\lambda_1}$, &c.

$$\left.\begin{array}{llllll} \text{Now the number of numbers for mod} & P_1^{\lambda_1} & \text{is the power of } p \text{ in} & \Phi(P_1^{\lambda_1}) \\ \quad ,, \qquad ,, \qquad ,, & P_2^{\lambda_2} & \quad ,, \qquad ,, & \Phi(P_2^{\lambda_2}) \\ \quad ,, \qquad ,, \qquad ,, & Q_1^{\mu_1} & \quad ,, \qquad ,, & \Phi(Q_1^{\mu_1}) \\ & \&c. & & \end{array}\right\}\text{Prop. (xxi.).}$$

Hence the number of numbers with exponent a power of p is $p^{\Sigma(l)}$.

In particular the number of numbers with exponent a power of 2 is $2^{\Sigma(\kappa)}$.

(xxiii.) The number of numbers having a given exponent p^s for modulus m, p^s being a divisor of the greatest exponent.

Any such number has a power of p or unity for its exponent for each of the moduli $(1+i)^\kappa$, $P_1^{\lambda_1}$, . . . &c., and the greatest power of p among these exponents must be p^s.

As in (23) the number of numbers, modulus $P_1^{\lambda_1}$, which have as exponent a power of p not greater than p^s is (if P_1 is not ρ) $p^{(l_1)_s}$.

In the particular case when P_1 is the prime p, if

$$\left.\begin{array}{l} \lambda_1 - 1 \text{ is } \geqq s \text{ there are } p^{2s} \text{ numbers, mod } P_1^{\lambda_1} \text{ with exp a power of } p \ngtr p^s \\ \text{and if} \\ \lambda_1 - 1 \text{ is } < s \text{ there are } p^{2(\lambda_1-1)} \quad ,, \qquad ,, \qquad ,, \qquad ,, \end{array}\right\}$$

In either case the number is $p^{(\overline{\lambda_1-1}+\overline{\lambda_1-1})_s}$.

Now we have arranged that in $\Phi(P_1^{\lambda_1})$ the power of P_1 (p in this case) shall be written $p^{l_1}p^{l_1}$, and not p^{2l_1}: thereby we easily express the number of numbers with exponent a power of $p \ngtr p^s$ for modulus $P_1^{\lambda_1}$, which is $p^{(l_1)_s}p^{(l_1)_s}$ or $p^{(l_1+l_1)_s}$, but which is *not* $p^{(2l_1)_s}$.

For modulus $Q_1^{\mu_1}$ the number of numbers with exponent a power of $p \ngtr p^s$ is $p^{(l_1')_s}$.

Multiplying these numbers we get $p^{(\Sigma l)_s}$ as the number of numbers with exponent a power of $p \ngtr p^s$.

Hence the number of numbers with exponent p^s modulus m is

$$p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}.$$

Example.—Mod $3^3(3+2i)$. The greatest exponent is the L.C.M. of $3^2(3^2-1)$ and $12, = 72$.

We will find how many numbers have exponent 3^2, and how many numbers have exponent 3.

$$\Phi(3^3) = 2^3 . 3^2 . 3^2.$$
$$\Phi(3+2i) = 2^2 . 3.$$

The number of numbers with exponent 3^2 is

$$3^{(2+2+1)_2} - 3^{(2+2+1)_1}$$
$$= 3^5 - 3^3.$$

The number of numbers with exponent 3 is

$$3^{(2+2+1)_1} - 1$$
$$= 3^3 - 1.$$

Hence

$$\left.\begin{array}{lll} & 3^5 - 3^3 & \text{numbers have exp } 3^2. \\ & 3^3 - 1 & \text{numbers have exp } 3. \\ \text{and} & & \\ & 1 & \text{number has exp } 1. \end{array}\right\}$$

(xxiii*a*.) By writing $\Phi(1+i)^\kappa$ in the form $2^{\kappa_0}2^{\kappa_0'}2^{\kappa_0''}$ we are enabled to apply exactly the same method to this case as we have to the case of any odd prime p. The result we obtain is that the number of numbers with exponent 2^s for mod m is

$$2^{(\Sigma\kappa)_s} - 2^{(\Sigma\kappa)_{s-1}}.$$

Example.—Mod $12 + 4i = i(1+i)^5(1+2i)$. Highest exponent $= 2^2$. $\Phi(m) = 2^6$.

$$\Phi(1+i)^5 = 2^2 . 2 . 2.$$
$$\Phi(1+2i) = 2^2.$$

The number of numbers with exponent 2^2

$$= 2^{(2+1+1+2)_2 - (2+1+1+2)_1}$$
$$= 2^6 - 2^4.$$

The number of numbers with exponent 2

$$= 2^{(2+1+1+2)_1} - 1$$

$$= 2^4 - 1.$$

Thus there are

$$\left.\begin{array}{llll} 2^6 - 2^4 & \text{numbers with exp} & & 4. \\ 2^4 - 1 & \quad ,, & ,, & 2. \\ 1 & \quad ,, & ,, & 1. \end{array}\right\}$$

Corollary to xxiii. and xxiii*a*.—The number of numbers that belong to any exponent is simply the product of the number of numbers that belong to each of its principal factors as exponents.

Thus, for mod m, the number of numbers with exp $t = p^s q^\sigma \ldots$ is

$$[p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}][q^{(\Sigma m)_\sigma} - q^{(\Sigma m)_{\sigma-1}}] \ldots$$

Example.—Mod $(1+i)^6 . 3^2 (3+2i)(4+i) = -i . 72 (10 + 11i)$.

$$\Phi(1+i)^6 = 2. 2^2. 2^2.$$

$$\Phi(3^2) = 2^3. 3. 3.$$

$$\Phi(3+2i) = 2^2. 3.$$

$$\Phi(4+i) = 2^4.$$

The highest exponent is $2^4. 3^3$.

$$\Phi(m) = 2^{14}. 3^3.$$

The numbers κ are 1. 2. 2. 3. 2. 4.

$$(\Sigma\kappa)_4 = 14,$$

$$(\Sigma\kappa)_3 = 13,$$

$$(\Sigma\kappa)_2 = 11,$$

$$(\Sigma\kappa)_1 = 6,$$

$$(\Sigma\kappa)_0 = 0.$$

Therefore

$$\begin{array}{llllr} 2^{14} - 2^{13} & \text{numbers have exp} & & & 2^4, \\ 2^{13} - 2^{11} & \quad ,, & ,, & ,, & 2^3, \\ 2^{11} - 2^6 & \quad ,, & ,, & ,, & 2^2, \\ 2^6 - 1 & \quad ,, & ,, & ,, & 2. \end{array}$$

The numbers l for the prime 3 are 1. 1. 1.

$$(\Sigma l)_1 = 3,$$
$$(\Sigma l)_0 = 0,$$

therefore $3^3 - 1$ numbers have exp 3.

From these we deduce that

$(3^3 - 1)(2^{14} - 2^{13})$ numbers have exp $3.\ 2^4$,
$(3^3 - 1)(2^{13} - 2^{11})$,, ,, ,, $3.\ 2^3$,
$(3^3 - 1)(2^{11} - 2^{6})$,, ,, ,, $3.\ 2^2$,
$(3^3 - 1)(2^{6} - 1)$,, ,, ,, $3.\ 2$.

Including 1 which has exponent 1, this makes the complete set of $2^{14}.\ 3^3$ numbers.

(xxiv.) A special set of generators which generate the $\Phi(m)$ numbers, modulus m.

$$m = (1 + i)^{\kappa} \mathrm{P}_1^{\lambda_1} \dots \mathrm{Q}_1^{\mu_1} \dots$$

Three numbers ϕ, ϕ', ϕ'', generate the residues for modulus $(1 + i)^{\kappa}$. (Propositions xiii. and xvi.) Two numbers g_1, f_1, generate the residues for modulus $\mathrm{P}_1^{\lambda_1}$. (Proposition xii*a*.), &c. One number g'_1 generates the residues for modulus $\mathrm{Q}_1^{\mu_1}$. (Proposition xii.), &c.

Suppose any number a modulus m is

$$\equiv \alpha_0 \,[\mathrm{mod}\ (1 + i^{\kappa})]$$
$$\equiv \alpha_1 \,(\mathrm{mod}\ \mathrm{P}_1^{\lambda_1}),$$
$$\&\text{c.}$$
$$\equiv \alpha'_1 \,(\mathrm{mod}\ \mathrm{Q}_1^{\mu_1}),$$
$$\&\text{c.}$$

Then

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \dots + \alpha'_1\xi'_1 + \dots \ (\mathrm{mod}\ m). \quad \text{(Proposition xix.)}$$

If now

$$\alpha_0 \equiv \phi^{i_0}\phi'^{i'_0}\phi''^{i''_0} \,[\mathrm{mod}\ (1 + i)^{\kappa}]$$
$$\alpha_1 \equiv g_1^{i_1} f_1^{j_1} \,(\mathrm{mod}\ \mathrm{P}_1^{\lambda_1}),$$
$$\&\text{c.}$$
$$\alpha'_1 \equiv g'^{\,i'_1}_1 \,(\mathrm{mod}\ \mathrm{Q}_1^{\mu_1}),$$
$$\&\text{c.}$$

Thus

$$\begin{aligned} a &\equiv \phi^{i_0}\phi'^{i'_0}\phi''^{i''_0}\xi_0 + g_1^{i_1} f_1^{j_1}\xi_1 + \dots + g'^{\,i'_1}_1\xi'_1 + \dots \ (\mathrm{mod}\ m) \\ &\equiv [\phi\xi_0 + \xi_1 + \dots + \xi'_1 + \dots]^{i_0} [\phi'\xi_0 + \xi_1 + \dots + \xi'_1 + \dots]^{i'_0} \\ &\qquad [\phi''\xi_0 + \xi_1 + \dots + \xi'_1 + \dots]^{i''_0} [\xi_0 + g_1\xi_1 + \dots + \xi'_1 + \dots]^{i_1} \\ &\qquad\quad [\xi_0 + f_1\xi_1 + \dots]^{j_1} \dots [\xi_0 + \xi_1 + \dots + g'_1\xi'_1 + \dots]^{i'_1} \dots \ (\mathrm{mod}\ m) \\ &\equiv [(\phi - 1)\,\xi_0 + 1]^{i_0} [(\phi' - 1)\,\xi_0 + 1]^{i'_0} [(\phi'' - 1)\,\xi_0 + 1]^{i''_0} [(g_1 - 1)\,\xi_1 + 1]^{i_1} \\ &\qquad\qquad [(f_1 - 1) + 1]^{j_1} \dots [(g'_1 - 1) + 1]^{i'_1} \dots \ (\mathrm{mod}\ m), \end{aligned}$$

where the numbers in the square brackets are a set of generators whose exponents are, in order, κ_0, κ'_0, κ''_0, $P_1^{\lambda_1-1}(P_1^2-1)$, $P_1^{\lambda_1-1}$, . . . $\Phi(Q_1^{\mu_1})$ . . . whose product is equal to $\Phi(m)$.

Example.—Mod $10 = -(1+i)^2(2+i)(1+2i)$

$$\begin{array}{llllllll} i & \text{with exp } 2 & \text{generates the } 2 & \text{numbers mod} & (1+i)^2 \\ 2 & \quad ,, \quad 4 & \quad ,, \quad 4 & \quad ,, & 2+i \\ 2 & \quad ,, \quad 4 & \quad ,, \quad 4 & \quad ,, & 1+2i. \end{array}$$

Hence the following will generate the $\Phi(10) = 32$ numbers, mod 10, viz.,

$$\left.\begin{array}{r} i\xi_1 + \xi_2 + \xi_3 \\ \xi_1 + 2\xi_2 + \xi_3 \\ \xi_1 + \xi_2 + 2\xi_3 \end{array}\right\} \text{mod } 10,$$

where

$\xi_1 = 5 \quad \xi_2 = 8+6i \quad \xi_3 = 8+4i.$ (See Example, Proposition xix.)

$$\left.\begin{array}{l} 5i + 8+6i + 8+4i \equiv 6+5i \\ 5 + 2(8+6i) + 8+4i \equiv 9+6i \\ 5 + 8+6i + 2(8+4i) \equiv 9+4i \end{array}\right\} (\text{mod } 10).$$

Hence

$$\left.\begin{array}{lll} 9+4i & \text{with exp} & 4 \\ 9+6i & \quad ,, & 4 \\ 6+5i & \quad ,, & 2 \end{array}\right\} \text{generate the 32 residues.}$$

The indices corresponding to each of the 32 numbers are given in the following table.

Numbers.	Indices of			Numbers.	Indices of		
	6 + 5*i*.	9 + 6*i*.	9 + 4*i*.		6 + 5*i*.	9 + 6*i*.	9 + 4*i*.
1	0	0	0	6 + 5*i*	1	0	0
9 + 4*i*	0	0	1	4 + 9*i*	1	0	1
5 + 2*i*	0	0	2	7*i*	1	0	2
7 + 8*i*	0	0	3	2 + 3*i*	1	0	3
9 + 6*i*	0	1	0	4 + *i*	1	1	0
7	0	1	1	2 + 5*i*	1	1	1
3 + 8*i*	0	1	2	8 + 3*i*	1	1	2
5 + 4*i*	0	1	3	9*i*	1	1	3
5 + 8*i*	0	2	0	3*i*	1	2	0
3 + 2*i*	0	2	1	8 + 7*i*	1	2	1
9	0	2	2	4 + 5*i*	1	2	2
1 + 6*i*	0	2	3	6 + *i*	1	2	3
7 + 2*i*	0	3	0	2 + 7*i*	1	3	0
5 + 6*i*	0	3	1	*i*	1	3	1
1 + 4*i*	0	3	2	6 + 9*i*	1	3	2
3	0	3	3	8 + 5*i*	1	3	3

E.g., $6 + i \equiv (6+5i)(9+6i)^2(9+4i)^3$ (mod 10).

(xxv.) Proposition (25) is completely applicable to the case of complex numbers and moduli. The result arrived at we may re-state as follows:—The most general set of numbers which generate the $\Phi\,(m)$ residues for modulus m must always be constructed in this manner, viz., we must form

$$\left.\begin{array}{c}\text{a set of } p\text{-power-exponent generators}\\ \text{a set of } q\text{-power-exponent generators}\\ \text{\&c.} \qquad\qquad \text{\&c.}\end{array}\right\}\ p,\ q,\ \ldots \text{ being the prime factors of } \Phi\,(m),$$

each generator must then be formed by taking numbers from these sets, not more than one from each, and forming their product. Each number, moreover, in these subsidiary sets is to appear once and once only as a factor of one of the generators that are thus formed.

It remains to investigate the most general mode of formation of a set of p-power-exponent generators.

(xxvi.) The proof of Proposition (26) holds good for complex numbers and moduli. It shows that the exponents of any set of p-power-exponent generators must be the same set of powers of p as those which occur in the Φ's of the principal factors of m; *i.e.*, they are what have been denoted by $p^{l_1}, p^{l_2}, \ldots p^{l'_1}, p^{l'_2}, \ldots$ (see Proposition xxii., in which the convention stated must be strictly attended to). The convention of Proposition xxii. makes the treatment of the 2-power-exponent numbers uniform with that of the p-power-exponent numbers; hence the same result is true for the 2-power-exponent numbers, viz., that the exponents of any set of 2-power-exponent generators must be $2^{\kappa_0}, 2^{\kappa'_0}, 2^{\kappa''_0}, 2^{\kappa_1}, \ldots 2^{\kappa'_1} \ldots$

The least number of generators for a given modulus m.

As in (26) we see that the least number of generators is equal to the number of terms of that row which contains most among the following:—

$$\begin{array}{cccccccc}\kappa_0 & \kappa'_0 & \kappa''_0 & \kappa_1 & \kappa_2 \ldots & \kappa'_1 & \kappa'_2 \ldots\\ & & & l_1 & l_2 \ldots & l'_1 & l'_2 \ldots\\ & & & m_1 & m_2 \ldots & m'_1 & m'_2 \ldots\end{array}$$

Consider first the set of numbers, $\kappa_0\ \kappa'_0\ \kappa''_0\ \kappa_1 \ldots \kappa_1 \ldots$

Since P_1 is an odd prime, therefore

$$\Phi\,(P_1^{\lambda_1}) = P_1^{2(\lambda_1-1)}\,(P_1^2 - 1)$$

is even, and therefore κ_1 always occurs.

Similarly in

$$\Phi\,(Q_1^{\mu_1}) = (\alpha_1^2 + \beta_1^2)^{\mu_1-1}\,(\alpha_1^2 + \beta_1^2 - 1)$$

$\alpha_1^2 + \beta_1^2$ is a real odd prime number, and therefore

$$\Phi\,(Q_1^{\mu_1})$$

is even, and therefore κ'_1 always occurs.

For the numbers, $\kappa_0\ \kappa'_0\ \kappa''_0$, all three occur if

$$\kappa > 4 \qquad [m = (1+i)^\kappa P_1^{\lambda_1} \ldots Q_1^{\mu_1} \ldots],$$

two occur if

$$\kappa = 4,$$

one occurs if

$$\kappa = 3,\ 2,\ 1,$$

none occur if

$$\kappa = 0.$$

Hence, if we denote by n the number of different prime factors, $P_1 P_2 \ldots Q_1 Q_2 \ldots$ of m (excluding $1 + i$) then the number of terms in the set of numbers $\kappa_0\ \kappa'_0\ \kappa''_0\ \kappa_1 \ldots \kappa'_1 \ldots$ is

$$\begin{array}{lll} n+3 & \text{if} & \kappa > 4, \\ n+2 & \text{if} & \kappa = 4, \\ n+1 & \text{if} & \kappa = 3,\ 2,\ 1, \\ n & \text{if} & \kappa = 0. \end{array}$$

These numbers give an inferior limit to the least number of generators. In all but one exceptional case the least number of generators coincides with these.

Consider next the set of numbers, $l_1\ l_2 \ldots l_1'\ l_2' \ldots$

l_1' occurs if p divides

$$\Phi\,(P_1^{\lambda_1}) = P_1^{2(\lambda_1 - 1)}(P_1^2 - 1).$$

If it divides

$$P_1^2 - 1$$

it occurs once. If it is identical with P_1 then it occurs twice ($l_1 = \lambda_1 - 1$), and the set of numbers would be written $l_1\ l_1\ l_2 \ldots l_1'\ l_2' \ldots$

l_2 occurs if p divides

$$\Phi\,(Q_1^{\mu_1}).$$

Therefore the number of terms in the set exceeds n (and is equal to $n + 1$) only when m is such that

$$\Phi\,(Q_1^{\mu_1}),\ \Phi\,(Q_2^{\mu_2}), \ldots\ \Phi\,(P_1^{\lambda_1}),\ \Phi\,(P_2^{\lambda_2}) \ldots$$

are all divisible by p and also one of the primes, $P_1, P_2, \ldots$ is equal to p.

If this be the case when $\kappa = 0$ the least number of generators is $n + 1$.

The result may be stated thus :—

If n be the number of different prime factors of $m = (1 + i)^\kappa P_1^{\lambda_1} \ldots Q_1^{\mu_1}, \ldots$ excluding $(1 + i)$, then the least number of generators is

$$\begin{aligned} &n + 3 \text{ if } \kappa > 4, \\ &n + 2 \text{ if } \kappa = 4, \\ &n + 1 \text{ if } \kappa = 3, 2, 1, \\ &n \quad\quad \text{ if } \kappa = 0, \end{aligned}$$

unless one of the primes P_1 divides $P_2^2 - 1$, $P_3^2 - 1, \ldots$ and $\Phi(Q_1^{\mu_1})$, $\Phi(Q_2^{\mu_2}), \ldots$ in which case it is $n + 1$.

(xxvii.) The formation of a set of *unitary* p-power-exponent generators for mod m.

Let a be any p-power-exponent number, and

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots + \alpha'_1\xi'_1 + \alpha'_2\xi'_2 + \ldots \pmod{m}.$$

Then $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha'_1, \ldots$ must each have as exponent, for its own modulus, either unity or a power of p.

Suppose first that p is not $= 2$, but an odd (real) prime.

Take

$$\begin{aligned} &\gamma_1 \text{ with exp } p^{l_1} \pmod{P_1^{\lambda_1}}, \\ &\gamma_2 \quad ,, \quad ,, \quad p^{l_2} \pmod{P_2^{\lambda_2}}, \\ &\quad \&c., \qquad\qquad \&c., \end{aligned}$$

if in any case p is not a factor of $\Phi(P^\lambda)$ γ is $\equiv 1$.

If for one of these, say P_1, $p = P_1$, then take also β_1 with exponent p^{l_1}, modulus $P_1^{\lambda_1}$, as in Proposition xii., so that β_1 and γ_1, each with exponent $l_1 = \lambda_1 - 1$, generate all the p-power-exponent numbers, modulus $P_1^{\lambda_1}$.

Take also

$$\begin{aligned} &\gamma'_1 \text{ with exp } p^{l'_1}, \text{ mod } Q_1^{\mu_1}, \\ &\gamma'_2 \quad ,, \quad ,, \quad p^{l'_2}, \quad ,, \quad Q_2^{\mu_2}, \\ &\quad \&c., \qquad\qquad \&c. \end{aligned}$$

Then we have

$$\alpha_0 \equiv 1 \, [\text{mod } (1 + i)^\kappa],$$

and can put

$$\begin{aligned} &\alpha_1 \equiv \gamma_1^{i_1}\beta_1^{j_1} \pmod{P_1^{\lambda_1}}, \\ &\alpha_2 \equiv \gamma_2^{i_2} \pmod{P_2^{\lambda_2}}, \\ &\quad \&c., \qquad \&c., \\ &\alpha'_1 \equiv \gamma'^{\,i'_1}_1 \pmod{Q_1^{\mu_1}}, \\ &\alpha'_2 \equiv \gamma'^{\,i'_2}_2 \pmod{Q_2^{\mu_2}}, \\ &\quad \&c., \end{aligned}$$

and so

$$a \equiv [(\gamma_1 - 1)\,\xi_1 + 1]^{i_1}\,[(\beta_1 - 1)\,\xi_1 + 1]^{j_1}\,[(\gamma_2 - 1)\,\xi_2 + 1]^{i_2} \ldots$$
$$[(\gamma'_1 - 1)\,\xi'_1 + 1]^{i'_1}\,[(\gamma'_2 - 1)\,\xi'_2 + 1]^{i'_2} \ldots \pmod{m}.$$

The quantities in the brackets we shall denote by $g_1 g_2 \ldots g_\mu$. Each is congruent to unity for all but one of the principal factors of m as moduli. Their exponents are the exponents of any set of p-power-exponent generators (Proposition xxvi.), viz., $p^{l_1}, p^{l_2}, \ldots$

Next let $p = 2$.

Then take ϕ, ϕ', ϕ'', generators for mod $(1 + i)^\kappa$

γ_1 a number with exp 2^{κ_1} (mod $P_1^{\lambda_1}$),

&c. &c.

γ'_1 a number with exp $2^{\kappa'_1}$ (mod $Q_1^{\mu_1}$),

&c. &c.

Then

$$\alpha_0 \equiv \phi^j \phi'^{j'} \phi''^{j''} \; [\text{mod } (1 + i)^\kappa]$$
$$\alpha_1 \equiv \gamma_1^{i_1} \pmod{P_1^{\lambda_1}},$$
&c.,
$$\alpha'_1 \equiv \gamma'^{\,i'_1}_1 \pmod{Q_1^{\mu_1}},$$
&c.

and, therefore,

$$a \equiv [(\phi - 1)\,\xi_0 + 1]^j\,[(\phi' - 1)\,\xi_0 + 1]^{j'}\,[(\phi'' - 1)\,\xi_0 + 1]^{j''}\,[(\gamma_1 - 1)\,\xi_1 + 1]^{i_1} \ldots$$
$$[(\gamma'_1 - 1)\,\xi'_1 + 1]^{i'_1} \ldots \pmod{m},$$

and the numbers in brackets are unitary generators of the numbers, modulus m, with exponent powers of 2.

Each is congruent to unity for all but one of the principal factors of m, and their exponents are $2^{\kappa_0}\, 2^{\kappa'_0}\, 2^{\kappa''_0}\, 2^{\kappa_1}\, 2^{\kappa_2} \ldots 2^{\kappa'_1}\, 2^{\kappa_2'} \ldots$

Thus, in either case, whether p is an odd prime or equal to 2, we can form a set of p-power-exponent generators (having the exponents found to be necessary in Proposition xxvi.), such that each is congruent to unity for all but one of the principal factors of m $[(1 + i), P_1^{\lambda_1}, \ldots Q_1^{\mu_1} \ldots]$ as moduli.

(xxviii–xxxi.) Propositions 28–30 are concerned with a discussion of the most general mode of formation of a set of p-power-exponent generators. They are throughout completely applicable to the case of complex numbers and moduli. The result we may state as follows. Let $g_1, g_2, \ldots g_\mu$ be a set of unitary p-power-exponent generators.

Then $\Gamma_1\, \Gamma_2 \ldots \Gamma_\mu$ (with the same set of powers of p as exponents) as given by

$$\left.\begin{array}{l}\Gamma_1 \equiv g_1^{i_{11}}\, g_2^{i_{21}}\, g_3^{i_{31}}\, , \ldots g_\mu^{i_{\mu 1}} \\ \Gamma_2 \equiv g_1^{i_{12}}\, g_2^{i_{22}}\, g_3^{i_{32}} \ldots g_\mu^{i_{\mu 2}} \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \Gamma_\mu \equiv g_1^{i_{1\mu}}\, g_2^{i_{2\mu}}\, g_3^{i_{3\mu}} \ldots g_\mu^{i_{\mu\mu}} \end{array}\right\} \pmod{m}$$

will be independent generators provided that the determinants formed by the indices i,

$$\left.\begin{array}{l}(i_{11}\, i_{22} \ldots i_{aa}) \\ (i_{a+1\ a+1} \ldots i_{bb}) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (\ldots\ldots\ldots\, i_{\mu\mu}) \end{array}\right\} \text{are all prime to } p.$$

The indices i which occur in any one of these determinants, are those which occur as indices of generators g all with the same exponent; the generators Γ, in which they occur, having also this same exponent.

The summary of Proposition (31) is also true of generators for complex moduli.

(xxxii. and xxxiii.) With one modification, Propositions (22) and (23) hold good for complex numbers and primes.

If in

$$ax^n \equiv b \pmod{m}$$

a and m have G.C.M κ, then

$$\frac{a}{\kappa}\, x^n \equiv \frac{b}{\kappa} \left(\text{mod } \frac{m}{\kappa}\right).$$

Each solution, x, of the second congruence gives N (κ) solutions of the first, viz., all the numbers $x + s\,\frac{m}{\kappa}$, where s is any one of the N (κ) incongruent residues for modulus κ.

The solutions of the congruences which follow are intended as examples of these propositions and also as illustrations of the Tables placed in the Appendix.

Example 1.

$$7x^3 \equiv 3 \pmod{4 + 2i}.$$

From the tables

$$\begin{aligned} 3 &\equiv (\overset{(4)}{3}.\ \overset{(2)}{0}) \\ 7 &\equiv (1.\ 0), \end{aligned}$$

therefore

$$x^3 \equiv (2.\ 0),$$

therefore

$$\begin{aligned} x &\equiv (2.\ 0) \\ x &\equiv 9 \pmod{4 + 2i}. \end{aligned}$$

Example 2.

$$(2+i)\,x^3 \equiv 3+2i \pmod{9}.$$

From the tables

$$3+2i \equiv (\overset{(24)}{22}.\ \overset{(3)}{0})$$

$$2+i \equiv (1.\ 0),$$

therefore

$$x^3 \equiv (21.\ 0),$$

therefore, if

$$x \equiv (a.\ b),$$

$$3a \equiv 21 \pmod{24},$$

therefore

$$a \equiv 7,\ 15,\ 23 \pmod{24}$$

$$3b \equiv 0 \pmod{3},$$

therefore

$$b \equiv 0,\ 1,\ 2 \pmod{3}.$$

There are thus nine solutions, viz. :

$$\begin{array}{lll}
(7.\ 0) \equiv 1+7i & (15.\ 0) \equiv 7+7i & (23.\ 0) \equiv 4+7i \\
(7.\ 1) \equiv 1+4i & (15.\ 1) \equiv 7+4i & (23.\ 1) \equiv 4+4i \\
(7.\ 2) \equiv 1+i & (15.\ 2) \equiv 7+i & (23.\ 2) \equiv 4+i
\end{array}$$

Example 3.

$$3x^3 \equiv 3+2i \pmod{10}.$$

From the tables

$$3+2i \equiv (\overset{(4)}{1}.\ \overset{(4)}{2}.\ \overset{(2)}{0})$$

$$3 \equiv (3.\ 3.\ 0),$$

therefore

$$x^3 \equiv (2.\ 3.\ 0),$$

and therefore

$$x \equiv (2.\ 1.\ 0)$$

$$x \equiv 3+8i.$$

Example 4.

$$3x^3 \equiv 3+2i \pmod{30+20i},$$

and so

$$3\,\frac{x^3}{3+2i} \equiv 1 \pmod{10}.$$

Let

$$x = (3+2i)\,\xi.$$

Then

$$3\,(3+2i)^2\,\xi^3 \equiv 1 \pmod{10}$$

Now

$$3 + 2i \equiv (\overset{(4)}{1}.\ \overset{(4)}{2}.\ \overset{(2)}{0}),$$

therefore

$$(3 + 2i)^2 \equiv (2.\ 0.\ 0);$$

also

$$3 \equiv (3.\ 3.\ 0),$$

therefore

$$3\,(3 + 2i)^2 \equiv (1.\ 3.\ 0),$$

therefore

$$\xi^3 \equiv (3.\ 1.\ 0),$$

therefore, if

$$\xi \equiv (a.\ b.\ c)$$

$$\left.\begin{array}{l} 3a \equiv 3 \pmod{4} \\ 3b \equiv 1 \pmod{4} \\ 3c \equiv 0 \pmod{2} \end{array}\right\} \begin{array}{l} a = 1 \\ b = 3 \\ c = 0 \end{array}$$

$$\begin{aligned} \xi &\equiv (1.\ 3.\ 0) \\ &\equiv 5 + 6i \pmod{10}, \end{aligned}$$

and therefore

$$\begin{aligned} x &\equiv (3 + 2i)(5 + 6i) \pmod{10\ \overline{3 + 2i}} \\ &\equiv 3 + 28i \\ &\equiv 103 + 8i \pmod{10\,.\,\overline{3 + 2i}}. \end{aligned}$$

Example 5.

$$4x^5 \equiv 14 + 7i \pmod{15 + 10i}.$$

This is

$$4x^5 \equiv 7\,(2 + i) \pmod{\overline{2 + i}.\ \overline{1 + 2i}.\ \overline{3 + 2i}},$$

therefore

$$4\,\frac{x^5}{2 + i} \equiv 7 \pmod{8 + 1}.$$

Let

$$x = (2 + i)\,\xi.$$

Then

$$4\,(2 + i)^4 \xi^5 \equiv 7 \pmod{8 + i}.$$

From the tables

$$\begin{aligned} 7 &\equiv (\overset{(12)}{11}.\ \overset{(4)}{2}), \\ (2 + i) &\equiv 59 \pmod{8 + i}, \\ &\equiv (11.\ 3), \end{aligned}$$

therefore

$$\begin{aligned} (2 + i)^4 &\equiv (8.\ 0), \\ 4 &\equiv (2.\ 0), \end{aligned}$$

therefore

$$(2+i)^4 . 4 \equiv (10.\ 0),$$

therefore

$$\xi^5 \equiv (1.\ 2),$$

therefore, if

$$\xi \equiv (a.\ b),$$

$$5a \equiv 1 \pmod{12}, \quad a \equiv 5,$$

$$5b \equiv 2 \pmod{4}, \quad b \equiv 2,$$

$$\xi \equiv (5.\ 2),$$

$$\equiv 58 \pmod{8+i}.$$

So

$$x \equiv 58\,(2+i) \pmod{15+10i}$$

$$\equiv 1 + 3i.$$

Example 6.

$$2x^2 \equiv 26 + 32i \pmod{40},$$

and therefore

$$x^2 \equiv 13 + 16i \pmod{20}.$$

Each solution of the latter gives four of the former, viz.,

$$x, \quad x+20, \quad x+20i, \quad x+20+20i \pmod{40}.$$

The congruence is

$$x^2 \equiv -i\,(1+2i)^2\,(4+i)\,[\text{mod}\ (1+i)^4.\ (2+i).\ (1+2i)],$$

therefore

$$\frac{x^2}{1+2i} \equiv -i\,(1+2i)\,(4+i)\,[\text{mod}\ (1+i)^4.\ (2+i)].$$

Let

$$x \equiv (1+2i)\,\xi \pmod{20}.$$

Then

$$(1+2i)\,\xi^2 \equiv -i\,(1+2i)\,(4+i)\,[\text{mod}\ (1+i)^4.\ (2+i)],$$

and therefore

$$\xi^2 \equiv -i\,.\,(4+i) \pmod{8+4i},$$

$$\equiv 1 - 4i \pmod{8+4i},$$

$$\xi^2 \equiv 9 \pmod{8+4i}.$$

From the tables

$$9 \equiv (\overset{(4)}{2}.\ \overset{(4)}{0}.\ \overset{(2)}{0}),$$

and therefore

$$\xi^2 \equiv (2.\ 0.\ 0).$$

This gives eight solutions

$$\begin{aligned} \xi &\equiv (1.\ 0.\ 0) \equiv 17, \\ &\equiv (3.\ 0.\ 0) \equiv 13, \\ &\equiv (1.\ 2.\ 0) \equiv 7, \\ &\equiv (3.\ 2.\ 0) \equiv 3, \\ &\equiv (1.\ 0.\ 1) \equiv 1 + 2i, \\ &\equiv (3.\ 0.\ 1) \equiv 17 + 2i, \\ &\equiv (1.\ 2.\ 1) \equiv 11 + 2i, \\ &\equiv (3.\ 2.\ 1) \equiv 7 + 2i. \end{aligned}$$

Hence there are eight solutions of the congruence

$$x^2 \equiv 13 + 16i \pmod{20},$$

viz.,

$$\begin{array}{cccc} 7 + 4i & 17 + 4i & 7 + 14i & 17 + 14i \\ 3 + 6i & 13 + 6i & 3 + 16i & 13 + 16i \end{array}$$

and hence there are 32 solutions of the congruence

$$2x^2 \equiv 26 + 32i \pmod{40},$$

viz.,

$$\begin{array}{rrrr} 7 + 4i & 17 + 4i & 7 + 14i & 17 + 14i \\ 27 + 4i & 37 + 4i & 27 + 14i & 37 + 14i \\ 7 + 24i & 17 + 24i & 7 + 34i & 17 + 34i \\ 27 + 24i & 37 + 24i & 27 + 34i & 37 + 34i \\ 3 + 6i & 13 + 6i & 3 + 16i & 13 + 16i \\ 23 + 6i & 33 + 6i & 23 + 16i & 33 + 16i \\ 3 + 26i & 13 + 26i & 3 + 36i & 13 + 36i \\ 23 + 26i & 33 + 26i & 23 + 36i & 33 + 36i \end{array}$$

APPENDIX.

TABLES OF INDICES FOR ALL MODULI WHOSE NORMS DO NOT EXCEED 100.

Description of Tables.

The following Appendix contains Tables of Indices for all the moduli whose norms do not exceed 100. For each modulus two tables are given: the first arranged so as to show readily the number that corresponds to given indices, the second to show what indices correspond to a given number. At the foot of any column, or the end of any row in which indices are tabulated, is placed in a bracket the exponent of the generator to which those indices refer. With each table are noted the formulæ necessary for finding to which of the numbers in the table any given number is congruent. There are also given for convenience the prime factors of the modulus, the norm, the highest exponent, the value of Φ expressed in factors which show the exponents of the generators, and the generators used in the table. All these will be found collected in the reference table next following. In this are also noted, for each modulus, the least possible number of generators and the values of the numbers ξ of Proposition xix. *E.g.*, for the modulus $5+5i$ we read thus:—

$$5+5i=-i(1+i)(2+i)(1+2i),\quad \mathrm{N}(5+5i)=50,\quad \Phi(5+5i)=16=2^2.2^2.$$

Highest exponent $=4$. Least number of generators $=2$, those used in the table being $4+i$ and $9+4i$.

Also if

$$\left.\begin{aligned} a &\equiv \alpha_0 \,(\mathrm{mod}\ 1+i) \\ &\equiv \alpha_1 \,(\mathrm{mod}\ 2+i) \\ &\equiv \alpha_2 \,(\mathrm{mod}\ 1+2i) \end{aligned}\right\}$$

then

$$a \equiv 5\alpha_0 + (3+i)\,\alpha_1 + (8+4i)\,\alpha_2 \,(\mathrm{mod}\ 5+5i).$$

The reducing formulæ (see preface, Part II.) are

$$\begin{aligned} y &\equiv \mathrm{Y}\,(\mathrm{mod}\ 5), \\ x &\equiv \mathrm{X}+\mathrm{Y}-y\,(\mathrm{mod}\ 10). \end{aligned}$$

The tables of indices for powers of $1+i$ as moduli, up to $(1+i)^8$, are placed at the end.

[The tables have been calculated with some care, but they have not been revised.]

m.	N (m).	Φ (m).	H.E.	No. of gens.	Gens. used.	The numbers ξ.	d $(\alpha^2+\beta^2)$.	d.	$-(\alpha\alpha'+\beta\beta')$.
$1+i$	2	1	1	1	1	..	2	..	1
$2 \quad =-i(1+i)^2$	4	2	2	1	i	..	2	2	
$2+i$	5	$4=2^2$	4	1	2	..	5	..	3
$1+2i$									2
$2+2i=-i(1+i)^3$	8	$4=2^2$	4	1	i	..	4	2	1
3.	9	$8=2^3$	8	1	$1+i$	..	3	3	
$3+i \ =-i(1+i)(1+2i)$	10	$4=2^2$	4	1	3	5, 6	10	..	7
$1+3i= \quad (1+i)(2+i)$									3
$3+2i$	13	$12=2^2.3$	12	1	2	..	13	..	5
$2+3i$									8
$4 \quad =-\ (1+i)^4$	16	$8=2.2^2$	4	2	i $1+2i$	..	4	4	
$4+i$	17	$16=2^4$	16	1	3	..	17	..	13
$1+4i$									6
$3+3i= \quad (1+i).3$	18	$8=2^3$	8	1	$1+2i$	3, 4	6	3	1
$4+2i=-i(1+i)^2(2+i)$	20	$8=2.2^2$	4	2	$7, 8+i$ $7, 4+i$	5, 6	10	2	3
$2+4i=-i(1+i)^2(1+2i)$									7
$5 \quad =-i(2+i)(1+2i)$	25	$16=2^2.2^2$	4	2	$4+i, 4+4i$	$3+i, 3+4i$	5	5	
$4+3i=-i(1+2i)^2$	25	$20=2^2.5$	20	1	2	..	25	..	7
$3+4i= \quad (2+i)^2$									18
$5+i \ =-i(1+i)(2+3i)$	26	$12=2^2.3$	12	1	7	13, 14	26	..	21
$1+5i= \quad (1+i)(3+2i)$									5
$5+2i$	29	$28=2^2.7$	28	1	2	..	29	..	12
$2+5i$									17
$4+4i=-\ (1+i)^5$	32	$16=2.2.2^2$	4	3	i $1+2i$ 3	..	8	4	1
$5+3i=-i(1+i)(1+4i)$	34	$16=2^4$	16	1	3	17, 18	34	..	21
$3+5i= \quad (1+i)(4+i)$									13
$6 \quad =-i(1+i)^2.3$	36	$16=2.2^3$	8	2	$1+4i, 4+3i$	3, 4	6	6	
$6+i$.	37	$36=2^2.3^2$	36	1	2	..	37	..	31
$1+6i$									6
$6+2i=-\ (1+i)^3(1+2i)$	40	$16=2^2.2^2$	4	2	$3, i$ $3, 14+i$	5, 16	20	2	7
$2+6i=-i(1+i)^3(2+i)$									13
$5+4i$	41	$40=2^3.5$	40	1	6	..	41	..	9
$4+5i$									32
$6+3i= \quad (2+i).3$	45	$32=2^2.2^3$	8	2	$1+i, i$ $10+2i, 9+2i$	6, 10	15	3	3
$3+6i= \quad (1+2i).3$									12
7.	49	$48=2^4.3$	48	1	$2+i$	..	7	7	
$7+i \ =-i(1+i)(2+i)^2$	50	$20=2^2.5$	20	1	3	25, 26	50	..	43
$1+7i=-i(1+i)(1+2i)^2$									7
$5+5i=-i(1+i)(2+i)(1+2i)$	50	$16=2^2.2^2$	4	2	$4+i, 9+4i$	$5, 3+i, 8+4i$	10	5	1

m.	N (m).	Φ (m).	H.E.	No. of gens.	Gens. used.	The numbers ξ.	$d\,(\alpha^2+\beta^2)$.	d.	$-(\alpha\alpha'+\beta\beta')$.
$6+4i=-i\,(1+i)^2\,(3+2i)$	52	$24=2.\,2^2.\,3$	12	2	$15,\ 22+i$	13, 14	26	2	5
$4+6i=-i\,(1+i)^2\,(2+3i)$					$15,\ 6+i$				21
$7+2i$	53	$52=2^2.\,13$	52	1	3	..	53	..	23
$2+7i$									30
$7+3i=-i\,(1+i)\,(2+5i)$	58	$28=2^2.\,7$	28	1	3	29, 30	58	..	17
$3+7i=\ (1+i)\,(5+2i)$									41
$6+5i$	61	$60=2^2.\,3.\,5$	60	1	2	..	61	..	11
$5+6i$									50
$8\ =\ i\,(1+i)^6$	64	$32=2.\,2^2.\,2^2$	4	3	$i,\ 1+2i,\ 3$	..	8	8	
$8+i\ =-i\,(1+2i)\,(3+2i)$	65	$48=2^2.\,2^2.\,3$	12	2	27, 2	26, 40	65	..	57
$1+8i=\ (2+i)\,(2+3i)$									8
$7+4i=-i\,(1+2i)\,(2+3i)$									47
$4+7i=\ (2+i)\,(3+2i)$									18
$8+2i=-i\,(1+i)^2\,(4+i)$	68	$32=2.\,2^4$	16	2	$3,\ i$	17, 18	34	2	13
$2+8i=-i\,(1+i)^2\,(1+4i)$					$3,\ 26+i$				21
$6+6i=-i\,(1+i)^3.\,3$	72	$32=2^2.\,2^3$	8	2	$2+i,\ i$	9, 4	12	6	1
$8+3i$	73	$72=2^3.\,3^2$	72	1	5	..	73	..	46
$3+8i$									27
$7+5i=-i\,(1+i)\,(1+6i)$	74	$36=2^2.\,3^2$	36	1	5	37, 38	74	..	43
$5+7i=\ (1+i)\,(6+i)$									31
$8+4i=-\ (1+i)^4\,(2+i)$	80	$32=2.\,2^2.\,2^2$	4	3	$17,\ 8+i,\ 5+2i$	5, 16	20	4	3
$4+8i=-\ (1+i)^4\,(1+2i)$					$17,\ 3i,\ 17+2i$				17
$9\ =\ 3^2$	81	$72=2^3.\,3.\,3$	24	2	$2+i,\ 4+3i$	..	9	9	
$9+i\ =-i\,(1+i)\,(4+5i)$	82	$40=2^3.\,5$	40	1	7	41, 42	82	..	73
$1+9i=\ (1+i)\,(5+4i)$									9
$9+2i=-i\,(2+i)\,(1+4i)$	85	$64=2^2.\,2^4$	16	2	3, 52	51, 35	85	..	38
$2+9i=\ (1+2i)\,(4+i)$									47
$7+6i=\ (2+i)\,(4+i)$									13
$6+7i=-i\,(1+2i)\,(1+4i)$									72
$8+5i$	89	$88=2^3.\,11$	88	1	3	..	89	..	34
$5+8i$									55
$9+3i=-i\,(1+i)\,(1+2i).\,3$	90	$32=2^2.\,2^3$	8	2	$2+i,\ i$	15, 6, 10	30	3	7
$3+9i=\ (1+i)\,(2+i).\,3$					$23+2i,\ 21+2i$				23
$9+4i$	97	$96=2^5.\,3$	96	1	5	..	97	..	22
$4+9i$									75
$7+7i=\ (1+i).\,7$	98	$48=2^4.\,3$	48	1	$2+i$	7. 8	14	7	1
$8+6i=-\ (1+i)^2\,(1+2i)^2$	100	$40=2.\,2^2.\,5$	20	2	$3,\ 44+i$	25, 26	50	2	7
$6+8i=-i\,(1+i)^2\,(2+i)^2$					$3,\ 8+i$				43
$10\ =-i\,(1+i)^2\,(2+i)\,(1+2i)$	100	$32=2.\,2^2.\,2^2$	4	3	$9+4i,\ 9+6i,\ 6+5i$	$5,\ 8+6i,\ 8+4i$	10	10	

$m = 2 + i$ $\quad \text{N}(m) = 5 \quad \Phi(m) = 4 = 2^2 \quad \text{H.E} = 4 \quad x \equiv \text{X} + 3\text{Y} \pmod{5}.$

$m = 1 + 2i$ $\quad \text{N}(m) = 5 \quad \Phi(m) = 4 = 2^2 \quad \text{H.E} = 4 \quad x \equiv \text{X} + 2\text{Y} \pmod{5}.$

Generator 2.

N	2	4	3	1
I	1	2	3	0

(4)

$m = 3 \quad \text{N}(m) = 9 \quad \Phi(m) = 8 = 2^3 \quad \text{H.E} = 8 \quad x \equiv \text{X} \pmod{3} \quad y \equiv \text{Y} \pmod{3}.$

Generator $1 + i$.

0	1
1	$1+i$
2	$2i$
3	$1+2i$
4	2
5	$2+2i$
6	i
7	$2+i$

(8)

i	6
$2i$	2
1	0
$1+i$	1
$1+2i$	3
2	4
$2+i$	7
$2+2i$	5

(8)

$m = 3 + i = -i(1 + i)(1 + 2i) \quad \text{N}(m) = 10 \quad \Phi(m) = 4 = 2^2 \quad \text{H.E} = 4$

$x \equiv \text{X} + 7\text{Y} \pmod{10}.$

$m = 1 + 3i = (1 + i)(2 + i) \quad \text{N}(m) = 10 \quad \Phi(m) = 4 = 2^2 \quad \text{H.E} = 4$

$x \equiv \text{X} + 3\text{Y} \pmod{10}.$

Generator 3.

N	3	9	7	1
I	1	2	3	0

(4)

$m = 3 + 2i \quad \mathrm{N}(m) = 13 \quad \Phi(m) = 12 = 2^2.3 \quad \mathrm{H.E} = 12 \quad x \equiv \mathrm{X} + 5\mathrm{Y} \pmod{13}.$

$m = 2 + 3i \quad \mathrm{N}(m) = 13 \quad \Phi(m) = 12 = 2^2.3 \quad \mathrm{H.E} = 12 \quad x \equiv \mathrm{X} + 8\mathrm{Y} \pmod{13}.$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	(12)
N	1	2	4	8	3	6	12	11	9	5	10	7	

N	1	2	3	4	5	6	7	8	9	10	11	12	
I	0	1	4	2	9	5	11	3	8	10	7	6	(12)

$m = 4 + i \quad \mathrm{N}(m) = 17 \quad \Phi(m) = 16 = 2^4 \quad \mathrm{H.E} = 16 \quad x \equiv \mathrm{X} + 13\mathrm{Y} \pmod{17}.$

$m = 1 + 4i \quad \mathrm{N}(m) = 17 \quad \Phi(m) = 16 = 2^4 \quad \mathrm{H.E} = 16 \quad x \equiv \mathrm{X} + 4\mathrm{Y} \pmod{17}.$

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(16)
N	1	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
I	0	14	1	12	5	15	11	10	2	3	7	13	4	9	6	8	(16)

$m = 3 + 3i = (1 + i)3 \quad \mathrm{N}(m) = 18 \quad \Phi(m) = 8 = 2^3 \quad \mathrm{H.E} = 8 \quad y \equiv \mathrm{Y} \pmod{3}$

$x \equiv \mathrm{X} + 3(\mathrm{Y} - y) \pmod{6}.$

Generator $1 + 2i$.

0	1
1	$1+2i$
2	i
3	$4+i$
4	5
5	$2+i$
6	$3+2i$
7	$5+2i$
(8)	

i	2
1	0
$1+2i$	1
$2+i$	5
$3+2i$	6
$4+i$	3
5	4
$5+2i$	7
	(8)

$m = 4 + 2i = - i (1 + i)^2 (2 + i)$ $\quad \mathrm{N}(m) = 20 \quad \Phi(m) = 8 = 2.\,2^2$ $\quad$ H. E. = 4

$y \equiv \mathrm{Y} \pmod{2} \quad x = \mathrm{X} + 3(\mathrm{Y} - y) \pmod{10}$.

Generators 7, $8 + i$.

0	0	1
0	1	$8+i$
1	0	7
1	1	$4+i$
2	0	9
2	1	$6+i$
3	0	3
3	1	i
(4)	(2)	

i	3	1
1	0	0
3	3	0
$4+i$	1	1
$6+i$	2	1
7	1	0
$8+i$	0	1
9	2	0
	(4)	(2)

$m = 2 + 4i = - i (1 + i)^2 (1 + 2i)$ $\quad \mathrm{N}(m) = 20 \quad \Phi(m) = 8 = 2.\,2^2$ $\quad$ H. E. = 4

$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 7(\mathrm{Y} - y) \pmod{10}$.

Generators 7, $4 + i$.

0	0	1
0	1	$4+i$
1	0	7
1	1	i
2	0	9
2	1	$2+i$
3	0	3
3	1	$6+i$
(4)	(2)	

i	1	1
1	0	0
$2+i$	2	1
3	3	0
$4+i$	0	1
$6+i$	3	1
7	1	0
9	2	0
	(4)	(2)

$m = 5 = - i (2 + i)(1 + 2i)$ $\quad \mathrm{N}(m) = 25 \quad \Phi(m) = 16 = 2^2.\,2^2$ $\quad$ H. E. = 4

$x \equiv \mathrm{X} \pmod{5} \quad y \equiv \mathrm{Y} \pmod{5}$.

Generators $4 + i$, $4 + 4i$.

0	0	1
0	1	$4+4i$
0	2	$2i$
0	3	$2+3i$
1	0	$4+i$
1	1	2
1	2	$3+3i$
1	3	$4i$
2	0	$3i$
2	1	$3+2i$
2	2	4
2	3	$1+i$
3	0	$2+2i$
3	1	i
3	2	$1+4i$
3	3	3
(4)	(4)	

i	3	1
$2i$	0	2
$3i$	2	0
$4i$	1	3
1	0	0
$1+i$	2	3
$1+4i$	3	2
2	1	1
$2+2i$	3	0
$2+3i$	0	3
3	3	3
$3+2i$	2	1
$3+3i$	1	2
4	2	2
$4+i$	1	0
$4+4i$	0	1
	(4)	(4)

$$m = 4 + 3i = -i(1 + 2i)^2 \quad \mathrm{N}(m) = 25 \quad \Phi(m) = 20 = 2^2 . 5 \quad \mathrm{H. E.} = 20$$

$$x \equiv \mathrm{X} + 7\mathrm{Y} \pmod{25}.$$

$$m = 3 + 4i = (2 + i)^2 \quad \mathrm{N}(m) = 25 \quad \Phi(m) = 20 = 2^2 . 5 \quad \mathrm{H. E.} = 20$$

$$x \equiv \mathrm{X} + 18\mathrm{Y} \pmod{25}.$$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(20)
N	1	2	4	8	16	7	14	3	6	12	24	23	21	17	9	18	11	22	19	13	

N	1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24	
I	0	1	7	2	8	5	3	14	16	9	19	6	4	13	15	18	12	17	11	10	(20)

$$m = 5 + i = -i(1 + i)(2 + 3i) \quad \mathrm{N}(m) = 26 \quad \Phi(m) = 12 = 2^2 . 3 \quad \mathrm{H. E.} = 12$$

$$x \equiv \mathrm{X} + 21\mathrm{Y} \pmod{26}.$$

$$m = 1 + 5i = (1 + i)(3 + 2i) \quad \mathrm{N}(m) = 26 \quad \Phi(m) = 12 = 2^2 . 3 \quad \mathrm{H. E.} = 12$$

$$x \equiv \mathrm{X} + 5\mathrm{Y} \pmod{26}.$$

Generator 7.

I	0	1	2	3	4	5	6	7	8	9	10	11	(12)
N	1	7	23	5	9	11	25	19	3	21	17	15	

N	1	3	5	7	9	11	15	17	19	21	23	25	
I	0	8	3	1	4	5	11	10	7	9	2	6	(12)

$m = 5 + 2i \quad N(m) = 29 \quad \Phi(m) = 28 = 2^2 . 7 \quad$ H. E. $= 28 \quad x \equiv X + 12Y \pmod{29}$.

$m = 2 + 5i \quad N(m) = 29 \quad \Phi(m) = 28 = 2^2 . 7 \quad$ H. E. $= 28 \quad x \equiv X + 17Y \pmod{29}$.

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	(28)
N	1	2	4	8	16	3	6	12	24	19	9	18	7	14	
I	14	15	16	17	18	19	20	21	22	23	24	25	26	27	(28)
N	28	27	25	21	13	26	23	17	5	10	20	11	22	15	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
I	0	1	5	2	22	6	12	3	10	23	25	7	18	13	(28)
N	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
I	27	4	21	11	9	24	17	26	20	8	16	19	15	14	(28)

$m = 5 + 3i = -i(1 + i)(1 + 4i) \quad N(m) = 34 \quad \Phi(m) = 16 = 2^4 \quad$ H. E. $= 16$

$x \equiv X + 21Y \pmod{34}$.

$m = 3 + 5i = \quad (1 + i)(4 + i) \quad N(m) = 34 \quad \Phi(m) = 16 = 2^4 \quad$ H. E. $= 16$

$x \equiv X + 13Y \pmod{34}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(16)
N	1	3	9	27	13	5	15	11	33	31	25	7	21	29	19	23	

N	1	3	5	7	9	11	13	15	19	21	23	25	27	29	31	33	
I	0	1	5	11	2	7	4	6	14	12	15	10	3	13	9	8	(16)

$m = 6 = -i(1+i)^2 . 3 \quad \mathrm{N}(m) = 36 \quad \Phi(m) = 16 = 2 . 2^3 \quad \mathrm{H.E.} = 8$

$x \equiv \mathrm{X} \pmod{6} \quad y \equiv \mathrm{Y} \pmod{6}.$

Generators $1+4i$, $4+3i$.

0	0	1
0	1	$4+3i$
1	0	$1+4i$
1	1	$4+i$
2	0	$3+2i$
2	1	$5i$
3	0	$1+2i$
3	1	$4+5i$
4	0	5
4	1	$2+3i$
5	0	$5+2i$
5	1	$2+5i$
6	0	$3+4i$
6	1	i
7	0	$5+4i$
7	1	$2+i$
(8)	(2)	

i	6	1
$5i$	2	1
1	0	0
$1+2i$	3	0
$1+4i$	1	0
$2+i$	7	1
$2+3i$	4	1
$2+5i$	5	1
$3+2i$	2	0
$3+4i$	6	0
$4+i$	1	1
$4+3i$	0	1
$4+5i$	3	1
5	4	0
$5+2i$	5	0
$5+4i$	7	0
	(8)	(2)

$m = 6+i \quad \mathrm{N}(m) = 37 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \mathrm{H.E.} = 36 \quad x \equiv \mathrm{X} + 31\mathrm{Y} \pmod{37}.$

$m = 1+6i \quad \mathrm{N}(m) = 37 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \mathrm{H.E.} = 36 \quad x \equiv \mathrm{X} + 6\mathrm{Y} \pmod{37}.$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(36)
N	1	2	4	8	16	32	27	17	34	31	25	13	26	15	30	23	9	18	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(36)
N	36	35	33	29	21	5	10	20	3	6	12	24	11	22	7	14	28	19	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
I	0	1	26	2	23	27	32	3	16	24	30	28	11	33	13	4	7	17	(36)
N	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
I	35	25	22	31	15	29	10	12	6	34	21	14	9	5	20	8	19	18	(36)

$$m = 6 + 2i = -(1 + i)^3 (1 + 2i) \quad \mathrm{N}(m) = 40 \quad \Phi(m) = 16 = 2^2 . 2^2 \quad \mathrm{H.E.} = 4.$$

$$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 7(\mathrm{Y} - y) \pmod{20}.$$

Generators 3, i.

0	0	1
0	1	i
0	2	19
0	3	$6+i$
1	0	3
1	1	$14+i$
1	2	17
1	3	$12+i$
2	0	9
2	1	$16+i$
2	2	11
2	3	$10+i$
3	0	7
3	1	$2+i$
3	2	13
3	3	$4+i$
(4)	(4)	

i	0	1
1	0	0
$2+i$	3	1
3	1	0
$4+i$	3	3
$6+i$	0	3
7	3	0
9	2	0
$10+i$	2	3
11	2	2
$12+i$	1	3
13	3	2
$14+i$	1	1
$16+i$	2	1
17	1	2
19	0	2
	(4)	(4)

$$m = 2 + 6i = -i(1 + i)^3 (2 + i) \quad \mathrm{N}(m) = 40 \quad \Phi(m) = 16 = 2^2 . 2 \quad \mathrm{H.E.} = 4.$$

$$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 13(\mathrm{Y} - y) \pmod{20}.$$

Generators 3, $14 + i$.

0	0	1
0	1	$14+i$
0	2	19
0	3	i
1	0	3
1	1	$8+i$
1	2	17
1	3	$6+i$
2	0	9
2	1	$10+i$
2	2	11
2	3	$4+i$
3	0	7
3	1	$16+i$
3	2	13
3	3	$18+i$
(4)	(4)	

i	0	3
1	0	0
3	1	0
$4+i$	2	3
$6+i$	1	3
7	3	0
$8+i$	1	1
9	2	0
$10+i$	2	1
11	2	2
13	3	2
$14+i$	0	1
$16+i$	3	1
17	1	2
$18+i$	3	3
19	0	2
	(4)	(4)

$m = 5 + 4i$ $\mathrm{N}(m) = 41$ $\Phi(m) = 40 = 2^3 . 5$ H. E. $= 40$ $x \equiv \mathrm{X} + 9\mathrm{Y}$ (mod 41).

$m = 4 + 5i$ $\mathrm{N}(m) = 41$ $\Phi(m) = 40 = 2^3 . 5$ H. E. $= 40$ $x \equiv \mathrm{X} + 32\mathrm{Y}$ (mod 41).

Generator 6.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
N	1	6	36	11	25	27	39	29	10	19	32	28	4	24	21	3	18	26	33	34	(40)
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
N	40	35	5	30	16	14	2	12	31	22	9	13	37	17	20	38	23	15	8	7	(40)

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
N	0	26	15	12	22	1	39	38	30	8	3	27	31	25	37	24	33	16	9	34	(40)
I	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	
N	14	29	36	13	4	17	5	11	7	23	28	10	18	19	21	2	32	35	6	20	(40)

$m = 6 + 3i = (2 + i)\,3 \qquad \mathrm{N}\,(m) = 45 \qquad \Phi\,(m) = 32 = 2^2 .\ 2^3$

H. E. = 8 $\qquad y \equiv \mathrm{Y}\,(\mathrm{mod}\ 3) \qquad x \equiv \mathrm{X} + 3\,(\mathrm{Y} - y)\,(\mathrm{mod}\ 15).$

Generators $1 + i,\ i$.

0	0	1
0	1	i
0	2	14
0	3	$6+2i$
1	0	$1+i$
1	1	$14+i$
1	2	$5+2i$
1	3	$7+2i$
2	0	$2i$
2	1	13
2	2	$6+i$
2	3	2
3	0	$13+2i$
3	1	$4+i$
3	2	$8+i$
3	3	$2+2i$
4	0	11
4	1	$12+2i$
4	2	4
4	3	$9+i$
5	0	$8+2i$
5	1	$1+2i$
5	2	$13+i$
5	3	$5+i$
6	0	$3+i$
6	1	8
6	2	$3+2i$
6	3	7
7	0	$11+i$
7	1	$11+2i$
7	2	$10+2i$
7	3	$10+i$
(8)	(4)	

i	0	1
$2i$	2	0
1	0	0
$1+i$	1	0
$1+2i$	5	1
2	2	3
$2+2i$	3	3
$3+i$	6	0
$3+2i$	6	2
4	4	2
$4+i$	3	1
$5+i$	5	3
$5+2i$	1	2
$6+i$	2	2
$6+2i$	0	3
7	6	3
$7+2i$	1	3
8	6	1
$8+i$	3	2
$8+2i$	5	0
$9+i$	4	3
$10+i$	7	3
$10+2i$	7	2
11	4	0
$11+i$	7	0
$11+2i$	7	1
$12+2i$	4	1
13	2	1
$13+i$	5	2
$13+2i$	3	0
14	0	2
$14+i$	1	1
	(8)	(4)

$$m = 3 + 6i = (1 + 2i)\,3 \qquad \mathrm{N}(m) = 45 \qquad \Phi(m) = 32 = 2^2 . 2^3$$

$$\text{H. E.} = 8 \qquad y \equiv \mathrm{Y} \pmod{3} \qquad x \equiv \mathrm{X} + 12(\mathrm{Y} - y) \pmod{15}.$$

Generators $10 + 2i$, $9 + 2i$.

0	0	1
0	1	$9+2i$
0	2	14
0	3	i
1	0	$10+2i$
1	1	$8+2i$
1	2	$14+i$
1	3	$1+i$
2	0	$9+i$
2	1	13
2	2	$2i$
2	3	2
3	0	$7+i$
3	1	$13+2i$
3	2	$2+2i$
3	3	$11+i$
4	0	11
4	1	$6+i$
4	2	4
4	3	$3+2i$
5	0	$2+i$
5	1	$10+i$
5	2	$7+2i$
5	3	$14+2i$
6	0	$12+2i$
6	1	8
6	2	$12+i$
6	3	7
7	0	$5+2i$
7	1	$5+i$
7	2	$4+i$
7	3	$4+2i$
(8)	(4)	

i	0	3
$2i$	2	2
1	0	0
$1+i$	1	3
2	2	3
$2+i$	5	0
$2+2i$	3	2
$3+2i$	4	3
4	4	2
$4+i$	7	2
$4+2i$	7	3
$5+i$	7	1
$5+2i$	7	0
$6+i$	4	1
7	6	3
$7+i$	3	0
$7+2i$	5	2
8	6	1
$8+2i$	1	1
$9+i$	2	0
$9+2i$	0	1
$10+i$	5	1
$10+2i$	1	0
11	4	0
$11+i$	3	3
$12+i$	6	2
$12+2i$	6	0
13	2	1
$13+2i$	3	1
14	0	2
$14+i$	1	2
$14+2i$	5	3
	(8)	(4)

$m = 7$ N $(7) = 49$ $\Phi(7) = 48 = 2^4.3$. H. E. $= 48$. $x \equiv$ X (mod 7)
$y \equiv$ Y (mod 7).

Generator $2 + i$.

0	1	24	6	i	12	$3+4i$	2
1	$2+i$	25	$5+6i$	$2i$	44	$3+5i$	9
2	$3+4i$	26	$4+3i$	$3i$	4	$3+6i$	11
3	$2+4i$	27	$5+3i$	$4i$	28	4	16
4	$3i$	28	$4i$	$5i$	20	$4+i$	35
5	$4+6i$	29	$3+i$	$6i$	36	$4+2i$	33
6	$2+2i$	30	$5+5i$	1	0	$4+3i$	26
7	$2+6i$	31	$5+i$	$1+i$	22	$4+4i$	38
8	5	32	2	$1+2i$	19	$4+5i$	39
9	$3+5i$	33	$4+2i$	$1+3i$	23	$4+6i$	5
10	$1+6i$	34	$6+i$	$1+4i$	17	5	8
11	$3+6i$	35	$4+i$	$1+5i$	37	$5+i$	31
12	i	36	$6i$	$1+6i$	10	$5+2i$	18
13	$6+2i$	37	$1+5i$	2	32	$5+3i$	27
14	$3+3i$	38	$4+4i$	$2+i$	1	$5+4i$	45
15	$3+2i$	39	$4+5i$	$2+2i$	6	$5+5i$	30
16	4	40	3	$2+3i$	21	$5+6i$	25
17	$1+4i$	41	$6+3i$	$2+4i$	3	6	24
18	$5+2i$	42	$2+5i$	$2+5i$	42	$6+i$	34
19	$1+2i$	43	$6+5i$	$2+6i$	7	$6+2i$	13
20	$5i$	44	$2i$	3	40	$6+3i$	41
21	$2+3i$	45	$5+4i$	$3+i$	29	$6+4i$	47
22	$1+i$	46	$6+6i$	$3+2i$	15	$6+5i$	43
23	$1+3i$	47	$6+4i$	$3+3i$	14	$6+6i$	46
(48)		(48)			(48)		(48)

$m = 7 + i = -i(1+i)(2+i)^2$ N $(m) = 50$ $\Phi(m) = 20 = 2^2.5$ H. E. $= 20$
$x \equiv$ X $+$ 43Y (mod 50).

$m = 1 + 7i = -i(1+i)(1+2i)^2$ N $(m) = 50$ $\Phi(m) = 20 = 2^2.5$ H. E. $= 20$
$x \equiv$ X $+$ 7Y (mod 50).

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(20)
N	1	3	9	27	31	43	29	37	11	33	49	47	41	23	19	7	21	13	39	17	

N	1	3	7	9	11	13	17	19	21	23	27	29	31	33	37	39	41	43	47	49	
I	0	1	15	2	8	17	19	14	16	13	3	6	4	9	7	18	12	5	11	10	(20)

$$m = 5 + 5i = -i(1+i)(2+i)(1+2i) \quad N(m) = 50 \quad \Phi(m) = 16 = 2^2 . 2^2$$

$$\text{H. E.} = 4 \quad y \equiv Y \pmod{5} \quad x \equiv X + Y - y \pmod{10}.$$

Generators $4+i$, $9+4i$.

0	0	1
0	1	$9+4i$
0	2	$5+2i$
0	3	$2+3i$
1	0	$4+i$
1	1	7
1	2	$8+3i$
1	3	$5+4i$
2	0	$3i$
2	1	$3+2i$
2	2	9
2	3	$6+i$
3	0	$7+2i$
3	1	i
3	2	$1+4i$
3	3	3
(4)	(4)	

i	3	1
$3i$	2	0
1	0	0
$1+4i$	3	2
$2+3i$	0	3
3	3	3
$3+2i$	2	1
$4+i$	1	0
$5+2i$	0	2
$5+4i$	1	3
$6+i$	2	3
7	1	1
$7+2i$	3	0
$8+3i$	1	2
9	2	2
$9+4i$	0	1
	(4)	(4)

$$m = 6 + 4i = -i(1+i)^2(3+2i) \quad N(m) = 52 \quad \Phi(m) = 24 = 2 . 2^2 . 3 \quad \text{H. E.} = 12$$

$$y \equiv Y \pmod{2} \quad x \equiv X + 5(Y - y) \pmod{26}.$$

Generators 15, $22+i$.

0	0	1
0	1	$22+i$
1	0	15
1	1	$10+i$
2	0	17
2	1	$12+i$
3	0	21
3	1	$16+i$
4	0	3
4	1	$24+i$
5	0	19
5	1	$14+i$
6	0	25
6	1	$20+i$
7	0	11
7	1	$6+i$
8	0	9
8	1	$4+i$
9	0	5
9	1	i
10	0	23
10	1	$18+i$
11	0	7
11	1	$2+i$
(12)	(2)	

i	9	1
1	0	0
$2+i$	11	1
3	4	0
$4+i$	8	1
5	9	0
$6+i$	7	1
7	11	0
9	8	0
$10+i$	1	1
11	7	0
$12+i$	2	1
$14+i$	5	1
15	1	0
$16+i$	3	1
17	2	0
$18+i$	10	1
19	5	0
$20+i$	6	1
21	3	0
$22+i$	0	1
23	10	0
$24+i$	4	1
25	6	0
	(12)	(2)

$m = 4 + 6i = -i\,(1+i)^2\,(2+3i)$ $\quad$ $\mathrm{N}(m) = 52$ $\quad$ $\Phi(m) = 24 = 2.\,2^2.\,3$ $\quad$ H. E. = 12

$y \equiv \mathrm{Y} \pmod 2$ $\quad$ $x \equiv \mathrm{X} + 21\,(\mathrm{Y} - y) \pmod{26}$.

Generators 15, $6 + i$.

0	0	1
0	1	$6+i$
1	0	15
1	1	$20+i$
2	0	17
2	1	$22+i$
3	0	21
3	1	i
4	0	3
4	1	$8+i$
5	0	19
5	1	$24+i$
6	0	25
6	1	$4+i$
7	0	11
7	1	$16+i$
8	0	9
8	1	$14+i$
9	0	5
9	1	$10+i$
10	0	23
10	1	$2+i$
11	0	7
11	1	$12+i$
(12)	(2)	

i	3	1
1	0	0
$2+i$	10	1
3	4	0
$4+i$	6	1
5	9	0
$6+i$	0	1
7	11	0
$8+i$	4	1
9	8	0
$10+i$	9	1
11	7	0
$12+i$	11	1
$14+i$	8	1
15	1	0
$16+i$	7	1
17	2	0
19	5	0
$20+i$	1	1
21	3	0
$22+i$	2	1
23	10	0
$24+i$	5	1
25	6	0
	(12)	(2)

$m = 7 + 2i$ $\quad$ $\mathrm{N}(m) = 53$ $\quad$ $\Phi(m) = 52 = 2^2.13$ $\quad$ H. E. = 52 $\quad$ $x \equiv \mathrm{X} + 23\mathrm{Y} \pmod{53}$.

$m = 2 + 7i$ $\quad$ $\mathrm{N}(m) = 53$ $\quad$ $\Phi(m) = 52 = 2^2.13$ $\quad$ H. E. = 52 $\quad$ $x \equiv \mathrm{X} + 30\mathrm{Y} \pmod{53}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	(52)
N	1	3	9	27	28	31	40	14	42	20	7	21	10	30	37	5	15	45	29	
I	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36		(52)
N	34	49	41	17	51	47	35	52	50	44	26	25	22	13	39	11	33	46		
I	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51					(52)
N	32	43	23	16	48	38	8	24	19	4	12	36	2	6	13					

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	0	49	1	46	15	50	10	43	2	12	34	47	32	7	16	40	22	51	45	(52)

N	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	
I	9	11	31	39	44	30	29	3	4	18	13	5	37	35	19	25	48	14	(52)

N	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	
I	42	33	6	21	8	38	28	17	36	24	41	20	27	23	26	(52)

$m = 7 + 3i = -i(1+i)(2+5i) \quad \mathrm{N}(m) = 58 \quad \Phi(m) = 28 = 2^2.7 \quad \mathrm{H.\,E.} = 28$

$$x \equiv \mathrm{X} + 17\mathrm{Y} \pmod{58}.$$

$m = 3 + 7i = \quad (1+i)(5+2i) \quad \mathrm{N}(m) = 58 \quad \Phi(m) = 28 = 2^2.7 \quad \mathrm{H.\,E.} = 28$

$$x \equiv \mathrm{X} + 41\mathrm{Y} \pmod{58}.$$

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
N	1	3	9	27	23	11	33	41	7	21	5	15	45	19	(28)

I	14	15	16	17	18	19	20	21	22	23	24	25	26	27	
N	57	55	49	31	35	47	25	17	51	37	53	43	13	39	(28)

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
I	0	1	10	8	2	5	26	11	21	13	9	4	20	3	(28)

N	31	33	35	37	39	41	43	45	47	49	51	53	55	57	
I	17	6	18	23	27	7	25	12	19	16	22	24	15	14	(28)

$m = 6 + 5i$ $\quad N(m) = 61$ $\quad \Phi(m) = 60 = 2^2.3.5$ $\quad$ H. E. = 60 $\quad x \equiv X + 11Y \pmod{61}$.

$m = 5 + 6i$ $\quad N(m) = 61$ $\quad \Phi(m) = 60 = 2^2.3.5$ $\quad$ H. E. = 60 $\quad x \equiv X + 50Y \pmod{61}$.

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(60)
N	1	2	4	8	16	32	3	6	12	24	48	35	9	18	36	11	22	44	27	54	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	(60)
N	47	33	5	10	20	40	19	38	15	30	60	59	57	53	45	29	58	55	49	37	
I	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	(60)
N	13	26	52	43	25	50	39	17	34	7	14	28	56	51	41	21	42	23	46	31	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
I	0	1	6	2	22	7	49	3	12	23	15	8	40	50	28	4	47	13	26	24	(60)
N	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	
I	55	16	57	9	44	41	18	51	35	29	59	5	21	48	11	14	39	27	46	25	(60)
N	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
I	54	56	43	17	34	58	20	10	38	45	53	42	33	19	37	52	32	36	31	30	(60)

$$m = 8 + i = -i\,(1+2i)\,(3+2i) \quad \mathrm{N}\,(m) = 65 \quad \Phi\,(m) = 48 = 2^2.\,2^2.\,3$$
$$\mathrm{H.\,E.} = 12 \quad x \equiv \mathrm{X} + 57\mathrm{Y}\ (\mathrm{mod}\ 65).$$

$$m = 1 + 8i = \quad (2+i)\,(2+3i) \quad \mathrm{N}\,(m) = 65 \quad \Phi\,(m) = 48 = 2^2.\,2^2.\,3$$
$$\mathrm{H.\,E.} = 12 \quad x \equiv \mathrm{X} + 8\mathrm{Y}\ (\mathrm{mod}\ 65).$$

$$m = 7 + 4i = -i\,(1+2i)\,(2+3i) \quad \mathrm{N}\,(m) = 65 \quad \Phi\,(m) = 48 = 2^2.\,2^2.\,3$$
$$\mathrm{H.\,E.} = 12 \quad x \equiv \mathrm{X} + 47\mathrm{Y}\ (\mathrm{mod}\ 65).$$

$$m = 4 + 7i = \quad (2+i)\,(3+2i) \quad \mathrm{N}\,(m) = 65 \quad \Phi\,(m) = 48 = 2^2.\,2^2.\,3$$
$$\mathrm{H.\,E.} = 12 \quad x \equiv \mathrm{X} + 18\mathrm{Y}\ (\mathrm{mod}\ 65).$$

Generators 27, 2.

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	(12)
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	(4)
1	27	14	53	2	54	28	41	4	43	56	17	8	21	47	34	16	42	29	3	32	19	58	6	
6	6	6	6	7	7	7	7	8	8	8	8	9	9	9	9	10	10	10	10	11	11	11	11	(12)
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	(4)
64	38	51	12	63	11	37	24	61	22	9	48	57	44	18	31	49	23	36	62	33	46	7	59	

1	2	3	4	6	7	8	9	11	12	14	16	17	18	19	21	22	23	24	27	28	29	31	32	
0	1	4	2	5	11	3	8	7	6	0	4	2	9	5	3	8	10	7	0	1	4	9	5	(12)
0	0	3	0	3	2	0	2	1	3	2	0	3	2	1	1	1	1	3	1	2	2	3	0	(4)
33	34	36	37	38	41	42	43	44	46	47	48	49	51	53	54	56	57	58	59	61	62	63	64	
11	3	10	7	6	1	4	2	9	11	3	8	10	6	0	1	2	9	5	11	8	10	7	6	(12)
0	3	2	2	1	3	1	1	1	1	2	3	0	2	3	1	2	0	2	3	0	3	0	0	(4)

$m = 8 + 2i = -i(1+i)^2(4+i)$ $\quad N(m) = 68$ $\quad \Phi(m) = 32 = 2.2^4$ $\quad$ H. E. = 16

$y \equiv Y \pmod{2} \quad x \equiv X + 13(Y - y) \pmod{34}.$

Generators 3, i.

0	0	1
0	1	i
1	0	3
1	1	$26+i$
2	0	9
2	1	$2+i$
3	0	27
3	1	$32+i$
4	0	13
4	1	$20+i$
5	0	5
5	1	$18+i$
6	0	15
6	1	$12+i$
7	0	11
7	1	$28+i$
8	0	33
8	1	$8+i$
9	0	31
9	1	$16+i$
10	0	25
10	1	$6+i$
11	0	7
11	1	$10+i$
12	0	21
12	1	$22+i$
13	0	29
13	1	$24+i$
14	0	19
14	1	$30+i$
15	0	23
15	1	$14+i$
(16)	(2)	

i	0	1
1	0	0
$2+i$	2	1
3	1	0
5	5	0
$6+i$	10	1
7	11	0
$8+i$	8	1
9	2	0
$10+i$	11	1
11	7	0
$12+i$	6	1
13	4	0
$14+i$	15	1
15	6	0
$16+i$	9	1
$18+i$	5	1
19	14	0
$20+i$	4	1
21	12	0
$22+i$	12	1
23	15	0
$24+i$	13	1
25	10	0
$26+i$	1	1
27	3	0
$28+i$	7	1
29	13	0
$30+i$	14	1
31	9	0
$32+i$	3	1
33	8	0
	(16)	(2)

$$m = 2 + 8i = -i(1+i)^2(1+4i) \quad \mathrm{N}(m) = 68 \quad \Phi(m) = 32 = 2.\,2^4 \quad \text{H. E.} = 16$$
$$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 21(\mathrm{Y} - y) \pmod{34}.$$

Generators 3, $26 + i$.

0	0	1
0	1	$26+i$
1	0	3
1	1	$18+i$
2	0	9
2	1	$28+i$
3	0	27
3	1	$24+i$
4	0	13
4	1	$12+i$
5	0	5
5	1	$10+i$
6	0	15
6	1	$4+i$
7	0	11
7	1	$20+i$
8	0	33
8	1	i
9	0	31
9	1	$8+i$
10	0	25
10	1	$32+i$
11	0	7
11	1	$2+i$
12	0	21
12	1	$14+i$
13	0	29
13	1	$16+i$
14	0	19
14	1	$22+i$
15	0	23
15	1	$6+i$
(16)	(2)	

i	8	1
1	0	0
$2+i$	11	1
3	1	0
$4+i$	6	1
5	5	0
$6+i$	15	1
7	11	0
$8+i$	9	1
9	2	0
$10+i$	5	1
11	7	0
$12+i$	4	1
13	4	0
$14+i$	12	1
15	6	0
$16+i$	13	1
$18+i$	1	1
19	14	0
$20+i$	7	1
21	12	0
$22+i$	14	1
23	15	0
$24+i$	3	1
25	10	0
$26+i$	0	1
27	3	0
$28+i$	2	1
29	13	0
31	9	0
$32+i$	10	1
33	8	0
	(16)	(2)

$m = 6 + 6i = -i(1+i)^3 . 3 \quad N(m) = 72 \quad \Phi(m) = 32 = 2^2 . 2^3 \quad \text{H. E.} = 8$

$y \equiv Y \pmod{6} \quad x \equiv X + Y - y \pmod{12}.$

Generators $2+i$, i.

0	0	1
0	1	i
0	2	11
0	3	$6+5i$
1	0	$2+i$
1	1	$11+2i$
1	2	$4+5i$
1	3	$7+4i$
2	0	$3+4i$
2	1	$8+3i$
2	2	$3+2i$
2	3	$10+3i$
3	0	$8+5i$
3	1	$1+2i$
3	2	$10+i$
3	3	$5+4i$
4	0	5
4	1	$5i$
4	2	7
4	3	$6+i$
5	0	$10+5i$
5	1	$1+4i$
5	2	$8+i$
5	3	$5+2i$
6	0	$9+2i$
6	1	$4+3i$
6	2	$9+4i$
6	3	$2+3i$
7	0	$4+i$
7	1	$11+4i$
7	2	$2+5i$
7	3	$7+2i$
(8)	(4)	

i	0	1
$5i$	4	1
1	0	0
$1+2i$	3	1
$1+4i$	5	1
$2+i$	1	0
$2+3i$	6	3
$2+5i$	7	2
$3+2i$	2	2
$3+4i$	2	0
$4+i$	7	0
$4+3i$	6	1
$4+5i$	1	2
5	4	0
$5+2i$	5	3
$5+4i$	3	3
$6+i$	4	3
$6+5i$	0	3
7	4	2
$7+2i$	7	3
$7+4i$	1	3
$8+i$	5	2
$8+3i$	2	1
$8+5i$	3	0
$9+2i$	6	0
$9+4i$	6	2
$10+i$	3	2
$10+3i$	2	3
$10+5i$	5	0
11	0	2
$11+2i$	1	1
$11+4i$	7	1
	(8)	(4)

$m = 8 + 3i$ $\quad N(m) = 73$ $\quad \Phi(m) = 72 = 2^3 . 3^2$ $\quad$ H. E. $= 72$ $\quad x \equiv X + 46Y \pmod{73}$

$m = 3 + 8i$ $\quad N(m) = 73$ $\quad \Phi(m) = 72 = 2^3 . 3^2$ $\quad$ H. E. $= 72$ $\quad x \equiv X + 27Y \pmod{73}$

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(72)
N	1	5	25	52	41	59	3	15	2	10	50	31	9	45	6	30	4	20	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(72)
N	27	62	18	17	12	60	8	40	54	51	36	34	24	47	16	7	35	29	
I	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	(72)
N	72	68	48	21	32	14	70	58	71	63	23	42	64	28	67	43	69	53	
I	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	(72)
N	46	11	55	56	61	13	65	33	19	22	37	39	49	26	57	66	38	44	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
I	0	8	6	16	1	14	33	24	12	9	55	22	59	41	7	32	21	20	(72)
N	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
I	62	17	39	63	46	30	2	67	18	49	35	15	11	40	61	29	34	28	(72)
N	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	
I	64	70	65	25	4	47	51	71	13	54	31	38	66	10	27	3	53	26	(72)
N	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	
I	56	57	68	43	5	23	58	19	45	48	60	69	50	37	52	42	44	36	(72)

$m = 7 + 5i = -i(1+i)(1+6i) \quad \text{N}(m) = 74 \quad \Phi(m) = 36 = 2^2.3^2 \quad \text{H.E.} = 36$

$$x \equiv \text{X} + 43\text{Y} \pmod{74}$$

$m = 5 + 7i = \quad (1+i)(6+i) \quad \text{N}(m) = 74 \quad \Phi(m) = 36 = 2^2.3^2 \quad \text{H.E.} = 36$

$$x \equiv \text{X} + 31\text{Y} \pmod{74}.$$

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(36)
N	1	5	25	51	33	17	11	55	53	43	67	39	47	13	65	29	71	59	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(36)
N	73	69	49	23	41	57	63	19	21	31	7	35	27	61	9	45	3	15	

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	
I	0	34	1	28	32	6	13	35	5	25	26	21	2	30	15	27	4	29	(36)
N	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	
I	11	22	9	33	12	20	3	8	7	23	17	31	24	14	10	19	16	18	(36

$$m = 8 + 4i = -(1+i)^4(2+i) \quad \mathrm{N}(m) = 80 \quad \Phi(m) = 32 = 2.2^2.2^2 \quad \mathrm{H.E.} = 4$$

$$y \equiv \mathrm{Y} \pmod 4 \quad x \equiv \mathrm{X} + 3(\mathrm{Y} - y) \pmod{20}.$$

Generators 17, $8+i$, $5+2i$.

0	0	0	1
0	0	1	$5+2i$
0	1	0	$8+i$
0	1	1	$18+i$
0	2	0	11
0	2	1	$15+2i$
0	3	0	$12+3i$
0	3	1	$2+3i$
1	0	0	17
1	0	1	$1+2i$
1	1	0	$4+i$
1	1	1	$14+i$
1	2	0	7
1	2	1	$11+2i$
1	3	0	$8+3i$
1	3	1	$18+3i$
2	0	0	9
2	0	1	$13+2i$
2	1	0	$16+i$
2	1	1	$6+i$
2	2	0	19
2	2	1	$3+2i$
2	3	0	$3i$
2	3	1	$10+3i$
3	0	0	13
3	0	1	$17+2i$
3	1	0	i
3	1	1	$10+i$
3	2	0	3
3	2	1	$7+2i$
3	3	0	$4+3i$
3	3	1	$14+3i$
(4)	(4)	(2)	

i	3	1	0
$3i$	2	3	0
1	0	0	0
$1+2i$	1	0	1
$2+3i$	0	3	1
3	3	2	0
$3+2i$	2	2	1
$4+i$	1	1	0
$4+3i$	3	3	0
$5+2i$	0	0	1
$6+i$	2	1	1
7	1	2	0
$7+2i$	3	2	1
$8+i$	0	1	0
$8+3i$	1	3	0
9	2	0	0
$10+i$	3	1	1
$10+3i$	2	3	1
11	0	2	0
$11+2i$	1	2	1
$12+3i$	0	3	0
13	3	0	0
$13+2i$	2	0	1
$14+i$	1	1	1
$14+3i$	3	3	1
$15+2i$	0	2	1
$16+i$	2	1	0
17	1	0	0
$17+2i$	3	0	1
$18+i$	0	1	1
$18+3i$	1	3	1
19	2	2	0
	(4)	(4)	(2)

$$m = 4 + 8i = -(1+i)^4(1+2i) \quad \mathrm{N}(m) = 80 \quad \Phi(m) = 32 = 2.\,2^2.\,2^3 \quad \text{H. E.} = 4$$

$$y \equiv \mathrm{Y} \pmod 4 \quad x \equiv \mathrm{X} + 17(\mathrm{Y} - y) \pmod{20}.$$

Generators 17, $3i$, $17 + 2i$.

0	0	0	1
0	0	1	$17+2i$
0	1	0	$3i$
0	1	1	$10+3i$
0	2	0	11
0	2	1	$7+2i$
0	3	0	$4+i$
0	3	1	$14+i$
1	0	0	17
1	0	1	$13+2i$
1	1	0	$16+3i$
1	1	1	$6+3i$
1	2	0	7
1	2	1	$3+2i$
1	3	0	i
1	3	1	$10+i$
2	0	0	9
2	0	1	$5+2i$
2	1	0	$8+3i$
2	1	1	$18+3i$
2	2	0	19
2	2	1	$15+2i$
2	3	0	$12+i$
2	3	1	$2+i$
3	0	0	13
3	0	1	$9+2i$
3	1	0	$12+3i$
3	1	1	$2+3i$
3	2	0	3
3	2	1	$19+2i$
3	3	0	$16+i$
3	3	1	$6+i$
(4)	(4)	(2)	

i	1	3	0
$3i$	0	1	0
1	0	0	0
$2+i$	2	3	1
$2+3i$	3	1	1
3	3	2	0
$3+2i$	1	2	1
$4+i$	0	3	0
$5+2i$	2	0	1
$6+i$	3	3	1
$6+3i$	1	1	1
7	1	2	0
$7+2i$	0	2	1
$8+3i$	2	1	0
9	2	0	0
$9+2i$	3	0	1
$10+i$	1	3	1
$10+3i$	0	1	1
11	0	2	0
$12+i$	2	3	0
$12+3i$	3	1	0
13	3	0	0
$13+2i$	1	0	1
$14+i$	0	3	1
$15+2i$	2	2	1
$16+i$	3	3	0
$16+3i$	1	1	0
17	1	0	0
$17+2i$	0	0	1
$18+3i$	2	1	1
19	2	2	0
$19+2i$	3	2	1
	(4)	(4)	(2)

$m = 9 = 3^2 \quad \mathrm{N}(m) = 81 \quad \Phi(m) = 72 = 2^3 . 3 . 3 \quad \text{H. E.} = 24$

$x \equiv \mathrm{X} \pmod 9 \quad y \equiv \mathrm{Y} \pmod 9$.

Generators $2 + i$, $4 + 3i$.

0	0	1	8	0	$4+6i$	16	0	$7+3i$
0	1	$4+3i$	8	1	7	16	1	$1+6i$
0	2	$7+6i$	8	2	$1+3i$	16	2	4
1	0	$2+i$	9	0	$2+7i$	17	0	$2+4i$
1	1	$5+i$	9	1	$5+7i$	17	1	$5+4i$
1	2	$8+i$	9	2	$8+7i$	17	2	$8+4i$
2	0	$3+4i$	10	0	$6+7i$	18	0	i
2	1	$7i$	10	1	$3+i$	18	1	$6+4i$
2	2	$6+i$	10	2	$4i$	18	2	$3+7i$
3	0	$2+2i$	11	0	$5+2i$	19	0	$8+2i$
3	1	$2+5i$	11	1	$5+5i$	19	1	$8+5i$
3	2	$2+8i$	11	2	$5+8i$	19	2	$8+8i$
4	0	$2+6i$	12	0	8	20	0	$5+3i$
4	1	$8+3i$	12	1	$5+6i$	20	1	2
4	2	5	12	2	$2+3i$	20	2	$8+6i$
5	0	$7+5i$	13	0	$7+8i$	21	0	$7+2i$
5	1	$4+5i$	13	1	$4+8i$	21	1	$4+2i$
5	2	$1+5i$	13	2	$1+8i$	21	2	$1+2i$
6	0	$8i$	14	0	$6+5i$	22	0	$3+2i$
6	1	$3+5i$	14	1	$2i$	22	1	$6+8i$
6	2	$6+2i$	14	2	$3+8i$	22	2	$5i$
7	0	$1+7i$	15	0	$7+7i$	23	0	$4+7i$
7	1	$1+4i$	15	1	$7+4i$	23	1	$4+4i$
7	2	$1+i$	15	2	$7+i$	23	2	$4+i$
(24)	(3)		(24)	(3)		(24)	(3)	

i	18	0	$3+i$	10	1	$6+i$	2	2
$2i$	14	1	$3+2i$	22	0	$6+2i$	6	2
$4i$	10	2	$3+4i$	2	0	$6+4i$	18	1
$5i$	22	2	$3+5i$	6	1	$6+5i$	14	0
$7i$	2	1	$3+7i$	18	2	$6+7i$	10	0
$8i$	6	0	$3+8i$	14	2	$6+8i$	22	1
1	0	0	4	16	2	7	8	1
$1+i$	7	2	$4+i$	23	2	$7+i$	15	2
$1+2i$	21	2	$4+2i$	21	1	$7+2i$	21	0
$1+3i$	8	2	$4+3i$	0	1	$7+3i$	16	0
$1+4i$	7	1	$4+4i$	23	1	$7+4i$	15	1
$1+5i$	5	2	$4+5i$	5	1	$7+5i$	5	0
$1+6i$	16	1	$4+6i$	8	0	$7+6i$	0	2
$1+7i$	7	0	$4+7i$	23	0	$7+7i$	15	0
$1+8i$	13	2	$4+8i$	13	1	$7+8i$	13	0
2	20	1	5	4	2	8	12	0
$2+i$	1	0	$5+i$	1	1	$8+i$	1	2
$2+2i$	3	0	$5+2i$	11	0	$8+2i$	19	0
$2+3i$	12	2	$5+3i$	20	0	$8+3i$	4	1
$2+4i$	17	0	$5+4i$	17	1	$8+4i$	17	2
$2+5i$	3	1	$5+5i$	11	1	$8+5i$	19	1
$2+6i$	4	0	$5+6i$	12	1	$8+6i$	20	2
$2+7i$	9	0	$5+7i$	9	1	$8+7i$	9	2
$2+8i$	3	2	$5+8i$	11	2	$8+8i$	19	2
	(24)	(3)		(24)	(3)		(24)	(3)

$m = 9 + i = -i(1+i)(4+5i) \quad N(m) = 82 \quad \Phi(m) = 40 = 2^3 . 5$
H. E. $= 40 \quad x \equiv X + 73Y \pmod{82}$.

$m = 1 + 9i = \quad (1+i)(5+4i) \quad N(m) = 82 \quad \Phi(m) = 40 = 2^3 . 5$
H. E. $\equiv 40 \quad x \equiv X + 9Y \pmod{82}$.

Generator 7.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(40)
N	1	7	49	15	23	79	61	17	37	13	9	63	31	53	43	55	57	71	5	35	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	(40)
N	81	75	33	67	59	3	21	65	45	69	73	19	51	29	39	27	25	11	77	47	

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	
I	0	25	18	1	10	37	9	3	7	31	26	4	36	35	33	12	22	19	8	34	(40)
N	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	81	
I	14	28	39	2	32	13	15	16	24	6	11	27	23	29	17	30	21	38	5	20	(40)

$$m = 9 + 2i = -i(2+i)(1+4i) \quad 2+9i = (1+2i)(4+i) \quad 7+6i = (2+i)(4+i)$$
$$\text{or } 6+7i = -i(1+2i)(1+4i).$$

For all $N(m) = 85 \quad \Phi(m) = 64 = 2^2 . 2^4 \quad$ H.E. $= 16$.

$$m = 9+2i \quad x \equiv X + 38Y \pmod{85} \quad m = 2+9i \quad x \equiv X + 47Y \pmod{85}$$
$$m = 7+6i \quad x \equiv X + 13Y \pmod{85} \quad m = 6+7i \quad x \equiv X + 72Y \pmod{85}.$$

Generators 3, 52.

0	0	1
0	1	52
0	2	69
0	3	18
1	0	3
1	1	71
1	2	37
1	3	54
2	0	9
2	1	43
2	2	26
2	3	77
3	0	27
3	1	44
3	2	78
3	3	61
4	0	81
4	1	47
4	2	64
4	3	13
5	0	73
5	1	56
5	2	22
5	3	39
6	0	49
6	1	83
6	2	66
6	3	32
7	0	62
7	1	79
7	2	28
7	3	11
(16)	(4)	

8	0	16
8	1	67
8	2	84
8	3	33
9	0	48
9	1	31
9	2	82
9	3	14
10	0	59
10	1	8
10	2	76
10	3	42
11	0	7
11	1	24
11	2	58
11	3	41
12	0	21
12	1	72
12	2	4
12	3	38
13	0	63
13	1	46
13	2	12
13	3	29
14	0	19
14	1	53
14	2	36
14	3	2
15	0	57
15	1	74
15	2	23
15	3	6
(16)	(4)	

1	0	0
2	14	3
3	1	0
4	12	2
6	15	3
7	11	0
8	10	1
9	2	0
11	7	3
12	13	2
13	4	3
14	9	3
16	8	0
18	0	3
19	14	0
21	12	0
22	5	2
23	15	2
24	11	1
26	2	2
27	3	0
28	7	2
29	13	3
31	9	1
32	6	3
33	8	3
36	14	2
37	1	2
38	12	3
39	5	3
41	11	3
42	10	3
	(16)	(4)

43	2	1
44	3	1
46	13	1
47	4	1
48	9	0
49	6	0
52	0	1
53	14	1
54	1	3
56	5	1
57	15	0
58	11	2
59	10	0
61	3	3
62	7	0
63	13	0
64	4	2
66	6	2
67	8	1
69	0	2
71	1	1
72	12	1
73	5	0
74	15	1
76	10	2
77	2	3
78	3	2
79	7	1
81	4	0
82	9	2
83	6	1
84	8	2
	(16)	(4)

$m = 8 + 5i$ $\mathrm{N}(m) = 89$ $\Phi(m) = 88 = 2^3 . 11$ H. E. $= 88$ $x \equiv \mathrm{X} + 34\,\mathrm{Y} \pmod{89}$.

$m = 5 + 8i$ $\mathrm{N}(m) = 89$ $\Phi(m) = 88 = 2^3 . 11$ H. E. $= 88$ $x \equiv \mathrm{X} + 55\,\mathrm{Y} \pmod{89}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	(88)
N	1	3	9	27	81	65	17	51	64	14	42	37	22	66	20	60	2	6	18	54	73	41	
I	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	(88)
N	34	13	39	28	84	74	44	43	40	31	4	12	36	19	57	82	68	26	78	56	79	59	
I	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	(88)
N	88	86	80	62	8	24	72	38	25	75	47	52	67	23	69	29	87	83	71	35	16	48	
I	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	(88)
N	55	76	50	61	5	15	45	46	49	58	85	77	53	70	32	7	21	63	11	33	10	30	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
I	0	16	1	32	70	17	81	48	2	86	84	33	23	9	71	64	6	18	35	14	82	12	(88)
N	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	
I	57	49	52	39	3	25	59	87	31	80	85	22	63	34	11	51	24	30	21	10	29	28	(88)
N	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	
I	72	73	54	65	74	68	7	55	78	19	66	41	36	75	43	15	69	47	83	8	5	13	(88)
N	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	
I	56	38	58	79	62	50	20	27	53	67	77	40	42	46	4	37	61	26	76	45	60	44	(88)

$$m = 9 + 3i = -i\,(1 + i)\,(1 + 2i)\,3 \quad \mathrm{N}\,(m) = 90 \quad \Phi\,(m) = 32 = 2^2 .\, 2^3$$

$$\mathrm{H.\ E.} = 8 \quad y \equiv \mathrm{Y}\ (\mathrm{mod}\ 3) \quad x \equiv \mathrm{X} + 7\,(\mathrm{Y} - y)\,(\mathrm{mod}\ 30).$$

Generators $2 + i$, i.

0	0	1
0	1	i
0	2	29
0	3	$9+2i$
1	0	$2+i$
1	1	$29+2i$
1	2	$7+2i$
1	3	$10+i$
2	0	$24+i$
2	1	17
2	2	$15+2i$
2	3	13
3	0	$5+2i$
3	1	$19+2i$
3	2	$4+i$
3	3	$20+i$
4	0	11
4	1	$3+2i$
4	2	19
4	3	$6+i$
5	0	$25+2i$
5	1	$16+i$
5	2	$14+i$
5	3	$23+2i$
6	0	$27+2i$
6	1	7
6	2	$12+i$
6	3	23
7	0	$22+i$
7	1	$26+i$
7	2	$17+2i$
7	3	$13+2i$
(8)	(4)	

i	0	1
1	0	0
$2+i$	1	0
$3+2i$	4	1
$4+i$	3	2
$5+2i$	3	0
$6+i$	4	3
7	6	1
$7+2i$	1	2
$9+2i$	0	3
$10+i$	1	3
11	4	0
$12+i$	6	2
13	2	3
$13+2i$	7	3
$14+i$	5	2
$15+2i$	2	2
$16+i$	5	1
17	2	1
$17+2i$	7	2
19	4	2
$19+2i$	3	1
$20+i$	3	3
$22+i$	7	0
23	6	3
$23+2i$	5	3
$24+i$	2	0
$25+2i$	5	0
$26+i$	7	1
$27+2i$	6	0
29	0	2
$29+2i$	1	1
	(8)	(4)

$$m = 3 + 9i = (1 + i)(2 + i)\,3 \quad \mathrm{N}(m) = 90 \quad \Phi(m) = 32 = 2^2 . \, 2^3$$

$$\text{H.E.} = 8 \quad y \equiv \mathrm{Y} \pmod{3} \quad x \equiv \mathrm{X} + 23\,(\mathrm{Y} - y) \pmod{30}.$$

Generators $23 + 2i$, $21 + 2i$.

0	0	1
0	1	$21+2i$
0	2	29
0	3	i
1	0	$23+2i$
1	1	$20+i$
1	2	$28+i$
1	3	$1+2i$
2	0	$15+2i$
2	1	17
2	2	$6+i$
2	3	13
3	0	$26+i$
3	1	$10+i$
3	2	$25+2i$
3	3	$11+2i$
4	0	11
4	1	$24+i$
4	2	19
4	3	$27+2i$
5	0	$16+i$
5	1	$7+2i$
5	2	$5+2i$
5	3	$14+i$
6	0	$18+i$
6	1	7
6	2	$3+2i$
6	3	23
7	0	$13+2i$
7	1	$17+2i$
7	2	$8+i$
7	3	$4+i$
(8)	(4)	

i	0	3
1	0	0
$1+2i$	1	3
$3+2i$	6	2
$4+i$	7	3
$5+2i$	5	2
$6+i$	2	2
7	6	1
$7+2i$	5	1
$8+i$	7	2
$10+i$	3	1
11	4	0
$11+2i$	3	3
13	2	3
$13+2i$	7	0
$14+i$	5	3
$15+2i$	2	0
$16+i$	5	0
17	2	1
$17+2i$	7	1
$18+i$	6	0
19	4	2
$20+i$	1	1
$21+2i$	0	1
23	6	3
$23+2i$	1	0
$24+i$	4	1
$25+2i$	3	2
$26+i$	3	0
$27+2i$	4	3
$28+i$	1	2
29	0	2
	(8)	(4)

$m = 9 + 4i$ $\quad$ $N(m) = 97$ $\quad$ $\Phi(m) = 96 = 2^5 . 3$ $\quad$ H. E. = 96 $\quad$ $x \equiv X + 22Y \pmod{97}$.

$m = 4 + 9i$ $\quad$ $N(m) = 97$ $\quad$ $\Phi(m) = 96 = 2^5 . 3$ $\quad$ H. E. = 96 $\quad$ $x \equiv X + 75Y \pmod{97}$.

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(96)
N	1	5	25	28	43	21	8	40	6	30	53	71	64	29	48	46	36	83	27	38	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		(96)
N	93	77	94	82	22	13	65	34	73	74	79	7	35	78	2	10	50	56	86		
I	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57		(96)
N	42	16	80	12	60	9	45	31	58	96	92	72	69	54	76	89	57	91	67		
I	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76		(96)
N	44	26	33	68	49	51	61	14	70	59	4	20	3	15	75	84	32	63	24		
I	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95		(96)
N	23	18	90	62	19	95	87	47	41	11	55	81	17	85	37	88	52	66	39		

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
I	0	34	70	68	1	8	31	6	44	35	86	42	25	65	71	40	89	78	81	69	(96)
N	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39		
I	5	24	77	76	2	59	18	3	13	9	46	74	60	27	32	16	91	19	95		(96)
N	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58		
I	7	85	39	4	58	45	15	84	14	62	36	63	93	10	52	87	37	55	47		(96)
N	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77		
I	67	43	64	80	75	12	26	94	57	61	51	66	11	50	28	29	72	53	21		(96)
N	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96		
I	33	30	41	88	23	17	73	90	38	83	92	54	79	56	49	20	22	82	48		(96)

$$m = 7 + 7i = (1 + i)\,7 \quad \mathrm{N}\,(m) = 98 \quad \Phi\,(m) = 48 = 2^4.\,3 \quad \mathrm{H.\,E.} = 48$$

$$y \equiv \mathrm{Y} \pmod{7} \quad x \equiv \mathrm{X} + \mathrm{Y} - y \pmod{14}.$$

Generator $2 + i$.

0	1	24	13
1	$2+i$	25	$5+6i$
2	$3+4i$	26	$4+3i$
3	$9+4i$	27	$12+3i$
4	$3i$	28	$7+4i$
5	$11+6i$	29	$10+i$
6	$9+2i$	30	$12+5i$
7	$9+6i$	31	$12+i$
8	5	32	9
9	$10+5i$	33	$11+2i$
10	$1+6i$	34	$6+i$
11	$3+6i$	35	$4+i$
12	i	36	$7+6i$
13	$13+2i$	37	$8+5i$
14	$10+3i$	38	$11+4i$
15	$3+2i$	39	$4+5i$
16	11	40	3
17	$1+4i$	41	$6+3i$
18	$5+2i$	42	$2+5i$
19	$1+2i$	43	$6+5i$
20	$5i$	44	$7+2i$
21	$2+3i$	45	$5+4i$
22	$8+i$	46	$13+6i$
23	$8+3i$	47	$13+4i$
(48)		(48)	

i	12	$7+2i$	44
$3i$	4	$7+4i$	28
$5i$	20	$7+6i$	36
1	0	$8+i$	22
$1+2i$	19	$8+3i$	23
$1+4i$	17	$8+5i$	37
$1+6i$	10	9	32
$2+i$	1	$9+2i$	6
$2+3i$	21	$9+4i$	3
$2+5i$	42	$9+6i$	7
3	40	$10+i$	29
$3+2i$	15	$10+3i$	14
$3+4i$	2	$10+5i$	9
$3+6i$	11	11	16
$4+i$	35	$11+2i$	33
$4+3i$	26	$11+4i$	38
$4+5i$	39	$11+6i$	5
5	8	$12+i$	31
$5+2i$	18	$12+3i$	27
$5+4i$	45	$12+5i$	30
$5+6i$	25	13	24
$6+i$	34	$13+2i$	13
$6+3i$	41	$13+4i$	47
$6+5i$	43	$13+6i$	46
	(48)		(48)

$$m = 8 + 6i = -(1+i)^2(1+2i)^2 \quad N(m) = 100 \quad \Phi(m) = 40 = 2.2^2.5.$$

$$\text{H. E.} = 20 \quad y \equiv Y \pmod{2} \quad x \equiv X + 7(Y - y) \pmod{50}.$$

Generators 3, $44 + i$.

0	0	1
0	1	$44+i$
1	0	3
1	1	$46+i$
2	0	9
2	1	$2+i$
3	0	27
3	1	$20+i$
4	0	31
4	1	$24+i$
5	0	43
5	1	$36+i$
6	0	29
6	1	$22+i$
7	0	37
7	1	$30+i$
8	0	11
8	1	$4+i$
9	0	33
9	1	$26+i$
(20)	(2)	

10	0	49
10	1	$42+i$
11	0	47
11	1	$40+i$
12	0	41
12	1	$34+i$
13	0	23
13	1	$16+i$
14	0	19
14	1	$12+i$
15	0	7
15	1	i
16	0	21
16	1	$14+i$
17	0	13
17	1	$6+i$
18	0	39
18	1	$32+i$
19	0	17
19	1	$10+i$
(20)	(2)	

i	15	1
1	0	0
$2+i$	2	1
3	1	0
$4+i$	8	1
$6+i$	17	1
7	15	0
9	2	0
$10+i$	19	1
11	8	0
$12+i$	14	1
13	17	0
$14+i$	16	1
$16+i$	13	1
17	19	0
19	14	0
$20+i$	3	1
21	16	0
$22+i$	6	1
23	13	0
	(20)	(2)

$24+i$	4	1
$26+i$	9	1
27	3	0
29	6	0
$30+i$	7	1
31	4	0
$32+i$	18	1
33	9	0
$34+i$	12	1
$36+i$	5	1
37	7	0
39	18	0
$40+i$	11	1
41	12	0
$42+i$	10	1
43	5	0
$44+i$	0	1
$46+i$	1	1
47	11	0
49	10	0
	(20)	(2)

$$m = 6 + 8i = -i(1+i)^2(2+i)^2 \quad N(m) = 100 \quad \Phi(m) = 40 = 2.2^2.5$$

$$\text{H. E.} = 20 \quad y \equiv Y \pmod{2} \quad x \equiv X + 43(Y - y) \pmod{50}.$$

Generators 3, $8 + i$.

0	0	1
0	1	$8+i$
1	0	3
1	1	$10+i$
2	0	9
2	1	$16+i$
3	0	27
3	1	$34+i$
4	0	31
4	1	$38+i$
5	0	43
5	1	i
6	0	29
6	1	$36+i$
7	0	37
7	1	$44+i$
8	0	11
8	1	$18+i$
9	0	33
9	1	$40+i$
(20)	(2)	

10	0	49
10	1	$6+i$
11	0	47
11	1	$4+i$
12	0	41
12	1	$48+i$
13	0	23
13	1	$30+i$
14	0	19
14	1	$26+i$
15	0	7
15	1	$14+i$
16	0	21
16	1	$28+i$
17	0	13
17	1	$20+i$
18	0	39
18	1	$46+i$
19	0	17
19	1	$24+i$
(20)	(2)	

i	5	1
1	0	0
3	1	0
$4+i$	11	1
$6+i$	10	1
7	15	0
$8+i$	0	1
9	2	0
$10+i$	1	1
11	8	0
13	17	0
$14+i$	15	1
$16+i$	2	1
17	19	0
$18+i$	8	1
19	14	0
$20+i$	17	1
21	16	0
23	13	0
$24+i$	19	1
	(20)	(2)

$26+i$	14	1
27	3	0
$28+i$	16	1
29	6	0
$30+i$	13	1
31	4	0
33	9	0
$34+i$	3	1
$36+i$	6	1
37	7	0
$38+i$	4	1
39	18	0
$40+i$	9	1
41	12	0
43	5	0
$44+i$	7	1
$46+i$	18	1
47	11	0
$48+i$	12	1
49	10	0
	(20)	(2)

$$m = 10 = -(1+i)^2(1+2i)(2+i) \quad N(m) = 100 \quad \Phi(m) = 32 = 2.2^2.2^2. \quad \text{H. E.} = 4$$

$$x \equiv X \pmod{10} \quad y \equiv Y \pmod{10}.$$

Generators $9+4i$, $9+6i$, $6+5i$.

0	0	0	1
0	0	1	$6+5i$
0	1	0	$9+6i$
0	1	1	$4+i$
0	2	0	$5+8i$
0	2	1	$3i$
0	3	0	$7+2i$
0	3	1	$2+7i$
1	0	0	$9+4i$
1	0	1	$4+9i$
1	1	0	7
1	1	1	$2+5i$
1	2	0	$3+2i$
1	2	1	$8+7i$
1	3	0	$5+6i$
1	3	1	i
2	0	0	$5+2i$
2	0	1	$7i$
2	1	0	$3+8i$
2	1	1	$8+3i$
2	2	0	9
2	2	1	$4+5i$
2	3	0	$1+4i$
2	3	1	$6+9i$
3	0	0	$7+8i$
3	0	1	$2+3i$
3	1	0	$5+4i$
3	1	1	$9i$
3	2	0	$1+6i$
3	2	1	$6+i$
3	3	0	3
3	3	1	$8+5i$
(4)	(4)	(2)	

i	1	3	1
$3i$	0	2	1
$7i$	2	0	1
$9i$	3	1	1
1	0	0	0
$1+4i$	2	3	0
$1+6i$	3	2	0
$2+3i$	3	0	1
$2+5i$	1	1	1
$2+7i$	0	3	1
3	3	3	0
$3+2i$	1	2	0
$3+8i$	2	1	0
$4+i$	0	1	1
$4+5i$	2	2	1
$4+9i$	1	0	1
$5+2i$	2	0	0
$5+4i$	3	1	0
$5+6i$	1	3	0
$5+8i$	0	2	0
$6+i$	3	2	1
$6+5i$	0	0	1
$6+9i$	2	3	1
7	1	1	0
$7+2i$	0	3	0
$7+8i$	3	0	0
$8+3i$	2	1	1
$8+5i$	3	3	1
$8+7i$	1	2	1
9	2	2	0
$9+4i$	1	0	0
$9+6i$	0	1	0
	(4)	(4)	(2)

Mod $(1 + i)$.

1	1

Mod $(1 + i)^2$.

Generator i.

i	1
1	0
	(2)

Mod $(1 + i)^3$.

Generator i.

i	1
3	2
$2+i$	3
1	0
	(4)

Mod $(1 + i)^4$

Generators i, $1 + 2i$.

0	0	1
0	1	$1+2i$
1	0	i
1	1	$2+i$
2	0	3
2	1	$3+2i$
3	0	$3i$
3	1	$2+3i$
(4)	(2)	

i	1	0
$3i$	3	0
1	0	0
$1+2i$	0	1
$2+i$	1	1
$2+3i$	3	1
3	2	0
$3+2i$	2	1
	(4)	(2)

Mod $(1 + i)^5$.

Generators i, $1 + 2i$, 3.

0	0	0	1
0	0	1	3
0	1	0	$1+2i$
0	1	1	$7+2i$
1	0	0	i
1	0	1	$3i$
1	1	0	$6+i$
1	1	1	$2+3i$
2	0	0	7
2	0	1	5
2	1	0	$3+2i$
2	1	1	$5+2i$
3	0	0	$4+3i$
3	0	1	$4+i$
3	1	0	$6+3i$
3	1	1	$2+i$
(4)	(2)	(2)	

i	1	0	0
$3i$	1	0	1
1	0	0	0
$1+2i$	0	1	0
$2+i$	3	1	1
$2+3i$	1	1	1
3	0	0	1
$3+2i$	2	1	0
$4+i$	3	0	1
$4+3i$	3	0	0
5	2	0	1
$5+2i$	2	1	1
$6+i$	1	1	0
$6+3i$	3	1	0
7	2	0	0
$7+2i$	0	1	1
	(4)	(2)	(2)

Mod $(1+i)^6$.

Generators i, $1+2i$, 3.

0	0	0	1
0	0	1	3
0	1	0	$1+2i$
0	1	1	$3+6i$
0	2	0	$5+4i$
0	2	1	$7+4i$
0	3	0	$5+6i$
0	3	1	$7+2i$
1	0	0	i
1	0	1	$3i$
1	1	0	$6+i$
1	1	1	$2+3i$
1	2	0	$4+5i$
1	2	1	$4+7i$
1	3	0	$2+5i$
1	3	1	$6+7i$
2	0	0	7
2	0	1	5
2	1	0	$7+6i$
2	1	1	$5+2i$
2	2	0	$3+4i$
2	2	1	$1+4i$
2	3	0	$3+2i$
2	3	1	$1+6i$
3	0	0	$7i$
3	0	1	$5i$
3	1	0	$2+7i$
3	1	1	$6+5i$
3	2	0	$4+3i$
3	2	1	$4+i$
3	3	0	$6+3i$
3	3	1	$2+i$
(4)	(4)	(2)	

i	1	0	0
$3i$	1	0	1
$5i$	3	0	1
$7i$	3	0	0
1	0	0	0
$1+2i$	0	1	0
$1+4i$	2	1	1
$1+6i$	2	3	1
$2+i$	3	3	1
$2+3i$	1	1	1
$2+5i$	1	3	0
$2+7i$	3	1	0
3	0	0	1
$3+2i$	2	3	0
$3+4i$	2	2	0
$3+6i$	0	1	1
$4+i$	3	2	1
$4+3i$	3	2	0
$4+5i$	1	2	0
$4+7i$	1	2	1
5	2	0	1
$5+2i$	2	1	1
$5+4i$	0	2	0
$5+6i$	0	3	0
$6+i$	1	1	0
$6+3i$	3	3	0
$6+5i$	3	1	1
$6+7i$	1	3	1
7	2	0	0
$7+2i$	0	3	1
$7+4i$	0	2	1
$7+6i$	2	1	0
	(4)	(4)	(2)

Mod $(1 + i)^7$.

Generators i, 3, $1 + 2i$.

0	0	0	1	2	0	0	15
0	0	1	$1+2i$	2	0	1	$7+6i$
0	0	2	$13+4i$	2	0	2	$11+4i$
0	0	3	$13+6i$	2	0	3	$11+2i$
0	1	0	3	2	1	0	13
0	1	1	$3+6i$	2	1	1	$5+2i$
0	1	2	$15+4i$	2	1	2	$9+4i$
0	1	3	$7+2i$	2	1	3	$1+6i$
0	2	0	9	2	2	0	7
0	2	1	$9+2i$	2	2	1	$15+6i$
0	2	2	$5+4i$	2	2	2	$3+4i$
0	2	3	$5+6i$	2	2	3	$3+2i$
0	3	0	11	2	3	0	5
0	3	1	$11+6i$	2	3	1	$13+2i$
0	3	2	$7+4i$	2	3	2	$1+4i$
0	3	3	$15+2i$	2	3	3	$9+6i$
1	0	0	i	3	0	0	$8+7i$
1	0	1	$14+i$	3	0	1	$10+7i$
1	0	2	$4+5i$	3	0	2	$4+3i$
1	0	3	$2+5i$	3	0	3	$6+3i$
1	1	0	$3i$	3	1	0	$8+5i$
1	1	1	$10+3i$	3	1	1	$14+5i$
1	1	2	$4+7i$	3	1	2	$4+i$
1	1	3	$14+7i$	3	1	3	$10+i$
1	2	0	$8+i$	3	2	0	$7i$
1	2	1	$6+i$	3	2	1	$2+7i$
1	2	2	$12+5i$	3	2	2	$12+3i$
1	2	3	$10+5i$	3	2	3	$14+3i$
1	3	0	$8+3i$	3	3	0	$5i$
1	3	1	$2+3i$	3	3	1	$6+5i$
1	3	2	$12+7i$	3	3	2	$12+i$
1	3	3	$6+7i$	3	3	3	$2+i$
(4)	(4)	(4)		(4)	(4)	(4)	

i	1	0	0	$8+i$	1	2	0
$3i$	1	1	0	$8+3i$	1	3	0
$5i$	3	3	0	$8+5i$	3	1	0
$7i$	3	2	0	$8+7i$	3	0	0
1	0	0	0	9	0	2	0
$1+2i$	0	0	1	$9+2i$	0	2	1
$1+4i$	2	3	2	$9+4i$	2	1	2
$1+6i$	2	1	3	$9+6i$	2	3	3
$2+i$	3	3	3	$10+i$	3	1	3
$2+3i$	1	3	1	$10+3i$	1	1	1
$2+5i$	1	0	3	$10+5i$	1	2	3
$2+7i$	3	2	1	$10+7i$	3	0	1
3	0	1	0	11	0	3	0
$3+2i$	2	2	3	$11+2i$	2	0	3
$3+4i$	2	2	2	$11+4i$	2	0	2
$3+6i$	0	1	1	$11+6i$	0	3	1
$4+i$	3	1	2	$12+i$	3	3	2
$4+3i$	3	0	2	$12+3i$	3	2	2
$4+5i$	1	0	2	$12+5i$	1	2	2
$4+7i$	1	1	2	$12+7i$	1	3	2
5	2	3	0	13	2	1	0
$5+2i$	2	1	1	$13+2i$	2	3	1
$5+4i$	0	2	2	$13+4i$	0	0	2
$5+6i$	0	2	3	$13+6i$	0	0	3
$6+i$	1	2	1	$14+i$	1	0	1
$6+3i$	3	0	3	$14+3i$	3	2	3
$6+5i$	3	3	1	$14+5i$	3	1	1
$6+7i$	1	3	3	$14+7i$	1	1	3
7	2	2	0	15	2	0	0
$7+2i$	0	1	3	$15+2i$	0	3	3
$7+4i$	0	3	2	$15+4i$	0	1	2
$7+6i$	2	0	1	$15+6i$	2	2	1
	(4)	(4)	(4)		(4)	(4)	(4)

Mod $(1+i)^8$.

Generators $1+2i$, 3, i.

0	0	0	1	2	0	0	$13+4i$	4	0	0	$9+8i$	6	0	0	$5+12i$
0	0	1	i	2	0	1	$12+13i$	4	0	1	$8+9i$	6	0	1	$4+5i$
0	0	2	15	2	0	2	$3+12i$	4	0	2	$7+8i$	6	0	2	$11+4i$
0	0	3	$15i$	2	0	3	$4+3i$	4	0	3	$8+7i$	6	0	3	$12+11i$
0	1	0	3	2	1	0	$7+12i$	4	1	0	$11+8i$	6	1	0	$15+4i$
0	1	1	$3i$	2	1	1	$4+7i$	4	1	1	$8+11i$	6	1	1	$12+15i$
0	1	2	13	2	1	2	$9+4i$	4	1	2	$5+8i$	6	1	2	$1+12i$
0	1	3	$13i$	2	1	3	$12+9i$	4	1	3	$8+5i$	6	1	3	$4+i$
0	2	0	9	2	2	0	$5+4i$	4	2	0	$1+8i$	6	2	0	$13+12i$
0	2	1	$9i$	2	2	1	$12+5i$	4	2	1	$8+i$	6	2	1	$4+13i$
0	2	2	7	2	2	2	$11+12i$	4	2	2	$15+8i$	6	2	2	$3+4i$
0	2	3	$7i$	2	2	3	$4+11i$	4	2	3	$8+15i$	6	2	3	$12+3i$
0	3	0	11	2	3	0	$15+12i$	4	3	0	$3+8i$	6	3	0	$7+4i$
0	3	1	$11i$	2	3	1	$4+15i$	4	3	1	$8+3i$	6	3	1	$12+7i$
0	3	2	5	2	3	2	$1+4i$	4	3	2	$13+8i$	6	3	2	$9+12i$
0	3	3	$5i$	2	3	3	$12+i$	4	3	3	$8+13i$	6	3	3	$4+9i$
1	0	0	$1+2i$	3	0	0	$5+14i$	5	0	0	$9+10i$	7	0	0	$13+6i$
1	0	1	$14+i$	3	0	1	$2+5i$	5	0	1	$6+9i$	7	0	1	$10+13i$
1	0	2	$15+14i$	3	0	2	$11+2i$	5	0	2	$7+6i$	7	0	2	$3+10i$
1	0	3	$2+15i$	3	0	3	$14+11i$	5	0	3	$10+7i$	7	0	3	$6+3i$
1	1	0	$3+6i$	3	1	0	$15+10i$	5	1	0	$11+14i$	7	1	0	$7+2i$
1	1	1	$10+3i$	3	1	1	$6+15i$	5	1	1	$2+11i$	7	1	1	$14+7i$
1	1	2	$13+10i$	3	1	2	$1+6i$	5	1	2	$5+2i$	7	1	2	$9+14i$
1	1	3	$6+13i$	3	1	3	$10+i$	5	1	3	$14+5i$	7	1	3	$2+9i$
1	2	0	$9+2i$	3	2	0	$13+14i$	5	2	0	$1+10i$	7	2	0	$5+6i$
1	2	1	$14+9i$	3	2	1	$2+13i$	5	2	1	$6+i$	7	2	1	$10+5i$
1	2	2	$7+14i$	3	2	2	$3+2i$	5	2	2	$15+6i$	7	2	2	$11+10i$
1	2	3	$2+7i$	3	2	3	$14+3i$	5	2	3	$10+15i$	7	2	3	$6+11i$
1	3	0	$11+6i$	3	3	0	$7+10i$	5	3	0	$3+14i$	7	3	0	$15+2i$
1	3	1	$10+11i$	3	3	1	$6+7i$	5	3	1	$2+3i$	7	3	1	$14+15i$
1	3	2	$5+10i$	3	3	2	$9+6i$	5	3	2	$13+2i$	7	3	2	$1+14i$
1	3	3	$6+5i$	3	3	3	$10+9i$	5	3	3	$14+13i$	7	3	3	$2+i$
(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)	

Mod $(1+i)^8$.

i	0	0	1	$4+i$	6	1	3	$8+i$	4	2	1	$12+i$	2	3	3
$3i$	0	1	1	$4+3i$	2	0	3	$8+3i$	4	3	1	$12+3i$	6	2	3
$5i$	0	3	3	$4+5i$	6	0	1	$8+5i$	4	1	3	$12+5i$	2	2	1
$7i$	0	2	3	$4+7i$	2	1	1	$8+7i$	4	0	3	$12+7i$	6	3	1
$9i$	0	2	1	$4+9i$	6	3	3	$8+9i$	4	0	1	$12+9i$	2	1	3
$11i$	0	3	1	$4+11i$	2	2	3	$8+11i$	4	1	1	$12+11i$	6	0	3
$13i$	0	1	3	$4+13i$	6	2	1	$8+13i$	4	3	3	$12+13i$	2	0	1
$15i$	0	0	3	$4+15i$	2	3	1	$8+15i$	4	2	3	$12+15i$	6	1	1
1	0	0	0	5	0	3	2	9	0	2	0	13	0	1	2
$1+2i$	1	0	0	$5+2i$	5	1	2	$9+2i$	1	2	0	$13+2i$	5	3	2
$1+4i$	2	3	2	$5+4i$	2	2	0	$9+4i$	2	1	2	$13+4i$	2	0	0
$1+6i$	3	1	2	$5+6i$	7	2	0	$9+6i$	3	3	2	$13+6i$	7	0	0
$1+8i$	4	2	0	$5+8i$	4	1	2	$9+8i$	4	0	0	$13+8i$	4	3	2
$1+10i$	5	2	0	$5+10i$	1	3	2	$9+10i$	5	0	0	$13+10i$	1	1	2
$1+12i$	6	1	2	$5+12i$	6	0	0	$9+12i$	6	3	2	$13+12i$	6	2	0
$1+14i$	7	3	2	$5+14i$	3	0	0	$9+14i$	7	1	2	$13+14i$	3	2	0
$2+i$	7	3	3	$6+i$	5	2	1	$10+i$	3	1	3	$14+i$	1	0	1
$2+3i$	5	3	1	$6+3i$	7	0	3	$10+3i$	1	1	1	$14+3i$	3	2	3
$2+5i$	3	0	1	$6+5i$	1	3	3	$10+5i$	7	2	1	$14+5i$	5	1	3
$2+7i$	1	2	3	$6+7i$	3	3	1	$10+7i$	5	0	3	$14+7i$	7	1	1
$2+9i$	7	1	3	$6+9i$	5	0	1	$10+9i$	3	3	3	$14+9i$	1	2	1
$2+11i$	5	1	1	$6+11i$	7	2	3	$10+11i$	1	3	1	$14+11i$	3	0	3
$2+13i$	3	2	1	$6+13i$	1	1	3	$10+13i$	7	0	1	$14+13i$	5	3	3
$2+15i$	1	0	3	$6+15i$	3	1	1	$10+15i$	5	2	3	$14+15i$	7	3	1
3	0	1	0	7	0	2	2	11	0	3	0	15	0	0	2
$3+2i$	3	2	2	$7+2i$	7	1	0	$11+2i$	3	0	2	$15+2i$	7	3	0
$3+4i$	6	2	2	$7+4i$	6	3	0	$11+4i$	6	0	2	$15+4i$	6	1	0
$3+6i$	1	1	0	$7+6i$	5	0	2	$11+6i$	1	3	0	$15+6i$	5	2	2
$3+8i$	4	3	0	$7+8i$	4	0	2	$11+8i$	4	1	0	$15+8i$	4	2	2
$3+10i$	7	0	2	$7+10i$	3	3	0	$11+10i$	7	2	2	$15+10i$	3	1	0
$3+12i$	2	0	2	$7+12i$	2	1	0	$11+12i$	2	2	2	$15+12i$	2	3	0
$3+14i$	5	3	0	$7+14i$	1	2	2	$11+14i$	5	1	0	$15+14i$	1	0	2
	(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)

V. *Ionic Velocities.*

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The most obvious and regular result of passing a current of electricity through the solution of a salt is the simultaneous appearance at the two electrodes of products of chemical decomposition. Since the intervening solution is unaltered, and the products appear only at the electrodes, the motion of the parts of the salt in opposite directions through the solution is a necessary hypothesis, and the idea of ionic velocity is at once suggested.

The facts of ordinary chemical action, as well as those of electrolysis, require us to suppose that a continual molecular interchange of partners is going on, but how this takes place, whether by means of "free ions" or by means of the fractional number of collisions which the mechanical theory teaches us will occur with sufficient violence to bring about separation and possible rearrangement, remains an open question, and one which it is not necessary to consider for our present purpose. The point is that such a motion in opposite directions must occur, and the ions move with a definite average velocity through any given solution. This is independent of any particular hypothesis as to the nature of the active electrolytic molecule, and need not commit us either to the "dissociation" or to the "chemical" theory.

In order to explain the facts of "migration" Hittorf ('Pogg. Ann.,' vols. 89, 98, 103, 106, 1853 to 1859) supposed that the velocities of opposite ions were different, which would produce the observed alterations of concentration round the electrodes according to the obvious and well-known law.

Later, Kohlrausch introduced the idea of a definite specific ionic velocity for each ion, independent of the nature of its combination ('Wied. Ann.,' vols. 6 and 26), except in so far as its motion is affected by the different resistance offered by the different media through which it travels. These differences will disappear at infinite dilution, and the values obtained by extrapolation then show that the specific ionic velocities of the same ion, as deduced from observations on various salts containing it, are, as nearly as can be observed, identical. As it is in examination of this idea that most of the experiments, presently to be described, were made, it will be convenient to reproduce Kohlrausch's theory in some detail. The experimental portion of his work consisted of a splendid series of determinations of the conductivities of salt solutions. Assuming that all the salt molecules dissolved are actively concerned in conveying the current, from simple notions of convection combined with OHM's law he calculated the relative velocity of the two ions which would be necessary to give the observed conductivity.

Thus (following Dr. LODGE's abstract in the 'B.A. Report,' 1886), let n^3 be the number of active molecules in 1 cub. centim., q the total charge of electricity of whichever kind possessed by the ions of each active molecule, and U the relative velocity with which the opposite ions are sheared past each other by a potential gradient, dv/dx; we shall then get from considerations of convection $n^3 q\mathrm{U}$ for the intensity of current (or quantity of electricity conveyed per second through unit area normal to the flow), and from OHM's law $k(d\mathrm{V}/dx)$ for the same quantity, where k is the conductivity of the unit cube, therefore

$$k\frac{d\mathrm{V}}{dx} = n^3 q\mathrm{U}.$$

If N be the number of monad gramme equivalents of the active substance present in the unit cube, and η the electro-chemical equivalent of hydrogen

$$\mathrm{N}\frac{1}{\eta} = n^3 q;$$

therefore

$$\gamma = k\frac{d\mathrm{V}}{dx} = n^3 q\mathrm{U} = \frac{\mathrm{NU}}{\eta};$$

therefore

$$\mathrm{U} = \frac{k}{\mathrm{N}}\eta\frac{d\mathrm{V}}{dx} = \frac{\eta}{\mathrm{N}}\gamma.$$

Whence the arithmetical sum of the opposite velocities of anion and kation (u and v) when urged by a slope of potential of one volt per centimetre through a solution containing N monad gramme equivalents of active substance per cubic centimetre, and of specific conductivity k, is

$$\mathrm{U} = u + v = k/\mathrm{N} \times \cdot 00010352 \times 10^8 = 10352k/\mathrm{N} \text{ centims. per second.}$$

In order to apply this to particular cases KOHLRAUSCH assumes that all the dissolved salt is active, so that N is known from the strength of the solutions. The ratio u/v is taken from HITTORF's migration experiments, and thus the absolute values of u and v can be obtained.

KOHLRAUSCH expresses his experimental results in terms of the specific molecular conductivity (*i.e.*, k/N), and finds that for increasing dilution this tends to a limiting value. Thus, by drawing curves with $\mathrm{N}^{\frac{1}{3}}$ (the reciprocal of the average distance between the molecules) as abscissæ, and k/N as ordinates, he is able to obtain the specific molecular conductivity for infinite dilution. When the values for u and v are worked out for these ideal solutions, it is found that the velocity of any given ion is independent of the nature of the salt of which it forms part, and hence KOHLRAUSCH assigns numbers to various ions which represent their specific ionic velocity.

From a table of these numbers we can at once deduce the conductivity of any given salt solution, and the agreement of the values so obtained with those experimentally observed, furnishes the first confirmation of KOHLRAUSCH'S theory. He calculates the conductivity of various salt solutions of strength 0·1 gramme equivalent per litre, and then experimentally determines it for solutions of that strength. Some of his results are as follows :—

Name of salt.	Molecular conductivity calculated from the ionic velocities.	Molecular conductivity observed.
KCl	1060	1047
NH_4Cl	1040	1035
NaCl	860	865
$\frac{1}{2}BaCl_2$	840	861
$\frac{1}{2}ZnCl_2$	780	768
KI	1070	1069
KNO_3	1000	983
$AgNO_3$	900	886
$KC_2H_3O_2$	780	784
HCl	3260	3244
KOH	1950	1986
NaOH	1750	1700

The tabulated numbers are $k/m \times 10^8$ where k is the conductivity in terms of that of mercury, and m is the number of gramme equivalents of salt per litre. The general agreement is very remarkable, though several exceptions show that the law is not of universal application ; for instance, acetic acid should give 2980, while the observed number is 43.

The next confirmation of the theory was given by Dr. OLIVER LODGE ('B. A. Report,' 1886), who actually observed the velocity of the hydrogen ion as it travelled along a glass tube and discolourized a solution of phenol-phthallein as it went. The tube contained a jelly in which was dissolved a little phenol-phthallein and common salt, with just enough soda to bring out the colour. The tube joined two vessels filled with sulphuric acid, and when a current was passed between them the hydrogen travelled with it, and liberated HCl as it went along the tube. This decolourized the phenol-phthallein, and could thus be traced. The velocity per unit potential gradient came out as ·0029, ·0026, and ·0024 centim. per second, while KOHLRAUSCH'S theoretical number was ·0030.

This agreement is striking evidence in favour of KOHLRAUSCH'S theory and leads us carefully to consider the data on which it is based. The only objection which can be raised is to the assumption that all the molecules are active, and on this the accuracy of the results deduced apparently depends, for if we suppose only a part of the salt to be active, it seems necessary to allow a greater ionic velocity *in order that the same current may be carried.* Now the work of ARRHENIUS, VAN'T HOFF, OSTWALD, and others, gives very strong evidence that only a certain fraction of the number of dissolved molecules are active (whatever we may assign as the cause of this activity), and it seems to have been generally supposed that this was quite inconsistent with KOHLRAUSCH'S hypothesis (see LODGE, 'B. A. Report,' 1886, p. 391, *et seq.*). I have shown, however, that by examining the matter a little more closely the two ideas can be reconciled. The proof has already been published ('Phil. Mag.,' July, 1891), but it will be convenient to reproduce it here.

Suppose that the ratio of the number of the active to the whole number of molecules, which measures the "ionization" (to use Professor FITZGERALD'S convenient name), represents in reality the fraction of each second during which each molecule is on the average active. Each molecule is in turn active, but *at any instant* only a certain fraction of the whole number of molecules are active. (In terms of the dissociation hypothesis this ratio measures the "mean free time" of each ion.) As far as statical effects, such as osmotic pressure are concerned, this is, of course, equivalent to supposing a certain fixed fraction of the whole number of molecules to be active, but when we consider the velocities of the ions the case is different.

KOHLRAUSCH gets the relative veloctity of the ions from the relation

$$U_1 = u + v = k/m ;$$

but if we suppose that only $1/n^{\text{th}}$ of the molecules are active, we should apparently have to put $U_2 = kn/m$ to get the same current through the solution, which would give $U_2 = nU_1$.

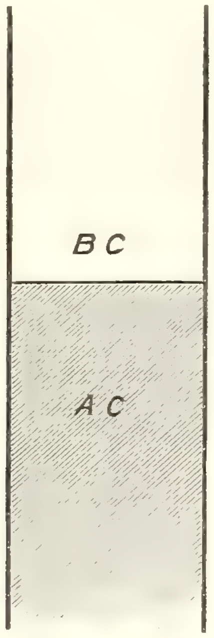
But this U_2 represents the actual velocity of the ions while they are active (or "free"), and if we take a dynamical view of the ionization equilibrium, they are active only for $1/n^{\text{th}}$ part of their time. While inactive (or "combined") they have no relative velocity, and so their average velocity for any *long* time is $1/n^{\text{th}}\ U_2 = U_1$ —the same as in KOHLRAUSCH'S hypothesis. The view that all the salt is active supposes a uniform ionic velocity—the view above detailed supposes a series of rests interposed with a series of intervals during which the ion is moving forward with a velocity which is, while it lasts, on the average $= nU_1$; but the final result is the same, the *effective* velocity is U_1.

We can, therefore, combine KOHLRAUSCH'S theory with the supposition that some of the molecules present in the solution are inactive, the result being that the presence of the inactive or non-electrolytic molecules, which decreases the molecular conductivity of the solution, shows itself by diminishing the effective velocities of the ions.

The agreement between Dr. LODGE's observed value for the velocity of hydrogen and that deduced by KOHLRAUSCH is too close to be accidental, but some further experiments in which the velocities of various ions were determined by watching the formation of precipitates as they travelled along did not give such good results. Thus, Dr. LODGE found Ba = ·00012, Sr = ·00015, and Cl = Br = I = ·00024 centims. per second, while KOHLRAUSCH gives Ba = ·00033, Cl = ·00053, I = ·00060 ('Wied. Ann.,' vol. 6, p. 206).

There are several disadvantages in this method. The greatest appears to be the impossibility of keeping the solutions of a definite strength—the mere fact of producing a precipitate must withdraw a certain amount of salt from the sphere of action, and so reduce the strength of the solution. This will increase the resistance of that portion of the tube in which the precipitate has been formed, and the potential gradient will cease to be uniform and calculable. When jelly is used, the experiments are easy to make, but it has a considerable conductivity of its own, and its great viscosity must offer a high resistance to the passage of the ions through it. Moreover, it gradually exudes from the tube in a direction opposite to that of the current —perhaps owing to electric endosmose. If ordinary liquid solutions are employed, the precipitates cause serious mechanical disturbances which prevent good results, while the use of indicators restricts the experiments to the cases of acids and alkalis and introduces a foreign substance which may itself serve to convey some of the current.

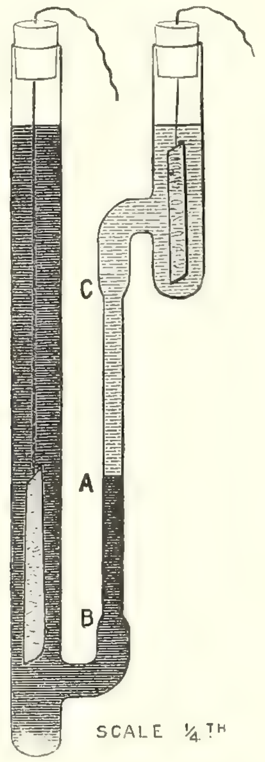
Fig. 1.



In order to get over these difficulties, and to submit KOHLRAUSCH's important and beautiful theory to adequate experimental tests, I undertook a series of experiments in which the substances used acted as their own indicators. The solutions can then be kept at a definite strength, no third substance need be introduced, no precipitate need disturb the uniform course of the ions, and no jelly bring in unknown and irregular conditions.

Consider the boundary of two salt solutions of slightly different density, which have one ion in common, but are of different colours (fig. 1). Let us denote the salts by AC and BC. When a current passes across the boundary there will be a transference of C ions in one direction (*against* the current, if we suppose the C's to be acid radicles, and, therefore, anions), and of A and B ions in the other (in the case supposed, *with* the current). Since the colour in this case depends on the kation A or B, the colour boundary will move with the current, and its velocity will indicate that of the ion causing the change in colour.

Fig. 2.



Postponing for the present the consideration of certain complications which may arise, I will describe the method of working and some preliminary observations. As small quantities of gas may be evolved at the electrodes they could not be sealed up, and finally a very simple apparatus was found to be the best (fig. 2). Two vertical glass tubes about 2 centims. in diameter were joined by a third, considerably narrower, which was bent parallel to the others for the greater part of its length. The longer tube was filled with the denser solution to about the level A, and then the lighter solution was run into the other tube from a burette, till it just began to trickle over the shoulder and run down to A. The proper solutions were then run into the two limbs at rates just sufficient to keep the junction near A; after some practice this could be done quite well, and the tubes filled without much disturbing the junction.

The current was passed from platinum electrodes, which could be connected with a battery of twenty-six accumulators by means of platinum wires passing through the corks. The corks fitted loosely, and a vertical groove, filed in each, allowed any gas which might be evolved to escape. The junction tube had a length, from B to C, of 12·6 centims., and a diameter of about 1·2 centims. If we add one-half the diameter as a correction for each end we get 13·8 centims. as the effective length of the tube, which must be divided into the total difference of potential between the electrodes to give the potential gradient.

The advantages of such an apparatus are shortly as follows :—

1. Any gas evolved at once escapes without interfering with the experiment.

2. Since the liberation of ions and decomposition of salt occur only at the electrodes, it is (in general) only near them that alterations of concentration occur. Since they are surrounded by a large mass of liquid, these alterations will not appreciably change the resistance of the tube as a whole, and will be entirely without effect on the resistance of the column of liquid in the narrow tube, which is the important point.

3. Changes in density at the electrodes will not appreciably accelerate or retard the velocity of the junction. Take the case of copper chloride when the current is passing down as a fairly typical example. The average current was $\frac{1}{2550}$ in C.G.S. units. The quantity of electricity which passed in one hour was, therefore, $\frac{60 \times 60}{2550}$, and the weight of copper deposited,

$$31{\cdot}5 \times \frac{60 \times 60}{2550} \times 10^{-4} = {\cdot}0045 \text{ grm.}$$

Supposing that all this copper comes from the kathode vessel, and that when deposited it occupies no volume (both of which suppositions would increase the disturbing effect we are considering), half this weight of solution would pass from anode to kathode to bring the liquids in the two tubes to their original levels. The area of cross-section of the tube is 0·430 sq. centim. (see p. 347). Therefore the passage of ·0023 cub. centim. of solution from anode to kathode in one hour would mean a velocity of ·0054 centim. per hour. The actual velocity observed in this case, due to the ionic velocities, was 1·6 centims. per hour. The correction for differences in density is thus utterly negligible.

4. A more important objection was pointed out by Professor J. J. Thomson. With salt solutions of equivalent strength there is, in general, a difference of specific resistance, although this is small in most cases. At the junction there will thus be a discontinuity of potential gradient, and, therefore, a distribution of electricity, which will be acted on by the electric force and introduce disturbances at the boundary. By a little consideration we shall see that this is a non-reversible effect, and should be eliminated by reversing the current and taking the mean of observations in each direction. Suppose that the current is passing in the direction which gives a distribution of positive electricity over the boundary. This will tend to make the

boundary move under the influence of the electric forces in the same direction as the current. Suppose the current reversed. It will now cause a distribution of negative electricity over the boundary, which will tend to move it in the direction opposite to that of the current. But the current has now also been reversed, and, therefore, the boundary tends to move in the same direction relatively to the tube as at first. This effect is therefore eliminated by reversing the current, and when the velocity is observed to be the same in each direction, within the limits of experimental error, it shows that the effect we are considering is negligible.

The first solutions used were those of copper and ammonium chlorides, with just enough ammonia added to each to bring out the deep blue colour of the copper. Their strength was about 0·18 grm. equivalent per litre.

The current was first sent upwards from the copper to the ammonium solution—the junction travelled upwards, with the current. The following readings of its position were made by a kathetometer :—

Voltmeter at grade 23·8.	Time.	Position.
	12.45	27·20
8·8		= 3·62 in 90′ = ·0402 in 1′
	2.15	23·58
8·8		= 4·09 in 100′ = ·0409 in 1′
	3.55	19·49

The current was then reversed—the junction travelled downwards.

Voltmeter at grade 23·8.	Time.	Position.
7·5	12.30	20·74
		= 6·09 in 160′ = ·0381 in 1′
7·7	3.10	26·83

If we calculate this for the same E.M.F. as before we get ·0441 centim. per minute for the velocity coming down. The mean is ·0423 centim. per minute.

The E.M.F. as measured by a voltmeter comes out $\frac{88}{23{\cdot}8} \times 10{\cdot}2 = 37{\cdot}7$ volts, which makes the potential gradient 2·73.

The velocity of the copper ion, under a potential gradient of 1 volt per centimetre, and through a solution of strength 0·18 grm. equivalent per litre, at a temperature of 15° C., comes out

0·00026 centim. a second.

The value deduced by KOHLRAUSCH from theory for a solution of infinite dilution at 18° C. is ('Wied. Ann.,' vol. 6, p. 206)

0·00031 centim. a second.

The next solutions were chosen to show the motion of the acid radicles, and potassium chloride and potassium permanganate were taken. The junction in this case, depending on the acid radicles, should, if its motion really shows the course of the ions, travel against the current. Such was found to be the case. The solutions had a strength of about ·046 grm. equivalent per litre.

Voltmeter at grade 23·8.	Time.	Position.	
10·8	4.11	21·42	·93 in 10′ = ·093 in 1′
	4.21	20·35	·62 in 5′ = ·124 in 1′
	4.26	19·73	
10·8	4.40	20·89	·55 in 5′ = ·110 in 1′
	4.45	20·34	·57 in 5′ = ·114 in 1′
	4.50	19·77	

A downward current from chloride to permanganate thus gives an upward velocity to the junction of 0·110 centim. per minute.

CURRENT reversed.

Voltmeter at grade 23·8.	Time.	Position.	
10 8	4.28	19·72	·61 in 5′ = ·122 in 1′
	4.33	20·33	·58 in 5′ = ·116 in 1′
	4.38	20·91	

The downward velocity of the junction is thus 0·119 centim. per minute.

The mean of the two is therefore 0·115 centim. per minute.

The potential gradient is 3·36, which makes the velocity for unit gradient

0·00057 centim. per second.

As the red permanganate disappears its place must be taken by chlorine, and hence this ought to give an indication of the velocity of the chlorine ion. For this KOHLRAUSCH gives

0·00053 centim. per second.

Solutions of one-tenth this strength were then set up (·0046 grm. equivalent per litre).

	Centim. per minute.
Average upward velocity	= ·1195
,, downward velocity	= ·1205
Mean	= ·120

Voltmeter, 10·9.

Therefore specific ionic velocity

$$= 0·00059 \text{ centim. per second,}$$

which is appreciably greater than for the stronger solutions.

The success of these preliminary determinations and the closeness of their agreement with theory led me to consider whether any improvement in the method was possible, by which further and more accurate results could be obtained.

In order to eliminate entirely the effect of the discontinuity of potential gradient it is necessary to use pairs of salts which have the same specific resistance for the same strength of solution. This very much limits one's choice, but an example will be found below. In such cases as that of copper and ammonium chlorides where the velocities in opposite directions are sensibly the same, it is fair to conclude that this effect is negligible, but it was worth while to find one case at any rate where it could not in the least affect the result.

The next improvement was in devising a means of getting over the necessity of directly estimating the potential gradient.

Consider the kation of one salt AC at the junction. Let its specific ionic velocity be v_1, and its actual velocity v, under a slope of potential $= d\mathrm{V}/dx$. Let r be the specific resistance of the solution. Let γ be the total current passing, and R the total resistance of the column of liquid from the anode to the junction.

Then we have $v = v_1 \, d\mathrm{V}/dx$.

Since γ is constant along the tube we get by OHM'S law

$$v = v_1 \gamma \frac{d\mathrm{R}}{dx}.$$

Now since the tube is of uniform bore near the junction $d\mathrm{R}/dx =$ the increase of resistance which would occur if we supposed the column of liquid from the electrode to the junction, whose resistance we have called R, to be increased in length by unity.

This increase of resistance $= r/\mathrm{A}$, where A represents the area of cross-section of the tube at the junction.

Therefore

$$\frac{d\mathrm{R}}{dx} = \frac{r}{\mathrm{A}},$$

and we get

$$v = v_1 \gamma \, r/\mathrm{A},$$

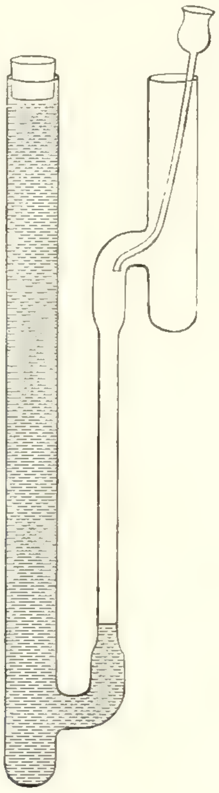
or

$$v_1 = v\mathrm{A}/\gamma r.$$

Now in this equation A can be determined once for all, γ can be read off in a graduated galvanometer of proper sensitiveness, r can be measured by separate experiments for the solution in question, and v, the velocity of the junction, is directly observed. Hence v_1, the specific ionic velocity, can be deduced without estimating the potential gradient directly, and introducing all the consequent uncertainties.

The method was first employed for the case of copper and ammonium chlorides, where the differences in specific resistance are not great, so that the velocities in opposite directions are approximately equal, and the correction for discontinuity is not important. The area of cross-section of the tube was determined as follows.

Fig. 3.



The longer limb was corked up so as to be air-tight, and filled with water till the surface stood just above the beginning of the narrow part of the junction tube. Its level was read off on the kathetometer. A flask containing water was then weighed, and liquid from it poured into the junction-tube through a weighed thistle-tube, till the surface nearly reached the top of the narrow part. The level was then again read off, giving the length of the column of liquid added, and the volume of water used was estimated by weighing the flask again, a correction for the small quantity retained on the walls of the thistle-tube being obtained by noticing the increase in its weight. Two determinations of the area of cross-sections by this section gave (1) 0·4300 sq. centim. and (2) 0·4297 sq. centim. Mean 0·4299, or correct to three places, **0·430** sq. centim.

The solutions were prepared by weighing out 0·670 grm. of pure cupric chloride and 0·534 grm. of ammonium chloride into two 100 cub. centims. flasks, and dissolving them in about 70 cub. centims. of water. An aqueous solution of ammonia was then added to the copper solution till the deep blue colour was formed. The same volume of ammonia was next run into the ammonium chloride solution, and both were made up to 100 cub. centims. This gave solutions whose strengths were 0 1 monad grm. equivalent per litre. Their specific resistances were found by the method used by KOHLRAUSCH and FITZPATRICK, and came out 157 × 10^9 (for the $CuCl_2$) and 117 × 10^9 (for the NH_4Cl) in C.G.S. units for the cubic centimetre.

The solutions were carefully run into the velocity tubes, and the current sent through them and a low-resistance galvanometer in series. The galvanometer had previously been graduated by sending the current from a single freshly-prepared Daniell cell through it and through a known resistance, and noticing the deflections as the resistance was varied from 11,000 ohms to 150 ohms.

GRADUATION of Galvanometer. Zero 1°·9.

Ohms.	Deflection.	Ohms.	Deflection.
11,000	= 5°·8	1,600	= 39°·5
10,000	6·1	1,400	42·5
9,000	6·7	1,200	45·9
8,000	7·4	1,000	49·8
7,000	8·2	900	52·1
6,000	9·8	800	54·5
5,000	11·5	700	57·0
4,000	15·4	600	59·9
3,500	18·7	500	63·2
3,200	21·0	400	67·1
3,000	22·9	350	69·1
2,700	25·6	300	71·7
2,500	28·0	250	74·1
2,200	31·3	200	76·8
2,000	33·7	150	79·4
1,800	36·6		

The resistances were then again increased to 400 ohms = 67°·1, 1200 = 45°·9, and 3500 = 18°·7—numbers identical with those first obtained.

COPPER and Ammonium Chlorides. Velocities. Current upwards.

Galvanometer.	Time.	Position.		
74°·7	10.50	32·4 32·2	32·3	1·70 centims. per hour.
	4.0	23·4 23·5	23·5	

CURRENT reversed at 4.0 downwards.

Galvanometer.	Time.	Position.
74°·9	6.25	26·9 27·2 } 27·0 = 1·45 centims. per hour.
72·8	11.30	27·2
	12.45	29·2 } 1·65 centims. per hour.
	1.30	30·5 } 1·70 ,, ,,
	1.45	30·9

CURRENT upwards.

Galvanometer.	Time.	Position.
73°·9	2.0	30·9
	4.15	27·3 } 1·53 centims. per hour.
73·0	5.0	26·3 } 1·43 ,, ,,
	7.15	23·0

Means going up G. = 73°·9 v = 1·57 centims. per hour.
,, ,, down 73°·9 1·60 ,, ,,

Means of all observations G. = 73°·9
v = **1·59 centims. per hour.**

We get γ from the table of graduations to be $\frac{1}{255}$, and find from our equation (p. 346)

$$v_1 = \frac{vA}{\gamma r} = \frac{1·59 \times ·430 \times 255}{157 \times 10^9 \times 60 \times 60} = 3·09 \times 10^{-12}.$$

This is for unit potential gradient on the C.G.S. system, and gives for a gradient of 1 volt per centim. a velocity of

$3·09 \times 10^{-4}$ = **0·000309** centim. per second.

This is the velocity of the copper ion in a solution of 0·1 grm. equivalent per litre, while KOHLRAUSCH deduced for infinite dilution

0·00031 centim. per second.

The next case I shall describe is that of a pair of salts whose 0·1 grm. equivalent solutions have specific resistances which are nearly the same. Mr. FITZPATRICK kindly supplied me with a long list of conductivities reduced to comparable units, and from those I was able to select two solutions which fulfilled the necessary conditions (viz., (i.) different colours; (ii.) different specific gravities; (iii.) nearly equal specific resistances). Potassium bichromate has a specific molecular conductivity (*i.e.*, k/m) of $9{\cdot}10 \times 10^{-12}$ (LENZ), and potassium carbonate $9{\cdot}34 \times 10^{-12}$ (KOHLRAUSCH), a difference of less than 3 per cent.

I thought it better to use solutions of slightly different strengths and exactly equal conductivities to get rid of even this small variation, and therefore adjusted the strength of the carbonate till its resistance just equalled that of the bichromate. Thus, in the resistance cell which I used, the bichromate gave 1485 ohms and 1490 ohms at 17°·4, and the carbonate 1475 ohms and 1480 ohms at 17°·5.

The first point I investigated with these solutions was the influence of change of potential gradient on the velocity. These should, on KOHLRAUSCH's theory, obviously be proportional to each other.

When the current was passed downwards, so that the junction travelled up the tube, two distinct surfaces of separation appeared, one between the orange of the strong bichromate and the yellow of that portion of the carbonate solution into which a little bichromate had passed by diffusion, and the other between this and the colourless carbonate. These kept fairly distinct and travelled upwards at equal rates. I append full particulars of the first series of observations.

VELOCITIES.—Potassium Bichromate and Carbonate. Current Downwards, Movement Upwards.

	Galvanometer.	Time.	Position.	Velocity in centims. per hour.
Orange	35·0	11.45	24·70	
		.46	·65	
		.47	·60	
Yellow	35·1	11.48	23·03	
		.49	22·92	
		.50	·85	
Orange	35·1	12.0	23·80	3·63
		.1	·74	
		.2	·69	
Yellow	35·1	12.3	22·18	3·39
		.4	·08	
		.5	·00	
Orange	35·4	12.15	22.89	3·65
		.16	·83	
		.17	·77	
Yellow	35·6	12.18	21·32	3·24
		.19	·28	
		.20	·23	

Mean upward velocity . . . . . = 3·48 centims. per hour.
,, galvanometer reading . . = 35°·3.

In the same way the observations of downward velocity came out 3·28 and 3·55 for galvanometer readings of 35·2 and 35·1, and when new solutions were set up 3·45 for 36·4.

Mean downward velocity . . . = 3·43 centims. per hour.
,, galvanometer reading . . = 35°·6.

The means in each direction are

$$v = 3·45 \qquad \text{G} = 35°·4,$$

which gives

$$v_1 = \frac{v\text{A}}{\gamma r} = \frac{3·45 \times ·430 \times 129 \times ·00910}{60 \times 60}$$
$$= 0·00048 \text{ centim. per second.}$$

The galvanometer was too sensitive, and had to be shunted and regraduated with a Daniell cell.

Resistance.	Resistance.
70 = 49°·0	140 = 32°·7
75 = 47·7	150 = 31·9
80 = 46·1	160 = 30·0
85 = 44·9	180 = 26·9
90 = 43·7	200 = 23·8
95 = 42·3	230 = 20·2
100 = 41·1	260 = 17·3
110 = 38·9	300 = 14·1
120 = 36·9	320 = 12·7
130 = 35·1	350 = 11·4

Resistance of battery + galvanometer + wires = 1 ohm.

An E.M.F. of about one-third that used above was then applied with the following results :—

CURRENT upwards. Movement downwards.

Galvanometer.	Time.	Position.	Galvanometer.	Time.	Position.	Result.	Centims. per hour.
12°·5	12.30	23·65	12°·5	12.45	24·04	·39	1·48
	.31	·67		.46	·04	·37	
	.32	·70		.47	·05	·35	
12·7	1.4	24·92	12·7	1.19	25·27	·35	1·39
	.5	·95		.20	·30	·35	
	.6	·98		.21	·32	·34	

CURRENT Downwards. Movement Upwards.

Galvanometer.	Time.	Position.	Galvanometer.	Time.	Position.	Result.	Centims. per hour.
°			°				
14·0	11.40	24·14	14·0	11.55	23·81	·33	1·28
	.41	·10		.56	·78	·32	
	.42	·07		.57	·76	·31	
13·8	3.25	24·14	13·7	3.55	23·46	·68	1·37
	.26	·11		.56	·43	·68	
	.27	·10		.57	·41	·69	
13·6	4.20	22·04	13·1	4.50	21·44	·60	1·23
	.21	·02		.51	·39	·63	
	.22	·00		.52	·38	·62	

Means—Downward velocity . . . . = 1·44 centims. per hour.
G. . . . . = $12^\circ{\cdot}6$
Upward velocity . . . . . = 1·29 centims. per hour.
G. . . . . = $13^\circ{\cdot}7$

This gives

$$v = 1{\cdot}37 \qquad G = 13^\circ{\cdot}2,$$

and

$$v_1 = vA/\gamma r = 0{\cdot}00047 \text{ centim. per second.}$$

Thus the value obtained for the specific ionic velocity is independent of the E.M.F. applied, *i.e., the velocity of the ions is proportional to the potential gradient.*

New solutions were afterwards set up and the experiments with the large E.M.F. repeated. The results were—

Movement upwards :

$$v = 4{\cdot}98,\ 4{\cdot}84,\ 4{\cdot}62,\ 4{\cdot}66,\ 4{\cdot}62.$$

$$G = 43{\cdot}8,\ 43{\cdot}8,\ 43{\cdot}0,\ 43{\cdot}2,\ 43{\cdot}2.$$

Means :

$$v = 4{\cdot}74 \qquad G = 43{\cdot}4 \qquad v_1 = {\cdot}00044.$$

Movement downwards :

$$v = 5{\cdot}02,\ 5{\cdot}16,\ 4{\cdot}78,\ 4{\cdot}70,\ 4{\cdot}66.$$

$$G = 42{\cdot}5,\ 43{\cdot}4,\ 43{\cdot}0,\ 43{\cdot}0,\ 42{\cdot}8.$$

Means :

$$v = 4{\cdot}86, \qquad G = 42^\circ{\cdot}0, \quad \text{therefore, } v_1 = {\cdot}00049.$$

Mean of v_1 in both directions = 0·00046.

Taking the mean value deduced from all three series of separate experiments, we get

$$v_1 = 0{\cdot}00047 \text{ centim. per second,}$$

as the specific ionic velocity of the bichromate group.

This is not given by KOHLRAUSCH, but we can calculate it by his method from known observations. Thus, LENZ has found the specific molecular conductivity of the solution to be $9{\cdot}10 \times 10^{-12}$, and the migration constant was determined by HITTORF, and came out ·502.

By KOHLRAUSCH's theory (see p. 338),

$$\begin{aligned} U = u + v &= 10352 \frac{k}{N} \\ &= 10352 \times 9{\cdot}10 \times 10^{-8} = {\cdot}000942, \end{aligned}$$

therefore,

$$\begin{aligned} v &= {\cdot}000942 \times {\cdot}502 \\ &= 0{\cdot}000473. \end{aligned}$$

This close agreement between theory and observation led me to examine a case in which the specific resistances of the solutions were not the same, in order to see whether the method could be used when no solutions could be found with the exact resistance needed. I used potassium bichromate and potassium chloride, whose conductivities are $9{\cdot}10 \times 10^{-12}$ and $11{\cdot}13 \times 10^{-12}$ respectively. If the extension of method is practicable, these solutions should give the same velocity for the bichromate group as the former pair—bichromate and carbonate. The galvanometer was moved, and, therefore, had to be regraduated.

Resistance.	Resistance.
60 = 54°·0	90 = 45°·4
65 = 52·4	100 = 43·0
70 = 51·0	110 = 40·7
75 = 49·4	120 = 38·6
80 = 48·0	130 = 36·5
85 = 46·7	

The following results were obtained—

CURRENT Downwards. Motion Upwards.

$v =$ 4·04 4·18 4·18 4·22. Mean, 4·15.
$G =$ 40·0 40·0 40·2 40·2. „ 40°·1.

Therefore

$$v_1 = 0{\cdot}000516.$$

CURRENT Upwards. Motion Downwards.

$v =$ 5·19 5·18. Mean, 5·19.
$G =$ 50·5 49·2. „ 49°·9.

$$v_1 = 0{\cdot}000394.$$

Thus, the effect of the discontinuity of potential gradient is to increase the velocity in one direction and to decrease it in the other, as it should do (see p. 343). The mean of the two numbers comes out

$$v_1 = 0{\cdot}000455 \text{ centim. a second,}$$

a value which, though it is of the same order of magnitude as the one obtained from the other pair of solutions, is appreciably less. A second series of observations gave

$$0{\cdot}000483 \quad \text{and} \quad 0{\cdot}000402. \quad \text{Mean, } 0{\cdot}000443.$$

These results show that the mean value of the velocities in opposite directions gives a number which is nearly, but not quite, the same as that obtained from a pair of solutions whose specific resistances are equal to each other. We may, therefore, use solutions whose resistances are not identical, to give, at any rate, some indication of the value of the specific ionic velocity, provided the differences are not great. But this extension of the method must be used with caution.

While working with solutions of different resistances it was often observed that when travelling in one direction the boundary got vague and uncertain, and when travelling in the other hard and sharp. Similar phenomena at the junction of liquids through which a current is passing have been previously described (see GORE, 'Roy. Soc. Proc.,' 1880 and 1881). Many of them can be explained as follows:—

Suppose that the coloured solution has greater resistance than the other, and that the junction is travelling from the coloured to the colourless solution. Any wandering ion which happens to be in advance of the main body, finds itself in a region where the potential gradient is less. It is therefore gradually overtaken, and the boundary becomes sharp. When the current is reversed, so that the junction travels in the opposite direction, any straggling ion, which lags behind the retreating column and so gets into the region of smaller potential gradient, finds itself left further and further behind, while others are continually falling out of the ranks. In this way the boundary becomes vague. If the coloured solution has less resistance than the other, a solitary ion is acted on by a *greater* potential gradient, and the order of these phenomena is reversed.

As far as I am aware, no attempt has hitherto been made to apply KOHLRAUSCH's theory to the case of solutions of salts in solvents other than water. The conductivity of alcoholic solutions is much less than that of the corresponding aqueous ones, and the question whether KOHLRAUSCH's theory still held good seemed of great interest.

The method described above can easily be applied, but the comparison of the results with theory offered some difficulty, as no data for the migration constants are available. In order to prevent the necessity of experimentally determining these constants, I thought it better to observe the velocities of both ions of a salt and compare their sum with the value deduced from the conductivity. The salts used were (i.) the chloride and nitrate of cobalt (the colours of which in alcoholic solutions are blue and red respectively) giving the velocity of the acid radicles, and (ii.) cobalt and calcium chlorides (blue and colourless) and cobalt and calcium nitrates (red and colourless), giving the velocity of the cobalt ion.

It was found that the observed velocities were very small, and in order to increase them a shorter tube was used of rather greater cross-section. This enabled a greater potential gradient to be applied. The area of cross-section was determined as before and came out 0·746 sq. centim.

A series of solutions of cobalt chloride and cobalt nitrate of different strengths was made up and the specific resistances determined. They were then placed in pairs of equivalent strength in the velocity apparatus, and a long investigation made on their behaviour. It was found that with weak solutions, whose strength was below 0·08 grm. equivalent per litre, the phenomena were quite regular. The junction travelled against the current, as it ought, since the change of colour depends on the acid radicles. As the strength of solution increases new phenomena appear. When the concentration reaches 0·1 grm. equivalent per litre the junction sometimes divides into two parts, which often travel in opposite directions, producing a broad purple band between the red and the blue solutions.

The junction sometimes seems to travel entirely the wrong way, but there is always either a fainter band which goes the right way, or, if this cannot be seen, the colour of the solution in that direction gradually changes. These phenomena more and more disturb the normal course of the experiment as the concentration of the solutions becomes greater, and with the strongest solutions which can conveniently be used (0·15 grm. equivalent) are very marked indeed.

In order to examine the applicability of the theory to these cases, it is necessary to choose solutions so dilute that the motion of the ions is normal, but still strong enough to show their colours plainly, and give an appreciable difference of density. The best solutions to use were found to be those whose strength was about 0·05 grm. equivalent per litre.

Solutions of cobalt nitrate and cobalt chloride in alcohol of ·05 strength were, therefore, set up in the velocity tube, and the following results were obtained :—

CURRENT from red to blue. Temperature, 17°·8.

$v =$	·39	·35	·43	·43.	Mean,	·40 centim. per hour.
$G =$	49·5	49·4	48·8	48·5.	„	49°·1

CURRENT from blue to red.

$v =$	·42	·41.	Mean,	·42 centim. per hour.
$G =$	49·1	48·6.	„	48°·9

CURRENT from red to blue.

$v =$ ·50 ·47 ·41 ·40. Mean, ·44 centim. per hour.
$G =$ 49·8 49·8. ,, 49°·8

CURRENT from red to blue.

$v =$ ·41 ·45 ·48 ·47 ·34 ·43. Mean, ·43 centim. per hour.
$G =$ 49·6 49·1 48·8 49·0 48·7. ,, 49°·0

Thus the velocity is practically the same in both directions, and the mean values are $v = 0·42$ centim. per hour and $G = 49°·2$.

The conductivity of the chloride was found by FITZPATRICK's method to be $2·86 \times 10^{-13}$ C.G.S. units, which gives

$$v_1 = v\text{A}/\gamma r_1 = 0·000026 \text{ centim. per second}$$

for the velocity of chlorine in this solution. We see at once from this result, that conductivity and specific ionic velocity for different solvents are, at any rate approximately, proportional to each other. The conductivity of these alcohol solutions is about one-tenth, or rather less, that of an aqueous solution of a neutral salt of equivalent strength, and we see that the ionic velocity of chlorine is reduced to rather less than one-tenth of its usual value for aqueous solution. This confirms the result at which we have already arrived, viz., that all the molecules are concerned, at any rate in turn, in the process of electrolysis, and that the cause of the increased resistance is a decrease in the *average* ionic velocity.

In order to obtain the velocity of the cobalt ion, solutions of cobalt and calcium chlorides were set up of strength 0·05 grm. equivalent.

CURRENT upwards.

$v =$ ·30 ·26 ·20 ·24. Mean, 0·25.
$G =$ 45·8 45·4 45·6 45·4. ,, 45°·6.

CURRENT downwards.

$v =$ ·26 ·24. Mean, 0.25.
$G =$ 46·9 48·0. ,, 47°·5.

CURRENT upwards.

$v =$ ·30 ·21 ·26 ·28. Mean, 0·26.
$G =$ 47·9 47·0 ,, 47°·4.

(After this, the velocity began to fall off, and the kathode solution ($CaCl_2$) got thick and faintly blue, perhaps indicating that some current is conveyed by the water of crystallisation of the $CoCl_2 6H_2O$, which deposits a hydrate when it meets the calcium, and that some cobalt goes faster than the rest and colours the solution. Finally, the junction went the wrong way, and other secondary actions occurred. All this shows that the effect of too great concentration has not yet been eliminated.)

Taking the mean velocity while it remained constant we get $G = 46^\circ{\cdot}8$, $v = 0{\cdot}25$ centim. per hour.

Therefore $v_1 = vA/\gamma r = 0{\cdot}000022$ centim. per second.

Adding this to the velocity of the chlorine (viz., 0·000026) we get

$$V = u + v = 0{\cdot}000048 \text{ centim. per second.}$$

The conductivity is $2{\cdot}86 \times 10^{-13}$, and on KOHLRAUSCH's theory this gives

$$V = 10352 \frac{k}{N} = 0{\cdot}000060 \text{ centim. per second.}$$

Considering the difficulties of the method and the effect of too great concentration, these numbers must be considered to agree within the limits of experimental error.

The conductivity of the nitrate of cobalt is rather greater than that of the chloride. This means that the number of active molecules is larger, and that the salt more nearly approaches the normal state. We should therefore expect the agreement to be closer than in the case of the chloride.

The velocity of the cobalt was found by using cobalt and calcium nitrate solutions, the first being red, the second colourless.

Upward velocity—Means :	$G = 50^\circ{\cdot}3$	$v = 0{\cdot}44$.
Downward ,, ,,	$G = 43^\circ{\cdot}6$	$v = 0{\cdot}41$.
Means of both directions :	$G = 46^\circ{\cdot}9$	$v = 0{\cdot}43$.

No irregularities (such as reversals, &c.) were detected with these salts. This is what would be expected from the better conductivity.

The result is

$$v_1 = vA/\gamma r = 0{\cdot}000044 \text{ centim. per second.}$$

The velocity of the NO_3 group can be calculated from the numbers in p. 356. The conductivity is $3{\cdot}80 \times 10^{-13}$, and this gives

$$v_1 = 0{\cdot}000035 \text{ centim. per second.}$$

Therefore

$$V = u + v = 0{\cdot}000079 \text{ centim. per second.}$$

From KOHLRAUSCH's theory we can calculate

$$V = 10352 \frac{k}{N} = 10352 \frac{3 \cdot 80 \times 10^{-13}}{\cdot 05} = 0 \cdot 000079 \text{ centim. per second.}$$

From the close agreement of these numbers, it is, I think, fair to conclude that KOHLRAUSCH's theory holds good in the case of alcoholic solutions, at all events when they are moderately dilute.

The case of the chloride, where the agreement is not so good, seems even more interesting than that of the well-behaved nitrate, and to be worthy of closer examination. The irregularities only appear when the concentration of the solutions is considerable, and we may assume that the agreement with theory would be closer for more dilute solutions if the difficulties of observing them could be surmounted. But how are we to explain the phenomena shown by strong solutions? The velocity of the junction becomes uncertain, several distinct lines often appear, bounding different shades between red and blue, and some of these may travel in the opposite direction to that indicated by theory and actually observed in the case of dilute solutions. On any theory of ionic charges, it seems impossible that an ion, as such, should travel in a direction against the electric forces, and we are forced to the supposition that the part of the salt which we see travelling thus is composed of non-electrolytic molecules, and is in reality attached to the opposite ion to form a complex structure which moves with the electric forces. The only other possible explanation seems to be the supposition that there is a proper motion of the solvent on which the ionic velocities are superposed, and that in some cases the motion of the solvent in one direction is greater than that of the ion in the other. Even neglecting the mechanical disturbances which would then appear, this view is untenable, firstly, because a motion of the solvent is disproved by another series of experiments, which I hope soon to publish, and, secondly, because the reversed motion is observed in strong solutions of both cobalt nitrate-cobalt chloride, and cobalt chloride-calcium chloride. The motion of the boundary in these two cases is in opposite directions with reference to the current, hence we cannot explain matters by a motion of the solvent in one direction.

The hypothesis of complex ions, advanced to explain the abnormal migration constant of cadmium iodide in alcohol must, therefore, be extended to these solutions. The cases are not quite parallel, for the phenomena shown by cadmium iodide can be explained if we imagine molecules of CdI_2 joined to the anion only, while in our case, since both pairs of solutions used show reversed motion when sufficiently strong, we must suppose that molecules of undecomposed salt are connected both to the anion and to the kation.

I hope soon to be able to publish an account of some further experiments bearing on the question of complex ions, and of others connected with several points in the subject of electrolysis.

The following table shows at a glance the results of this paper, and the figures obtained by calculation from KOHLRAUSCH's theory :—

SPECIFIC Ionic Velocities.

I. Aqueous Solutions.

Ion.	Velocity observed.	Velocity calculated from KOHLRAUSCH's theory.
Copper	·00026* ·000309	·00031
Chlorine	·00057* ·00059*	·00053
Bichromate group (Cr_2O_7) . .	·00048 ·00047 ·00046	·000473

II. Alcoholic Solutions.

Salt.	Velocity of anion (observed).	Velocity of kation (observed).	Sum of velocities (observed).	Sum of velocities (calculated).
Cobalt chloride .	·000026	·000022	·000048	·000060
„ nitrate .	·000035	·000044	·000079	·000079

* Preliminary determinations.

VI. *The Value of the Mechanical Equivalent of Heat, deduced from some Experiments performed with the view of establishing the relation between the Electrical and Mechanical Units; together with an Investigation into the Capacity for Heat of Water at different Temperatures.*

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Communicated by R. T. GLAZEBROOK, *F.R.S.*

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[PLATES 2–4.]

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INTRODUCTION.

THE necessity for a re-determination of the value of the mechanical equivalent may not be obvious at first sight. The classic determinations by JOULE have undergone but little alteration at the hands of succeeding observers, and the researches of ROWLAND (1879) into this matter were of such an exhaustive nature that there would appear to be little room left for further investigation. It should, however, be remembered that even JOULE's later determinations differ by as much as ·1 part in 100[*]; and that marvellous as is the agreement, amongst themselves, of the results obtained by ROWLAND, they, since his method of investigation was unaltered throughout, stand in need of confirmation by different methods of observation. Again, ROWLAND, as far as I have been able to ascertain, stands practically alone in his conclusion that the specific heat of water diminishes as the temperature rises from 0° to 30° C. It is difficult to conceive of a more important investigation (for the purposes of accurate physical measurements) than the determination of the capacity for heat of water at different temperatures, and it is to me a matter of extreme surprise that greater efforts have not been made to trace the variation (if any) in its value. The science of calorimetry must be regarded as in its infancy so long as its fundamental unit is a matter of doubt.

Other observers who have attempted to obtain the value of the mechanical equivalent, by means of the work done by an electric current, have been hampered by constant perplexities as to the absolute values of the electrical units adopted. The science of electrical measurements has now arrived at such a stage that its units may be regarded as sufficiently established,[†] and, therefore, the time seems particularly appropriate for an enquiry into the relation between those units and the mechanical ones.

The difficulties of such an investigation are, of course, great, as is shown by the divergence of the results obtained by those observers who, in recent years, have adopted the electrical method. One cause of inaccuracy has been present in all determinations I have examined, viz., the increase in temperature of the conductor above the temperature of the medium in which it was placed, and the consequent alteration in its resistance. ROWLAND[‡] writes as follows :—"There can be no doubt that experiments depending on the heating of a wire give too small a value of the equivalent, seeing that the temperature of the wire during heating must always be higher than that of the water surrounding it, and hence more heat will be generated than there should be."

An account is given in the following pages of the manner in which this difficulty has been overcome, and I think it will be seen that this objection to the electrical method of investigation is now removed.

The difficulties with regard to measurement of temperature are not peculiar to the electrical method of investigation, and, therefore, I need not dwell upon them. I would, however, venture to add my expressions of astonishment to those of ROWLAND that so many enquirers have attached so little importance to this point; many investigators, whose methods have otherwise been of a high order of accuracy, having contented themselves with the mercurial thermometer as a standard. The importance of close attention to the thermometry is shown by the fact that the difference between the air and mercury readings, even at low temperatures, will more than account for a change of decrease into increase in the specific heat of water as the temperature rises. An indirect result of my present investigation is to add further proof (if that is necessary) to the accuracy of the measurements of temperature by platinum thermometers, and it is remarkable that so exact and easy a method is not more generally applied to that most difficult subject, thermometry.

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[*] See Table XLV., *infra.*

[†] 'Report Brit. Assoc.,' 1892.

[‡] 'Proceedings American Academy,' June, 1879, p. 153.

Another difficulty which meets any enquirer into calorimetric measurements, is the uncertain nature of the thermal unit ordinarily adopted. I take it that the thermal unit is usually defined as the "amount of heat required to raise unit mass of water from 0° to 1° C." The obvious objections to this unit are that no one has more than a vague idea as to its magnitude, and that in choosing such a range of temperature we have selected, without doubt, the most difficult of all temperatures at which to ascertain it. Again, it is not improbable that over such a range water may exhibit greater changes in its properties than it does at other temperatures. The consequence is that investigators, such as ROWLAND, are compelled to express their results in terms of a variable unit, the theoretical one being an unascertained and indefinite quantity. To escape this difficulty I have taken as my standard the quantity of heat required to raise unit mass of water through 1° C. of the air-thermometer at 15° C. (*i.e.*, from 14°·5 to 15°·5 C.); in other words, I assume the specific heat of water at 15° C. to be 1, and I venture to suggest that (for want of a better) the above definition be accepted as that of the thermal unit.*

I have not included in this paper any prolonged analysis of the work of previous observers. ROWLAND has given a most complete summary, with references, of all results anterior to 1880, and the agreement between later observers is not sufficient to warrant any modification of his conclusions.†

The main portion of this communication is devoted to an account of the experiments performed in my laboratory during 1892, and, in order to avoid the introduction of redundant matter into that account, I here give a brief summary of my previous investigations.

I commenced this work in 1887, and the general principle on which I proposed to proceed was that of eliminating the effects of conduction, radiation, &c., rather than of ascertaining the actual loss or gain due to such causes. If a calorimeter is suspended in a chamber whose walls are kept at a constant temperature, it is obvious that if the initial temperature of the calorimeter is below that of the external envelope, and if the calorimeter has its temperature steadily raised by means of a current, a time will come when the loss by radiation, &c., must equal the gain experienced by the calorimeter during the time it was below the temperature of the surrounding space. The temperature at which the effects of radiation, &c., are eliminated, I term the "null point," and an investigation into its position indicates that if t_1 be the time that the calorimeter was beneath the external temperature, and t_2 the time required to raise it from its initial temperature to the same number of degrees above, that it was previously beneath, the temperature of the surrounding envelope, then the calorimeter would arrive at its null point at a time somewhere between $2t_1$ and t_2. In the Appendix will be found an investigation into the exact position of this point.

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* Personally I should have preferred to select a higher temperature, since a temperature above that of our ordinary surroundings is, at all times, more easily obtainable than a lower one; but as 15° C. has become so commonly accepted as a standard of temperature in electrical measurements, I have adopted it as my standard.

† MICULESCU gives all results published before January, 1892. See 'Annales de Chimie et de Physique,' vol. 27, p. 206.

1887–1889.—During these years my whole attention was directed to an effort to ascertain the heat developed by the current while the calorimeter was passing from its initial temperature to this null point. Attractive as the method appeared, I ultimately relinquished it, for I wished to direct my attention as much to the investigation of the changes in the specific heat of water as to the determination of the mechanical equivalent, and, although the method is an admirable one for the latter, it is not so suitable for the purposes of the former enquiry.

My general method of conducting the experiments was the same as that observed in 1892 and described in the succeeding pages. From the outset, I had determined to vary all the conditions as much as possible, believing that it was only by such means that constant sources of error could be detected. The agreement amongst individual experiments taken under the same conditions was, if anything, at times more marked than in the experiments of 1892; nevertheless, when the final reduction of the results took place, fatal discrepancies invariably showed themselves. For example, the water equivalent, as deduced from the experiments, increased when the mass of the water was increased, and experiments conducted with a high electromotive force invariably gave too great a value for the time as compared with that obtained when a lower electromotive force was used. These discrepancies, as it now appears, may be attributed to two causes:—

(1.) To insufficient stirring;
(2.) To the rise in temperature of the wire, previously referred to.

Throughout these experiments I was conscious of the error introduced by ignorance of the actual temperature of the wire, but it was not until the summer of 1892 that a satisfactory method of estimating this difference of temperature was adopted.

Unfortunately, many of my experiments in past years were conducted with wires coated with a thick insulating covering, and it is, therefore, impossible to apply the correction obtained this year to the reduction of the earlier results. The observations were excellent in themselves, and some hundreds of experiments were performed. They all give too high a value of J (almost invariably above $4\cdot2 \times 10^7$), and are of no value except for the experience gained by their means and the improvements they suggested in the apparatus.

Again, the thermometry was not of a sufficiently satisfactory nature. The thermometers were standardized by direct comparison with B.A. thermometers calibrated at Kew.[*] As these B.A. thermometers were only divided into $\frac{1}{10}$ of a degree, the $\frac{1}{100}$ of a degree had to be estimated,[†] and the order of accuracy was not sufficient.

1890.—Throughout this year I devoted my attention entirely to the measurement of temperatures by platinum thermometers, and I, at that time, proposed to use such thermometers in place of mercurial ones during my J experiments. Excellent, however, as platinum thermometers are, for the accurate determination of temperature, they are not suitable when the observations have to be taken at exact intervals of time. A thermometer, whose readings are at all times visible, must be used in such a case.

1891.—By the kind permission of the Master and Fellows of Sidney College, the Chemical Laboratory was placed at my disposal during this summer. This building was unsuitable for physical work, for the traffic in the adjoining street was so great that galvanometers became at times unmanageable, and many of the observations had to be taken during the night, under conditions somewhat conducive to personal errors.

The steel regulating chamber, described in Section III., was first used in this year (1891), and the results are therefore of more value than the preceding ones; but the same persistent error in the measurement of R still remained, and the form of stirrer adopted was unsatisfactory. The summer's work was, however, of some value, as it gave sufficient data for calibrating the mercury thermometer *when rising.*

A platinum wire coil, having a single coating of amber varnish, was used, and it is reasonable to suppose that its increase of temperature above the surrounding water was of a similar order to that ascertained to be the case in the coil of 1892. Assuming the increase of its resistance to bear the same ratio to its total resistance as in the case of the 1892 coil, and, applying a similar correction to our results, we obtain

$$J = 4{\cdot}188 \times 10^7$$

expressed in terms of water at 18°·88 C, and assuming the coefficient of the change in specific heat of water obtained in 1892, we get

$$4{\cdot}192 \times 10^7\ddagger$$

as the value expressed in terms of the thermal unit as previously defined.

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[*] The accuracy of the Kew corrections for the inequality of the bore has been previously discussed, see 'B.A. Report,' 1890.

[†] I find that many observers, by means of such thermometers, determine temperatures (without using a micrometer scale) to $\frac{1}{1000}$ of a degree. I envy, but cannot lay claim to, such powers of observation.

[‡] The wires connecting the lid of the calorimeter with the steel chamber (see fig. 2, *infra*) had (in 1891) a slightly higher resistance than those used in 1892. I am unable to make the necessary correction, as I have not sufficient data. The effect of the correction (which would certainly be less than in 5000) would be to slightly increase the value of J here given.

This agrees within 1 in 2000 with our later determinations.*

1892.—The difficulties experienced in obtaining an altogether suitable room for the work compelled me to have a special building erected in which to carry on the investigation.

The apparatus is so involved and complicated that it requires weeks to put it together and to get it into working order, and such spare time as I had in the early part of this year was devoted to these preparations. On previous occasions the constant day and night changes in temperature of the rooms in which experiments were conducted had been a cause of uncertainty; for the numbers, obtained by using resistance boxes or Clark cells whose temperature is constantly changing, are always of doubtful accuracy. I therefore designed a regulator which, except in cases of severe frost or extreme heat, automatically maintained the room at a constant temperature. I found this arrangement so effective that I have, in the hope that it may be useful to others, given a short description of it in Appendix II.

If I endeavoured to adequately describe how much I owe to the assistance of others during this investigation I should unduly increase the length of this introduction. I must, however, express my thanks to the Council of the Royal Society, to Lord KELVIN, Lord RAYLEIGH, Professor G. DARWIN, Professor HICKS, Professor J. J. THOMSON, Dr. FEUSZNER, Dr. GUILLAUME, Mr. CALLENDAR, Mr. VERNON HARCOURT, Mr. HEYCOCK, Mr. LARMOR, and Mr. NEVILLE for the encouragement and assistance they have afforded me. More especially am I indebted to Mr. GLAZEBROOK, not only for his advice and help, but for the careful and laborious comparison of my resistance box with the standards of the B.A.; and also to Mr. SKINNER for his repeated comparison of my Clark cells with the Cavendish standards. My thanks are also due to Mr. A. IVATT, B.A., Christ's College, Cambridge, for his assistance in the experimental work during 1887 and part of 1888.

Nearly the whole of the apparatus (some of it of a novel and intricate character) was constructed by, or under the direction of, Mr. F. THOMAS, Jesus Lane, Cambridge. Had it not been for his unremitting attention and skill, the results of my investigation would have been far from satisfactory.

From 1888 onwards I was assisted by Mr. G. M. CLARK, B.A., Sidney College, Cambridge, who has from the commencement of our joint work been indefatigable in his exertions. Our respective contributions are so intimately associated that I find it impossible to make the remainder of this communication in the first person, and although, by his own wish, his name does not appear on the title-page, it should, in justice, be regarded as a joint contribution.

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* The Clark cells used as our standards of E.M.F. are those given in Table IX. of Messrs. GLAZEBROOK and SKINNER'S paper, 'Phil. Trans.,' A, 1892, p. 605.

SECTION I.—GENERAL DESCRIPTION.

Let R be the resistance of a conductor at a certain standard temperature θ, then if the ends of the conductor be kept at a constant difference of potential E

$$\frac{dQ_\epsilon}{dt} = \frac{E^2}{J \, . \, R} \quad \ldots \ldots \ldots \quad (1),$$

where Q_ϵ is the quantity of heat generated by the current, t is time, and J the mechanical equivalent of heat.

If the wire is immersed in water, and if the capacity for heat of the water and the calorimeter at the standard temperature θ is M, then

$$\frac{dQ}{dt} = M \, . \, \frac{d\theta}{dt} \quad \ldots \ldots \ldots \quad (2).$$

It is necessary to distribute the heat generated in the wire throughout the whole calorimeter as quickly as possible, otherwise we cannot accurately observe $d\theta/dt$, hence, since work must be done when stirring, we have to deal with a mechanical as well as with an electrical supply. Let the rise in temperature per second due to the mechanical supply be σ, and let $Q_{\epsilon, \sigma}$ denote the quantity of heat generated by both electrical and mechanical sources. Then

$$\frac{dQ_{\epsilon, \sigma}}{dt} = \frac{E^2}{J \, . \, R} + \sigma M = M \, . \, \frac{d\theta}{dt} \quad \ldots \ldots \ldots \quad (3).$$

It is certain that some heat will be gained or lost by radiation, conduction, and convection. Let ρ be the rise or fall per second in temperature due to radiation, &c., when the difference between the external and internal temperatures is 1° C. Denote the temperature of the surrounding envelope by θ_0, and the temperature of the calorimeter at any time by θ_1; also suppose that M becomes M′ and R becomes R′ when θ becomes θ_1.

We now have

$$\frac{dQ_{\epsilon, \sigma, \rho}}{dt} = \frac{E^2}{J \, . \, R'} + \sigma M' - \rho M' (\theta_1 - \theta_0)^* = M' \frac{d\theta_1}{dt} \quad \ldots \ldots \quad (4),$$

where $Q_{\epsilon, \sigma, \rho}$ denotes the gain of heat due to all causes, hence

$$\frac{E^2}{J \, . \, R' \, . \, M'} + \{\sigma - \rho (\theta_1 - \theta_0)\} = \frac{d\theta_1}{dt} \quad \ldots \ldots \ldots \quad (5).$$

---

* It is shown in Section XII. that we are justified in assuming that NEWTON'S law of cooling holds true over our range of temperature.

If E = 0, this becomes

$$\sigma - \rho(\theta_1 - \theta_0) = \left(\frac{d\theta_1}{dt}\right)^{*}_{\sigma,\,\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Provided that the mechanical supply is regular, and θ_0 is kept constant, we can, by a sufficient number of observations over small ranges, determine the values of $(d\theta_1/dt)_{\sigma,\,\rho}$ for different values of θ_1, and by substituting in equation 5, we obtain the value of $E^2/J.R'.M' = A$, where A is known.

Assuming that over small ranges the values of R′ and M′ are linear functions of θ_1,[†] we have

$$R' = R\,\{1 + k(\theta_1 - \theta)\}^{\ddagger}, \text{ and } M' = M\,\{1 + l\,(\theta_1 - \theta)\},$$

hence

$$\frac{E^2}{J\,.\,R\,.\{1 + k\,(\theta_1 - \theta)\}\,M\,.\,\{1 + l\,(\theta_1 - \theta)\}} = A \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now the values of E and R at the standard temperature can be ascertained by comparisons with the standards, and the value of k can be ascertained by direct measurements of R′ at different temperatures, hence

$$J\,.\,M\,.\,\{1 + l\,(\theta_1 - \theta)\} = \frac{E^2}{A\,.\,R\,.\,\{1 + k\,(\theta_1 - \theta)\}} = B, \text{ where B is known} \quad . \quad (8),$$

we have thus one equation connecting the three quantities, J, M, and l.

Let w and w_x be the capacities for heat of the water and the calorimeter respectively at the standard temperature θ, and let the temperature coefficients of their specific heats be f and g respectively, then

$$M\,\{1 + l\,(\theta_1 - \theta)\} = w\,\{1 + f(\theta_1 - \theta)\} + w_x\,\{1 + g\,(\theta_1 - \theta)\},$$

hence equation (8) becomes

$$J\,[w\,\{1 + f(\theta_1 - \theta)\} + w_x\,\{1 + g\,(\theta_1 - \theta)\}] = B \quad . \quad . \quad . \quad . \quad (9).$$

If observations are taken with different weights (w_1 and w_2) of water, we obtain B_1, B_2 the corresponding values of B, thus

$$J\,[w_1\,\{1 + f(\theta_1 - \theta)\} + w_x\,\{1 + g\,(\theta_1 - \theta)\}] = B_1$$

and

$$J\,[w_2\,\{1 + f(\theta_1 - \theta)\} + w_x\,\{1 + g\,(\theta_1 - \theta)\}] = B_2\,;$$

---

[*] In this, and similar cases, we use the suffix to denote the sources of heat.

[†] The value of $d\theta$, during our experiments, was about 1° C.

[‡] The true value of $R' = R\,\{1 + k\,(\theta_1 + \beta - \theta)\}$ where β is the excess of the temperature of the wire above θ_1 the temperature of the calorimeter. See Section XIV., p. 478.

by subtraction we obtain

$$J(w_1 - w_2)(1 + f\overline{\theta_1 - \theta}) = B_1 - B_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Hence, when $\theta_1 = \theta$, the value of J can be found without ascertaining the value of the water equivalent or the temperature coefficient of the specific heat of water. As J is a constant quantity we can, by repeating the observations at different temperatures, obtain from equation (10) the value of f.

Or we can obtain f without finding J, for repeating the observations at temperatures θ_1 and θ_2

$$J(w_1 - w_2)(1 + f\overline{\theta_1 - \theta}) = B_{1,1} - B_{2,1}$$

and

$$J(w_1 - w_2)(1 + f\overline{\theta_2 - \theta}) = B_{1,2} - B_{2,2}$$

the second suffix of B denoting the temperatures.

Hence, by division, we obtain

$$\frac{1 + f\overline{\theta_1 - \theta}}{1 + f\overline{\theta_2 - \theta}} = \frac{B_{1,1} - B_{2,1}}{B_{1,2} - B_{2,2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

An alternative method of finding J is to first find f, by equation (11), and then to find the value of w_x and g.* The value of J can then be deduced from a single experiment by means of equations (8) or (9), and as this method enables us more easily to compare the results of individual experiments we have, as a rule, adopted it in our reductions.

In the remainder of this section we briefly indicate the manner in which we have determined the various quantities grouped together and denoted in the above equations by B.

The calorimeter was suspended by means of glass tubes in an air-tight chamber whose walls were kept at a constant temperature. The pressure of the dry air in this chamber was reduced to under ·5 millim. The water in the calorimeter was stirred at a very rapid rate throughout the experiments, and the value of $(d\theta/dt)$ ascertained at all parts of the range 14° to 26° C.; firstly, when the work was done by the stirrer only, secondly, when the rise was due to both the electrical and the mechanical supply of heat.

The masses of water used during the experiments varied in the ratio of about 1 to 3, and the difference of potential at the ends of the coil was so altered that the heat developed by the current was changed in the proportion of 1 to 9.

As shown by equation (10) *supra*, there was no necessity to ascertain the water equivalent of the calorimeter, although it was found convenient, as a check upon the calculations, to deduce it from the differences in time caused by differences in the mass of the contained water.

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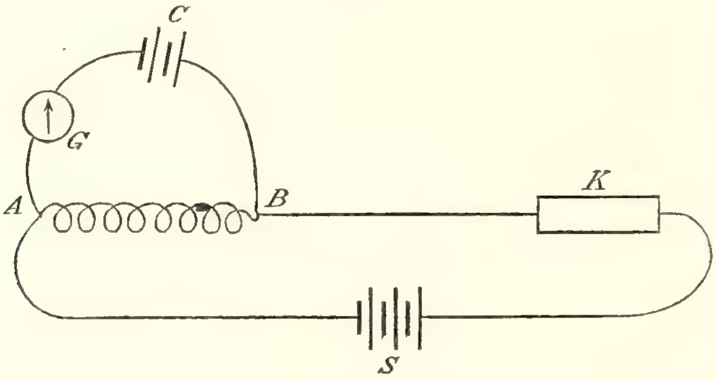
* These values can be obtained from the preceding equations without first determining J. The process is fully explained in Section XIV.

The walls of the chamber containing the calorimeter were maintained at a constant temperature throughout each experiment, this being, in general, the mean temperature of the above-mentioned range. Direct observations of the changes in $(d\theta_1/dt)_{\sigma,\rho}$, caused by changes in θ_1, enabled us to ascertain the values of both σ and ρ.

(1.) *Potential* (E).—A detailed account is given in Section VI. of the method by which the ends of the calorimeter coil were maintained at a constant potential difference. Briefly, the principle on which our method depended was as follows.

The extremities of the wires *CA*, *CB*, will (when the galvanometer *G* shows no deflection) be maintained at a difference of potential equal to that due to the cells at *C*.

Fig. 1.



Let *K* be an adjustable resistance placed in the circuit which communicates with *S* (the storage cells), and let R be the resistance of the wire *AB*. Whatever variations may take place in R and in the E.M.F. at *S* it is always possible (provided the adjustment of *K* is sufficiently under control) to maintain, by close attention to the indications of the galvanometer, the points *A* and *B* at a constant difference of potential.

A special rheostat was designed by means of which it was found possible, in spite of variations in the resistance of *AB*, to maintain the potential-difference unchanged throughout an experiment, and we believe that in no case did the variations exceed $\frac{1}{10,000}$ of the mean difference of potential during each experiment.

The Clark cells (which were placed at *C*) were constructed by us according to the directions of Messrs. GLAZEBROOK and SKINNER, and have, on several occasions, been directly compared with the Cavendish and indirectly with the Berlin standard.

Their differences from the standard are small, and their mean E.M.F. at 15° C. = 1·4344 volts (see Section VI.).

(2.) *Resistance* (R).—Had it been possible to maintain a constant value for R it would have greatly simplified both the experimental work and the calculations. In the year 1890 we devoted much time to the examination of the various copper-manganese-nickel alloys, and we performed a series of determinations with a coil whose tempera ture-coefficient was practically zero. The reasons which led us to reject these alloys and adopt a platinum wire will, we think, be found sufficient (see section VII.).

The value of R was first determined by a dial-box (legal ohms) constructed by Messrs. Elliott.* Mr. Glazebrook has been so kind as to perform a complete standardization of this box by means of the B.A. standards. The resulting corrections have been applied, and the values of R expressed in true ohms, as defined by the B.A. Report, 1892.

The method by which we ascertained the true temperature, and therefore, the resistance of the wire when traversed by a current, is fully described in Section VII. We found that the difference in temperature between the wire and that indicated by the thermometer could be accurately determined, and, other conditions being equal, varied as C^2.

The arrangement of our connections enabled us to dispense with exterior resistance, as the ends of the coil itself were maintained at a constant potential-difference.

(3.) *Current* (C).—The current was maintained by storage cells of the (L) type, supplied to us by Messrs. Bailey and Grundy. The E.M.F. of these cells was wonderfully constant. This constancy, although not a necessity, was a convenience. As previously pointed out, C was not directly measured.

(4.) *Time* (T).—An electrical clock with a seconds pendulum was used as our standard for time. It was carefully compared at intervals with a chronometer by Dent. A chronograph was controlled by this clock. The rate of the clock was a losing one until August 21st, but after that date its error was less than 1/25,000, and no correction was necessary.

(5.) *Mass* (M).—A balance sensitive to a change of 1/100,000 of the least mass measured by us, and a set of Oertling's weights,† were used in our determination of mass.

(6.) *Temperature* (θ).—At the time of writing (December, 1892) our thermometry is based on measurements made by platinum-thermometers. We propose to make, at an early date, a direct comparison of our standard thermometer with the air thermometer by means of the apparatus described by Mr. Callendar,‡ who has been so kind as to promise his assistance.

In a previous paper§ the details of a careful comparison of the platinum and the air-thermometer have been given. It was then shown that the platinum-air difference curve $\delta\,[\overline{\theta/100}|^2 - \theta/100]$ gave values of θ at all temperatures from 0° to 100° C., accurate within ·01° of the real value of θ, and that discrepancies appeared to have an experimental origin. The experience of two years' work with platinum-thermometers has but increased our confidence in them.‖ Should a direct comparison with the air-thermometer modify our determinations of differences of temperature (and it is only differences which are important to us), our results will be modified accordingly. This will not, however, necessitate any repetition of the experimental work, as the corrections involved will be of a numerical order only.

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* Particulars of this box have been given in a previous paper. 'Phil. Trans.,' A., 1891, p. 44.

† These weights were re-standardized by Messrs. Oertling in August, 1892.

‡ 'Roy. Soc. Proc.,' Jan., 1891.

§ See 'Phil. Trans.,' 1891, A, p. 155.

‖ 'Philosophical Magazine,' December, 1892.

Measurement of the Heat generated by Stirring.

The difficulties that we have met with in this portion of our work have been very great. As will be shown in Section XI., we have, against our will, been compelled to increase the work done by the stirrer until the heat thus generated became about 1/10 of the whole supply. We found continuous and very rapid stirring to be a necessity. Variations in R and in the thermometer readings and in the water equivalent of the calorimeter invariably followed if the mixing was insufficient. Commencing in 1888 with a stirrer revolving at the rate of about 100 revolutions per minute, the rate was gradually increased until, in our last series, we aimed (with, it is true, a different form of stirrer) at a rate of nearly 2000 revolutions per minute.

The accurate measurement of the heat thus generated is only second in importance to the determination of the heat due to the current, and, accordingly, a great portion of our time and attention was directed to it. Fortunately, we were able to completely establish the law that, with our form of stirrer, the work done varied directly as r^3, r being the rate of rotation. From this time our difficulties rapidly diminished.

Radiation, Conduction, Convection, &c.

We endeavoured, as far as possible, to diminish loss or gain due to the above causes.

Bottomley[*] has shown that there is a rapid fall in the rate of loss or gain by radiation and convection when the pressure of the surrounding gas falls below 1 millim. Most of our observations were taken with the surrounding pressure under ·5 millim.

The calorimeter itself had a bright polished surface, and the walls of the enclosing chamber were clothed with bright speculum metal. In Section XI., we show that throughout our range of temperature, Newton's law of radiation holds—*i.e.*, the loss or gain varies directly as the differences of temperature. It must, however, be remembered that we have not endeavoured to separate the partial effects of the radiation, conduction, and convection, and hence we can only say that the combined effect obeys this law. The total loss or gain, due to the above causes, was determined separately for each change in the contained mass.

Any error in this part of our work would have but a small effect on the mean value of J, since, in each experiment (if taken as a whole), the total loss or gain by conduction, convection, and radiation, must have been small.

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[*] 'Phil. Trans.,' 1887, A, p. 445.

SECTION II.—THE CALORIMETER AND COIL. (See Plate 2, fig. 2.)

The calorimeter, which was of a cylindrical form, was made of "gilding metal"—73 per cent. copper and 27 per cent. zinc.

The weight of the lower part was 501·84 grms.; of the lid 245·67 grms.

The internal diameter and the depth were each 8 centims.

Its capacity was therefore about 400 cub. centims.

The weight of the calorimeter may appear excessive for this capacity. However, as any "panting" of the lid or base would have thrown the stirrer out of adjustment and as the pressure on the flat base of the calorimeter would amount to about 120 lbs., when the space surrounding it was vacuous, it was evident that considerable thickness was necessary to secure rigidity. The two parts of the calorimeter were held together by eight screw bolts; a lead ring previously placed between the parts was crushed by the tightening of the screws and thus the join was rendered air-tight. It was therefore necessary to make both the lid and the projecting lip of the calorimeter sufficiently strong to bear the strain without bending.

As we expected to determine the water equivalent with as much accuracy as other quantities, the large mass of the calorimeter did not appear to be a serious objection.

The whole of the calorimeter was heavily gilded both inside and out, and the surfaces polished. The lead ring* was covered with gold leaf because considerable quantities of $PbCO_3$ were formed if the water had access to it; thus, no metal except gold was exposed to the action of the water. An advantage of the gold exterior was that its surface did not tarnish, and thus its coefficient of radiation remained constant.

Fixed on the interior of the lid were eight metal tubes, 1 centim. in length. Three of these were 18 millims. in diameter and served to hold the glass tubes by which the calorimeter was suspended. The others were 5 millims. in diameter and supported the glass rods on which the coil was wound. These rods reached to within a couple of millimeters of the base when the lid was placed in position. Two copper rods, 7·5 centims. in length (14 B.W.G.) passed from the insulated junctions at *M* and *N* (fig. 2, p. 382) to nearly the bottom of the calorimeter. These were also heavily gilded, and the ends of the platinum wires were fastened to their lower extremities. The last coil used by us was about 13 inches in length, and was wound round the lower end of the glass rods in such a manner, that the difference of potential of any two adjacent points could never be great. Thus the first 6 inches were wound on one side, the remainder on the further side of the calorimeter. A depth of 1·5 centim. of water completely covered the whole coil.

The glass tubes passing through the lid were held in position by perforated india-rubber corks. Great pressure had to be exerted to force the tubes "home," and several were broken in the process. The external diameter of the glass tubes was 14 millims., the interior diameter of the gold tubes 18 millims.; thus the annulus of india-rubber was very thin. The lower ends of the glass tubes were turned outwards so as to nearly cover the lower end of the india-rubber, of which the small exposed surface was originally coated with gutta-percha; but (as explained elsewhere) this was removed after J 19, and replaced by amber varnish.* The india-rubber corks, where they appeared above the lid, were, from the commencement, coated with amber varnish. No diffusion appeared to take place through these corks, and, as other observers seem to have been troubled by such diffusion when using india-rubber, it is probable that the coating of amber varnish (which adheres very firmly to the rubber) is an effective remedy. In any case it prevented any contact between the rubber and the contained water. We may thus assert that the water came into contact with the following substances only—gold, amber, glass, agate, gutta-percha (before experiment J 20) and (after experiment J 20) some hard shellac,† very small quantities of which were used to prevent slipping of the coil on the glass rods, the substance previously used for this purpose being gutta-percha. The platinum coil itself was coated with amber varnish, but it is possible, that owing to cracks in its coating caused by movements of the wire, the water had access to it in places. The small slabs of ebonite used at the junctions *M* and *N* (fig. 4, p. 396) were also covered with several layers of the same invaluable varnish.

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* A pure gold ring was tried, but it was found impossible to crush it sufficiently to render the join air-tight.

SECTION III.—THE METHOD OF MAINTAINING THE WALLS SURROUNDING THE CALORIMETER AT A CONSTANT TEMPERATURE.

The necessity of maintaining at a constant temperature the walls of the chamber containing the calorimeter is so obvious that it seems unnecessary to dwell upon it. In our earlier experiments, the calorimeter was suspended in a copper vessel which was immersed in a large tank, and the uniformity of the temperature depended on the skill of the observer in adjusting the temperature of the tank water by altering the supply of hot and cold water admitted. The method adopted by us in 1891 and 1892 worked automatically, and enabled us to maintain a chamber at any temperature within the range of our experiments, whether that temperature was above or below that of the room. Briefly, the arrangement may be described as an enormous thermometer-bulb within which was the calorimeter.

A section and plan, on a scale of $\frac{1}{3}$, are shown on Plate 3. The steel chamber, *A*, *B*, *C* (a very fine piece of work) was constructed by Messrs. WHITWORTH and Co. A detailed description of its various parts is unnecessary, as all particulars are given in the plate referred to. The annular space (coloured black) was filled with mercury.

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* *See Note*, p. 392, *infra*.

† The shellac was melted on, and not deposited from its solution. A hard surface is rarely obtained by the latter method.

As it was advisable to free that space from air bubbles, the screw plug *P* was removed before the insertion of the mercury, and replaced by a tube leading to a GEISSLER'S mercury pump. A tube from *D* dipped into a basin of hot mercury, and in this tube was a tap which was at first closed. The air in the annular space was withdrawn until the pressure was a small fraction of a millimeter, and the hot mercury was then slowly admitted—the exhaust pump being worked at intervals. When the mercury became visible within the tube at *P*, the tube was cautiously removed, and the air-tight screw plug replaced. The mercury necessary to fill the space weighed 70 lbs.

So true were the surfaces of the two parts of the metal chamber, which were screwed together by the bolts at *A* and *C*, that, although no packing except the finest possible coating of grease had been placed upon them, they proved absolutely air-tight. These two portions of the chamber were each formed from a single block of steel and, when placed together, somewhat resembled a double hat-box. The inner and outer surfaces were turned smooth and true, and then were nickel-plated. Two holes on opposite sides were bored down between the mercury chamber and the inner surface, as shown at *E* and *F*. These holes were filled with mercury and, their inner walls being very thin, the thermometer placed in them would accurately indicate the temperature of the inner surface of the chamber. This inner surface, as also the base of the chamber, was, in our more recent experiments, covered with highly-polished speculum metal. A ring of lead wire was placed round the top edge of the chamber, on which was laid that portion of the lid shown in section at *G* and *H*, and in plan at *G'* and *H'*, and the twelve bolts securing it were screwed home. Many as have been our difficulties with regard to air-tight joints, those made in this manner, viz., by the crushing of a lead ring between two surfaces, have never given us any anxiety. The second lid, from which the calorimeter was suspended, and whose section is shown at *K L*, and plan at *K' L'*, was secured in a similar manner.

The necessity for this double lid is not, at first sight, obvious, but it must be remembered that it was impossible to separate the calorimeter from the portion of the lid to which it was attached, and, therefore, a heavy steel plate would have been inconvenient, and have rendered any direct weighings impossible. On the other hand, the total pressure on this lid, when the chamber was exhausted, was very great, and, had the whole of it consisted of one thin piece, it is probable that it would not have withstood the strain without "panting," and, as it was necessary on account of the stirring rod, &c., that the whole apparatus should be absolutely rigid, it was essential to avoid any phenomenon of this kind. The first lid, therefore, is of considerable thickness, and the orifice in it only just sufficient to allow for the introduction of the calorimeter. Again, if the lid had been of one piece, it would have been impossible to remove it without disturbing the tubes *M* and *N*.

The tube *M* communicated with a five-way Sprengel pump and a McLeod gauge; the tube *N* with a Geissler pump and a water pump. The arrangements for drying and exhausting by means of these tubes is described in Section V.

The steel chamber, together with the mercury, weighed nearly 2 cwts. It stood on a small tripod in a galvanized iron tank, whose capacity was about 20 gallons. The tank itself was placed on a bed of Portland cement before the cement had hardened, and thus the whole erection was firm and stable. When the outer tank was filled, the surface of the water was about 3 inches above the lid of the steel chamber and was maintained at that level by a constant inflow of water, which entered almost above the screw Q and left by an overflow pipe shown at W. The screw at Q rotated in such a direction that the water was forced downwards, passed under the steel chamber, and returned across the lid. The screw revolved about 800 times a minute, and the stirring was very thorough, so that it was impossible to detect any difference of temperature in different portions of the tank, even when hot water was being admitted, unless the thermometer was held immediately against the entrance pipe. A glass tube with narrow bore (Sprengel tubing) was fixed into the steel tube at D and communicated with the regulating apparatus shown in Plate 4, fig. 1. R is the regulator—the gas entering at A and leaving at B. The tube R was about 1 inch in diameter and was drawn to a fine opening at its lower end. The thin glass partition C passed down the centre of this tube and projected at its lowest extremity into the narrow opening. To render the partition air-tight, its edges were coated with hard shellac, and after it had been thrust into position, the tube was warmed until the shellac melted. The top of the tube was closed with shellac, thus the gas, entering at A, could only arrive at B by passing round the lower end of this partition, and a very small movement of the mercury in the narrow neck sufficed to cut off the supply. We adopted this form of regulator after considerable experience of the ordinary form, in which the supply passes down an inner tube and up the annular space around it, which labours under the defect that the adhesion between the mercury and the inner tube causes the gas to remain cut off for some time after the mercury has commenced to contract. The form here shown is free from this defect, as it is found that the mercury does not adhere to the thin glass partition. Beneath the narrow tube a small reservoir (capacity about 3 cub. centims.) at E allowed considerable contraction to take place, and yet permitted the mercury to be kept in view. A side tube passed from E to the three-way taps F and H, and thus, by means of the tap at F, mercury could be withdrawn when necessary. The tubes HK and HL were about 4 feet in length—K terminating in a thistle-funnel, in which a supply of mercury was placed.

The tube L was connected with the three-way tap M, by which communication could be made with the exhaust water-pump, or with the open air. Behind the tube L was fixed a vertical scale of millimeters, and the whole of the tubing was rather wide-bore Sprengel. Through the tap at H mercury from the arm KH could be passed into the arm KL to any convenient height. By means of the taps H and F any portion of this mercury could be passed into the regulator, and, if necessary, the *same amount* could at any time be withdrawn by connecting L with the exhaust (by means of tap *M*) and opening *F* and *H*—independently of the level of the mercury in the regulator. This contrivance, therefore, rendered it possible to "set" the instrument so as to produce any required change of temperature in the tank; for it was found that the transference of about 312 millims. of the column *HL* into the regulator caused a difference of 1° C. in the temperature of the tank. It was thus possible to adjust the temperature to a very small fraction of a degree.

The connecting tube from *E* to *D* was bent into a spring of sufficient length to allow of some play to the lower end, and thus it was possible to withdraw the end from *D* without moving the whole edifice. The glass work was supported on a vertical board, about 5 feet in height, and the various taps were placed within easy reach of the observer. All gas admitted at *A* was dried by passing through tubes containing quicklime, as it was found that the regulator worked in a much more satisfactory manner when the surface of the mercury was clean. After leaving *B*, the gas entered a tube about 2 feet long, containing a very large number of small pin-holes. The tube was connected by two cross pieces containing similar small holes, with two pilot lights fed by an independent supply. A change of $\frac{1}{500}$ of a degree in the tank visibly affected the brilliancy of the tiny gas jets, and a change of less than $\frac{1}{100}$ made all the difference between total eclipse and a complete series of bright blue jets. The supply of gas through the regulator was, however, sufficient when the mercury had receded far enough to maintain the jets at a length of nearly $\frac{3}{4}$-inch.

Placed above this row of jets, but so far from the pilot lights as to be unaffected by them, was a flat silver tube, the section of whose bore was a rectangle of about 1 inch wide and $\frac{1}{16}$ inch deep, and, through this tube, the water passed on its way to the tank. The supply was regulated by a small "constant level" tank of the usual pattern, suspended from the ceiling by a string. Alterations in the level of this tank afforded a fine adjustment—the temperature depending on the rate of flow. The water was supplied direct from the main, and hence, even in the hottest summer weather, it was possible to maintain the tank at a temperature as low as 14°; in winter, as low as 3° or 4° C. Any temperature above this minimum was, of course, obtainable, but we considered it unadvisable to raise it above 40° C. on account of the ebonite, &c. This regulator proved itself to be a most satisfactory instrument. Although, when set to give a new temperature, it was slow in its action (the capacity for heat of the tank and its contents being great) it was none the less sure, and, after about half-an-hour from the readjustment, the temperature would be found to have become steady. If set to give a higher temperature, the thermometer at *E* (Plate 3) would pass that temperature by about $\frac{1}{15}$ of a degree, and, after this preliminary oscillation, would become steady, although it is probable that this first was followed by other oscillations, which were too small to be apparent. It is possible that the temperature of the inner surface of the steel chamber was, to some extent, affected by the temperature of the calorimeter. The change must, however, have been very small, since it was insufficient to affect the thermometer at *E*. It must be remembered that any such change would at once produce an effect on the mercury, and, by its action on the regulator, reproduce the original condition.

SECTION IV.—THE MOTOR AND ITS CONNECTIONS.

A belt passed from the motor to a vertical shaft fixed near the back of the tank. This shaft carried "speed-wheels," from which ran two cords, one to the stirrer in the outer tank, the other to the stirrer in the calorimeter. In order to prevent any "slipping," with which we were at one time much troubled, all these cords ran through loose pulleys hung by strings, which, passing over small fixed pulleys, were connected with pans containing shot. The tension of the cords could thus be regulated by altering the weight of shot, and any change in their length, due to stretching, &c., was of no consequence. The shaft of the calorimeter stirrer was connected with the revolving piece above it by a double HOOKE's joint. Until experiment J 19 a short piece of brass tubing with a cross wire at each end served as the HOOKE's joint, with the exception of J's 17 and 18, when only a single HOOKE's joint was used. A difference in the "stirring supply" was shown in these two experiments (see p. 450), probably due to some difference in pressure between the agate cylinder at the bottom of the stirring-rod and the ring surrounding it. After experiment J 19 the connection was made by a small circle of thin wire, which passed through a hole in the revolving shaft and dropped into a slit in the top of the stirring-rod. Thus all rigidity in the connection was avoided, and the stirring effect was more regular after the change. Above the stirring-shaft was a HARDING's counter, whose axis carried at its upper extremity a V-pulley, and at its lower extremity the HOOKE's joint (see Plate 2, fig. 2). These counters move in steps, and not with a continuous sliding motion, and, as it was essential that we should be able to read with considerable accuracy the time of each 1000 revolutions, this form of counter was the best adapted to the purpose. The wear that these instruments will stand is surprising, for, on many occasions, nearly a quarter of a million revolutions per day were recorded, and the same counter was used by us throughout our experiments of 1891 and 1892, with the exception of experiments J 20 to 34, when the original counter, having shown some slight signs of wear, was replaced by another.

The motor (one of the fan type) was by BAILEY and Co., and of the usual pattern. When working at the Sidney Laboratory in 1891, the changes in pressure, and therefore in the speed of the motor, were so frequent that we were compelled to relinquish our attempts to perform our J experiments during the day, and the observations had, therefore, to be taken between 10 p.m. and 2 a.m. Even during these hours, the regularity was far from satisfactory.

As has been explained in the introduction a special room was built for the experiments of 1892. Here the supply pipe to the motor led directly from the main, but the irregularities of pressure were still so great as to render some improvement imperative. The form of regulator devised, although it did not insure that absolute uniformity which we could have wished, was yet so effective that we venture to describe it at length, in the hope that it may prove useful to others who have met with the same difficulty.

AB (see Plate 4, fig. 2) was a flexible diaphragm made of india-rubber of one ply. This was bolted between two brass saucer-shaped discs with tinned surfaces. The lower of these discs was connected with a tube leading to the motor; the upper, by means of *E*, *F*, *G*, with an iron reservoir (a mercury bottle) placed on the floor. A thin metal rod was attached to a plate in the centre of the diaphragm, and at its lower extremity were fixed two cylinders placed so as to form a balance valve. The ends of these cylinders were grooved in such a manner that any upward movement diminished, and any downward movement increased the size of the channels through which the water had to pass. The water from the main entered at *C*, and after passing through the grooved channels followed the path indicated by the arrows, until it arrived at the injector of the motor. An iron tube about 6 feet high was supported by a vertical plank, which reached from the floor to the roof of the laboratory. The lower end of this tube entered the iron reservoir *G* at the bottom, and to the upper extremity was fastened about 4 feet of high pressure tubing of which the free end was lashed to a mercury jar *H*, similar to the hand one used in a GEISSLER'S pump. This jar rested on a movable shelf which, by means of a rope and pulley, could be raised from about 4 feet to 10 feet above the vessel *G*. Sufficient mercury was placed in this jar to fill the flexible tube and the lower 3 or 4 inches of the iron vessel *G*. The taps at *K*, *F*, and *L*, were then opened and water passed into the tube *E*, *K*, and the vessel *G*. As the pressure increased, the mercury was forced up the tube *M*, *N* until its surface became visible in *H*. The tap at *F* was then closed and the screw plug at *E* opened to allow any air in the pipes, or above the diaphragm, to escape. *E* was then closed and first the tap *L* was opened, then the tap *F*, until the mercury again appeared in *H*. *F* was then finally closed and would not again be used unless it was necessary to refill the apparatus. The pressure above *AB* could now be adjusted at will by altering the elevation of *H*, and the effective pressure could thus be changed from 20 lbs. to 60 lbs. per sq. inch. When the tap *L* was opened, if the pressure due to the tap-water on the lower side of *AB* exceeded the pressure on the top, the valve was lifted and the flow to the motor diminished until the pressure of the water which had passed the valves became the same as that above the diaphragm. Conversely, if the water pressure diminished, the flow to the motor increased. The tap at *K* was only used to remove the pressure of the mercury column from the top of the diaphragm when the instrument was not in use. Within certain limits, this regulator acted excellently; but if the water pressure fell so low that the diaphragm assumed its lowest position, it ceased to be effective. By altering the elevation of the movable shelf at *H*, the rate of the motor could be adjusted at will, and a scale being fixed to the vertical plank, the extent of the adjustment necessary to produce a given alteration was approximately known after a few observations.

[Note by E. H. G. Added April, 1893. Lord RAYLEIGH has discussed the principles which should be observed when using a motor of the fan type ('Proc. Soc. Roy.,' 1881, p. 109), and has shown that when the motor is run at high speed the curves of work and of resistance cut each other at a greater angle than is possible when the rate of revolution is small, and thus a more uniform motion is obtained in the former, than in the latter, case. Our speed wheels were so arranged that the motor under ordinary circumstances ran at about 16 revolutions per second, which appeared to be as high a speed as was necessary.]

SECTION V.—THE PRESSURE IN THE SURROUNDING SPACE.

As already mentioned, the tubes *M* and *N* (see Plate 3) led to the pumps and McLeod gauge. The circuit embraced several yards of glass tubing, a large number of junctions and four taps, of which three had to be absolutely air-tight. In addition there were the four insulated junctions in the steel and the two in the calorimeter lid, and also the six openings (three in the steel and three in the calorimeter), through which the glass tubes which supported the calorimeter passed. All who have worked with low pressures will appreciate the difficulty of making such a series of joints sufficiently air-tight.

The manner in which the difficulty was overcome in the openings through the steel and calorimeter is described in Section VII. In what we may term the external circuit the greatest difficulty was experienced in the joints between the glass and the steel tubes *M* and *N*. Throughout 1891 we used telescopic joints. The glass tubes, which closely fitted into the steel ones, were coated with hard shellac, and, when both glass and steel had been warmed, were pressed into position. There is no doubt that such joints can be made absolutely air-tight; but they cannot be trusted to last for any length of time. It is possible that shellac is sufficiently viscous to yield to long continued pressure, or, on the other hand, it may crack when exposed to sudden changes of temperature. Whatever the cause may be, we have found such joints most treacherous. Their very perfection when first made renders them the more dangerous, for they inspire the observer with a false confidence.

Our experiences during 1891 led us to reject the use of shellac in any form.* In 1892 the connection between the glass and steel was made by platinum tubes of small bore, soldered into the steel at one end and fused into the glass at the other. However, the latter form of junction could not be relied on unless the diameter of the platinum was small, and this constriction of both the passages leading to the pumps greatly increased the labour of exhausting, for the process of diffusion through such narrow tubes is very slow when the difference of pressure at the ends falls below 1 millim. As we could find no way out of this difficulty we had to content ourselves with a higher pressure than we had proposed to work with.

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* From the commencement we avoided the use of india-rubber tubing.

The taps used were of a kind that has a diagonal hole through the stopper and a mercury cup at each end; they appear to be very perfect, and any leakage from either end can be detected by its effect on the mercury. One of these taps was in each glass tube leading from *M* and *N*, and, by their means, all the rest of the circuit could be cut off from the calorimeter.

The tube connected with *M* led to a five-way Sprengel and a McLeod gauge (both by HICKS). The tube from *N*, after passing the first tap, branched into two arms, one of which led to a Geissler pump, while in the other was a three-way tap establishing connection with the water pump, or with a series of drying bottles, containing H_2SO_4. A tube containing $Ba(HO)_2$ was fixed between the drying bottles and the calorimeter, as we thought it just possible that, otherwise, sufficient H_2SO_4 vapour might pass into the chamber to produce some action on the steel surfaces.

When the apparatus was fixed together the temperature of the tank was raised to about 40° C.; the air was then exhausted down to about 16 millims. by the water pump, dry air was passed in slowly, and this process was repeated several times. Finally the pressure was reduced as far as possible by the water pump (to about 12 to 14 millims.), and the Sprengel and Geissler brought into play.

It took about 100 double strokes with these pumps to bring the pressure down to 1 millim., so the labour involved was great, and want of time alone prevented our reducing the pressure further than we did. By increasing the number of the platinum tubes we hope to be able in future to work with much better vacua.

The absence of leakage is indicated by the fact that the lapse of a week made no perceptible difference in the reading of the McLeod gauge, although a change of ·01 millim. could have been easily detected.

It was long before this degree of perfection was attained, for the difficulty of ascertaining the position of a slight leakage, in such a circuit, was very great, and many weary hours were expended in leak hunting and in consequent taking to pieces and putting together again the various portions of the apparatus.

SECTION VI.—THE METHOD OF MAINTAINING A CONSTANT E.M.F.

General plan of the electrical connections* (fig. 2, p. 382):—

Leads passing through points numbered 1 and 3 belong to the Clark cell circuit.

Leads passing through points numbered 2 and 4 belong to the storage cell circuit.

The leads 1 and 2, as also 3 and 4, unite at the lid of the calorimeter itself, as shown in the small sketch.

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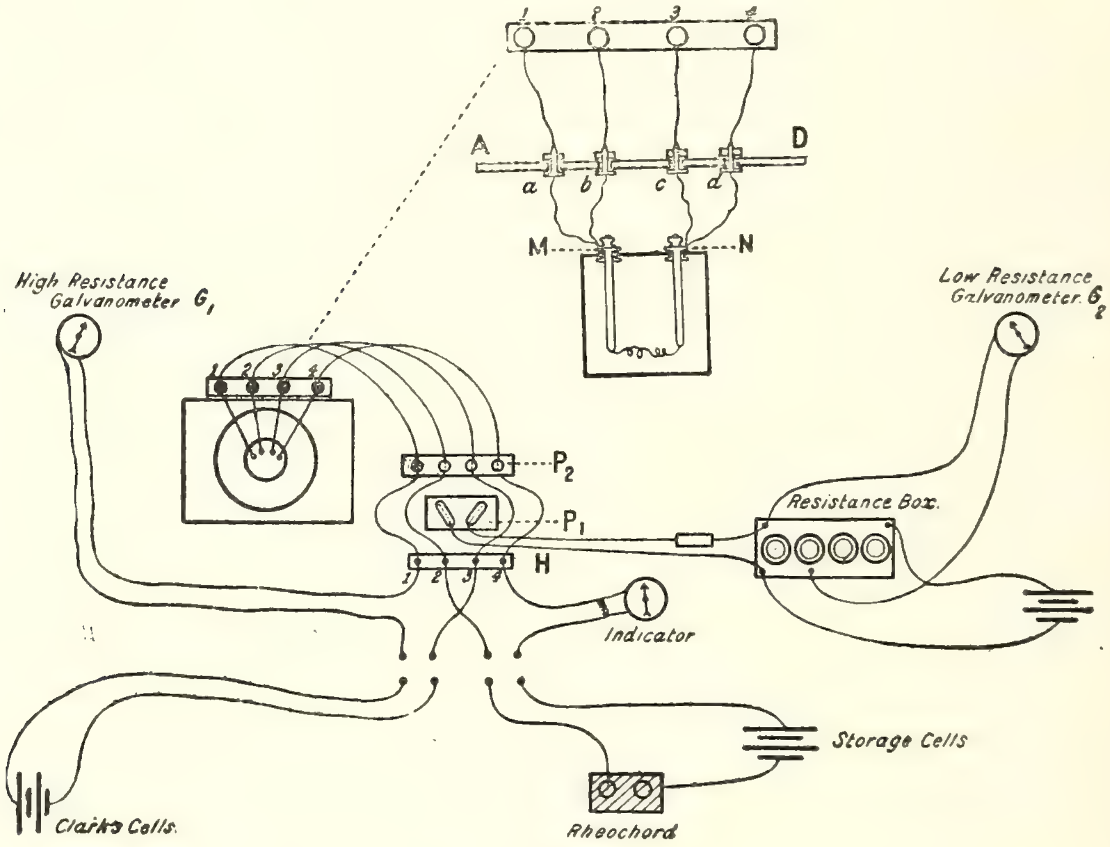
* The actual position of the galvanometers, &c., was different from the sketch, which is only intended to indicate the nature of the arrangement.

All connecting wires of any length were carried overhead, and hung by silk threads from a net suspended about 7 feet above the table. Thus all were visible, and could be traced without difficulty. All keys and instruments through which communication with earth could be made were placed on slabs of paraffin or ebonite, and the accumulators were also insulated. Each lead was placed as close as possible to its return to avoid any electro-magnetic action. The resistance of the galvanometer G_1 was about 9000 ohms. Instruments of such high resistance speedily indicate any leakage in the circuit, and it was not until we had given considerable time and attention to the insulation that we were free from trouble on this account.

Fig. 2.



The galvanometer (an astatic one manufactured by the Scientific Instrument Company, and the field so adjusted that the time of a single oscillation was about 4 seconds) showed no deflection unless connected with the Clark cells, although, at times, the difference of potential between the screws 2 and 4 amounted to 10 volts. If the fingers were placed on the ends of the wires communicating with it, the spot of light could be thrown nearly off the scale. The galvanometer was placed at the farther end of a tunnel (4 feet in length) of which the nearer end was closed by a sheet of ground glass. The beam of light was thrown in through the side of the tunnel, reflected on to the galvanometer mirror and thence on to the ground glass, which was protected by a hood from other illumination. Thus the "spot" was easily visible, even in bright day-light, by either of the observers when in the situations they occupied during the experiments. The extreme sensitiveness of this galvanometer will be illustrated when dealing with the comparison of our Clark cells.

The method of maintaining the ends of the coil at a constant potential difference has been indicated in Section 1. The experimental accuracy of the method depends entirely upon the action of the rheostat at *K* (fig. 1, p. 370). This instrument should be capable of a fine, as well as a rough, adjustment, and, above all, there should be no possibility of uncertain contact.

The majority of rheostats establish connection by the pressure of a sliding piece upon a wire, which is, at the best, an unreliable form of contact. The instrument devised by us has, however, proved to be so free from this defect that we describe it at some length.

It is well known that the electrical contact between a moving platinum wire and mercury is, under ordinary circumstances, an unsatisfactory one; its uncertain nature being probably due to the air-film attached to the wire.* We found that in vacuo the contact was in every way satisfactory.

A and *B* (see Plate 2, fig. 3) are two glass tubes about 7 feet long, with a platinum wire running through the interior of each—the ends of the wires projecting at *A* and *B*. The resistance of the wire in *A* is about 6 ohms per foot, and of that in *B* 1·7 ohm per foot. *C* and *D* are two wide tubes about $3\frac{1}{2}$ feet long, closed at the lower ends.

A and *B* were carefully filled through a capillary tube with hot mercury, and inverted so as to stand in *C* and *D*. Practically, *A* and *B* are two barometers having tubes of about twice the usual length. *C* and *D* are firmly fixed in a stand, and *A* and *B* are raised or lowered by handles at the side, connected with gearing wheels which move two racks, fixed upon the rods carrying *A* and *B*. In their lowest position the barometer column reaches almost to the top of the tubes; in their highest, about 4 feet of the wire is left above the mercury. Spiral wires are attached to the projecting ends of the platinum wire at the top, and the ends of a copper fork, carrying a screw connector (*E*) at the bend, communicate with the mercury in *C* and *D*.

The current enters at *F*, passes down the wires in *A* and *B* (which are thus in parallel arc), and leaves the instrument at *E*. The wire from *F* to *B* is so arranged that additional resistance can be placed in it if required.

The mean resistance of the calorimeter coil used by us this year was about $8·5\omega$; the minimum external resistance about $1·3\omega$. The E.M.F. of each storage cell was about 1·4 times the E.M.F. of one Clark cell. Denoting our external resistance by r, and the coil resistance by R, it is evident that (when the number of storage and Clark cells were equal) an approximate balance would be obtained if $(\mathrm{R} + r)/\mathrm{R} = 1·4$, which gives $r = 3·4$, *i.e.*, when the rheostat resistance was about $2·1\omega$.

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* If the wire is amalgamated the mercury is apt to cling to it when the wire is withdrawn.

It was found that by raising A about 6 inches, B about 24 inches, and introducing a resistance of 4ω into FB, an approximate balance was obtained. The resistance of A was then about $\cdot 3\omega$, and B $7\cdot 30\omega$. B thus acted as a shunt, and a considerable alteration in it caused only a small change in r. By altering their relative positions the ratio A/B could be changed, and the sensitiveness of the instrument could thus be altered when required. When arranged as above, a movement of 1 inch in B would throw the galvanometer spot off the scale, while the effect of a change of 1/25 inch was distinctly visible.

Thus a rise of 1 inch in B would change r from $(\cdot 3 + 7\cdot 3) / (\cdot 3 \times 7\cdot 3)$ to $(\cdot 3 + 7\cdot 44) / (\cdot 3 \times 7\cdot 44)$, *i.e.*, a change of $\cdot 0003\omega$.

And thus the ratio r/R would change from 1·3 / 8·5 to 1·3003 / 8·5, a change of less than 1 / 50,000.

The scale on the galvanometer screen had divisions of about 1 inch in length, and a deviation of 1 scale division, during a whole experiment, was exceptional. It thus appears that the variations in E were certainly within 1 / 100,000 of the mean value during the experiments, and, therefore, changes in E, consequent on changes in R or S, might be disregarded.* It was necessary that throughout each experiment one observer should give his entire attention to the adjustment of the rheostat. As the temperature of the coil rose steadily the value of r had to be increased, its total increase (over our range) being about $\cdot 005\omega$, *i.e.*, a rise of about 36 inches in B.† No doubt this adjustment took place in steps. The spot was observed to cross the zero line and was then thrown back to the further side—a rise of 1 millim. in B being, as a rule, sufficient for this purpose. As the handle had a large radius, this small change was effected without difficulty. The effect of such small oscillations about the zero position would probably "mean out," and if not, would be negligible.

Other possible causes of alterations in the value of E are—

(1) Polarization, as a consequence of the current required to affect the galvanometer. The high resistance of the galvanometer rendered any perceptible polarization extremely improbable. In our earlier experiments an additional resistance of $10,000\omega$ was placed in the circuit; contact was intermittent and established by a key which was worked by a water motor.

Throughout our 1891 experiments we made it a rule to compare before and immediately after each experiment the E.M.F. of the cells used‡ with others which were kept as standards, and, as we were unable to detect any change, these excessive precautions were abandoned. On twenty-six occasions when no extra resistance was introduced these cells were compared before, and at the close of, an experiment.

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* When the external circuit became heated by the current at the commencement of an experiment considerable adjustment was required, but it was made without difficulty.

† This increase is in *addition* to the increase in r due to the heating of the external circuit.

‡ The Clark cells used in 1891 belonged to the Cavendish Laboratory, and are the ones referred to in Table IX., 'Phil. Trans.,' A, 1892, p. 605.

Poggendorff's method was used ; a change of one plug indicated a change of E/6000, each cell was balanced against the standard ones in the manner adopted by Messrs. Glazebrook and Skinner, and in no case could any alteration whatever be detected. This point having been satisfactorily established, we considered it unnecessary to repeat the observations in a similar manner this year, but contented ourselves with occasional comparison of the cells used with others which we kept as standards.

(2) The second difficulty was not surmounted so satisfactorily. As before mentioned (Introduction) the laboratory was maintained in cold weather at a constant temperature of about 16° C., by an automatic regulator. However, during the months of August and September, 1892, while the nights were cool, the temperature during the day was extremely high, and occasionally rose in the laboratory to over 20° C. The Clark cells were contained in a tank of water, but their temperature at times changed by 3° or 4° C. ; and under such circumstances the difficulty was to know what temperature to take.

Messrs. Glazebrook and Skinner have shown that the E.M.F. does not change with the thermometer, but lags considerably. A difference between the "virtual" and "real" temperature of 1° C. would cause a difference of 1 in 1000 in our corrected times, and we believe that many of the irregularities in our results, especially in experiments before August 15, are attributable to this cause. On that date we got into working order a regulator which much diminished this source of error. A large bulb of mercury was immersed in the tank, and was connected by a narrow tube with a trap of the ordinary gas regulator pattern, through which cold tap-water, instead of gas, was kept flowing. When the mercury expanded, the water ceased to flow through the regulator, and diverged by a bye-way into the tank ; when the mercury contracted, the cold water passed into a waste pipe, instead of into the tank. The pressure of the water was kept constant by a small supply cistern, with an overflow pipe, which was suspended by a string, and the head of water could thus be regulated. This apparatus worked most satisfactorily—a change of $\frac{1}{4}$° C. turning the inflowing water from the tank to the waste. Unfortunately the weather became so warm that even the tap-water at that end of the laboratory rose to about 16° C. The changes, after the introduction of the regulator, were, however, much diminished, and rarely amounted to 0°·5 C. As regards the errors thus introduced, although they seriously affect individual experiments, they must tend to eliminate when a large number of observations are taken.

The Clark cells used for the 1892 experiments were constructed during January and February of that year. Particulars of their construction are given in an Appendix to Messrs. Glazebrook and Skinner's paper ; therefore it is unnecessary to give here more than a brief summary.

Although the method of preparing the solutions for different batches of six was varied, the E.M.F. of all the cells is practically the same. In all cases, the platinum wires were amalgamated by being dipped in mercury when red hot. The mercury, which was freshly distilled, was about 1 centim. in depth, and was nearly touched by the ends of the zinc rods. About 1 centim. of the "paste" was placed on the mercury, then, about 1·5 centim. of $ZnSO_4$ solution and some crystals of $ZnSO_4$. Above this solution was fixed a thin cork, on which was a thin layer of paraffin wax. On this was placed about 6 centims. of glass-wool and then about 2 centims. of marine glue. The diameter of the cell was about 2·5 centims. The platinum wire was sealed into a thin glass tube, the end of which projected above the marine glue. In most cases the upper end of the zincs was completely buried.

Nos.[*] 31 to 42.—Hg_2SO_4[†] unwashed but shaken up with Hg and the $ZnSO_4$ solution filtered at 30° C., zincs not amalgamated.

Nos. 43 to 48.—Prepared by following out the Board of Trade directions exactly.

Nos. 49 to 54.—Boiled the $ZnSO_4$ solution with ZnO, allowed to cool to 30° C. Filtered at 30° C. Added to this a small quantity of Hg_2SO_4 paste. Filtered off resulting black precipitate, warmed filtrate, and again filtered at 30° C. The cells made with this solution appeared to settle down to their final E.M.F. immediately after their manufacture, unlike the others, which were some time before they became steady.

About thirty comparisons of these cells were made between January and September, 1892. Nos. 31 to 42 remained at the Cavendish Laboratory for some time, and were repeatedly compared with the Rayleigh Standard by Mr. SKINNER, to whom we owe our best thanks for the trouble he took in this matter.

We select as examples comparisons made on three different dates. The results of the comparisons on other dates are as close, but it is unnecessary to give them at length.

No discrepancy showed itself except after rapid changes of temperature, and in the most extreme case the difference from the mean did not amount to 1 in 2000. The comparison of cells 31 to 36 with the Cavendish Standard, on February 13th and 20th, gave precisely similar results. These comparisons were only the last of a series. On both dates the temperature of the standard was slightly higher than that of the cells, and therefore the equality was very close. Mr. SKINNER, writing on February 20th, said, "You may take it that Nos. 31 to 36 are now equal to our standard, and no further comparison is necessary." In our comparisons the cells were, in each case, placed in opposition to No. 31.

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[*] These cells are numbered 131 to 160 in Messrs. GLAZEBROOK and SKINNER's paper, 'Phil. Trans.,' A, 1892, pp. 622 to 624. Numbers 154 to 160 were not used by us during this investigation.

[†] Chemicals supplied by Messrs. HARRINGTON.

TABLE I.

	Feb. 13 and 20, 1892. Comparison with Cavendish Standard. (Diff. in 6000 parts.)	May 1, 1892. Comparison with No. 31. (Diff. in 6000 parts.)	July 3, 1892. Comparison with No. 31. (Diff. in 6000 parts.)	Sept. 6, 1892. Comparison with No. 31.* (By potentiometer.)
31	+ 1 +	0	0	1·00000
32	+ 1	+ 2 +	+ 2 −	+ ·00002
33	+ 1	+ 1	0 +	− ·00003
34	+ 1	− 1	− 1 −	− ·00013
35	+ 2 +	− 1	0	− ·00011
36	+ 2 +	0	+ 1	·0
37		0	+ 1	+ ·00001
38		0	+ 1	+ ·00002
39		0 −	+ 1	+ ·00001
40		0	+ 1	− ·00006
41		0 −	0 +	− ·00006
42		0 −	0 +	− ·00007
43		+ 2	+ 3	+ ·00022
44		+ 1	+ 2	+ ·00013
45		0 +	+ 2 +	+ ·00014
46		+ 2 +	+ 3	+ ·00020
47		+ 4	+ 3	+ ·00014
48		+ 3 +	+ 5	+ ·00016
49		0 +	0 +	− ·00001
50		+ 1	+ 1 −	·0
51		− 1	0 +	− ·00002
52		− 1	0 +	− ·00009
53		0 −	+ 1 −	− ·00007
54		− 1	0 +	·0
	Temperature . . .	12°·2	17°·2	16°·04

On May 1 and July 3 the total resistance of the two boxes used was 12,000ω, so the numbers given represent the differences in 6000 parts. In the comparison on September 6 a CLARK'S potentiometer (constructed by Messrs. ELLIOTT) was used. Readings can be taken with an instrument of this kind, with much greater ease and certainty than when using two resistance boxes.

It may be worth mentioning that on September 6, the instrument was adjusted by one observer and the readings taken by the other, who then moved the handle, thereby compelling a fresh adjustment by the first observer. The galvanometer used was the high resistance one, G_1, previously referred to, and gave a distinct deflection for a change of $1/10^{-6}$ in R.

Several comparisons were also made at different dates with some cells belonging to the Cavendish Laboratory.* These cells† were (early in August) in close agreement with No. 31, and also with the Cavendish. They have since been compared with the Berlin Standard, and appear to exceed that standard by about ·0002 volt.

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* We have not reduced the potentiometer readings to the same form as that adopted in the previous columns, as a difference of ·00017 corresponds to 1 in 6000, and the differences shown on September 6 would only amount to + 1 + in the most extreme case.

It will be noticed that the results of the comparison on September 6th are in much closer agreement than those of previous dates. This is partially due to the potentiometer, since the adjustment in this case has not to proceed by steps. It must be remembered also that the cells had been kept (by the regulator above described) at a steady temperature for a considerable time; and the great importance of a constant temperature is rendered evident by the figures given in Table I., and illustrates the truth of Mr. GLAZEBROOK'S remark that many discrepancies are due to the different "lag" of the cells when their temperature is changing.

A consideration of the above facts will, we believe, bear out the conclusion that the mean value of our Clark cells may be regarded as exceeding that of the Cavendish standard by not more than ·0002 volt (*i.e.*, by about 1 in 7000). During our J experiments we always used a considerable number of these cells, placing several in parallel arc, and in no case did we use less than nine of them at any one time. Thus no correction for individual cells is necessary.

A reference to GLAZEBROOK and SKINNER'S paper will show that but a short time elapsed between their determination of the absolute value of the Cavendish standard, and the comparison of our cells with that standard. The absolute value of the standard is given by them as 1·4342 volt;‡ we are therefore justified in assuming the mean value of cells No. 31 to 54, as

$$1{\cdot}4344\ \{1 + {\cdot}00077\ (15 - t)\}.$$

[Note by E. H. G., added April, 1893. All the cells mentioned above practically maintain their relative positions at this date. I have recently taken a series of observations in the hope of throwing some light on the gradual approximation of the cells, as shown by Table I. I find that if the temperature is suddenly raised, and then kept quite constant, the initial differences steadily decrease with lapse of time, and that the cells continue to converge after the lapse of many days, and even weeks. The cells prepared by my method of adding some Hg_2SO_4 to the solution before filtering, appear to settle into their final state more rapidly than the others.]

SECTION VII.—THE MEASUREMENT OF RESISTANCE.

The coil used by us in 1888 was of platinum-iridium wire, and had a mean R of about 19 ω.

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* We take this opportunity of thanking Professor THOMSON for his kindness in permitting us to borrow many pieces of apparatus belonging to the Cavendish Laboratory.

† These are the cells Nos. 65, 69, 70, 77, 78, and 79, mentioned in Messrs. GLAZEBROOK and SKINNER'S paper. No. 65 may be regarded as equal to our No. 31. See 'Phil. Trans.,' A, 1892, pp. 605, 623, and 625.

‡ 'Phil. Trans.,' A, 1892, p. 582.

In 1889 we used two coils of 2 and 4 ω, one of German silver and one of platinum, both covered with a layer of gutta-percha.

In 1891 we used a coil of about 10 ω covered with amber varnish and made of copper-manganese-nickel alloy, and also a platinum wire of about the same value.

This year we have contented ourselves with a platinum wire of about 8·5 ω.

Since $H \propto E^2/R$ and E is, throughout each experiment, a constant, therefore $H \propto 1/R$. Hence if a coil of large resistance is used, the quantity of heat developed will be small, unless E is great.

It appeared to us desirable to keep the value of E as low as possible, for the greater the difference of potential at the ends of the coil, the greater is the loss arising from imperfect insulation, &c. On the other hand, if the coil is too short the development of heat is too localized, and its even distribution throughout the calorimeter rendered more difficult. Our experience in the earlier experiments, as also a consideration of the masses of water, &c., led us to the conclusion that a difference of potential equal to that of from 3 to 5 Clark cells* and a resistance of 8 to 10 ω would balance the opposing influences in the manner best adapted for our purpose.

Platinum would at first sight appear to be an unsuitable metal of which to form the coil, for its temperature coefficient is very great, and the element of change thus introduced involves much arithmetic in the final calculations. For the same reason the error due to its rise in temperature above the surrounding water is likely to be more serious than if the coil was formed of an alloy. These considerations carried so much weight that we undertook (in the summer of 1890) a series of investigations into the resistance of the various copper-manganese-nickel alloys, which had been recently introduced by Dr. FEUSZNER of Berlin. Some of these alloys were made for us by Messrs. JOHNSON and MATTHEY, and several specimens were given to us by Dr. FEUSZNER, to whom we owe our best thanks for his kindness; others were obtained from Messrs. WOLFF, Berlin.

The behaviour of these alloys was investigated over a range of from − 20° C. to 100° C. The different wires passed down a tube about 4 feet long, together with a platinum wire, cut from the same coil as that from which one of our platinum thermometers was made. The resistance of this wire (*in situ*) was determined in ice and steam, and the value of δ being known, by its determination when in our platinum thermometer, the mean temperature of the wires surrounding it could be deduced with great accuracy.

We hoped to proceed with an analysis of the composition of all the wires used but time has not been sufficient.

The results are summarized in the following table.

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* Had we felt certain that the insulation of the wire was perfect, we should have had no hesitation in increasing the potential difference, in which case we could have worked with larger masses of water. The uncertainty as to the temperature of the wire when thickly coated would, however, more than counterbalance any advantage gained by the use of larger masses.

DATE—July 24 to September 18, 1890.

A = Platinum wire, diameter ·005, same as thermometer H (previously annealed).
B = Copper wire (commercial).
C = Copper-manganese alloy made by Messrs. JOHNSON and MATTHEY—approximate composition—·4 per cent. manganese (previously annealed in carbon).*
D = Copper and manganese (J and M), 16 per cent. manganese (previously annealed in carbon).
E = Copper and manganese alloy presented to us by Dr. FEUSZNER (Mn 10 per cent. and Cu 90 per cent.).
F = Copper-manganese-nickel alloy from Dr. FEUSZNER (diameter ·17 millim.), rather more Mn than G and H.
G = Copper-manganese-nickel alloy from Dr. FEUSZNER (diameter ·14 millim.), Mn 12 per cent., Ni 3 per cent., Cu 85 per cent.
H = Copper-manganese-nickel alloy from Dr. FEUSZNER (diameter ·10 millim.), Mn 12 per cent., Ni 3 per cent., Cu 85 per cent.
J = Copper-manganese-nickel alloy supplied by WOLFF (diameter ·24 millim.), Mn, Ni, and Cu, but composition uncertain.
K = Copper-manganese-nickel alloy supplied by WOLFF (diameter ·17 millim.) Mn, Ni, and Cu, but composition uncertain.
L = Copper-manganese-nickel alloy supplied by WOLFF (diameter ·10 millim.), Mn, Ni, and Cu, but composition uncertain.
M = German silver.

TABLE II.—(Temperatures by Air Thermometer.)

	– 20°	0	10	15	20	25	50	75	100	
A (Pt)	·9288	1	1·0352	1·0525	1·0700	1·0874	1·1745	1·2608	1·3468	
B (Cu)	·9382	1	1·0306	1·0468	1·0627	1·0790	..	..	1·3054	
C	·9805	1	1·0097	1·0147	1·0195	1·0244	1·0485	1·0733	1·0977	
D	·9968	1	1·0014	1·0021	1·0027	1·0033	1·0058	1·0078	1·0096	
E	·9976	1	1·0009	1·0013	1·0017	1·0020	1·0032	1·0036	1·0031	Max. at 75° C.
F	·9983	1	1·0008	1·0012	1·0015	1·0019	1·0032	1·0042	1·0052	
G	·9966	1	1·0014	1·0022	1·0030	1·0037	1·0084	1·0104	1·0134	
H	·9999	1	1·0000	·9999	·9999	·9998	·9991	·9979	·9967	Max. at – 5° C.
J	.9987	1	1·0005	1·0007	1·0009	1·0011	1·0015	1·0014	1·0004	,, 55° C.
K	·9989	1	1·0006	1·0009	1·0011	1·0014	1·0023	1·0026	1·0027	,, 90° C.
L	·9986	1	1·0005	1·0007	1·0008	1·0010	1·0014	1·0014	1·0011	,, 65° C.
M (calculated)	·9920	1	1·0040	1·0060	1·0080	1·0100	1·0200	1·0300	1·0400	

The above resistances were taken with an E.M.F. of four Leclanché cells, and the arms of the bridge were 1000/10. A double set of electrodes was used, and the external resistance determined for each observation. Also the resistance of the platinum wire was observed before and after each determination of the other wires. The current was reversed on each occasion to eliminate thermal effects. The total swing of the galvanometer averaged about 150 scale divisions for a difference of ·01 ω, and thus the next figure could be determined with accuracy. The actual resistance of the wires used varied from 1 to 50 ω; but in the above table the resistance of each at 0° C. is assumed as 1 ω. It must be remembered that the temperatures are expressed in the *air* (not mercury) scale.

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* We have to thank Mr. C. T. HEYCOCK for his kindness in making an analysis of this alloy.

According to the analysis given by Dr. FEUSZNER, G and H have the same composition, and differ only in their diameters, yet the difference in their behaviour is very marked. It is possible that this difference is due to the effect of the "drawing."

The *calculated* value of German silver alloy is given for the sake of comparison. We regret that we did not include a wire of that substance. It is, of course, improbable that its increase is proportional to its temperature—from 0° to 100° C.

Our observations on these wires[*] extended from July 24 to September 18, 1890, and we were unable to detect any change in their resistances during that time.

The behaviour of the alloy C is curious. Our figures, which were carried to a place beyond that given in the above table, prove that, over the above range the increase in resistance varied strictly as θ. In order to test the straightness of its line beyond the above range, we constructed a thermometer which had this wire for its coil, but which was, in all other respects, constructed in the same manner as a platinum thermometer with double electrodes.

This thermometer gave $R_0 = 12{\cdot}972$, $R_1 = 14{\cdot}255$, and R_s (*i.e.*, resistance in sulphur vapour at 760 millims.) $= 18{\cdot}677$. Assuming the straight line we have

$$\theta_s = (R_s - R_0)/(R_1 - R_0) \times 100 = 444{\cdot}6.$$

According to an investigation subsequently conducted as to the boiling point of sulphur[†] the true air temperature ought to be 444°·53 C. It is difficult to imagine a closer approximation. A thermometer thus constructed is, therefore, a very simple and accurate instrument. It would, however, be exceedingly difficult to make another specimen of the alloy of a precisely similar composition, and the above table shows that the effect of small changes may be considerable.

A consideration of the results led to the conclusion that the wire marked H was the one most suitable for our purpose. Its change in resistance from 10° to 25° C. was only 1 in 5000, and since the resistance diminished as θ increased, it was easy to compensate by the addition of a little of the C wire. All difficulties connected with change of resistance consequent on the rise of temperature of the water, or of the difference in temperature of the wire and the water, appeared, therefore, to have been overcome. This conclusion, however, led to the loss of many weeks' work, and was the cause of much disappointment and delay. In the summer of 1891 we commenced a series of direct determinations of J, the arrangement of the apparatus being very nearly the same as that of 1892. Our coil consisted of about 30 inches of the above wire, H, which was first annealed at a temperature of about 200° C. in an atmosphere of CO_2. The fine white silk covering was allowed to remain, and it was painted with Mr. LAURIE's amber varnish.*

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[*] All these wires had been previously heated to a temperature of about 150° C.

[†] It must be borne in mind that the date of this determination preceded the work of CALLENDAR and GRIFFITHS on this subject, and at this time (July 27, 1890) the boiling point of sulphur was accepted as 448°·4 C. (REGNAULT).

The resistance of this coil, and of the leads from it to the calorimeter lid when *in situ* and immersed in rapidly stirred water, is shown in the following table.

TABLE III.

Temperatures . . .	14°	16°	18°	20°	22°	24°
August 21, 1891 . .	R=10·5214	10·5220	10·5223	10·5225	10·5226	10·5225

RESISTANCE of the coil at temperature 14° C., on different dates.

Aug. 22	10·5228		
,, 23, 11 A.M. . .	10·5275		
,, 23, 11 P.M. . .	10·5299		
,, 24	10·5342		
,, 25	10·5398		
,, 27, before exp. J_3	10·5559 ;	after exp. J_3	10·5600
,, 28, ,, ,, J_4	10·5570 ;	,, ,, J_4	10·5632
,, 29, ,, ,, J_5	10·5618 ;	,, ,, J_5	10·5696
,, 30, ,, ,, J_{6a}	10·5760 ;	,, ,, J_{6a}	10·5860
,, 31, ,, ,, J†	10·5867 ;	,, ,, J†	10·5962
,, 31, ,, ,, J_{6b}	10·5953 ;	,, ,, J_{6b}	10·5977
Sept 1, 10 A.M. . .	10·5968 ;	(some current passed) 11 P.M.	10·5982

The resistance of the wire at different temperatures was also ascertained in the observations taken August 21 to 25.

It is unnecessary to give the resistance at temperatures other than 14° C., since the resulting curves are all parallel to the one on August 21. The alteration caused by a rise of 10° C. being only 1 in 10,000, the steady diurnal rise in the resistance (at the same temperature) amounts to nearly 1 in 500 in the above interval of time. We trusted that this might be a result of the annealing, and that the wire would assume a stable condition in course of time.

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* Mr. LAURIE was so kind as to allow us to have a bottle of this varnish prepared by himself ; for particulars concerning this varnish, see 'Proceedings Cambridge Phil. Soc.,' vol. 7, part 2, p. 52.

It should be noticed that on the above dates we were constantly passing currents through the coil in order to adjust the temperature of the calorimeter during our stirring experiments. As, however, there was but little current when ascertaining the resistance of the wire at different temperatures, the curves were not appreciably altered, and we did not recognize the fact that the rise was a consequence of the current.

On August 27, we performed one of our "J" experiments. We then had a current of about half ampère passing through the coil for about $1\frac{1}{2}$ hours. We had determined the resistance at the commencement of the experiment, and, on repeating our observations at its close, we found that the value of R had risen from 10·556 to 10·560. It would thus appear that the rise was a function of the current rather than of the time. We continued the series of experiments until September 1, when we decided to abandon the use of this alloy.* The labour thus expended was not altogether lost, for, as will be explained, in the section on the measurement of temperature, the results were of considerable value to us.

Had the rise been a function of the time it would not have been so disastrous, since we could have deduced the resistance during each experiment. As, however, the change was caused by the current, it was impossible to determine (with sufficient accuracy) the actual value of R at each stage of the experiment. It appeared probable that electrolysis of some kind took place. If, after a current had been passing for some time, the ends of the wires from the coil were connected with the galvanometer, a considerable deflection was produced, and it was impossible to determine the resistance of the coil for about half an hour after the completion of an experiment, owing to the disturbance of the galvanometer thus caused. The effect was apparently due to polarization rather than to thermal, or "Thomson" effects, and decreased regularly until it might be neglected, although the differences of temperature in the circuit were maintained unaltered. Much time and energy having already been expended in experimenting with these various alloys, we decided to retrace our steps and revert to a platinum wire coil. Another consideration which influenced us, was that we had, during the past few years, given considerable time and attention to the behaviour of platinum wires when used in platinum thermometers, and we considered that the experience thus gained of the changes in the resistance of such wires under varied conditions, might enable us to proceed with more confidence.

The smaller the diameter of the wire the less its capacity for heat and, in comparison, the greater its cooling surface. The wire selected for this year's experiments was supplied by Messrs. Johnson and Matthey. It had a diameter of ·004 in., and was of as pure platinum as could be obtained. The length used was about 13 in., and the wire was so disposed that parts near to each other should have only a small difference of potential. This wire was covered with a coating of the amber varnish before referred to. Several coatings form an excellent insulating covering for such a wire, but we contented ourselves with a thin skin. The better the insulation between the wire and the water the slower the flow of heat across the insulator, and the greater the difference in temperature between the surfaces which it separates.

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* Mr. Skinner has recently conducted some experiments with the same coil immersed in paraffin instead of in water; and although he has passed far greater currents than any used by us, he was unable to trace any signs of consequent increase in R. The increase shown during our experiments was, therefore, probably due to some action between the wire (when carrying a current) and the water, this is not unlikely when we remember that one of the metals present was Mn.

If the insulation was not perfect we have little doubt but that it was sufficient. In order to throw more light on this point we decided to make a careful determination of the value of R, with the coil immersed in a pure paraffin (pentane), but otherwise under the same conditions as those of the experiment. We selected pentane (1) because it would be difficult to find a more perfect non-conductor; (2) if pure, it would be easy to free the calorimeter from all traces by passing through it a current of warm, dry air. The latter was an important consideration, for any substance such as ordinary impure paraffin might so alter the interior surfaces of the calorimeter as to change the stirring coefficient.

On December 8, 1892, about 250 cub. centims. of pentane* were introduced into the calorimeter, which had previously been dried with extreme care, and the resistance of the coil was taken at twelve different temperatures. A summary of the results is given at the end of this section (see p. 413), and a very slight increase in the resistance is observable, amounting to one part in 22,000. It is satisfactory to find that, whatever may be the effect of conduction through the water, it is, in this case, so small as to be unimportant.

The measurements of R were made when the coil was immersed and the water stirred in the same manner as during the experiments, hence it follows that if the water acted merely as a conductor it would in no way affect our results.

Even if its conductivity changed with the temperature our values of R include the effects of such change, and the manner in which R is built up at each temperature is of no consequence.

Again, if electrolysis takes place (of which, however, we have had no sign), polarization must ensue. The best proof of the absence of polarization is the fact that when the storage and Clark cell circuits were reversed (and this was invariably done more than once during each J experiment) no readjustment of the rheochord was required when using the platinum coil, although when using the manganine coil a considerable readjustment was necessary.

Had electrolysis occurred, it is almost certain that HNO_3 would have been formed. In our earlier experiments (both in the platinum and silver calorimeters), although the same water was never used for two experiments, we found that it became acid—in some cases strongly so.

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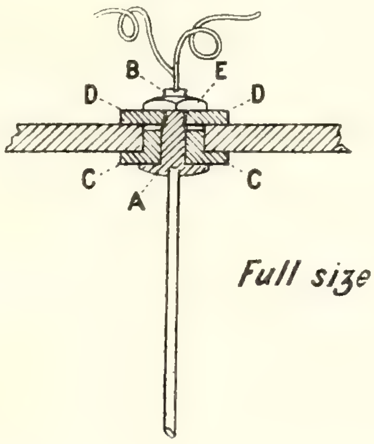
* Great difficulty was experienced in obtaining this specimen of pentane, and it was only through the kindness of Mr. VERNON HARCOURT that we were able to procure it. It was prepared under his own direction and was free from all impurities, except some traces of tetrane and hexane.

This year the water remained unchanged for twenty consecutive experiments and we were unable to detect any acidity, although all water withdrawn from the calorimeter was invariably subjected to a careful examination.

The insulation of the leads passing through the steel cover and the lid of the calorimeter was a troublesome matter, as there was great difficulty in making the insulating junctions absolutely air-tight.

Fig. 3.



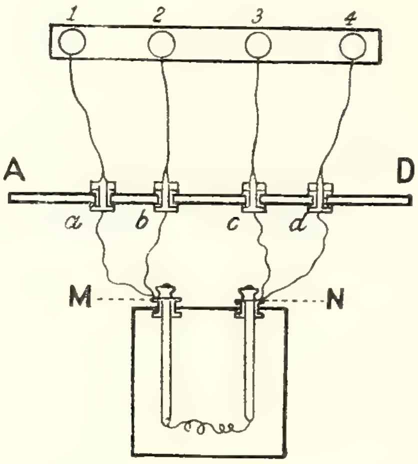
A, *B* was a brass pin, which had a screw thread on the end *B* and terminated in a nut at *A*. *C* was a ring of ebonite fitting tightly into the hole in the metal plate and having a projecting flange at the lower end.

The surface of the ebonite was moistened with gutta-percha solution (made by dissolving gutta-percha in chloroform); the ring, *C*, was then forced into the hole and the brass pin passed through it. A ring of ebonite, *D*, was then threaded on to the pin and a brass nut, *E*, firmly screwed down—the ends of the wire being soldered on at *A* and *B*.

This form of joint proved to be an excellent one, and when once adopted (after the failure of many other contrivances) gave no further trouble. The steel lid was pierced by such junctions in four places, and the calorimeter lid in two. As tap-water was continually flowing over the top of the steel lid it was necessary to insulate the exposed parts at *B*. A piece of wide india-rubber tubing enveloped the wire, the lower end fitted over the ebonite ring *D*, and was wired firmly on to it. The upper end of this tubing projected above the surface of the water in the tank. The tubes themselves were filled with distilled water to prevent any rise in temperature of the leads. The general arrangement is shown in Plate 2, fig. 2.

Leads 1 and 3 (fig. 4, p. 396) were connected with the Clark cells, and thus no current passed along them. 2 and 4 carried the current. Any heat developed in the parts above *AD* passed into the water of the outside tank, and might be disregarded. A portion of the heat developed in the wires *bM* and *dN* would pass to the water in the calorimeter. By sufficiently diminishing the resistance of *bM* and *dN* the heat thus developed could be diminished indefinitely; on the other hand, the conduction of heat between the calorimeter and the steel chamber would be increased. The wire selected was of pure copper (No. 21, B.W.G.), and each arm was about 10 centims. in length. The resistance of $bM + dN$ was carefully determined, and found to be ·0068ω. It is probable that half the heat generated in these wires would find its way to the inner, and half to the outer, vessel. We may, therefore, in order to determine the heat thus added to the calorimeter, assume this external resistance (r) as ·0034ω.

Fig. 4.



Let R be the coil resistance. Then in any time T, the quantity of heat developed in the calorimeter, will be $J.H = C^2(R + r)T$. Now $C = E/R$, and since M and N are maintained at a constant difference of potential, therefore

$$J.H = \frac{E^2}{R}\left(1 + \frac{r}{R}\right)T,$$

therefore the effective value of the resistance $= R - r$ (neglecting terms involving r^2/R and higher powers of r). Since the mean value of R is about 8·7, the correction is, in any case, a small one.

In order to test the insulation an arm of the bridge was constructed, which included wire No. 4, the insulators, and the steel casings—good connection with the steel being insured by passing a wire into the mercury contained in the thermometer-hole. High battery power was used, and the resistance was found to be greater than we could measure—that is, greater than 10 megohms. Since No. 4 was in connection with all the junctions in the plate *AD*, the insulation of all the junctions through the steel was thus established.

The arm was then made to include No. 4, the insulators in the lid of the calorimeter, and a copper wire, which was passed down the glass tubes until in contact with the floor of the calorimeter. The resistance of the two calorimeter junctions was, in the same way, found to be greater than 10 megohms.

Method of finding R.

P_1 and P_2 (see fig. 2, p. 382) were blocks of paraffin. The wires leading from the calorimeter to the junctions on the ebonite rod H passed through the block P_2, down which holes were then bored until the wires were exposed. These exposed portions were carefully amalgamated and the holes filled with mercury. Two thick pieces of copper were so bent that communication could be established between any of the holes in P_1 and the two troughs in P_2. When placed in position, the ends of these connecting strips rested on the conductors at the bottom of the holes. Two thick wires led from the troughs in P_1 to the resistance box, and, in order to bring their resistance up to a convenient quantity, a coil was introduced into one of these leads.

Denoting the total resistance of the conductors from the box to the lid of the calorimeter by r_1, r_2, r_3, r_4, it is obvious that all the following resistances could be taken by a movement of the connectors between P_1 and P_2, viz., $r_1 + r_2$, $r_1 + \mathrm{R} + r_3$, $r_2 + \mathrm{R} + r_4$, $r_3 + r_4$. The determinations were always made in the order thus given.

If N_1, N_2, N_3, and N_4 are the resulting numbers, we have

$$\mathrm{R} = \{(\mathrm{N}_2 + \mathrm{N}_3) - (\mathrm{N}_1 + \mathrm{N}_4)\}/2.$$

The galvanometer used had a resistance of about 8ω, and was fitted with a microscope containing a micrometer eyepiece by ZEISS. The image of a black line placed on a slip of paper in front of the mirror was thrown on to the micrometer scale.

A galvanometer fitted in this way is a very convenient instrument, for the swings can, with practice, be read with greater ease and accuracy than when the ordinary lamp and screen arrangement are used.

When a resistance was taken (whether of the coil or of a platinum thermometer) the order of the observations was as follows:—

Having determined between which plugs the resistance lay, the swing corresponding to the lower one was taken, the battery inverted and the swing in the opposite direction read. This process was repeated when the higher plug was in place. In this way all thermal or other effects are practically eliminated, and the method is independent of small changes in the zero position of the spot.

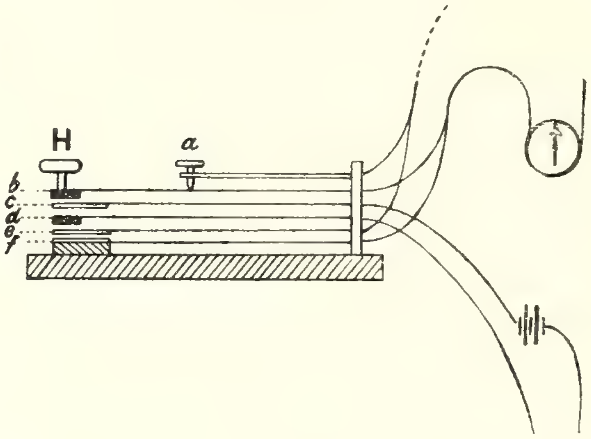
In some cases, as when working with platinum thermometers at high temperatures, the thermal effects are considerable, and the position of the zero point becomes so changed that the spot is thrown off the scale when connection with the galvanometer is made, although the battery circuit is not connected. To meet this difficulty a special key was constructed.

The ends of a and b (fig. 5) and also of e and f are connected with the galvanometer circuit, the ends of c and d with the battery. The screw at a is so arranged that, when the key is untouched, the galvanometer circuit from the box is completed.

The position of the spot at any time, therefore, is that due to the influence of any

E.M.F. in the circuit exterior to the battery circuit. Thus the spot can be placed in a convenient position by the control magnet, although a perceptible current is passing through the galvanometer. When *H* is sharply pressed down the connection at *a* is first broken; then the battery circuit is established by *c* and *d* and the galvanometer circuit by *e* and *f* (the projecting blocks on *b* and *d* being insulators). By proper adjustment of the key, the whole operation, performed by one movement, is accomplished in a small fraction of time. Thus the observed swing is due to the difference of potential caused by the battery and bridge, and is independent of any electrothermal effects.

Fig. 5.



We have described this form of key at some length because, simple as it is, we have not seen any similar arrangement, and we have found it a great convenience, especially when working with platinum thermometers.

As an example of our method of observation, we give one of our determinations of the value of R.* It should be remembered that, although the exterior leads were made as equal as possible, the measurement of R does not depend on their equality and N_1 and N_4; N_2 and N_3 are not necessarily equal. Arms of bridge 1000/10. E.M.F. that of 2 Leclanché cells. When measuring N_1 and N_4 a resistance of 1400ω was thrown into battery circuit, in order to make the total swing a convenient size.

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* A summary of our determinations of R is given at the end of this section (see Table XVI., p. 413); the value of R in the last column but one of Table IV. has yet to be corrected for box errors, &c.

TABLE IV.

	Galv. swings.		Obs. R.	Temp. E_m.*	Temp. box.	R.	R corrected for temp. of box.	Temp. C.
	272	53		311·2	16·65	8·73205	8·73058	19·560
N_1	227	398	$59\frac{45}{390}$					
	45	345						
	333	110						
N_2	183	411	$932\frac{150}{452}$					
	150	301						
	348	124						
N_3	200	426	$932\frac{148}{450}$					
	148	302						
	330	138						
N_4	276	477	$59\frac{54}{393}$					
	54	339						

$$R = \{(N_2 + N_3) - (N_1 + N_4)\}/2.$$

A large number of similar observations were taken at temperatures ranging from 13° to 26° C. Observations at certain temperatures were repeated at regular intervals, and no change of any kind could be detected. The resulting resistance curve was the characteristic platinum one, and in no way differed from those given by our platinum thermometers. This in itself seems to us to be an adequate proof that the insulation of the coil was sufficient.

On September 4, the calorimeter was taken to pieces, and the method of fastening the wire to the glass rods within the calorimeter was altered. The wire was a little strained during the operation; but the only effect was to slightly increase its resistance at all temperatures. A new resistance curve, however, had to be drawn, and new values of R used after that date.

It may be mentioned that this coil was often used by us as a platinum thermometer to determine the temperature of the calorimeter when below, or above, the range of E_m.

The reversing keys in the storage and Clark cell circuit (see fig. 2, p. 332) were of the ordinary "rocker" form, but so constructed that as the wires on one side passed out of the mercury cups, the wires on the other almost immediately made contact; the interval of time between the break and make being less than we could measure.

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* See Section X. The mercury thermometer E_m was used throughout all observations on the temperature of the calorimeter.

If a current be passed through a naked wire, immersed in water, it is certain that the temperature of the wire will be greater than that of the water, even when the latter is rapidly stirred.

Had we been able to use the alloy wire previously referred to this would have been a matter of small importance. Since, however, the value of R, when a platinum wire is used, varies greatly with the temperature of the wire, we felt it necessary to commence a series of investigations into the extent of the change. Hence we proceeded to determine the resistance of the coil when there was a considerable difference of potential at its ends. Our first trial was a modification of POGGENDORFF's method of comparing E.M.F.'s. The storage cells were put in circuit through the coil, and a potentiometer arranged in series. One end of the Clark cell circuit was connected to the storage end of the coil, the other forming a sliding contact on the potentiometer; thus when the galvanometer in this circuit showed no deflection,

$$\frac{\text{E.M.F. of Clark cells}}{\text{E.M.F. of storages}} = \frac{R_1}{R_1 + \rho},$$

when R_1 is the resistance of the coil and part of the potentiometer, and ρ the resistance of the rest of the storage circuit. We had supposed that the internal resistance of the storages might be neglected so that ρ could be determined directly after adjustment by throwing the storages out and completing the circuit by a short length of wire. The ratio of the E.M.F.'s could also be obtained by the usual method through the potentiometer, and thus, by noting the temperature of the coil when adjustment was perfect, we had sufficient data from which to calculate the resistance of the coil. The results obtained by this method were very unsatisfactory, owing probably to the following causes:—

(1.) The heating up of the circuit ρ in the same manner as the coil itself.

(2.) The internal resistance of the storages, although almost zero when no current is passing, increases when the circuit is established, and hence

(3.) The ratio of the E.M.F.'s is not the true one.

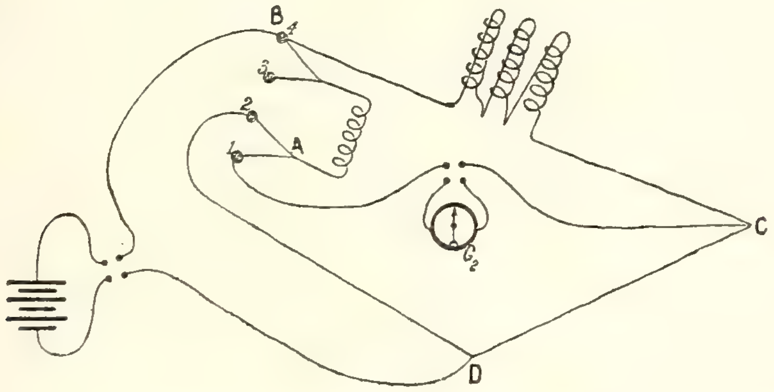
Of these, the first was probably the greatest source of error, although this circuit was entirely composed of naked wires, which had a large cooling surface.

We therefore relinquished the attempt to find the whole resistance of the coil, and devised a method whereby we could find the increment of the resistance due to increasing electromotive force; the resistance when E = ·004 volt having been found at all points of our range as described above (Table IV.). For this purpose a temporary WHEATSTONE's bridge was arranged, constructed as follows:—

The arm *AB* (fig. 6) was formed by the coil alone. *BC* contained three large coils of uncovered wire, the first of which (Cavendish Laboratory coil) was of German silver and had a resistance of 5ω; the second, also of German silver (belonging to the

British Association), had a resistance of $2{\cdot}5\omega$; and the third, of copper, had a resistance of $\cdot 5\omega$.*

Fig. 6.



The arms *AD*, *DC* (about 6 feet each) were made of uncovered copper wire. A preliminary rough adjustment could be made at *D* and the final adjustment of the whole bridge was made by altering the temperature of the coil in the arm *AB*. This alteration in temperature was produced by the heat developed in the wire itself, and thus the whole calorimeter was under the same conditions as those prevailing during our determination of J.

The temperature on thermometer E_m was read at the moment when the galvanometer (G_2) showed no deflection. The behaviour of the galvanometer indicated that the flow of heat from the coil was not quite steady, but took place in gushes, owing probably to the coating of insulating varnish on the wire; for when the key was depressed, the mirror did not swing steadily back to its zero, but oscillated about it, and the moment had to be estimated when the mean of these oscillations coincided with the zero. That this could be done with considerable accuracy is shown by the fact that the individual observations agreed well amongst themselves.

The general conduct of a series of these experiments was as follows.

We first found the temperature of the coil when the bridge was in adjustment, the E.M.F. at the ends of the coil being that used when taking a resistance in the ordinary way. When the calorimeter was cooled down, the E.M.F. was raised and the coil allowed to warm up until a new balance was obtained and the temperature at that moment noted.

As the temperature and consequently the resistance of the arm *BC* remains constant, and since the arms *AD*, *DC* are practically equal, their ratio remains unchanged. The resistance of the arm *AB* (the coil) must also be the same as before, hence the difference in the two observed temperatures gives us the number of degrees that the coil is hotter than the surrounding water; or we can express the result in terms of increase in resistance, as the resistance (when $E = \cdot 004$) at all temperatures is known.

---

* The total mass of metal in this arm of the bridge amounted to several pounds.

Table V.

The following Table gives a series of observations taken on August 22, 1892 :—

Time.	Battery in circuit.	Temp. E_m.	Temp. C.	R when E = ·004.	∂R.
		At which bridge balances.			
5.35	1 Leclanché	344·7			
6.22	,, ...	343·9	20·397	8·7519	
6.47	1 Storage	337·8	20·240	8·7471	·0048
6.55	2 Storages	316·5	19·698	8·7311	·0208
6.58	,, ...	316·7			
7.15	1 Leclanché	343·7	20·395	8·7518	
7.40	3 Storages	277·0	18·684	8·7010	·0507
7.45*	,, ...	279·0 (?)			
7.55	,, ...	276·8			
8.17	1 Leclanché	344·4	20·409	8·7021	
9.9	,, ...	342·5	20·361	8·7508	
9.40	4 Storages	221	17·286	8·6600	·0899
9.50	,, ...	221·2			
10.13	5 Storages	137	15·231	8·5995	·1485
10.22	,, ...	139			
10.44	1 Leclanché	337·9	20·243	8·7472	

When the readings obtained by using 1 Leclanché differ (the change being probably due to changes in the temperature of the bridge), intermediate values are deduced from the times.

It seemed absolutely immaterial whether the current was on for only a few seconds or indefinitely. In the above set, when three storage cells were being used, the current was not broken at all, the calorimeter being cooled down by the ether apparatus[†] in defiance of the heat developed in the coil. Although a quarter of an hour elapsed between the observations, they were in very good agreement.[‡]

To determine the E.M.F. at the ends of the coil, the various batteries used were compared with the Clark cells. This method was therefore open to the sources of error mentioned on p. 400, Nos. (2) and (3). The resistance of the other parts of the bridge were then determined and were as follows :—

Resistance of whole bridge 4·30ω.
,, large coils 7·69ω.
,, coil and its ratio arm 9·17ω. Hence coil = 8·75, arm ·42.
,, large coils and ratio arm 8·06ω. Hence coil = 8·75, arm ·369.
,, wires forming battery circuit ·030ω.

The E.M.F.'s were determined by the ordinary **Poggendorff's** method in terms of the Clark cell at temperature 16°·8 C., hence E.M.F. = 1·434 volts; another set of storage cells being used to give the current through the potentiometer.

---

* This observation was uncertain; the observer at the galvanometer remarking so, before he knew that the reading of the temperature was different.

[†] See p. 419, *infra*.

[‡] This would show that the increase in resistance of the other arms of the bridge was not appreciable.

TABLE VI.

Battery.	Potentiometer readings.		E.M.F.
	For battery.	For Clark cells.	
		No.	
1 Leclanché	32·232 32·225	1 . 33·832	1·366
1 Storage	23·915 23·936	1 . 17·188	1·994
2 "	47·735 47·734	2 . 34·195	4·004
3 "	72·088 72·083	3 . 51·347	6·039
4 "	64·105	4 . 45·602	8·064
5 "	79·844 79·857	5 . 57 029	10·040

To determine the internal resistance of the batteries, the current from them was sent through the coil and balanced in our usual manner with the mercury rheostat (see Section VI.). The temperature being noted, when the galvanometer was steady, the resistance of the whole battery circuit external to the coil, was immediately taken; then if x is the internal resistance of the battery, R the resistance of the coil approximately corrected for the rise due to the current, then,

$$\frac{\text{E.M.F. of Clark cell}}{\text{E.M.F. of battery}} = \frac{\mathrm{R}}{\mathrm{R} + \rho + x},$$

but this ratio was given by the potentiometer comparison, hence we found x, the internal resistance of the storages, = ·018ω, and of the Leclanchés 2·24ω.

Hence we have the following results:—

TABLE VII.

Battery.	E*. At end of coil.	∂R. 1 Leclanché being called 0.	∂R. E = ·004.
1 Leclanché . .	1·035	0	·0016
1 Storage . . .	1·90	·0048	·0064
2 "	3·81	·0208	·0224
3 "	5·74	·0507	·0523
4 "	7·67	·0899	·0915
5 "	9·55	·1485	·1501

---

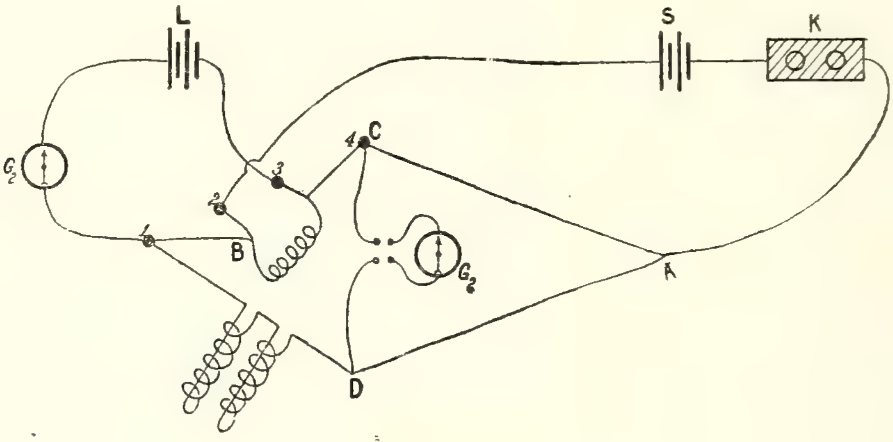
* It is probable that the values of E are too high, for (as before pointed out) these are the values of

These observations will be seen to agree fairly well with the parabola

$$\delta R = \cdot 00160\ E^2.$$

This method, although the results were concordant, was unsatisfactory in the manner in which the difference of potential at the ends of the coil had to be calculated. We therefore adopted the following arrangement (fig. 7), which possessed the great advantage of at once giving us the difference of potential at the ends of the coil in terms of our standards. The Clark cell circuit was connected to the ends 1 and 3 of the coil, and the mercury rheostat introduced into the storage circuit; thus, by adjustment of the rheostat the E.M.F. at the ends of the coil could be made equal to that of 1, 2, 3, &c., Clark cells. No difficulty was experienced in working the two galvanometers together, the observer at the low resistance one being entirely unaware whether the high resistance circuit was made or broken. As it was felt that the whole success of the method depended on the non-heating of the large coils, the two smaller were replaced by one large one of German silver, wound double, so as to expose a very great surface to the air. The resistance of this coil was selected so that when placed in series with the Cavendish large coil,* their united resistance was very nearly equal to that of the calorimeter coil; thus the remaining arms of the bridge were more nearly equal than before, and were made of stout German silver wire, so that each arm was about 5 feet in length and had a resistance of $\cdot 3\omega$. We believe that these results were quite satisfactory, since (1) there was no difficulty in the measurement of E; (2) the mass of metal in the large coils was very great, their cooling surface large (a thermometer placed on the coils never showed any tendency to rise) and their temperature coefficient small; (3) the other arms of the

Fig. 7.



E when the exterior resistance is infinitely great, hence the value of α (in the equation $\delta R = \alpha E^2$) is too low.

* The Cavendish coil was a triple strand one, containing nearly 1400 feet of single German silver wire, the other coil had a double strand and contained more than 400 feet of a stouter wire.

bridge were so nearly equal, and were so arranged, that they would be affected to the same degree by any temperature changes.

The figure shows the connections—

BC is the arm containing the coil.
BD contains the two large German silver coils.
AC, *AD* the two German silver wires forming the remaining arms of the bridge.
G_1, G_2 the high and low resistance galvanometers.
L, *S* the Clark cells and storages.
K the mercury rheostat.

Table VIII.—The following Table gives the observations taken September 20, 1892.

Time.	No. of Clark cells.	Temperature at which bridge is balanced.		R at C° (in Legal ohms).	δR.
		E_m.	C°.		
3.40	1	376·3	21·226	8·7790	x
3.46		376·3			
3.54	2	360·7	20·818	8·7671	$x+$·0119
3.56		360·2			
4. 3		360·4			
4.17	3	332·5	20·100	8·7458	$x+$·0332
4.21		332			
4.25		332·1			
4.33	4	292·5	19·080	8·7157	$x+$·0633
4.36		292·8			
4.38		292·3			
4.50	5	240	17·760	8·6771	$x+$·1019
4.59		240·5			
5.50	6	177·5	16·202	8·6316	$x+$·1474
6. 0		176·8			
7.12	1	379·4	21·306	8·7814	x
7.18	2	363·4	20·895	8·7692	$x+$·0122
7.26	3	333·4	20·130	8·7468	$x+$·0346
7.30		333·6			
7.45	4	292	19·067	8·7156	$x+$·0658
8. 0	5	240·4	17·765	8·6772	$x+$·1042

Of the second series of experiments, the first was taken immediately after entering the laboratory, when the arms *AC*, *AD* (being very near one observer, and the gas having been lighted) had not probably reached a state of thermal equilibrium, but they evidently did so during the series.

TABLE IX.*

The following numbers were plotted:—		δR (Legal ohms.)	δR deduced from $\delta R = \cdot00422n^2$.
No. of cells.	Increase.		
0	$-x$	0	0
1	0	·0042	·0042
2	·0120	·0163	·0168
3	·0333	·0376	·0378
4	·0638	·0681	·0675
5	·1023	·1066	·1055
6	·1478	·1516	·1519

Hence $x = \cdot0042$,

and the above agree within the limits of experimental error with the parabola

$$\delta R = \cdot00422 \times n^2;$$ where n is number of CLARK's cells.

In this case there can be no doubt about the actual D.P. at the ends of the coil. The results of the previous investigation (p. 404) are of great importance, in so far as they confirm the conclusion that $\delta R = \alpha E^2$, but are of little use for determining the value of α.

In order to complete these experiments variations in the rate of stirring were tried. It was found, as would be expected, that the more perfect the stirring, the less the wire became heated; but, within the limits of our rate of stirring, the change was so slight that no correction was thought necessary.

The observer at the high-resistance galvanometer could, however, always detect minute changes in the rate of stirring by the irregular behaviour of the spot. These changes, although thus rendered very evident, had an exceedingly small effect and thus they served as a proof that the oscillations observed during our J experiments also indicated variations too minute to affect our measurements.

[NOTE.—As illustrating the importance, as also the accuracy, of the correction rendered necessary by the difference between the temperature of the wire and of the water, we here give a summary of the results of J 9 and J 34. These two experiments were performed in order to subject the corrections (both for radiation, &c., and for the difference in the temperature of the wire) to a severe test, the heat developed per second in the wire during J 34 being nine times as great as that developed per second throughout J 9.

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* In comparing the last two columns, it must be remembered that a difference of ·0010 corresponds to a difference of but 1 in 8600 in R.

	J 9.	J 34.
E.M.F. (expressed in terms of a Clark cell at 15°)	2	6
Weight of water + water equivalent = (M) = ...	273·77 grms.	345·20 grms.
Mean rise per 1″ (after correction for stirring, radiation, &c.) = γ = ...	$8225° \times 10^{-7}$	$5778° \times 10^{-6}$
Mean temperature	19°·540 C.	19°·540 C.
Hence, thermal grms. generated per 1″, when E is that due to 1 Clark cell = $\left(\gamma \times \frac{M}{E^2}\right)$ = ...	56295×10^{-6}	55404×10^{-6}
Now, resistance of wire at 19·540, as determined by box in the ordinary manner = (R_0) = ...	8·7084	8·7122*
Hence, thermal grms. per 1″ if R = 1ω; $\left\{\left(\gamma \times \frac{M}{E^2}\right) R_0\right\}$ = .	49026×10^{-5}	48270×10^{-5}
These quantities (which ought to be equal, and from which we deduce the value of J) here differ by nearly 1 part in 60.		
If, however, we deduce the values of R_{2E} and R_{6E} from Table IX., we get 8·7084 + ·0168 and 8·7122 + ·1519	8·7252	8·8641
Hence $\left(\gamma \times \frac{M}{E^2}\right) R_1$ = ...	49111×10^{-5}	49110×10^{-5}

We did not see how to apply a more severe test, and we, therefore, considered it unnecessary to carry the investigation further, although we had supposed that a repetition would be necessary.]

Comparison of our Resistance Coils with the Standards.

The resistance box was taken to the Cavendish Laboratory on October 31, 1892, to be compared with the standard coils in possession of the British Association.

In order to be able to standardize each coil separately, two plugs were made, half ivory and half brass, so that good contact was secured with the outer circle of the dials through the brass, but no contact was possible with the inner circle through the ivory. The insulation of these plugs was, however, defective, the shunt so formed being sufficient to bring the 10 legal ohms down to 10 B.A. ohms.

Two brass bridges were then made which rested over the outer circles of the dials, one end of each bridge fitted into the plug-holes, but was cut away so as not to touch the central circle. By tightening up a screw in the other end of the bridge a good and regular contact was secured. To the centre of each bridge a binding screw was fixed, and connection was made by thick wire leads from these to the Carey-Foster bridge.

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* The resistance of the coil had been altered between these two experiments (see p. 411).

In the comparison of the units and tens, the Carey-Foster bridge belonging to the B.A. was used, and the difference between the standard and the box-unit was found in terms of a length of bridge-wire. In the comparison of the 100's and 1000's, the bridge described by GLAZEBROOK ('B.A. Report,' 1883) was always used.*

The standard coils used, and their values, are given in the following Table:—

TABLE X.

Coil.	Value.
ELLIOTT 269	1 + ·000282 (t − 15·1)
„ 270	1 + ·000286 (t − 15·3)
„ 288	10 + ·00265 (t − 16·2)
„ 289	10 + ·00260 (t − 15·4)
NALDER 3634	100 + ·0308 (t − 17·2)
„ 3637	100 + ·0308 (t − 17·05)
1000 *b.w.d.*	·05000 + ·0000715 (t − 18·2)

The same thermometer (HICKS, No. 352,601) that had been employed in all our determinations was again used in the standardization, and, since its bore was very nearly uniform, no correction was necessary. The B.A. thermometer (K.O., 75,149) was used for the temperatures of the standards; its correction was − ·10 at the temperatures of our observations.

The resistance of the connections between the box and bridge was taken in terms of the bridge-wire of the Carey-Foster bridge.

The results of observations on different days agree well, the average departure from the mean being less than $\frac{2}{1000000}$.

The differences could, in most cases, be easily accounted for by small temperature changes in the interior of the coils, which could not be indicated by the thermometer.

The accompanying Table gives the results.

(Although the coils in the 1000 dials were not used during the determinations of the resistance of the calorimeter coil, we have included them in this Table, since many of them were required when ascertaining the platinum temperatures, and also because a complete standardization of this kind is interesting, as it shows the order of accuracy of boxes of this pattern.)

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* In comparing the 100 and 1000 ohms coils, two Leclanché cells were used. The currents in the battery circuit were ·090 and ·023 ampère respectively.

TABLE XI.

Coil.	Unit dial.			Ten dial.			Hundred dial.					Thousand dial.				
	Nov. 16.	Nov. 17.	Mean.	Nov. 7.	Nov. 9.	Mean.	Nov. 10.	Nov. 10.	Nov. 11.	Nov. 14.	Mean.	Nov. 12.	Nov. 13.	Nov. 13.	Nov. 14.	Mean.
1	..	·99523	·99523	9·9764	9·9759	9·9762	99·796	99·793	99·801	..	99·797	998·31	..	998·40	..	998·35
2	·99888	·99894	·99891	9·9826	9·9826	9·9826	99·779	99·779	99·784	..	99·780	997·71	..	997·80	..	997·75
3	·99687	·99693	99690	9·9708	9·9709	9·9708	99·798	99·797	99·802	..	99·799	997·96	..	998 07	..	998·01
4	·99595	·99600	·99597	9·9792	9·9789	9·9791	99·784	99·783	99·790	99·789	99·787	997·69	..	997·78	..	997·75
5	·99793	·99808	·99800	9 9748	9·9747	9·9747	99·803	99·805	99·811	99·811	99·809	997·97	..	998·10	..	998·04
6	·99654	·99661	·99658	9·9774	9·9772	9·9773	99·725	99·721	99·736	..	99·727	997·94	..	998·01	..	997·98
7	·99820	·99822	·99821	8·9738	9·9735	9·9736	99·862	99·859	99·870	..	99·864	..	998·12	998·10	..	998·11
8	·99952	·99959	·99956	9·9717	9·9711	9·9714	99·823	99·823	99·830	..	99·825	..	998·38	998·38	..	998·38
9	·99714	·99723	·99719	9·9713	9·9711	9·9712	99·780	90·780	99·792	99·786	99·784	..	997·91	997·90	..	997·90
Bridge R	..	..	..	9·9811	9·9808	9·9810	..	..	99 841	..	99·841	997·79	997·95	997·92	997·82	997·87
,, L	..	..	..	9·9733	9·9735	9·9734	..	..	92·815	..	99·815	997·81	997·91	997·89	997·78	997·85

NOTE.—The box is marked by Messrs. ELLIOTT as "Legal ohms at 17° C.," hence the value of each coil, in terms of a True ohm, should be ·99757. The mean value of all the coils is ·99801, at a true temperature of 16°·92 C., or 17° by its own thermometer.

As the terminals were used for testing at the time the box was constructed, the resistance of the connections through the dials was then taken. This is sufficient to account for the apparently low value of the first unit, as the resistance of the box with all the plugs at 0 is ·00334. In testing the arms of the bridge, the centre and end terminals were used in the same manner as when the box itself formed the bridge. so that no plug resistance correction had to be applied.

A table was constructed, giving the total difference between the reading and the real value (in terms of legal ohms) for every position of the plugs in each dial. This difference we termed the "plug correction."* Having made this correction, we then had to correct for any inaccuracy in the ratio of the arms of the bridge, and as all determinations of the calorimeter coil resistance were made with the 1000 (right) and 10 (left) as the arms, it is here only necessary to give the correction for these coils.

Now $10L/1000R = 10{\cdot}0077/1000{\cdot}30 = {\cdot}0100047$. This we termed the "bridge correction."

The resulting values are expressed in legal ohms, and true ohms = reading in legal ohms $(1 - {\cdot}0024275)$.

By applying the corrections in the above form it was possible to perform all the operations by means of the slide rule,† as it was only necessary to determine the quantity to be added or subtracted in one case to three, and in the other to four figures.

(When working with platinum thermometers the last two corrections are omitted, as it is only necessary that the resistances should be expressed in terms of the same unit, the absolute value of that unit being unimportant).

Table XII. gives the values of R, when the calorimeter coil was in water, stirred in the same way as during a "J" determination. Each number in the column R was obtained in the manner given in full on p. 399, and includes the correction for temperature. The "plug correction" cannot be deduced from column R, because the difference between the correction on $R + r$ and r has to be used in each case. The plug correction however is, in every case, about the same as that required by the values of $R + r$ and r as given in Table IV.

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* The correction for the temperature of the box was made before applying the "plug correction." The values of the temperature coefficient for each dial had been previously determined by Messrs. ELLIOTT, and the accuracy of the coefficients has been exposed to severe tests.

As the temperature of the box was controlled by a special regulator, it never differed greatly from 17° C. (See 'Phil. Trans.,' vol. 182, A, p. 45.)

† FULLER's spiral slide rule gives results correct to 1 in 20,000 or 30,000.

TABLE XII.

Date.	C°.	R (at 17°).	R corrected for "plugs."	R corrected for bridge.	True ohms.
Aug. 6 . . .	18·701	8·70152	8·70514	8·70923	8·68809
" 6 . . .	13·133	8·53685	8·54041	8·54442	8·52368
" 10 . . .	25·217	8·89343	8·89701	8·90119	8·87958
" 17 . . .	19·231	8·71690	8·72046	8·72456	8·70338
" 24 . . .	20·337	8·74906	8·75260	8·75671	8·73545
" 31 . . .	17·066	8·65267	8·65627	8·66034	8·63932
Sept. 4 . . .	15·325	8·60104	8·60459	8·60863	8·58773

Between September 4 and 11 the coil was readjusted, and was probably somewhat strained in the process. The value of R was slightly increased.

TABLE XIII.

Date.	C°.	R (at 17°).	R corrected for "plugs."	R corrected for bridge.	True ohms.
Sept. 11 . . .	19·560	8·73058	8·73416	8·73827	8·71706
" 11 . . .	25·522	8·90563	8·90923	8·91342	8·89178
" 11 . . .	13·280	8·54471	8·54826	8·55228	8·53152
" 11 . . .	15·086	8·59816	8·60171	8·60575	8·58486
" 11 . . .	22·154	8·80618	8·80975	8·81389	8·79250
" 11 . . .	23·274	8·83966	8·84326	8·84742	8·82594
" 14 . . .	16·952	8·65268	8·65628	8·66035	8·63933
" 19 . . .	18·697	8·70415	8·70777	8·71186	8·69072

The sixth figure is, of course, unnecessary. The numbers in the first and last columns were plotted on a very large scale, so that the fifth figure could be read distinctly. The resulting curves differ in no respect from those given by platinum thermometers. The difference between the curve reading (at *any* of the above temperatures) and the corresponding value in the table in no case exceeds ·0005 (*i.e.*, 1 in 17,000).

The values of R after September 4 exceed the values before that date (when the temperature is the same) by ·0038.

The values of R (true ohms) at other temperatures than those in the above tables were obtained from the curves.*

---

* In order to trace the changes in the specific heat of water throughout our range, we had decided to find the time of rising 1° at each of the following temperatures, viz., 14°, 15°, 16°, 18°, 20°, 22°, 24°, and 25° C. It would have been difficult, if not impossible, to arrange our actual observations throughout our experiments so that they were exactly grouped about these points. The value of each millim. of our thermometer stem varied as the temperature changed, and the above temperatures in no case corresponded with any integral number. The values of our variables at these temperatures had to be obtained by

As mentioned (p. 394) the resistance of the coil when immersed in pentane was ascertained. The observations were taken in the same manner and with the same rate of stirring as those whose results are given in the above tables.

TABLE XIV.

C°.	Determination of R in pentane.	
	R at (17°).	True ohms (after "plug" and bridge corrections).
13·632	8·55513	8·54195
14·010	8·56516	8·55197
14·982	8·59544	8·58223
16·202	8·62941	8 61612
17·161	8·65892	8·64557
18·214	8·69058	8·67699
19·164	8·71857	8·70506
20·502	8·75805	8·74443
21·702	8·79354	8·77989
23·001	8·83186	8·81812
24·699	8·88218	8·86836
24·699	8·88204	8·86822
24·713	8·88275	8·86891

The results must be compared with those in Table XIII., since the observations were made after the change in the coil.

In order to find what difference, if any, was caused by the change from water to pentane, the values of R in water at the above temperatures were taken from the curve plotted from Table XIII.

large scale curves. No purpose would be served by reproducing any of these curves in this paper, because if drawn on a small scale they would be useless and misleading; for they would give an exaggerated idea of the accuracy of our results—the discrepancies being imperceptible. In every case we give data by which the curves could be reproduced, and where any marked departure is shown by an individual observation we mention it.

TABLE XV.

C°.	R (pentane).	R (water).	Difference.*
13·632	8·5420	8·5419	+·0001
14·010	8·5520 ?	8·5528	−·0008 ?
14·982	8·5822	8·5818	+·0002
16·202	8·6161	8·6148	+·0013
17·161	8·6456	8·6456	+·0000
18·214	8·6770	8·6766	+·0004
19·164	8·7051	8·7046	+·0005
20·502	8·7444	8·7444	+·0000
21·702	8·7799	8·7794	+·0005
23·001	8 8181	8·8176	+·0005
24·699	8·8683	8·8676	+·0007
24·713	8·8689	8·8680	+·0009

Mean difference = ·0004,

showing a rise of about $\frac{1}{22000}$ when in pentane.

The specific heat of pentane is so small that it is possible that the wire would be at a slightly higher temperature than when in water, as it would not be robbed so rapidly of the heat supplied by the current, and thus some portion of the small rise observable might be due to this cause.

The following tables give the resistances used by us for the reduction of our "J" experiments :—

R_0 denotes the resistance when the current is so small as to produce no visible change in R.

($R_0 = R$, as deduced from Table XIII., diminished by ·0034, see p. 396.)

R_E, R_{2E}, &c., the resistance when the difference of potential at the ends of the coil is that due to 1, 2, &c., Clark cells.

($R_{NE} = R_0 + \delta R_0$, where $\delta R_0 = {}$·00421 × n^2 true ohms, see p. 406.)

TABLE XVI.—Series II. Values of R. For Series I. subtract ·0038.

C°.	R_0.	R_{2E}.	R_{3E}.	R_{4E}.	R_{5E}.	R_{6E}.
14·477	8·5626	8·5794	8·6012	8·6307	8·6686	8·7149
15·581	8·5959	8·6127	8·6337	8·6632	8·7011	8·7474
16·682	8·6282	8·6450	8·6660	8·6955	8·7334	8·7797
17·683	8·6578	8·6746	8·6956	8·7251	8·7630	8·8093
18·688	8·6876	8·7044	8·7254	8·7549	8 7928	8·8391
19·835	8·7214	8·7382	8·7592	8·7887	8·8266	8·8729
21·115	8·7592	8·7760	8·7970	8·8265	8·8644	8·9107
22·409	8·7972	8·8140	8·8350	8·8645	8·9024	8·9487
23·862	8·8399	8·8567	8·8777	8·9072	8·9451	8·9914
25·006	8·8735	8·8903	8·9113	8·9408	8·9788	9·0250

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* The differences in the last column show a slight tendency to increase as the temperature rises, indicating a decrease in the resistance of water.

SECTION VIII.—THE MEASUREMENT OF TIME.

As previously mentioned, an electrical clock (made by Mr. THOMAS) was used as our standard. The energy required was supplied by two Leclanché cells. The pendulum (a seconds one) was very heavy, weighing about $\frac{1}{4}$ cwt. Its rod was of wood, and we were unable to detect that any appreciable change in rate was caused by the small changes of temperature to which it was exposed. The working parts of an electrical clock of this kind are so simple that, when once standardized, it is a very reliable instrument. The pendulum was fitted with a rough and a fine adjustment, and comparisons were made at regular intervals with a ("rated") DENT's chronometer. Until August 21, 1892, the rate of gain was − 1 in 2424; after that date the rate of gain was certainly less than 1 in 25,000, and such variations as were observed may have been due to temperature changes, as they were irregular and, in any case, so small as to render corrections unnecessary. All observations of time made before the above date (August 21) were corrected for clock error, before they were used in the calculations. The actual uncorrected observation is, however, the one given in all columns headed "time."

The chronograph, which was worked by a separate battery, was of a somewhat peculiar construction. It was contrived so that on "making" the circuit connecting it with the clock, the tape was at once started, and, on breaking that circuit, the tape came to rest. Thus the observer could, without moving his position, start or stop the chronograph by pressing a key.*

We rarely, however, used this contrivance, because, as our observations lasted over considerable intervals of time, the instrument had to be continuously at work. The regulator was so adjusted that the tape passed through about $\frac{3}{4}$ of an inch per second —the fractions of a second being rapidly read off by means of glass scales. The labour of numbering the seconds was great, for the instrument was not so contrived that it could make a distinguishing mark every tenth second.†

Immediately before the observer were placed two keys connected with the chronograph, one of which made permanent connection with the clock pendulum, the other was a spring key for recording the times of passing certain temperatures, or the time of each 1000 revolutions of the stirrer.

Some preliminary observations indicated that the personal error (*i.e.*, the difference between the actual time and those read off from the tape) amounted to about $\frac{3}{10}$ of a second. As, however, all times were recorded by the same observer, this constant "lag" was of no consequence.

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* It was found necessary to place the chronograph at a considerable distance from the galvanometer.

† The Table XIX., given in Section XII., shows that one experiment alone lasted more than 10,000 seconds.

SECTION IX.—THE MEASUREMENT OF MASS.

ROWLAND, in the paper to which reference has already been made, takes objection to WEBER'S determination of "J," on the ground that he used only 250 grms. of water. Also M. ARNOUX in the discussion which followed D'ARSONVAL'S paper said, "Its determination has always been carried out in diminutive apparatus . . . in which the losses by radiation attained a high figure. With large apparatus the loss by radiation becomes negligible since the volume increases as the cube and its radiating surface as the square only of the linear dimensions. With appliances capable of transforming into heat a few hundred kilogrammetres per second, we ought to succeed in determining the mechanical equivalent to three or four (*sic*) places."* Now as the quantity of water used by us was small, it will be well to point out the considerations by which we were influenced.

Besides the mass of water, which is used in the calorimeter, account must be taken of what is usually termed the "water equivalent" of the calorimeter. [We take this opportunity of remarking that when we, in subsequent statements, use the word "equivalent" we refer to the water equivalent of our calorimeter, and do not use the word in the sense in which it occurs in ROWLAND'S paper as meaning "the mechanical equivalent;" this we have designated by the letter "J."] Now it is the very difficulty of satisfactorily determining this equivalent, rather than the estimation of loss by radiation, which has led experimenters to prefer a large mass of water, when practicable, in order to make any error in the equivalent one of the second order.

Up to the time of writing, only two methods seem to have been generally adopted for determining the equivalent, (1) the method of mixtures, (2) the method by calculations from the specific heats and masses of the component parts of the calorimeter. Of the first of these we need say but little, for JOULE himself obtained values which varied amongst themselves by 12 per cent.† We did not, however, dismiss this method without trial, for having applied it, with every conceivable precaution, to one of our calorimeters, we obtained numbers whose differences were so great as to render them useless for the purposes of this investigation.

The second method is attractive on account of its simplicity, but since the specific heats have probably been determined by the application of the first method,‡ the accuracy is not of a superior order. Of greater weight is the consideration that a calorimeter has no definite limits, it being impossible to foretell to what extent and in what manner the necessary supports, connections, and surrounding envelopes have to be introduced. The method which we have adopted of making the experiments themselves give us the value of the equivalent appears, to us, to be the only way in which this difficulty can be successfully overcome, and, as our results will show, the equivalent was determined with the same accuracy as the other quantities.*

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* ROWLAND'S calorimeter, when full, weighed over 12 kilos., JOULE'S about 6 kilos.

† See Table, p. 639, 'JOULE'S Scientific Papers,' vol. 1.

‡ We may quote from Prof. EVERETT'S 'Units and Physical Constants,' "Specific Heat of Liquids." "We have omitted decimal figures after the fourth, as even the second figure is different with different observers." To this we may add that the specific heat of solids will bear no closer scrutiny.

There does not appear to be any argument in favour of using a small mass of water when the value of "J" is to be determined according to JOULE's original method by the friction of water; but the complexion of the case is completely changed if the mode of investigation is altered, for in each different method the accuracy of the result will be some function of the variables at our disposal, and, since we cannot apply the calculus, these ought to be so selected from general considerations, as to make that accuracy a maximum. In the method which we have adopted the variables are the mass of water, the resistance of the coil, the difference of potential at its ends, and, dependent upon these, is the rate of rise in temperature.

Now the most satisfactory method of ascertaining the rate of rise in temperature, consists in timing the transit of the mercury column across the spider-wire of a reading telescope. This wire, being set along the graduations of the thermometer, several successive transits are recorded at different parts of the range and the mean time is taken at each place. Although the rate of rise ought to be fairly great, in order to diminish the importance of radiation losses, if the rate is too rapid the observer has not sufficient time to set his cross-wire to the successive graduations, and only one observation can be recorded at each part of the range. The confusion which would be introduced by attempting to observe every other scale division need scarcely be mentioned. It was only by constant practice that one of the observers was able to take observations with accuracy at the rate we adopted.

If, on the other hand, the advance of the column is too slow, the time of apparent contact with the telescope wire is uncertain and, at the same time, the phenomenon of "sticking" is rendered more probable. As during these experiments the strain on the observer was very great, mere personal fatigue would be sufficient to prevent accurate observations being taken if the duration of the experiment was extended beyond an hour. In the case of the experiments on the heat developed by the stirrer, it will be seen by reference to the tables that these sometimes lasted three or four hours; but the nature of these experiments not being so complicated, the strain was less and the observer had short intervals of rest.

The rate of rise in temperature thus practically determines the values which we must assign to the other quantities at our disposal, for to make the rate great we must either diminish the mass or resistance, or else increase the difference of potential. The considerations which led us to give to E and R the limiting values assigned to hem during these experiments have already been mentioned on p. 389. We were hus restricted to masses of water varying from 130 to 300 grms.

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* The value of J can be obtained without ascertaining the value of the water equivalent (see p. 481), and the determination of the changes in the capacity for heat of water is entirely independent of it. Our conclusions are based on observations of differences in time caused by differences in the mass of water, and we see no reason why such differences could be ascertained with greater accuracy if the water equivalent was diminished. The total times would be decreased, but the differences would remain unaltered.

As the determination of the equivalent and the change in capacity of the whole ass due to increase of specific heat are fully discussed in another section, it is only ecessary here to deal with the mass of water added to the calorimeter. The balance sed by us was by **Becker**, carrying 500 grms. and turning with a milligramme. As his type of balance is not provided with a rider-arm, it was found convenient to add eights to the nearest 5 milligrms. above and below, and to interpolate by the ethod of oscillations. Nearly the whole of the measurements were conducted by he ordinary method of direct weighing, as the balance, when once in good adjustment, emained so. The adjustment was however, occasionally checked by the method of ouble weighing.

Holes were bored through the bottom of the balance case and the top of the cupboard n which it stood, and wires suspended from the pans for use when weighing flasks, &c. s it was necessary to open this cupboard whenever a flask had to be weighed, it would ave been difficult to keep the air within the cupboard in a state of dessication, and hus if proper precautions had not been taken a considerable error might have been ntroduced in the weighing of glass vessels of large area, owing to the hygroscopic ature of their surfaces. The usual simple expedient of taring one glass vessel against nother of approximately equal surface appeared eminently satisfactory, as the weighings could be repeated at varying intervals of time (generally a few hours, but in ne test case four days) and the results were always found to agree to a milligramme.

In our earlier series of experiments (1887–1890) the calorimeter could be rapidly taken to pieces, dried, any required mass of water added by means of a fine dropping-pipette, weighed, and put together again. As, however, those experiments had shown us the necessity of keeping the outer space perfectly dry, we now decided to reduce, if possible, the pressure to less than 1 millim., since, in this case, radiation, &c., would be considerably diminished, and any moisture present would immediately be detected.

It thus became impossible to take the calorimeter to pieces, and means had to be devised for introducing and withdrawing water through the supporting glass tubes.

The impossibility of removing the calorimeter without destroying our vacuum led to another difficulty, for, at the end of the experiment, the water had been heated to the highest point of our range, and before we could conduct a second experiment we had either to cool the whole apparatus down, which would have been an extremely slow process, owing to the smallness of the radiation, and would also have rendered it necessary to throw the regulator of the outer bath entirely out of adjustment; or else withdraw part of, or all, the water in the calorimeter, adding sufficient cold water to bring the temperature down below the point at which our observations commenced. During the earlier part of our 1891 series we withdrew a portion only of the water by means of a modified form of weighed pipette, and then, by the same means, added a weighed quantity of cold water. This method was in some respects satisfactory, except that every withdrawal involved four weighings, so that, after a short time, the mass of water in the calorimeter was dependent upon a very great number of weighings, and small errors in each, if cumulative, might have become serious. We, therefore, soon abandoned this as a general method, and only resorted to it occasionally. Since then we have always used the following method of adding a known mass of water to the calorimeter :—

Pairs of glass flasks (with india-rubber stoppers), of 100, 200, and 300 cub. centims. capacity, were selected and carefully cleaned. One of each pair was kept as a tare for its fellow. In introducing the water into the calorimeter it was necessary to avoid loss either by accident or by evaporation, and to guard against losses of this kind the following device was employed :—A thin glass tube, 30 centims. in length, had a wider piece, about 7 centims. long and 2 centims. in diameter, fused to its upper end, thus forming a funnel. Up the centre of this wide portion a piece of glass rod was fixed, its upper extremity projecting above the top of the wider tube, so that the lip of the flask, from which water was poured, could rest against it, and thus prevent any drops running down the outside of the flask or tube. By forming a slight support, it also somewhat relieved the difficulty of the operation, which, even with its aid, was a trying one. A cork was fixed to the stem of the funnel, in such a manner as to permit its lower end to dip into the calorimeter. This funnel was provided with a case, consisting of a length of glass tube closed permanently at the lower, and temporarily at the upper, end by an india-rubber stopper. A piece of similar tubing, closed at both ends, was used as a tare. The full flask and funnel having been weighed, the funnel was withdrawn from its case, placed in one of the tubes leading to the calorimeter, and the water poured down it from the flask. This is the only part of the process during which evaporation could possibly take place, and any such loss, during the few seconds the operation lasted, would be a matter of no consequence. The empty flask was immediately re-stoppered, the funnel withdrawn and returned to its case, and the weight ascertained at any convenient time. This method also enabled us to add water below the dew-point of the room without any error due to moisture condensed on the flask and funnel, for it was only necessary to first weigh the flask, then cool it, and pour its contents into the calorimeter. The exterior of the funnel was dried before returning it to its case, and the stoppered flask placed in a dessicator for some time before the final weighing. This method appeared to be satisfactory, and, although it was impossible to add exactly the same amount on different occasions, still, as the difference was only the fraction of a gramme, it was easy to apply any necessary correction to the resulting numbers. A glass tube, reaching to the bottom of the calorimeter, was connected with an exhaust pump, which enabled us to withdraw nearly all the water, and the remaining moisture was removed by passing a current of hot dry air through the calorimeter for several hours.

Owing to some difficulty with regard to the air-tight joints, we were obliged to take the calorimeter to pieces at the beginning of July, 1892, and to our astonishment we found, although we had, as we thought, dried completely, there still remained several cub. centims. of water. We subsequently ascertained that the calorimeter could be properly dried in the above manner, provided that sufficient time was allowed; but the hot air had to pass for several days before a drying tube, suspended within the calorimeter for 12 hours, showed no increase in weight. Now, as the first method involved too many weighings, and the second was unsafe, unless two or three days elapsed between each experiment, we were compelled to seek a different mode of procedure. Thus, as we could neither withdraw any portion of the water, nor cool the whole calorimeter down without great labour, means had to be devised for cooling it from within. A tube, through which a freezing mixture could be forced, was evidently objectionable, as there would always be uncertainty as to whether all the freezing mixture had been removed. We found the following arrangement worked admirably, for the temperature of the calorimeter fell as rapidly when cooling, as it rose when the current was passing. We were thus able to use the same water again and so save a numerical reduction.

A very thin glass tube, closed at its lower end, passed down the support tube *F* (Plate 2, fig. 1) of the calorimeter, just clearing the base, and the upper end, which was open, projected about $\frac{1}{2}$ inch above the top of the tube *F*. Care was taken to so fix this tube that no water could find its way up the annular space between the two tubes. When not in use, the upper end was closed by a cork, so as to prevent air-currents circulating down to the calorimeter. When used for cooling, this tube was partially filled with ether, through which a rapid current of *dried* air was sucked by means of the arrangement shown in Plate 2, fig. 1. The fine capillary point reached just to the bottom of the closed tube, so that, when the cooling had proceeded far enough, the remaining ether could be withdrawn by reversing the connections to the exhaust pump and the H_2SO_4 drying bottles. Any ether or impurity left at the bottom of this tube would have caused great irregularities in the experiments, so, as a precaution, only freshly distilled ether was used,* and when the temperature had been sufficiently lowered, *dry* air was passed for a considerable time and, finally, the cooling tube thoroughly cleaned by means of an absorbent mop. The joint at *G* was made by slipping a small ring of india-rubber tubing round the projecting end of the cooling tube, and, by slightly splaying out the mouth of the aspirating portion, a sufficiently good union was effected, the extreme thinness of the walls rendering a ground glass joint impracticable. The entrance tube *K* was used for adding fresh ether as occasion required.

As soon as we perceived that this method of cooling was a success, it was at once evident that it could be applied to finding the latent heat of evaporation of liquids, for by regulating the rate of evaporation we could make the cooling effect such as to counterbalance the heat developed by a known current. We have already made a few rough determinations, but prefer to postpone their publication until we have applied the method with greater precautions.

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* The ether was always redistilled by us immediately before use—at a temperature under 40° C.

SECTION X.—THE MEASUREMENT OF TEMPERATURE.

When commencing this account of our experiments, we proposed to give full details of the calibration and standardization of the mercury thermometer E_m, upon which all our measurements depend. Nearly half of our time and attention has been devoted to this part of the subject, and there are but two courses open to us, viz., to give a full account of our method of standardization, with the tables upon which our conclusions are based; or to reserve a detailed description for a further communication, and give a short summary of the results in the present paper. We have decided to adopt the latter course, for the following reasons:—

(1.) We are (with the help of Mr. CALLENDAR) now entering on a careful direct comparison of thermometer E_m with a new form of air thermometer, which, there is every reason to believe, will give very accurate results; but we are unable to assign any definite limit to the time that this investigation may take.

(2.) Dr. GUILLAUME is now* engaged in the comparison with the Paris standard of a thermometer specially constructed for us by M. TONNELOT, and some time must elapse before this comparison can be completed.

(3.) The dimensions of this communication are already very great, and if, in addition, we gave a full account of our thermometric comparisons, its length would become immoderate.

(4.) We consider that the success of the method of exact determination of temperature by platinum thermometers may now be regarded as established, and that the accuracy of the results obtained by platinum thermometers in our possession is proved by communications already published.†

(5.) The results of our J experiments may themselves be taken as a proof of the accuracy of the method of time-calibration adopted by us, and described on pp. 423 and 424, *infra*.

Assuming the accuracy of the calibration it is possible we are in error—

(1.) In the actual elevation;

(2.) In the value of our total range.

No. (1) may be disregarded, for it is only the changes in the specific heat of water which render the actual elevation of any consequence whatever. Now it is impossible that we are in error by 0°·2 C., for the comparisons we have made with the B.A. thermometers, which have been carefully standardized at Kew, would alone have been sufficient to detect any error of that kind, and an error of 0°·2 C. in elevation would mean an alteration of only 1 in 20,000 in the value of J.

No. (2) is, of course, of vital importance.

In 'Phil. Trans.,' vol. 182, A, p. 155, will be found the results of a comparison of one of our platinum thermometers (H) and the air thermometer over a range of 0° to 100°, and the numbers there given show that the probable limit of error at *any* point in the above range is 0°·01 C. This limit may be further diminished by the "plug corrections" (see p. 410), then neglected by us. It must be remembered that this limit of error refers to the actual elevation, and is probably decreased when we come to consider differences over a range of 11° C. Further, the table there given is a record of only a single series of observations, and must include experimental errors, which would tend to mean out if sufficient observations were taken.

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* December, 1892.

† 'B.A. Report,' 1890; 'Phil. Trans.,' vol. 182, A, pp. 43–72 and 119–157; 'Phil. Mag.,' December, 1891; 'Chem. Soc. Journ.,' 1890, p. 656.

Again as regards *differences*, the quantity which chiefly affects us (apart from experimental errors, etc.) is the value of δ in the equation

$$d = \theta - pt = \delta \{\overline{\theta/100}|^2 - \theta/100\} ;$$

and it is therefore important to point out to what extent the variations we have observed in the value of δ may influence our results.

We have standardized our mercury thermometer E_m by means of the platinum thermometer N.* (For particulars see 'Phil. Trans.,' 1891, A, p. 151.) This thermometer contained two wires, and by placing these in series a third resistance could be determined; it thus practically contained three coils, which we denote by A + B, A, and B.

The fixed points (R_s, R_1, and R_0) of this thermometer have been re-determined at regular intervals during the past two years, and no change of any consequence has been observed.

The values of its constants are as follows†:—

	A + B.	A.	B.
R_0	9·8636	5·9881	3·8762
R_1	13·2987	8·0743	5·2252
R_s	24·2703	14·7631	9·5343 hence
δ	1·641	1·645	1·640

Now, the value of *d* at 14° C. and 25° C. obtained by assuming the above values of δ are − ·198 and − ·308, and it is on the *difference* between these numbers (·110) that possible error in the value of our range depends. Assume it as possible that the true value of δ is as high as 1·70 or as low as 1·60 (our greatest differences in any determinations of this constant have been ·009) we should then get − ·205, − ·319, and − ·193 and − ·300 as the differences, thus causing changes of ·004 and ·003, respectively, in our range, although causing differences of ·010 and ·008 in the elevation.

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* The coil of thermometer H was so long that it was barely contained in the calorimeter, and, as it required complete immersion, it was not suitable for use in this case.

† The constants differ slightly from those previously published ('Phil. Mag.,' December, 1892), as the "plug correction" (see p. 410), has, for the first time, been applied to them. By assuming that R = 0 we get for the absolute zero − 270·66, − 270·55, and − 270·86 instead of the numbers there given.

Apart from errors of observation, we may, therefore, regard the limit of our range error as ± ·003 in 11° C., *i.e.*, 1 in 4000; and we believe it to be much less.

As to experimental errors, it must be remembered that we have made the comparison of the platinum and mercury thermometers under the most favourable circumstances. Both were immersed in the calorimeter, the cooling tube (Plate 2, fig. 2), having been removed to allow of the insertion of the platinum thermometer. The observer, with his eye at the telescope and his hand on the key could, by means of the current, set the temperature with extreme exactitude to any required reading E_m, stirring being maintained at the same rate as during an experiment. Observations could thus be repeated at the same reading as often as necessary.

Again, the observations taken by the observer at the galvanometer were entirely independent of the readings of the observer at the telescope, yet a difference of 0°·004 C., on the repetition of an observation, was very unusual.

Many hundreds of comparisons were made, the particulars of which we reserve for our second communication.

We may mention that the comparisons made in 1892 differ from those of 1891 by ·003 in range, but show a rise of about ·010 in the actual elevation of E_m.

E_m was one of three thermometers specially constructed for this work by HICKS. We set him a somewhat difficult task, as the dimensions of the bulb were strictly limited both in length and diameter, and the weight of mercury was also specified.

The stem was divided into millimetres, about 40 millims. to 1° C., and the lower 18 centims. were out of sight within the calorimeter tube, the graduation marked 0 being within this tube and the 26th millimetre just appearing above it.

The reading 87·5 = 13°·990 C.; the reading 537·5 = 25°·471 C.

The graduations continued to 580 millims.

Our method of observation obviated the necessity of attempting to divide the millimetres by eye. The thermometer was fixed in such a position that the view of the bore was unimpeded by any of the shorter graduations. When timing changes of temperature the spider-wire of the telescope was placed on these graduations and the time of contact with the spider-wire observed. No attempt was made to take any reading terminating in 0 or 5, as these graduations extended across the bore and obscured the edge of the mercury column for about 1/20th millim. Thus to obtain the time of passing a given point we were limited to the observation of four consecutive transits, and for this reason, our "fixed points" were determined at 87·5, 127·5, &c., and the time of passing these points deduced from the times at 86, 87, 88, and 89, &c.

A consideration of the behaviour of this thermometer (as well as of other narrow bore ones) rendered it evident that the ordinary method of calibration was insufficient if the thermometer was to be used for the determination of changing temperatures. We believe that no calibration can be considered satisfactory unless made under the *same circumstances* as those in which the thermometeris to be used. As an illustration of our meaning we may mention that the range 87·5 to 127·5 $E_m = 0^\circ{\cdot}975$ C. when the thermometer was stationary, but $= 0^\circ{\cdot}973$ C. when rising at the normal rate of our experiments, *i.e.*, in addition to calibration for inequalities in the bore, a thermometer used under circumstances similar to ours must be calibrated for "sticking" and for certain periodic alterations in the rate of advance which are probably due to the behaviour of the glass bulb. Whatever may be the case with other thermometers, a close and prolonged study of the thermometer E_m has rendered it certain that the rate of advance of the column (when the rise in temperature is uniform) is influenced by other conditions than mere irregularities in the bore, but that these changes in rate will, under the same conditions, almost always recur at the same places.

During 1891 the times of rising every 10 millims.* throughout the whole range were observed and the observations repeated on more than thirty occasions with varying rates of rise. Nearly half of these observations were taken with the alloy wire, whose resistance did not increase as the temperature rose, the remainder with a platinum wire. During this series of observations the weights of water were different, the supply of heat was trebled, the exterior temperature was shifted from the bottom to the top of our range, the pressure changed from 1 to 760 millims., and, in fact, all the conditions were altered. The times of rising 10 millims. under all these different circumstances were thus determined at 42 points in our range, and the results plotted; the mean path in each case was, of course, different, but the deviations from it were regular in their occurrence and similar in their nature. Such deviations as invariably occurred throughout all the altered conditions must have had their *origin in the thermometer itself.*

We hope to publish some of these curves in our further communication for, although the changes in the curves themselves are great, the steady repetition of lag and acceleration is both curious and interesting. This year these differences recurred with another coil and a different method of stirring.

The mean values of these deviations, which were repeated throughout the whole of the experiments, were expressed in terms of the curve-reading at each point and plotted as ordinates on a straight line,† and the remainder of the calibration completed in the usual manner. The effect of the application of the resulting corrections to the numbers obtained from our J experiments is a sufficient proof of the accuracy of the method. An error of $0^\circ{\cdot}001$ C. in the comparative value of any of our ranges would produce a difference of from ·4 to ·7 second between the "mean times" given in columns 5, 9, 14, 18, and 22, Table XL., and the curve numbers in the succeeding columns. Further, this difference would *recur in each experiment.* No persistent discrepancy of any consequence is observable. In our own work we have expressed the value of these ranges to another figure, but we refrain from printing it until we produce more evidence. One possible cause of error in our total range yet remains, viz., the difference caused by unequal lag at the beginning and end of that range, and this difficulty cannot be surmounted by means of the same method as that used for determining the comparative values of the smaller ranges.

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* 10 millims. on thermometer $E_m = 0^\circ{\cdot}25$ C. approximately.

† *E.g.*, if t'' was observed time over a small range $\theta_2 - \theta_1$, and if t' was the value over that range as given by the smooth curve, then t''/t' was plotted as ordinate and $\frac{1}{2}(\theta_2 - \theta_1)$ as abcissa, and the areas thus obtained were integrated, &c.

Our experiments on this point were of the following nature :—

A platinum thermometer E (Pt) of exceedingly small capacity for heat (the external diameter was only 3/16 inch) was placed in the calorimeter. The coil of this thermometer was pressed tightly against its glass envelope, which was of egg-shell thinness, and thus the thermometer was rendered extremely sensitive. Its stem passed down a condenser-tube, through which the tank water was continually drawn by means of the water pump. The stem-resistance, therefore, remained constant, and thus only *one* observation was required, whereas with N three observations had to be taken to accurately determine a temperature. The value of R when the temperature of the calorimeter was steady was determined at the initial and final points of our range. The temperature was then lowered to our usual starting-point (10° C.), and raised in the same manner as during an experiment. The value of R was ascertained when the reading of E_m (mercury thermometer) again indicated the initial and final points. No doubt the platinum thermometer also lagged, but we proved by independent experiments that it did so to an extremely small extent, and, if uniform, the lag was of no consequence, as it would only affect the elevation and not the range. Now in the platinum thermometer we have no "sticking," &c., and there is every probability that the lag *is* regular. The observer at the galvanometer called at the moment when the "spot" was seen to pass the zero point, the other observer taking the mercury readings at the same instant. This was done over four or five consecutive plugs, and the resistance deduced at the moment of passing the required points. The results indicated that the lag of the mercury thermometer was greater at the commencement than at the end of our range by (at our normal rate of rise) about 0°·008 C.* The "stationary range" would, therefore, require diminishing by about ·008 in 11° C., or by 1 in 1400.

In order to investigate this point more fully another platinum thermometer (Q), having a naked wire, was placed in the calorimeter. The coil of this thermometer had fifty turns to the inch, and thus the different parts of the wire were very close together. The results were not satisfactory, the conduction across the water, under such circumstances, being sufficient to render the galvanometer uncertain and to produce irregularities in our results. We propose to repeat these "lag" determinations, using the naked wire, but placing pentane instead of water within the calorimeter. In the meantime we have left our total range uncorrected for the difference in the initial and final lag, as our own data are not at present sufficiently accurate for us to apply the correction, which is certainly less than 1 in 1400. Its effect would be to increase the value of J,[*] but it would not affect the changes in the specific heat of water or the calorimeter.

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* The average lag of the mercury thermometer proved to be less than we anticipated. It, of course, depends, in a great measure, on the thoroughness of the stirring; but when the mass of mercury that has to be heated is considered, the extreme lag, observed at our rate of rise, viz., ·025, is surprisingly small.

The correction will only have to be applied to the final result.

The exterior temperature was determined by a mercury thermometer A (by HICKS), which was similar in most respects to E_m, but its range was not so open—about 27 millims. = 1° C. It was placed in the calorimeter with E_m and the readings compared at every 5 millims. of E_m. When conducting our experiments we only required to know accurately the difference between the internal and external temperature, and, for such a purpose, this comparison was sufficient. As A was to be used for the reading of stationary temperatures, a careful calibration of the usual kind was made by means of the Cavendish dividing microscopes.[†]

Our method of experimenting eliminated several of the errors which are associated with the use of mercury thermometers.

(1.) Changes of temperature of stem.

The exposed portion of the stems of A and E_m was always immediately above the surface of the water in our twenty-gallon tank. The water was always maintained at a constant temperature (see p. 454) and (as we ascertained by direct measurements) the temperature of the air above it varied but slightly. The lower portion of the stem passed through a tube immersed in this tank water, and thus the stem temperature remained steady when the temperature of the bulb was rising. The change in the temperature of even the upper portion of the stem never exceeded three or four degrees throughout the year, and the change in the average stem temperature must have been much smaller. At the time of determination of the values of the ranges, the stem was, therefore, under exactly the same conditions as those prevalent during an experiment.

(2.) The effect of changes of pressure on the bulb.

The only changes to which it was exposed were atmospheric ones, which, even if they affected the elevation, would not affect the range. Again, A and E_m were so similar in construction and dimensions that the result of such changes would be nearly the same in both. Thus the differences of temperature deduced from the readings would not be altered to any appreciable extent, and it was only with such differences that we had to deal.

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[*] ROWLAND appears to have paid no attention to the effect of this difference in initial and final lag. It is probable, therefore, that the correction, if it was possible to obtain it, would cause a greater increase in his value than it is likely to effect in ours, since his average rate of rise was three times as great.

[†] This calibration is in practical agreement with another calibration of A conducted by platinum thermometers; the inequalities as determined by the latter method are, however, always greater than those shown by the former. This is to be expected when the *length* of the mercury thread used in the ordinary method of calibration is considered.

(3.) The changes in zero caused by the sudden changes in temperature.

The history of thermometer E_m* was much the same from day to day. During the night it remained about the temperature of the tank. Before an experiment, its temperature was lowered to 10° C. and, during an experiment, it was steadily raised from 10° C. to 25° C., no readings being taken until it reached 13°·9 C. When determining the fixed points, the same routine was, as nearly as possible, observed; thus the conditions were, in all cases, so alike that the effects were probably similar. In any case, we have been unable to detect any irregularities arising from this cause.

TABLE XVII.—Values of "Fixed Points" on Thermometer E_m when the Temperature is Steady.

E_m.	C.
87·5	13·990
127·5	14·965
177·5	16·198
217·5	17·187
257·5	18·180
297·5	19·196
347·5	20·474
397·5	21·755
447·5	23·061
507·5	24·662
537·5	25·471

The following Table shows the difference between the comparative values of the ranges when the thermometer is steady and when rising.

Range E_m.	Range C. Thermometer steady.	Range C. Thermometer rising.
87·5–127·5	·975	·973
127·5–177·5	1·233	1·235
177·5–217·5	·989	·987
217·5–257·5	·993	·995
257·5–297·5	1·016	1·016
297·5–347·5	1·278	1·278
347·5–397·5	1·281	1·282
397·5–447·5	1·306	1·307
447·5–507·5	1·601	1·598
507·5–537·5	·809	·811

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* E_m remained in the calorimeter from the commencement to the close of our work except on the few occasions when water was added or withdrawn.

The calibration of those portions of the thermometer between our "fixed points" was conducted in the same manner as the calibration over the whole range.

[Note by E. H. GRIFFITHS, added March 31st, 1893.

On February 14 I received the thermometer by M. TONNELOT (No. 11048), which had been standardized both by Dr. GUILLAUME, and under his direction, at the Bureau International des Poids et Mesures.

This thermometer is of hard glass, with transparent stem, and is divided into tenths of a degree Centigrade. The scale is continuous from — 3° to 27° C. when there occurs a bulb, then a narrow portion giving readings 48° to 52° C., then a second bulb, and, finally, readings from 98°·5 to 101°·5 C. The length of each degree is about 10 millims.

The study of this thermometer has been a most exhaustive one; two separate calibrations were performed for the most useful part (10° to 26°·5 C.)—the *greatest* difference between any two values given by the separate determinations being 0°·0028 C. The fundamental interval and the coefficients of external and internal pressure were repeatedly determined. Several hours must have been occupied in the mere copying of the observations forwarded to me—although the record consists, almost entirely, of numerals. I mention this as indicating the labour which has been devoted to this standardization, and I owe my warmest thanks to the Bureau International for the care and attention bestowed upon it.

The papers include printed tables for the reduction of the corrected mercury readings both to the hydrogen and the nitrogen scale as resulting from the comparisons of Dr. CHAPPUIS.

The comparison between this thermometer (denoted hereafter by P) and the thermometer E_m was conducted in the outer tank, for had the comparison been made in the calorimeter, the lower readings of P would have been invisible. E_m was, therefore, under the same conditions as during our J experiments, except in the following particulars:—

(1.) The external pressure was increased by about 110 millims. of water. A series of observations proved that the resulting correction was — ·3 millim. (rather less than 0°·001 C.) and this correction was always made before entering the reading of E_m.

(2.) When in the calorimeter, the lower portion of the stem passed through tubes washed by the tank water. About 60 millims. were thus maintained at a constant temperature, whatever the extent of alteration in the bulb temperature. About 50 millims. intervened between the top of the calorimeter and the lower end of this constant temperature portion. We may consider that this 50 millims. assumed the mean temperature between the tank and the calorimeter. Thus a length of (60 + 25) millims. of the stem, which during our previous experiments was at the tank temperature 19°·26, assumed, during this comparison, the temperature of the bulb θ_1.

The correction in millimeters is given with sufficient accuracy by the formula

$$85\,(\theta_1 - 19{\cdot}3) \times {\cdot}00016,$$

and as the mean value of 1 millim. is 0°·025 C., the resulting correction is

$$85\,(\theta_1 - 19{\cdot}3) \times {\cdot}00016 \times 0^{\circ}{\cdot}025 \text{ C.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha).$$

This amounts in the greatest case to 0°·002 C.

The readings of P were taken by a microscope (re-levelled before each observation) which was fitted with a micrometer scale. The one hundredth of a degree Centigrade was thus read directly and no difficulty was experienced in estimating the one thousandth, the scale being an unusually clear one, made by the Cambridge Scientific Instrument Company.

The thermometer was placed in a tube, open at the lower extremity, and an opening at the upper end was connected with a rubber tube leading to the waste pipe. This arrangement was used as a siphon and the thermometer stem was thus maintained at the bulb temperature, and, therefore, as the mercury column rose, any alteration in the stem immersion was rendered unnecessary.

In order that the readings might be obtained under the exact conditions observed by Dr. GUILLAUME, I implicitly followed the instructions given in a printed note added to the certificate. After the tank had arrived at the required temperature at least half-an-hour was allowed to elapse before taking a reading, and immediately after the observation the thermometer P was transferred to ice, and a series of readings taken to obtain the greatest zero depression. The admirable methods adopted by the Bureau International are less known, or, if known, are less practised in this country than they deserve to be. I therefore give one example in full showing the various corrections.

COMPARISON of the Reading 87·5 on E_m with P (March 15, 1893).

	Temperature reading by P. millims.	Zero point. millims.
Barometer (corrected)	754·1	754·1
H_2O pressure (expressed in terms of Hg)	12·1	4·4
Total external pressure	766·2	758·5
Observed reading	14·025	− ·036
Calibration correction	− ·026	0
External pressure correction	− ·001	0
Internal ,, ,,	+ ·026	+ ·008
Zero	+ ·028	− ·028
Fundamental error correction	− ·006	
Sum corrections	+ ·021	
Correction for stem E_m [see (α), *supra*]	− ·002.	

Hence reading on the mercury scale = 14·044.
Correction to H scale − ·067 ; to N scale − ·059.

Hence

Reading E_m.	Hydrogen scale.	Nitrogen scale.
87·5	13·977	13·985

I confess that I did not know until I adopted in detail the precautions advised by Dr. GUILLAUME in his "Traité de Thermométrie," that it was possible to regard the mercury thermometer as an instrument of precision, and I had considered such corrections as too refined for so rough an instrument. I am, however, now convinced that, given the *right conditions,* it is possible to ascertain temperatures by a mercury thermometer with a limit of error of 0°·002 or 0°·003. It is rarely, however, that such conditions prevail. The labour involved in the previous standardization of the thermometer is enormous, and there are but few men capable of constructing thermometers on which such labour may be profitably expended. The actual operation of observing and reducing the results is also considerably greater than is the case when platinum thermometers are used.

The following table gives the results of all observations made with the micrometer eye-piece. I had previously made several comparisons whose mean result is in fair agreement with those here given ; but, as the last two figures of the readings had to be estimated, the discrepancies are greater, and the results of less value.

Comparison of E_m with the French Standards.

Reading E_m.	87·5	127·5	177·5	217·5	257·5	297·5	347·5	397·5	447·5	507·5	537·5
	θ_1 on the corrected mercury scale.										
March 14	—	—	—	—	—	—	20·541	21·828	—	—	—
„ 15	14·044	15·022	16·260	17·249	—	—	—	—	—	—	—
„ 16	14·044	15·020	—	—	—	—	—	—	—	—	—
„ 17	14·045	15·023	16·259	17·248	18·245	19·261	20·544	—	—	—	—
„ 18	—	—	—	—	—	—	20·545	21·832	23·140	24·740	25·552
„ 19	—	—	—	—	—	—	—	—	23·138	24·738	25·550
Mean . .	14·044	15·022	16·260	17·249	18·245	19·261	20·543	21·830	23·139	24·739	25·551
	Hence θ_1 on the hydrogen scale.										
H	13·977	14·952	16·186	17·172	18·165	19·178	20·457	21·741	23·048	24·645	25·455
	And θ_1 on the nitrogen scale.										
N	13·985	14·960	16·193	17·181	18·174	19·188	20·467	21·751	23·058	24·655	25·466
	θ_1 as determined from platinum thermometers (see Table XVII.).										
Table XVII.	13·990	14·965	16·198	17·187	18·180	19·196	20·474	21·755	23·061	24·662	25·471
Difference between N and Table XVII.	·005	·005	·005	·006	·006	·008	·007	·004	·003	·007	·005

Mean difference N—XVII = − ·0055.
Mean discrepancy from the mean ± ·0011.
Difference in range between 14° and 25° : ·001.

It thus appears that our temperatures, as deduced by platinum thermometers, exceed those of the Paris Hydrogen and Nitrogen Standards in actual elevation by 0°·014 C. and 0°·005 C. respectively, but that the value of our total temperature range is practically the same by both methods of standardization.*

It is true that slight discrepancies appear in the value of the smaller ranges, but here I am inclined to place greater reliance on our previous determinations than on those obtained by the comparison with P, for mistakes of observation are far less likely to occur when using platinum, than when reading mercury, thermometers, and also the former values are deduced from the observation of several different resistances at each temperature, all of which after the correction given by the standardization of the resistance-box (see p. 408, *supra*) are in practical agreement, although the actual values of R differ greatly.

The difference in actual elevation is of no consequence, although it is interesting to notice that the agreement with the Paris Nitrogen Scale is nearer than that with the Hydrogen Scale. This was to be expected, since the boiling-point of sulphur was ascertained by Mr. CALLENDAR and myself by means of an *air* thermometer, and our values of d depend on that determination.

We have not, as yet, been able to commence, by means of Mr. CALLENDAR's new form of air thermometer, our direct comparison with the nitrogen scale, for the calibrating and cleaning of the instrument have occupied much time. Pending the completion of that work the results of the above comparison are of great value, and, in the meantime, we are justified in assuming that the temperature range values, as determined by us, are in such close agreement with the Paris nitrogen thermometer as to render any alteration in our results unnecessary. The reduction of our temperature from the air, to the absolute, scale would diminish the value of our total temperature range by nearly ·002, but I do not consider the accuracy of our results of such an order as to render the resulting correction (about 1 in 5000) more than a matter of form.

I have forwarded this note to Dr. GUILLAUME with the request that he would be so good as to add his opinion as to the probable limit of error of the observations with thermometer P, together with any other comments which he may think advisable. His remarks are as follows :—

"Le résultat fort remarquable des comparaisons faites par M. GRIFFITHS est sans doute le meilleur critérium de la limite d'erreur, tant des observations du thermomètre que de son étude ; et l'estimation que j'en aurais faite *à priori* aurait été sensiblement plus élevée. En effet, dans le résultat final viennent s'ajouter toutes les causes d'erreurs suivantes :—

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* The range value on the corrected *mercury* scale is (in the case of this thermometer) 11·508, and this would *diminish* our value of J by 1 part in 367.

1°. *Étude du thermomètre* P.

a. Calibrage (y compris l'interpolation dans la supposition que la division est équi distante) ;

b. Détermination du coefficient de pression ;

c. Détermination de l'intervalle fondamental.

Les corrections résultant de ces deux dernières études n'agissent que sur le *range;* et sont de peu d'importance (moins de ·001 degré) sur un intervalle de 10 degrés. Le calibrage, au contraire, peut introduire de point en point des erreurs irrégulières. J'ai montré autrefois* que, dans un calibrage fait avec tous les soins possibles par un observateur très-exercé, l'extrême limite d'erreur, pour un thermomètre divisé en dixièmes de degré est de ·001 ; le calibrage du thermomètre P a été fait avec soin, mais par une méthode abrégée et l'on peut s'attendre à des erreurs de ·002 environ sur les points déterminés directement ; le tube étant assez irrégulier les erreurs des points interpolés peuvent atteindre ·003.

2°. *Réduction au thermomètre à azote.*

Cette réduction se fait au moyen des tables générales déterminées par la comparaison, faite une fois pour toutes, d'un groupe de thermomètres en verre dur avec le thermomètre à gaz. Dans l'application de cette table au thermomètre P, on suppose que les corrections sont les mêmes, c'est-à-dire que le verre et le mercure de ce thermomètre sont identiques à ceux des thermomètres-étalons. Cette identité thermique a été vérifiée approximativement par des comparaisons précises faites sur un grand nombre de thermomètres ; mais si l'on songe : (*a*) que le verre est un corps chimiquement compliqué, et dont la dilatation est soumise à des lois singulières ; (*b*) qu'il suffit d'une différence relative de $\frac{1}{400}$ ($2 \cdot 10^{-11}$ en valeur absolue) dans le coefficient de θ^2 dans la formule de la dilatation linéaire du verre pour provoquer une différence de ·001 degré dans les indications du thermomètre à 50°, on ne peut être que fort étonné d'une telle concordance, qui résout la question vitale du thermomètre à mercure, et détruit tous les préjugés que l'on avait contre cet instrument, aussi bien que sur la dilatation du verre.

3°. *Comparaisons.*

Celles-ci comprennent :—

(*a.*) les observations à diverses températures ;

(*b.*) les déterminations du zéro.

On peut vraisemblablement s'attendre à ce que les erreurs de température, les défauts de l'éclairage, les irrégularités du ménisque provenant en partie des irrégu-

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* "Etudes thermométriques," 'Travaux et Mémoires du Bureau International des Poids et Mesures,' tome 5, p. 23 (1886).

larités du tube, enfin les erreurs de l'estimation produisent au total des discordances de ·001 degré au minimum dans (*a*) et dans (*b*).

Toutes les causes mentionnées ci-dessus et ne concernant que le thermomètre P auraient pu, sans aucune doute, conduire dans la réduction au thermomètre à azote à des erreurs irrégulières atteignant ·005.

La *différence systématique* de θ_1 ·005 en moyenne s'explique suffisamment par deux causes :—

(*a.*) Pression capillaire moindre dans la colonne toujours descendante à zéro que dans la colonne à ménisque le plus souvent ascendant ou en équilibre moyen aux autres températures.

Dans les tubes de thermomètres l'angle de raccordement varie entre 46° et 75° environ, ce qui, dans les tubes de ·1 millim. de diamètre, correspond à une variation de pression de 50 millims. de mercure au maximum.

(*b.*) Le zéro se relève dès l'instant où le thermomètre est plongé dans la glace ; ce relèvement au moment initial est à peu près indépendant de la dépression, et dépasse ·001 degré par minute.

Ces deux causes réunies suffiraient à expliquer des différences constantes plus fortes même que celle qu'a trouvée M. GRIFFITHS ; pour la plupart des recherches, comme dans le cas actuel, elles sont sans nulle conséquence. Il me sera permis en terminant d'insister sur la sécurité que donne aux mesures de températures le contrôle si indirect et en même temps si parfait qui résulte des comparaisons de M. GRIFFITHS, et d'exprimer le vœu que la vérification soit bientôt étendue à un intervalle de température aussi large que possible. Jusqu'à ce jour, l'étude du thermomètre à mercure au Bureau International a été faite entre — 25° et + 100° ; l'échelle sera prochainement étendue jusqu'à 200°. Le thermomètre à toluène a été parfaitement étudié et comparé au thermomètre à hydrogène jusqu'à — 75° ; à cette température la discordance entre divers instruments étudiés individuellement n'atteint pas ·05 degré."]

SECTION XI.—THE HEAT DEVELOPED BY THE STIRRING.

During our earlier experiments (1887) we stirred by means of a screw which revolved about 200 times per minute. Throughout 1888 and 1889 we continued to use the same form of stirrer, but endeavoured to make the mixing more efficient by increasing the rate of revolution. As our experience increased we became strongly impressed with the necessity of *thorough* mixing. Any slackness in this respect makes itself evident (*a*) in changes in resistance of the wire, owing to changes in its rate of cooling ; (*b*) irregularities in the radiation and conduction, probably owing to varying temperatures of the lid ; (*c*) irregularities in the thermometer readings ; (*d*) in the water equivalent, which then is found to alter to some extent with the mass of the contained water.

One assumption underlies all the calculations, viz., that whatever the mass of the contained water, the water equivalent of the calorimeter remains constant.* We find this assumption is not correct, unless the water is thrown against every part of the calorimeter throughout each experiment. At first sight it would appear that, whatever might be the "lag" of any unwetted portion of the calorimeter when the flow of heat had become steady, the quantity of heat absorbed by it for each rise of 1° C. would be the same, and that it would only be at the commencement of each experiment that differences would be observable. It must, however, be remembered that the radiation coefficient of the whole calorimeter would be altered by the existence of this colder portion; and, as this cause of error would continue to exist throughout the whole experiment, the effect would be appreciable.

In 1891 we adopted a new form of stirrer. An ebonite shaft, passed down the central tube, and terminated at its lower extremity in an agate point, which rested on a metal bearing on the base of the calorimeter. Two gold tubes joined together in the form of a V were fastened to this shaft, the apex of the V being downwards and immediately above the agate point. This stirrer acted as a centrifugal pump, and, when revolving at a rate of about 1200 per minute, threw the water upwards and outwards with great rapidity—the whole contents of the calorimeter being removed in a few seconds when the lid was off. There could be no doubt that, provided there was sufficient water to at all times cover the lower openings of the tubes, the stirring was sufficient, and that no portion of the calorimeter could remain for any appreciable time at a different temperature to the water. This form of stirrer appeared to us so satisfactory that we did not attempt to improve upon it during our experiments in 1891; and it was this misplaced confidence which was the chief cause of our failure, in that year, to obtain satisfactory results.

Its defects may shortly be summed up as follows:—

(1.) As the depth of water in the calorimeter increased, the work expended in forcing the tubes through the water also increased; and the rate of rise in temperature (which ought to have diminished) was much greater when the mass of the water was large.

(2.) This change in the quantity of heat developed would not have been fatal, since it could have been determined separately for each mass of water used, and we persevered in our attempts to thus determine it. Unfortunately, however, it varied capriciously, even when the mass of water and rate of revolution were constant; and we have never arrived at any satisfactory explanation of these variations. It appears probable, however, that their origin was due to differences in pressure between the agate point and its bearing, the slightest variation in the position of the shaft of the stirrer causing very different quantities of heat to be generated. The quantity of heat generated by the stirrer varied from about one-sixth to one-fifteenth of the heat generated by the current, according to the electromotive force used; and as the capricious changes above referred to amounted, in some cases, to 5 or 6 per cent. of the heat due to the stirring, they might affect our ultimate result by as much as a half and, in some cases, nearly 1 per cent. The quantity of heat appeared to vary with the rate of revolution in such a manner that $tr^{5/2}$ was a constant, where t was the time of rising 1° C. and r the rate of revolution.

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* If the water equivalent varies, the fact is rendered very evident when the values of J deduced from different masses of water are compared. Until, in 1891, we adopted forms of stirrers which threw the water against the roof of the calorimeter, our experiments were rendered useless by this constant source of error. The agreement between the values obtained in 1892 is a satisfactory proof that this difficulty had been overcome.

Our experience of this stirrer led to the following conclusions:—

(1.) The ideal stirrer must be able to throw the water from the bottom to the top of the calorimeter whatever the mass of the water.

(2.) The "work done" by the stirrer should be the same, whatever the depth of the water.

(3.) The "work done" by the stirrer should be, as far as possible, employed in throwing the water upwards to the lid, rather than in simply causing a rotation of the water near the bottom of the calorimeter.

(4.) The whole of the energy should, if possible, be employed in lifting the water and not in overcoming the friction between solid surfaces, *i.e.*, the bearings should be outside the calorimeter.

Although we found it impracticable to devise a form which would entirely fulfil the above requirements, the one adopted by us during our 1892 experiments was satisfactory.

During the spring of 1892, we made a large number of trial experiments with different forms of stirrers, and it should be borne in mind the form it ultimately assumed was an example of the "survival of the fittest." A section will be found in Plate 2, fig. 2.

A cylindrical tube *AB*, 1 inch in diameter, closed at the lower end, rested on four little feet about 1/16 of an inch in length, which were fixed to the base of the calorimeter. The plate at the lower end was perforated in the centre, so as to allow the end of the revolving shaft to pass through without touching it. Round this centre hole were four slits through which the water in the calorimeter passed into the cylinder. The end of the revolving shaft was fitted with a small cylinder of agate, which hung loosely within a ring fixed to the base of the calorimeter at *C*. The end of the agate was not in contact with the base of the calorimeter. The bearing by which the stirrer was supported was fixed at the top of the central glass tube, and thus any increased downward thrust of the stirrer did not alter the friction within the calorimeter. The stirrer consisted of an inverted cone with nearly vertical paddles at its extremities (a section is shown at *D*). When the stirrer revolved, the water was drawn in through the base of the cylinder, thrown outwards by the paddles, and, being unable to escape at the lower end, mounted rapidly to *A*, whence it was ejected with considerable violence against the roof of the calorimeter. If the rate exceeded about twenty-six revolutions per second, it was found that when the lid was removed, the water was thrown many inches above the top of the cylinder, while the stirring-shaft remained dry. A small window at *E* caused a stream of water to be continually thrown on to the top of the thermometer bulb.

The bearings at the top of this shaft ultimately assumed the form shown at *H*. A small block of steel was fastened by a screw to the shaft, and the lower end of this steel terminated in a projecting ring, which revolved in a brass bearing of the form shown at *H*. This brass bearing was, as already mentioned, fixed to the top of the central glass tube. A drop of oil placed in the channel, on the top of the brass, rendered this bearing practically air-tight, at all events, sufficiently so to prevent any possibility of evaporation.* The shaft from *H* to the top of the calorimeter was about 8 inches long, and made of ebonite; the conduction of heat along this shaft was, therefore, very small. The heat generated in the bearing at *H* passed almost entirely into the water of the outer tank, for the brass tube which terminated there was continued downwards to the steel lid, and thus the lower three inches were immersed in the water.

The portion at *D* was at first shaped like a horizontal paddle-wheel. It was, however, found that although it would work excellently for a time, a gradual accumulation of air at the centre of the paddle-wheel greatly diminished its efficiency. Many attempts were made to surmount this difficulty, and it was only by the conversion of the upper portion into the form of an inverted cone that it was finally overcome.

To adjust the stirrer, the screw in *S* was loosened and the shaft lowered until the end of the agate touched the base of the calorimeter. The stirrer was then raised about 1 millim., and fixed in that position by means of the screw at *S*. When investigating the action of these stirrers, the lid of the calorimeter was replaced by a sheet of glass which prevented the contents from being ejected. As far as it was possible to see the interior through the shower of water that poured against the glass, the water without the cylinder assumed a slow circular movement only. At the same time, there can be but little doubt that the entire contents passed from the bottom to the top within the space of a second or two. As long as the volume of the contained water exceeded 130 cub. centims., the throw appeared to be the same, whatever the quantity, and, as the figures will show, the work done altered only slightly.

Our first series of experiments was performed with the object of ascertaining (1) If, under the same conditions, the work done by the stirrer was constant; (2) In what respect the work done varied, if at all, with the mass of water; (3) If there was any constant relation between the heat developed and the rate of revolution.

The results of our preliminary series are given in the following table, the rise in temperature being expressed in millimeters of thermometer E_m, since our object at this time was not to determine the actual quantity of heat developed, but to investigate the points mentioned in the above headings.

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* It was found that this "join" would support a pressure of 7 or 8 inches of water without leakage.

TABLE XVIII.—Experiments indicating that tr^3 = constant.
(t = time of rising 1 millim. r = rate of revolution in seconds.)

Date, 1892.	Weight.	Exp.	Range in millims. E_m.	Time in seconds.	r.	t.	$tr^3 \times 10^{-3}$.	Remarks.
July 12 . .	104·015	1	276–296	900·5	29·19	Preliminary.		Outer temp. 286·5 E_m.
„ 12 . .	..	2	276–296	912·9	28·76	Stirrer not in adjustment.		
„ 12 . .	..	3	279–294	832·3	27·34	55·49	1134	
„ 13 . .	..	4	279–294	533·9	31·71	35·60	1135	
„ 13 . .	..	5	279–294	845·6	27·27	56·37	1143	
„ 13 . .	..	6	279–294	907·0	26·62	60·47	1141	
„ 13 . .	..	7	279–294	466·9	32·95	31·13	1114	Motor very irregular.
„ 14 . .	..	8	279–294	803·0	27·70	53·53	1138	
„ 14 . .	..	9	279–294	628·2	30·05	41·88	1137	
							1134	
July 14 . .	194·200	10	281–292	642·1	30·13	58·37	1596	Outer temp. 286·5 E_m.
„ 14 . .	..	11	283–290	507·4	28·01	72·49	1593	
„ 14 . .	..	12	281–292	514·8	32·38	46·80	1590	
„ 14 . .	..	13	283–290	618·3	26·33	88·33	1511	
„ 15 . .	..	14	281–292	711·9	29·15	64·72	1603	
„ 15 . .	..	15	280–293	643·7	31·87	49·37	1603	
„ 15 . .	..	16	282–291	648·5	28·11	72·06	1600	
							1600	
July 18 . .	194·190	31	283–290	483·1	28·62	69·01	1619	Outer temp. 285·5 E_m.
July 15 . .	239·73	17	283–290	439·6	30·78	62·74	1830	Outer temp. 286·5 E_m.
„ 15 . .	..	18	283–290	628·2	27·44	89·74	1850	
„ 16 . .	..	19	283–290	544·6	28·73	77·80	1846	
							1842	
July 18 . .	239·27	30	283–290	606·0	27·69	86·57	1839	1843 for wt. 239·73.
July 16 . .	285·37	20	282–291	767·0	29·10	85·22	2100	Outer temp. 286·5 E_m.
„ 16 . .	..	21	280–293	909·2	30·98	69·95	2080	
„ 16 . .	..	22	283–290	793·7	26·60	113·4	2134	
„ 16 . .	..	23	283–290	711·2	27·57	101·6	2129	
„ 17 . .	..	24	282–291	841·1	28·39	93·45	2138	
„ 17 . .	..	25	282–291	703·8	30·02	78·20	2116	
„ 17 . .	..	26	282–291	735·3	29·68	81·70	2135	
							2119	
July 17 . .	275·40	27	283–290	608·9	28·82	87·00	2081	Outer temp. 286·5 E_m.
„ 17 . .	..	28	282–291	635·5	30·61	70·61	2025	
„ 18 . .	..	29	282–291	821·5	28·39	91·27	2089	
							2065	

From the above results we drew the following conclusions :—

(1.) That under the same conditions the quantity of heat developed was the same.

(2.) That the work done increased slightly with the mass of water.

(3.) That tr^3 was a constant quantity.*

If Nos. 1 and 3 were established, No. 2 was of little consequence, as we proposed to determine the heat developed for each mass of water.

When it is remembered that our "rate" varied in the above experiments from 26 to 34 revolutions per second, the close approximation of tr^3 to a constant is remarkable, and, as we proposed to maintain a rate of, as nearly as possible, 30 revolutions per second throughout our experiments, it was evident that any assumption based on this conclusion, could lead to no appreciable error. Again, a considerable variation in the value of tr^3 would have but little effect on the value of t, *i.e*, the time of rising 1° C.

These experiments were regarded as only preliminary, and no special care was taken with regard to external temperature or pressure, and, it is probable, that many of the apparent discrepancies arose from such causes. Again, when the movement of the mercury column was so slow that from 50 to 100 seconds were taken in rising 1 millim., it was difficult to estimate with great precision the time of apparent contact of the mercury column and the spider wire of the telescope. Personal errors of this description must, however, in the long run, tend to "mean out" and give a true average.

Had we been able to bring our mechanical arrangements to such perfection that the rate of revolution was absolutely constant, it would have rendered the above investigation unnecessary, for the stirring correction would have been constant whenever the mass of water was the same.

Although by the pressure regulator, described p. 378, the regularity of the motor was increased to an extent which those accustomed to the behaviour of these instruments hardly anticipated, we were unable to ignore the residual variations. It was, therefore, necessary to record the revolutions (the time per 1000 being taken in each case) throughout all our experiments. The labour, both of observing and calculating, was thereby greatly increased; but as we had no better means at our disposal, we had to submit to the inevitable.

The establishment of the law that (with this stirrer) tr^3 might be regarded as constant, enabled us to apply with certainty the necessary corrections.

As previously stated, our object was to deduce the rise in temperature per 1 second, at rate 30. If t is time of rise of 1° in any scale, then $1/t = \mathrm{K}r^3$, therefore, $\delta \,.\, 1/t = 3\mathrm{K}r^2 \,.\, \delta r$. Knowing $1/t_2$ (the rise per 1 second at any observed rate), and $1/t_1$ (the rise at rate 30), we know $1/t_1 - 1/t_2$, that is the term required to reduce the observation to the rate 30.

The accuracy of the correction can be estimated by means of the above table. For example, the mean value of tr^3 for the weight 194·20 is 1600, the greatest departure from the mean is 1611 in Experiment 13. By assuming $tr^3 = 1600$, we find the rise per 1 second at rate 30 = ·001687 (expressed in millims. of E_m), whereas if we assume that $tr^3 = 1611$ and correct Experiment 13 (rate 26·33) to rate 30, we get ·001676.* Again, the mean value of tr^3 for weight 285·37 is 2119, which gives the rise at rate 30 as ·001274. Experiment 22 (rate 26·60) is an extreme case; this would give ·001265 at rate 30.

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* t = time rising 1° (in any scale), in seconds.
r = rate of revolution, *i.e.*, number of revolutions per second.

Experiment 7 (weight 104·02) shows a greater divergence from the mean value, since it would give a rise of ·00242 per 1 second as against ·00238, the value as given by the mean tr^3. As, however, we discarded all J experiments where the weight of water was less than 130 grms., this extreme case did not affect our results.

During our J experiments our rate was always between 29 and 31 (generally between 29·8 and 30·2), hence the divergences in the cases, above selected, are far greater than any which occurred during our experiments.

The value of tr^3 appeared to increase very regularly as the quantity of water increased, and thus changes in its value caused by small changes in the mass could be deduced. Thus experiment 30 (239·27 grms.) gave a value of 1839. Hence we deduced that, if the weight of water was 239·73 (as in Nos. 17 to 19), the value of $tr^3 = 1843$ as against 1842, the mean of the actual experiments with the latter weight.

Had the work done by the stirrer been independent of the quantity of water, we could have deduced the water equivalent of the calorimeter from the values of tr^3, for, if K_1, K_2, etc., are the values of tr^3 when W_1, W_2 are the weights, we have $\frac{W_2K_1 - W_1K_2}{K_2 - K_1}$ as the value of the water equivalent. Using the values of K deduced from Table XVIII., we get

W.	K.	Water equivalent.	M/K.
104·02	1134	115·5	·1418
194·20	1600	105·7	·1740
239·73	1843	73·5	·1769
285·37	2119		·1753

We ultimately ascertained that the water equivalent was as nearly as possible 86·0. Assuming this value and denoting by M the mass of contained water + water equivalent, then, if the work done by the stirrer was constant, M/K would be constant The numbers in the last column show that the quantity of heat developed increases rapidly at first and arrives at a maximum.

This maximum, as well as we can estimate, occurs when the quantity of water is sufficient to fill the calorimeter to about the top of the tube up which the water is thrown, and it is possible that the conditions would, at this depth, change slightly.

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* We required but *three* significant figures when we applied a similar correction to our J experiments.

In any case, the work done by the stirrer alters but very little as the height of water changes, provided that 130 cub. centims. is regarded as the minimum quantity.*

The examples given above show that the limit of error introduced by applying this method of reduction to variations far exceeding any occurring during our experiments is less than 1 per cent., and as the heat generated by the stirrer during the majority of our 1892 experiments varied from about $\frac{1}{9}$ to $\frac{1}{16}$ of the heat due to the current, it is evident that any error must be less than $\frac{1}{1000}$ of the whole supply. Bearing in mind the small variations which actually occurred, we may safely regard the limits of error as a small fraction of this $\frac{1}{1000}$.

As will be seen later on, we performed an experiment in which the current was so diminished that the stirring supply was half that due to the current (see J 9), and another in which the current was so increased that the proportions were as 1 : 20 (see J 34). Any inaccuracy in the determination of the stirring supply would have caused considerable divergence in the corrected results; but the numbers obtained are themselves a proof of the accuracy of the method adopted. (See note, p. 406.)

It appeared possible that changes in the viscosity of water at different temperatures would have an influence on the results.

This may be the case, although our experiments do not lead to that conclusion (see p. 448). If so, however, it would not affect us, for the total heat lost or gained by the calorimeter at different temperatures (independently of the current supply) was directly determined for each weight of water, and therefore although our values of the radiation coefficient, &c., might be modified, the expression for the total loss or gain would remain unchanged.

Want of time alone prevented our investigating this matter more fully, but the consideration that neither the value of J, nor the changes in the specific heat of water would be affected thereby, justified its postponement to another season.

No use was made of the numbers obtained in Table XVIII.; our only object being to establish the law $tr^3 = \text{K}$. This was fortunate, for on afterwards examining the water used for this series we had reason to believe that it had become slightly soiled. We found that the lead ring used in the calorimeter lid had been insufficiently gilded, and also we had not allowed the gutta-percha varnish, which we used as a "size" for the gold leaf, to harden sufficiently before introducing the water.

The results of nearly ninety additional experiments of a similar kind indicate that the irregularities of the numbers in the above table are unusual, and that the actual values of tr^3, obtained from them, are unreliable, but that the conclusion drawn (viz., that under the same circumstances tr^3 is a constant) is correct.

Very small changes in the adjustment caused considerable changes in K, but its value remained constant so long as the adjustment was unaltered. Any alteration in the fixing of the stirring shaft necessitated a re-determination of K.

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* This minimum quantity, 130 cub. centims., is deduced from the results of later experiments.

After experiment J 19, the whole apparatus was taken to pieces. It was then found that small fragments of gutta-percha had, by the continued impact of the water, been detached from the ring of gutta-percha which had been placed between the cooling tube and the tube down which it passed. Many of these fragments had collected beneath the base of the cylinder at B* and must have considerably impeded the flow of water. In consequence, therefore, before putting the apparatus together again, we removed all remaining gutta-percha surfaces which were exposed to the action of the water. The values obtained for K were, from this time, very different from preceding ones, and, once obtained, remained constant as long as the same mass of water was used.

The experiments J 20 to 34, are in better agreement amongst themselves than the earlier ones, and the cause no doubt is that just indicated.

The mean result of the earlier experiments is satisfactory as likewise the mean time of each experiment, but the times over small ranges (especially in experiments Nos. J 5 to 12) leave much to be desired. The fact that the quantity of heat developed by the stirrer was increased by the removal of the gutta-percha fragments is a proof that the flow of water was appreciably diminished by the obstruction.

SECTION XII.—THE GAIN OR LOSS BY RADIATION, &c.

It has already been pointed out that the method of observation adopted enabled us to determine, with sufficient accuracy, the total loss or gain of heat per second at any temperature, apart from the supply due to current. We propose to use the phrase "non-electrical supply" to denote heat due to all other sources than the current.

By the method described in Section XI., we were enabled to deduce the rise per second when the rate of revolution of the stirrer was 30, although the actual rate differed slightly from our standard one. The rate of rise, when radiation, &c., was eliminated, was first ascertained by noting the time of a small rise across the outside temperature. The temperature of the calorimeter was then gradually raised through the whole range by means of stirring only, and the times ascertained of small changes at the different temperatures. Assuming the "stirring heat" to be the same at all temperatures (when the rate is the same), we can deduce the radiation coefficient. (By radiation coefficient we denote the loss or gain in temperature per second, due to the combined effects of radiation, conduction, and convection, for a difference in temperature of 1° C).

One advantage of this method is, that all observations of temperature are taken on a rising thermometer, for "sticking" is far less likely to occur when the mercury is advancing than when receding.

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* Plate 2, fig. 2.

We have found that all observations of slowly falling temperatures (when taken with a mercury thermometer) are unsatisfactory for the above reason. Again, the conditions resemble more nearly, in every respect, those prevalent during a "J" experiment. On this account, we did not desire to reduce the "stirring heat," even if we had been able to do so, since the loss, due to radiation, &c., would, at the higher portions of the range, have mastered the stirring supply, and the observations would have had to be taken on a falling thermometer, a mode of observation which, guided by the results obtained in previous years, we had decided to reject.

It appeared unlikely that NEWTON'S Law of Radiation would hold over our range, and we anticipated a decided curvature in the line showing our "non-electrical supply."

We first proceeded to determine isolated points on this curve (ordinate = rise in temperature per second, abscissa = difference in temperature between the calorimeter and the outside temperature), but these appeared to fall so nearly on a straight line, that we decided to follow its course over our range of temperature, using a small mass of water so that the changes in the value of the rise should be as marked as possible. A very large number of experiments were made in order to decide this point—for, if the exponential formula given by DULONG and PETIT holds, the curvature ought to have been marked. Although changes in the specific heat of water and of the calorimeter would influence the inclination of the line, it appeared unlikely that they would affect it in such a manner as to straighten it; and further, the effect of such changes would alter as the mass of water changed.*

The numbers alone resulting from these observations half fill a large note-book, and to give them in full would occupy too much space. As an example, we venture to give experiments 51 to 54, exactly as they appeared in our notes. As the method employed never varied, we will, in other cases, simply give results, and we select this series because it was the first done by us with the object of tracing the "non-electrical supply" curve throughout the greater part of our range.

The numbers in the columns marked "time," are those given by the chronograph tape. New tapes were used after the readings at 343, 407, and 458, and the times of rising between these temperatures were not noted; therefore, the experiments at those ranges are numbered as separate experiments although taken consecutively.

No temperatures were taken whose readings ended in 0 or 5 because, as these were denoted by longer lines on the thermometer scale, the time of contact with the "spider wire" was not sufficiently definite, and the thickness of the graduation would have made a considerable difference in the value of t.

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* We may state at once that (although some slight signs of curvature are visible when the mass of water is great) by assuming the non-electrical supply to be a linear function of the temperature (when the rate of stirring is constant), we introduce no error sufficient to affect our results, and this conclusion is based on the data supplied by more than 100 "stirring and radiation" experiments. If the results were expressed in terms of the mercury thermometer scale, the curvature would, of course, be marked.

We did not attempt to record the time for every individual 1000 revolutions, but sufficient observations were taken to enable us to detect any change in rate.

The pressure in the annular space round the calorimeter was high (·98 millim.) during experiments 51 to 54, as we had only recently commenced re-exhausting.

TABLE XIX.—Experiment LI. Aug. 11, 1892.

McLeod gauge = 62 millims. Outside temp. = 241·3 A, = 300E$_m$.
Pressure = ·98 millim. Mass of water (*in vacuo*) = 103·01 grm.
Clock error = + ·0004125.

Experiment.	Temperature E$_m$.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LI.	86	111·0	6	0·8
	87	137·9	8	67·6
	88	164·8	10	134·0
	89	193·0	1	167·4
	91	248·4	2	200·7
	92	275·9	3	234·3
	93	302·5	4	267·7
	94	330·2	5	300·9
	96	386·0	6	334·0
	97	412·5	7	367·3
	98	..	9	433·5
	99	469·3	21	501·0
	101	524·4	2	533·9
	102	552·8	3	567·2
	103	579·8	4	601·0
	104	609·8		
	..	..	38	1065·2
	121	1095·6	9	1098·4
	122	1124·2	40	1131·6
	123	1154·4	1	1164·8
	124	1183·3	2	1197·9
	126	1244·3	4	1264·6
	127	1274·0		
	128	1304·0	7	1364·7
	129	1334·2	8	1397·8
	131	1394·6	50	1463·9
	132	1425·4	1	1497·3
	133	1455·3	2	1530·5
	134	1485·0	53	1563·5
	136	1546·0		
	..	..	76	2332·4
	161	2345·9	7	2365·8
	162	2379·4	8	2399·3
	163	2412·0	9	2432·6
	164	2444·9	80	2466·0
	166	2511·2	1	2499·3
	167	2544·0	3	2565·8
	168	2578·0	4	2599·4
	169	2611·7	5	2632·5
	..	..	109	3435·0

Table XIX.—continued.

Experiment.	Temperature E_m.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LI.	193	3440·9	110	3468·3
	194	3477·5	1	3501·6
	196	3548·0	3	3568·2
	197	3583·8	4	3601·8
	198	3619·6	5	3635·0
	199	3656·6	6	3668·5
	201	3727·2	7	3701·7
	..	..	8	3735·0
	226	4665·0	145	4633·2
	227	4704·5	7	4699·9
	228	4743·0	8	4733·2
	229	4782·0	9	4766·6
	231	4859·9	150	4800·0
	232	4900·0	1	4833·3
	233	4939·2	2	4866·6
	234	4978·8	4	4932·9
	..	..	155	4966·2
	252	5719·8	178	5732·0
	253	5762·9	180	5798·9
	254	5806·7	2	5865·9
	256	5892·1	4	5932·4
	257	5935·8	6	5999·3
	259	6023·0	8	6066·0
	261	6110·6	190	6132·8
	262	6154·6	2	6199·3
	276	6787·7	210	6797·6
	277	6834·6	2	6864·3
	278	6881·9	5	6964·4
	279	6929·4	6	6997·5
	281	7021·3	8	7064·0
	282	7071·5	9	7097·3
	296	7763·0	240	7797·7
	297	7813·7	1	7831·0
	298	7864·7	3	7897·8
	299	7917·1	245	7964·5
	301	8020·9	6	7998·0
	302	8073·4	8	8064·9
	303	8127·0	250	8131·9
	304	8179·4	2	8198·6
	306	8285·9	4	8265·6
	322	9163·7	280	9133·5
	323	9220·8	3	9234·2
	324	9277·9	5	9301·4
	326	9393·5	7	9368·0
	327	9452·2	9	9435·4
	328	9510·6	291	9502·2
	337	10044·7	307	10038·0
	338	10104·8	310	10138·9
	339	10167·3	2	10206·0
	341	10290·8	4	10272·5
	342	10350·2	6	10339·2
	343	10411·5	8	10406·0

Table XX.—Experiment LII.–LIV. Aug. 11, 1892. The constants are the same as for LI.

Experiment.	Temperature E_m.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LII.	403	1·0	0	15·0
	404	93·7	3	116·4
	406	281·6	6	217·1
	407	378·5	10	352·3
	408	472·0	3	453·4
	409	465·5	6	554·2
	411	757·4	9	655·0
			21	722·0
LIII.	456	1·0	3	36·8
	457	149·6	6	137·4
	458	297·4	10	271·6
	459	438·5	4	405·5
	461	732·0	22	673·6
	462	883·7	8	875·6
LIV.	507	1·1	3	13·8
	508	290·5	10	247·8
	509	560·2	2	314·4
	511	1179·9	20	580·3
			39	1215·2

Method of reducing the results in the above Table.

The time over any range is the mean time obtained by taking the observations in pairs—thus the time over the range 87·5–97·5 E_m is the mean of the times taken in rising from 86–96, 87–97, 88–98, and 89–99. Thus each number in column 4 of the following Table is the mean of, at least, four intervals of time.

Table XXI.

Range E_m.	Range C.	Mean Temperature C.	Time.	Rate of stirring.	$\frac{\text{Range}}{\text{Time}} \times 10^6$	(Correction to rate 30) $\times 10^6$.	(Rise, in degrees, per second at rate 30) $\times 10^6$.	Calculated rise $[\sigma \pm \rho(\theta_1 - \theta)] \times 10^6$.	Difference between last two columns.
87·5– 97·5	·242	14·110	274·4	30·01	883	0	883	880	+ 3
92·5–102·5	·242	14·232	277·6	30·00	873	0	873	871	+ 2
102·5–122·5	·487	14·597	572·9	30·14	851	− 7	844	844	0
122·5–132·5	·246	14·963	300·8	30·07	816	− 3	813	817	− 4
127·5–162·5	·864	15·395	1106·9	29·96	780	+ 2	782	784	− 2
167·5–197·5	·740	16·321	1041·2	29·93	711	+ 4	715	716	− 1
197·5–227·5	·742	17·063	1122·1	30·03	661	− 1	660	661	− 1
227·5–232·5	·124	17·495	195·9	30·02	632	− 1	631	629	+ 2
232·5–252·5	·498	17·805	822·1	30·02	606	− 1	605	605	0
257·5–277·5	·504	18·432	901·2	30·02	560	− 1	559	559	0
277·5–297·5	·510	18·940	981·6	29·98	520	+ 1	521	521	0
297·5–302·5	·128	19·260	259·7	29·92	493	+ 4	497	497	0
302·5–322·5	·512	19·580	1092·7	29·92	469	+ 4	473	473	0
322·5–337 5	·382	20·028	883·2	29·83	433	+ 8	441	440	+ 1
327·5–342·5	·383	20·156	900·2	29·86	425	+ 7	432	431	+ 1
404·5–409·5	·130	22·002	473·3	29·69	275	+ 16	291	293	− 2
456·5–461·5	·133	23·367	732·9	29·74	181	+ 10	191	192	− 1
507·5–510·5	·081	24·701	879·5	29·96	92	+ 2	94	93	+ 1

Mean rise due to stirring alone = ·0004971.

Hence $tr^3 = 5420 \times 10^4$. (The "Correction to rate 30" is deduced from this value of tr^3.)

The outside temperature throughout the above experiments was 19°·260 C. The actual observation across that temperature was over a range of 0°·128 C. only, and an error of 0°·001 C. in the range would make a difference of 4 in the number in the last column. As the rise at this point was one of the most important of all, a straight line was drawn through the numbers obtained from the pairs of observations on each side of it, the resulting value being ·000497. Assuming this to be the rise due to stirring only (at rate 30), we get radiation coefficient $(\rho) = (\gamma - \cdot000497)/(19\cdot260 - t)$, where γ is rise (gain) per second due to all non-electrical sources.

The mean temperature of the first four observations is 14°·475 C., and the mean value of $\gamma = \cdot853 \times 10^{-6}$, hence

$$\rho = \cdot000744.$$

The straightness of this "non-electrical supply line" is a matter of extreme importance, as if there were any marked curvature it would seriously affect our conclusions as to the specific heat of water. In the last columns of the above Table we give the numbers obtained by direct calculation, assuming $\rho = \cdot000744$ and $\sigma = \cdot000497$, also the difference between the calculated and experimental results. The irregularities are obviously experimental ones, and are to be expected when it is remembered that an error of 0°·001 C. in the "range" would account for nearly all of them. Again, we are here dealing with our smallest mass of water, and the individual experiments are in better agreement when the depth of water is greater. The even distribution of the numbers about a straight line is rendered more evident to the eye when plotted on a large scale than when presented in rows of figures. The Table also illustrates the accuracy of the correction for differences in rate of stirring.*

Assuming the value of the water equivalent of the calorimeter as 85·70, the value of M (mass of water + water equivalent) = 103·01 + 85·70 = 188·71, we get ·0140 as the number of thermal grammes lost or gained by radiation, &c., per second for a difference in temperature of 1° C.

In order to test to what extent any change in the viscosity of the water caused a change in the stirring heat, we performed a few experiments with different external temperatures.

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* This "non-electrical supply curve" being a straight line, we are enabled to deduce the values of both ρ and σ by two experiments only, conducted at any two different temperatures.

TABLE XXII.

Date.	Exp.	Range E_m.	Range C.	Mean temperature and temperature of outer bath.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}} \times 10^6$.	(Rate correction) $\times 10^6$.	(Rise per second at rate 30) $\times 10^6$.
Aug. 12, 1892	LV.	87·5–108·5	·509	14·251	1009·2	29·99	504	+ 1	505
"	LVI.	394·5–409·5	·389	21·873	802·7	29·80	485	+10	495
"	LVII.	494·5–503·5	·241	24·433	470·9	30·21	512	−11	501

As we regarded these experiments only as preliminary, they were somewhat hurriedly performed, and we proposed investigating this point more fully later on. It has been previously pointed out (p. 440) that, in any case, the conclusions arrived at would not affect our results, and we only give them to show that no marked change is indicated. Some similar experiments performed in 1891 had led to the same conclusion.

During this year's (1892) experiments we were not troubled by any air, or water, leakages into the annular space between the calorimeter and the steel chamber. A great portion of our time and attention in previous years having been devoted to leak-hunting, we had considerable (unwished for) opportunities of investigating the effects of changes of pressure, or alterations in the dryness of the surrounding medium.

The smallest trace of moisture has a most astonishing effect on the radiation coefficient. The change in the rate of rise is most marked, and if the moisture present is sufficient to saturate the space when the calorimeter is cooled down, all attempts to obtain a radiation coefficient are hopeless. Any trace of aqueous vapour could immediately be detected by its effect on the rate of rise, and on one or two occasions was sufficient to prevent any rise whatever.

Considerable effect was also caused by changes in pressure, and when comparing the results obtained from different masses it must be remembered that the pressure was frequently changed. Although we were then not conscious of the full importance of this point, we endeavoured to maintain the pressure unaltered from the time that any set of stirring determinations were made until the corresponding J experiments were completed.

Results of the Stirring and Radiation Experiments.

Throughout the remainder of this section the following notation is adopted :—

W = weight of water (*in vacuo*).
M = W + water equivalent of calorimeter.
p = pressure in annular space.
t = time of rising 1° C.
r = rate of revolution of stirrer (revolutions per 1 second).
K = value of tr^3.
σ = rise per 1 second (in degrees C.) due to stirring only.
ρ = gain or loss (in degrees C.) per 1 second due to radiation, conduction, and convection, when the difference of temperature = 1° C.
Q = thermal grms. lost or gained per 1 second by radiation, conduction, and convection = $M\rho$.

Experiments 1 to 31 (see Table XVIII.).—These were all observations taken acros the outside temperature only, and were used to establish the relation $tr^3 = K$.

Experiments 32 to 40.—Are only of use for the same purpose, but they bear out the conclusion that $tr^3 = K$.

As the calorimeter was taken to pieces after Experiment 40 (before any of our J experiments were commenced) the results are of no special value.

Experiments 41 to 50 :—

$$W = 103{\cdot}01$$
$$\sigma = {\cdot}000498$$
$$\rho = {\cdot}0000743$$
$$Q = {\cdot}0104. \quad p = 1{\cdot}15 \text{ millims.}$$

Experiments 51 to 57.—(Tables XIX., XX., and XXII.) (Nos. 41 to 54 were used in the reduction of J 1 to 4.)

Experiments 58 to 65.—During this series the value of σ began to show signs of alteration. The cause has already been mentioned; flakes of gutta-percha having collected at the base of the inner cylinder, the "throw" of the stirrer became less regular. Also (as we were not at this time aware of the importance of maintaining the pressure unaltered), we both diminished and increased the pressure during this series. Fortunately, however, we performed sufficient radiation experiments to enable us to deduce the probable values of the variables appropriate to the respective J determinations. W = 188·065 throughout this series.

Experiments 58 and 59 :—

$$\left.\begin{array}{l}\sigma = {\cdot}000354\\ \rho = {\cdot}000490\\ Q = {\cdot}0134. \quad p = {\cdot}40 \text{ millim.}\end{array}\right\} \text{J 5 to 7.}$$

Experiments 60 and 61 :—

$$\left.\begin{aligned}\sigma &= \cdot000358\\ \rho &= \cdot000478\\ \mathrm{Q} &= \cdot0131. \quad p = \cdot37 \text{ millim.}\end{aligned}\right\} \text{ J 8 and 9.}$$

Experiments 62 to 65 :—

$$\left.\begin{aligned}\sigma &= \cdot000360\\ \rho &= \cdot000483\\ \mathrm{Q} &= \cdot0132. \quad p = \cdot37 \text{ millim.}\end{aligned}\right\} \text{ J 10 to 12.}$$

Experiments 66 to 80.—The cause of irregularity had not yet been ascertained, and therefore not removed. During this series we made several changes in the connections between the stirring rod and the revolving shaft above it, and also in the pressure around the calorimeter.

W = 277·971 throughout this series.

Experiments 66 to 71 :—

$$\left.\begin{aligned}\sigma &= \cdot000276\\ \rho &= \cdot000361\\ \mathrm{Q} &= \cdot0131, \quad p = \cdot37 \text{ millim.}\end{aligned}\right\} \text{ J 13 and 14.}$$

Experiments 72 and 73.—The position of the revolving shaft and counter was slightly altered, and the whole of this portion of the apparatus re adjusted.

$$\left.\begin{aligned}\sigma &= \cdot000272\\ \rho &= \cdot000374\\ \mathrm{Q} &= \cdot0136. \quad p = \cdot38 \text{ millim. ? [*]}\end{aligned}\right\} \text{ J 15 and 16.}$$

Experiments 74 to 78.—The double Hooke's joint was now removed and replaced by a single one in the hope of diminishing the vibration of the stirring rod, the pressure was at the same time somewhat reduced.

The value of σ showed considerable alteration.

$$\left.\begin{aligned}\sigma &= \cdot000267\\ \rho &= \cdot000363\\ \mathrm{Q} &= \cdot0132. \quad p = \cdot37 \text{ millim.}\end{aligned}\right\} \text{ J 17 and 18.}$$

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[*] There is probably some error here. We have only a single note of the pressure during this day (August 28), and we have no means of checking it since Experiment 74 was not performed until August 30. Now (as we find recorded in our notes), we gave twenty-three strokes to the mercury pumps on August 29, the resulting pressure on August 30 being ·37 millim.; it appears almost certain, therefore, that the pressure on August 28 must have exceeded ·38 millim.

Experiments 79 and 80.—The simple HOOKE's joint was now replaced by a light ring of wire, which passed through holes in the stirring rod and the revolving shaft.

$$\left.\begin{aligned}\sigma &= \cdot000272\\ \rho &= \cdot000374\\ Q &= \cdot0136. \quad p = \cdot44 \text{ millim.}\end{aligned}\right\} \text{J 19.}$$

It thus appeared that both σ and ρ had resumed their former values.

Experiments 81 and 82.—These two experiments were intended to be the first pair of a series. On commencing J 20, the same evening (September 4), it was discovered that the coil in the calorimeter had become loose. The calorimeter had, in consequence, to be taken to pieces, and a week elapsed before we were able to resume our J determination. These two experiments are consequently of no use, but they are interesting as showing the kind of accuracy obtainable.

$$\begin{aligned}W &= 140\cdot270.\\ \sigma &= \cdot000439.\\ \rho &= \cdot000578.\\ Q &= \cdot0130. \qquad p = \cdot36 \text{ millim.}\end{aligned}$$

Experiments 83 to 93.—After Experiment 82 a marked difference is observable in the value of σ. Between September 4 and 10 the calorimeter had been thoroughly cleaned, and all gutta-percha surfaces removed from its interior. The "throw" of the stirrer was evidently improved, and as a consequence the value of σ was raised.

The series of J Experiments (Series II., J 20 to 34), performed after this change, are so much superior to their predecessors that we base our conclusions almost entirely upon them. We feel it right, therefore, to give more particulars concerning their "non-electrical supply" determinations than we have considered necessary in the case of Series I. (J 1 to 19). The following table is a summary of the "stirring experiments" used in the reduction of J 20 to 25.

TABLE XXIII.—W = 139·776.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction × 10^6.	Rise at rate 30.	Mean temperature C.
Sept. 10 .	292·5–307·5	·384	752·05	29·93	·000510	+ 3	·000513	19·260
,, 10 .	87·5–127·5	·973	1173·43	30·35	·000829	− 18	·000811	14·476
,, 10 .	292·5–307·5	·384	746·75	30·07	·000514	− 4	·000510	19·260
,, 11 .	292·5–307·5	·384	733·6	30·18	·000523	− 9	·000514	19·260
,, 11 .	292·5–307·5	·384	753·22	29·96	·000510	+ 2	·000512	19·260
,, 11 .	442·5–452·5	·263	899·23	30·31	·000293	− 13	·000280	23·064
,, 13 .	292·5–307·5	·384	723·05	30·20	·000522	− 10	·000512	19·260
,, 13 .	87·5–127·5	·973	1168·85	30·39	·000832	− 20	·000812	14·476
,, 13 .	442·5–452·5	·263	890·65	30·30	·000296	− 15	·000281	23·064

Hence

$$\left.\begin{aligned} {}^{*}\sigma &= \cdot 000512, \text{ therefore } tr^3 = 5273 \times 10^4 \\ \rho &= \cdot 000620 \\ Q &= \cdot 0140. \qquad p = 1\cdot 15 \text{ millims.} \end{aligned}\right\} \text{J 20 to 25.}$$

Experiments 94 to 102.

TABLE XXIV.—W = 199·674.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction × 10^6.	Rise at rate 30.	Mean temperature C.
Sept. 14 .	292·5–307·5	·384	898·68	30·10	·000428	− 4	·000424	19·260
,, 14 .	442·5–452·5	·263	1161·05	29·81	·000227	+ 8	·000235	23·064
,, 14 .	442·5–452·5	·263	1110·62	30·04	·000237	− 2	·000235	23·064
,, 14 .	87·5–127·5	·973	1481·7	30·16	·000657	− 7	·000650	14·476
,, 15 .	292·5–307·5	·384	856·65	30·52	·000448	− 22	·000426	19·260
,, 15 .	87·5–127·5	·973	1519·8	29·77	·000640	+ 10	·000650	14·476
,, 16 .	292·5–307·5	·384	880·98	30·25	·000437	− 11	·000426	19·260

Hence

$$\left.\begin{aligned} \sigma &= \cdot 000426, \text{ therefore } tr^3 = 6353 \times 10^4 \\ \rho &= \cdot 000485 \\ Q &= \cdot 0138. \qquad p = \cdot 48 \text{ millim.} \end{aligned}\right\} \text{J 26 to 29.}$$

Experiments 103 to 110.

TABLE XXV.—W = 259·500.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction × 10^6.	Rise at rate 30.	Mean temperature C.
Sept. 16 .	292·5–307·5	·384	1081·6	30·03	·000355	− 1	·000354	19·260
,, 17 .	292·5–307·5	·384	1025·7	30·50	·000374	− 18	·000356	19·260
,, 17 .	87·5–127·5	·973	1816·5	29·78	·000536	+ 8	·000544	14·476
,, 18 .	87·5–127·5	·973	1784·2	30·05	·000545	− 2	·000543	14·476
,, 18 .	292·5–307·5	·384	1081·8	30·03	·000355	− 1	·000354	19·260
,, 18 .	292·5–307·5	·384	1078·7	30·03	·000356	− 1	·000355	19·260
,, 18 .	442·5–452·5	·263	1335·7	29·88	·000197	+ 4	·000201	23·064
,, 18 .	442·5–452·5	·263	1284·1	30·10	·000205	− 4	·000201	23·064
,, 18 .	452·5–462·5	·265	1278·6	29·78	·000207	− 14	·000193	23·327

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* The values of σ and ρ given at the end of the Tables are derived from large scale curves, and thus give the mean of all the observations.

Hence

$$\sigma = \cdot 000355 \qquad tr^3 = 7606 \times 10^4$$
$$\rho = \cdot 000399$$
$$Q = \cdot 0138 \qquad p = \cdot 48 \text{ millims.}$$

The reduction of our observations involved so much arithmetic that the whole series of experiments was concluded before we were able to get the results in the form now given.

Although we were prepared to find a rapid reduction in the value of Q as the pressure diminished, we had not realized the importance of small changes in the pressure. This was unfortunate, for, as the figures show, our average pressure was just that at which small alterations produced the greatest effect. We usually considered it sufficient to record the pressure each day, whereas it appears that a careful observation of the pressure during each stirring experiment ought to have been made. It is evident that we should have done better to select pressures of not less than 1 millim., and thus small variations would have had but little effect.

At the time of the observations we did not know the real value of the water equivalent. It was thus impossible for us to obtain Q with any accuracy, and it is only by the changes in the value of Q that the effects of changes in pressure are rendered evident.

The results compare very favourably with the conclusions arrived at by BOTTOMLEY.*

Collecting the numbers given in this section (with the exception of Experiments 72 and 73, see note, p. 450), we get

TABLE XXVI.

Date.	Experiment.	Mass of water.	Pressure in millims.	Thermal grms. per second.
Sept. 10–13 . . .	83– 93	139·78	1·15	·0140
Aug. 8–10 . .	41– 50	103·01	1·15	·0140
„ 11	51– 54	103·01	·98	·0140
Sept. 14–16 . . .	94–102	199·67	·48	·0138
„ 16–18 . . .	103–110	259·50	·48	·0138
Aug. 26	79– 80	277·93	·44	·0136
„ 14–26 . . .	58– 59	188·07	·40	·0134
„ 24, 25 . . .	62– 65	188·07	·37	·0132
„ 30, 31 . . .	74– 78	277·93	·37	·0132
„ 26, 27 . . .	66– 71	277·93	·37	·0131
„ 17, 18 . . .	60– 61	188·07	·37	·0131
Sept. 4	81– 82	140·27	·36	·0130

These numbers are plotted in fig. 8 the abscissæ denoting pressures and the ordinates the number of the thermal grms. gained per second. When it is remembered that adjacent points are obtained from very different masses of water, the close agreement is a proof of the accuracy of the observations. The sudden change in the value of Q when the pressure falls below ·50 millim. is remarkable; but a reference to BOTTOMLEY'S paper will show that the curve given by him is of a similar type, although the critical point is at a higher pressure and the bend in the curve is less sharp.

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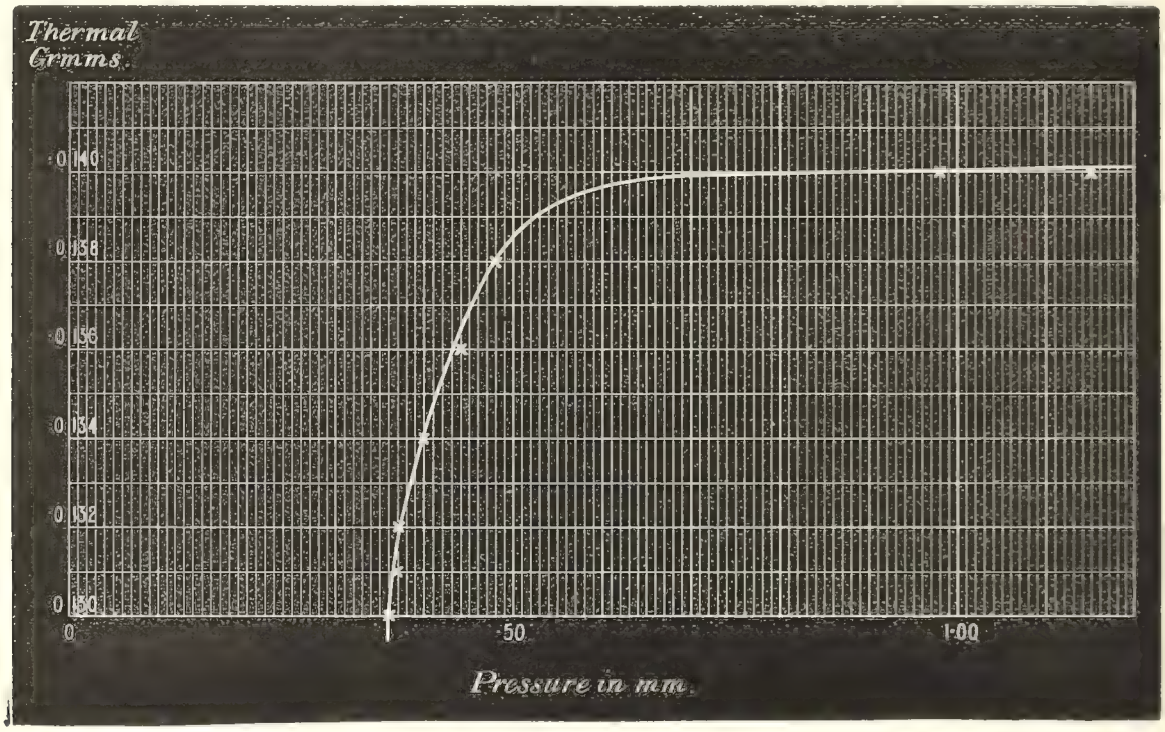
* 'Phil. Trans.,' 1887, A.

Fig. 8.



Although all the observations on the "non-electrical supply" have been brought together in this section for the purposes of comparison, they were not taken continuously but in groups. Each group of observations was used for the reduction of J experiments performed at the same time and on the same weight. The stirring experiments were usually performed during the daytime, the corresponding J experiments in the evening, and it was rarely that two sets of J observations were taken on the same day. Throughout the whole of the experiments J 1 to 34, as also their corresponding stirring experiments, the surrounding temperature was maintained at 300 E = 19·260 C.*

If θ_1 = temperature of calorimeter, then $(\theta_1 - 19{\cdot}260)\,\rho$ gives the rise or fall in temperature per second due to radiation, &c., hence $\sigma + (\theta_1 - 19{\cdot}260)\,\rho$, gives the rise in temperature per second due to the whole "non-electrical" supply. As previously pointed out (see note, p. 411) we proposed to reduce the numbers obtained from our J experiments at certain fixed temperatures and the following Table gives the changes in temperature per 1 second, due to the "non-electrical" supply at these points. The numbers were obtained from large-scale curves, but can be calculated from the values of σ and ρ given in this section.

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* In previous years we had performed several pairs of experiments in which the outside temperature was, in one case at the bottom, and in the other at the top, of our range. By "meaning" the reciprocals of the times of rising 1°, the effect of radiation, &c., could be eliminated. We hoped to repeat these experiments this year, but want of time prevented our doing so.

TABLE XXVII.—Rise per second due to non-electrical supply, at temperatures required for the reduction of the J experiments.

J (expts.) =	1–4.	5–7.	8–9.	10–12.	13–14.	15–16 and 19.	17–18.	20–25.	26–29.	30–34.
W =	103·010	188·065	188·065	188·065	277·931	277·931	277·931	139·776	199·674	259·500
$\sigma \times 10^6$ =	498	354	358	360	276	272	267	512	426	355
$\rho \times 10^6$ =	743	490	478	483	361	374	363	621	483	399
					Value of $\{\sigma + \rho\,(19{\cdot}260 - \theta_1)\} \times 10^6$.					
Temp. C.										
14·477	853	587	587	591	449	451	441	809	658	546
15·581	771	534	534	538	409	410	401	741	604	502
16·682	690	480	481	485	369	368	361	672	551	458
17·683	615	431	433	436	333	331	324	610	502	418
18·688	541	382	385	388	297	293	288	548	454	378
19·835	455	326	331	332	255	250	246	476	397	332
21·115	360	263	269	270	209	203	200	397	336	281
22·409	264	200	207	208	162	154	153	316	273	229
23·862	156	128	138	138	110	100	100	226	203	171
25·006	71	78	83	82	69	57	58	155	148	125

The above Table is deduced from "stirring" experiments Nos. 41 to 110. Th values of ρ and σ, obtained by repeating an experiment under the same conditions, rarely show any difference as great as ·000004, and thus a difference from the mean value of ·000002 may be regarded as our limit of experimental error. Now a change of two in the last digit would, when 5 Clark cells were used, cause a difference of less than 1 in 4000 in the resulting value of J; but as the numbers in each column are obtained from independent experiments, and as all the columns are used in the final determination of J, it is probable that the mean resulting error is of very small dimensions.

[Note by E. H. G. Added April, 1893. The statement that the loss or gain by radiation, convection, and conduction is, in our apparatus, a linear function of the difference in temperature may at first sight appear to differ from the conclusions arrived at by DULONG and PETIT, MACFARLANE, and other observers, but I think that the contradiction is more apparent than real. The curve which shows the absolute loss or gain (fig. 8, p. 454) indicates that, if we had reduced the pressure to something under one-thousandth of a millimetre, the loss or gain by radiation, convection, and conduction might almost have been disregarded over our range, and hence it is evident that convection plays by far the greatest part at such pressures as those with which we were working; and the wonderful experiments of Professor DEWAR (the account of which had not been published when the preceding section was written) appear to place this conclusion on a firm foundation. Now DULONG and PETIT make the loss or gain by convection vary directly as the difference in temperature when the density of the gas remains constant. In our apparatus the density of the air surrounding the calorimeter must have remained nearly constant throughout each experiment, and therefore our observations appear to corroborate, rather than to contradict, the conclusions arrived at by DULONG and PETIT.]

SECTION XIII.—THE METHOD OF PERFORMING A J EXPERIMENT.

The general method of procedure was as follows.

Freshly distilled ether having been placed in the cooling tube, the temperature of the calorimeter was lowered until the resistance of the coil indicated a temperature of 10° C., any remaining ether was then withdrawn and dried air passed through the cooling apparatus for about ten minutes. The aspirating portion of the apparatus was removed and the cooling tube cleaned by a small mop of absorbent cotton wool. The open end was closed by a long cork, and a metal tube placed over the whole of the projecting portion, the lower end of this tube being washed by the tank water.

The storage circuit was then completed and the rheostat adjusted until the balance with the Clark cells was obtained. About twenty minutes elapsed before the temperature rose to 86 E_m—the point at which we commenced our observations. During this interval the chronograph circuit was tested, and any final adjustment of the motor and external circuit completed.

The task of one observer was invariably that of maintaining the potential balance by means of the rheostat, and also inverting the currents at regular intervals. The attention of the second observer was directed to recording the group of transits about each fixed point, the chronograph key being placed at the base of the telescope stand. In the intervals he recorded the times of the revolutions of the stirrer, which were distinguished on the tapes from the temperature records by double marks. He also occasionally recorded the readings of thermometer A (the external temperature), but these were as a rule so regular as to render this observation a matter of form.

Thus, during the progress of an experiment, no notes had to be taken unless some exceptional incident occurred. We believe this to be an important matter during observations of this description, for the mechanical operation of note-taking greatly distracts the attention.

The chronograph required rewinding about every twenty minutes, and in the press of the other observations this was sometimes forgotten. Some of the omissions in the tables are due to this cause. The duration of a whole experiment varied from 40 to 80 minutes, according to the weight of water present and the E.M.F. used.

At the end of an experiment, the chronograph tape was counted and the results tabulated in the form shown in Tables XXVIII. to XXXIII. These six experiments are typical of the rest, and have not been in anyway selected, except that they are the first performed on each weight, in Series II., with different E.M.F. s.

TABLE XXVIII.—Series II. J 20. September 11, 1892.

Weight of water (*in vacuo*) = 139·776 grms.
Clark Cells.* $n = 4$. Nos. **(37, 38, 39)**, **(40, 41, 42)**, **(43, 44, 45)**, **(46, 47, 48)**.
Temperature = 15°·05 C.
External temperature throughout experiment = 241·2 A = 300 E = 19°·260 C.
Pressure = ·97 millim.
Calorimeter cooled to 10°·00 C. before commencing experiment. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	7	..	4	4	
87	11·8				
88	16·8				
89	21·8	..	5	36·8	
Mean					
126	206	199	10	200·2	
127	211	199·2			
128	216	199·2			
129	221·2	199·4	11	233·2	
Mean	..	199·2	..	..	30·54
176	464·2	258·2	17	431	
177	469·2	258·4			
178	474·7	258·7			
179	479·8	258·5	19	496·6	
†Mean	..	258·45	..	..	30·37
216	673·8	209·6			
217	679·1	209·7			
218	684·5	209·8			
219	689·9	210·1	25	693·5	
Mean	..	209·8	..	..	30·47
256	888·3	214·5	30	857·5	
257	893·8	214·7			
258	899·2	214·7			
259	904·8	214·9	32	922·8	
Mean	..	214·7	..	..	30·53

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* The cells in brackets were placed in parallel arc.

TABLE XXVIII.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1111·3	223	37	1087	
297	1116·9	223·1			
298	1122·7	223·5			
299	1128·3	223·5			
Mean	. .	223·38	. .	. .	30·45
346	1398	286·7	46	1383·5	
347	1403·9	287			
348	1409·6	286·9			
349	1415·4	287·1			
Mean	. .	286·93	. .	. .	30·35
396	1691·5	293·5			
397	1697·4	293·5			
398	1703·4	293·8			
399	1709·5	294·1	56	1712·4	
†Mean	. .	293·75	. .	. . .	30·41
446	1997·4	305·9			
447	2003·7	306·3			
448	2010·0	306·6			
449	2016·2	306·7	66	2039·6	
Mean	. .	306·38	. .	. .	30·56
506	2380·8	383·4	76	2366·7	
507	2387·5	383·8			
508	2394·2	384·2			
509	2400·8	384·6	78	2431·5	
Mean	. .	384	. .	. .	30·54
536	2579·8	199	82	2563·8	
537	2586·8	199·3			
538	2593·5	199·3			
539	2600·3	199·5	84	2629·7	
Mean	. .	199·38	. .	. .	30·42

TABLE XXIX.—Series II. J 23. September 12, 1892.

Weight of water (*in vacuo*) = 139·776 grms.

Clark cells. $n = 3$. Nos. **(37, 38, 39)**, **(40, 41, 42)**, **(43, 44, 45)**. Temperature 15°·21 C.

External temperature. 241·2 A = 300 E = 19°·260 C. Pressure = ·78 millim.

Calorimeter cooled to 10°·10 C. before commencing experiment. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolution.	Rate.
	seconds.		thousands.	seconds.	
86	− 9·5 ?				
87	− 1·7				
88	6				
89	14	..	7	21	
Mean					
126	305·2	314·7 ?			
127	313·2	314·9			
128	321	315			
129	329·5	315·5	17	354·2	
Mean	..	315·03	..	..	30·01
176	714·2	409			
177	723·6	409·4			
178	731	410			
179	739·2	409·7	29	753·4	
†Mean	..	409·52	..	..	30·06
216	1050·3	336·1			
217	1058·7	336·1			
218	1067·3	336·3			
219	1076	336·8	40	1120	
Mean	..	336·33	..	..	30·01
256	1397·2	346·9			
257	1406	347·3			
258	1415	347·7			
259	1423·9	347·9	50	1453	
Mean	..	347·45	..	..	30·02

TABLE XXIX.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolution.	Rate.
	seconds.		thousands.	seconds.	
296	1760·9	363·7	59	1753·6	
297	1770·2	364·2			
298	1779·7	364·7			
299	1789	365·1	61	1820·3	
Mean	..	364·43	..	..	29·95
346	2231·5	470·6	73	2219·9	
347	2240·9	470·7			
348	2250·5	470·8			
349	2260	471	75	2286·7	
Mean	..	470·78	..	..	30·02
396	2719	487·5	87	2686·6	
397	2729	488·1			
398	2739	488·5			
399	2749·3	489·3	89	2753	
†Mean	..	488·35	..	..	30·02
446	3233·9	514·9	103	3219·3	
447	3244·8	515·8			
448	3254·9	515·9			
449	3265·8	516·5	105	3286	
Mean	..	515·78	..	..	30·02

Owing to the breakage of a wire in the clock and chronograph circuit, the records of the times from 449 to 539 are so uncertain that we have decided to omit them.

TABLE XXX.—Series II. J 26. September 14, 1892.
Weight of water (*in vacuo*) = 199·674 grms.
Clark cells. $n = 4$. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48).**
Temperature 15°·12 C.
External Tempereture 241·2 A = 300 E = 19·260° C. Pressure = ·48 millim.
Calorimeter cooled to 10°·00 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	45·2	..	0	19·6	
87	51·7				
88	57·9				
89	64·1	..	2	86·1	
126	297·7	252·5	8	285·3	
127	304	252·3			
128	310	252·1			
129	316·8	252·7			
Mean	..	252·4	..	..	30·10
176	623·8	326·1	18	617·6	
177	630	326			
178	636·7	326·7			
179	643	326·2	19	650·8	
† Mean	..	326·25	..	..	30·12
216	888·6	264·8	26	883·2	
217	895·2	265·2			
218	902	265·3			
219	908·9	265·9	27	916·4	
Mean	..	265·3	..	..	30·12
256	1160·7	272·1	34	1148·8	
257	1167·1	271·9			
258	1174	272			
259	1181	272·1			
Mean	..	272·03	..	..	30·10

TABLE XXX.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1442·1	281·4	42	1414·6	
297	1449·4	282·3			
298	1456·5	282·5			
299	1463·8	282·8	44	1481	
Mean	. .	282·25	. .	. .	30·12
346	1804·3	362·2	53	1780·3	
347	1811·7	362·3			
348	1818·9	362·4			
349	1826·5	362·7	55	1847·3	
Mean	. .	362·4	. .	. .	30·02
396	2175	370·7	64	2147	
397	2182·9	371·2			
398	2190·5	371·6			
399	2198	371·5	66	2213·7	
† Mean	. .	371·25	. .	. .	30·02
446	2562	387	76	2546·4	
447	2569·9	387			
448	2577·7	387·2			
449	2585·6	387·6	78	2613	
Mean	. .	387·20	. .	. .	30·05
506	3046	484	90	3012	
507	3054·3	484·4			
508	3062·8	485·1			
509	3071·2	485·6	92	3078·5	
Mean	. .	484·78	. .	. .	30·08
536	3297·4	251·4	97	3244·7	
537	3305·9	251·6			
538	3314	251·2			
539	3322·6	251·4	98	3278	
Mean	. .	251·4	. .	. .	30·07

TABLE XXXI.—Series II. J 27. September 15, 1892.

Weight of water (*in vacuo*) = 199·674 grms.

Clark cells. $n = 5$. Nos. **(37, 38, 39)**, **(40, 41, 42)**, **(43, 44, 45)**, **(46, 47, 48)**, **(52, 53, 54)**

Temperature 15° C.

External Temperature 241·1 A = 299·8 E = 19°·258 C. Pressure = ·48 millim.

Calorimeter cooled to 10°·01 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	76·7	..	7	22·9	
87	80·9				
88	85·2				
89	89·4				
Mean					
126	249·7	173	13	223·7	
127	254	173·1			
128	258·4	173·2			
129	262·6	173·2	15	290·7	
Mean	..	173·13	..	..	29·88
176	471·8	222·1	20	457·5	
177	476	222			
178	480·6	222·2			
179	485	222·4	21	490·5	
Mean	..	222·18	..	..	30·13
†216	651·6	179·8	25	623·2	
217	656	180			
218	660·5	179·9			
219	665	180	27	689·5	
Mean	..	179·95	..	..	30·15
256	834·7	183·1	31	822·4	
257	839·5	183·5			
258	844	183·5			
259	848·8	183·8	32	855·6	
Mean	..	183·48	..	..	30·22

Table XXXI.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1024·4	189·7	37	1021·6	
297	1029	189·5			
298	1034	190			
299	1038·6	189·8	38	1054·8	
Mean	..	189·75	..	..	30·12
346	1267·2	242·8	44	1254·4	
347	1271·8	242·8			
348	1276·7	242·7			
349	1281·6	243	45	1287·5	
Mean	..	242·83	..	..	30·08
396	1513·6	246·4			
397	1518·6	246·8			
398	1523·6	246·9			
†399	1528·8	247·2	53	1553·2	
Mean	..	246·83	..	..	30·11
446	1769·3	255·7	59	1752·6	
447	1774·7	256·1			
448	1779·9	256·3			
449	1785	256·2			
Mean	..	256·08	..	..	30·10
506	2087·5	318·2	68	2051·1	
507	2092·8	318·1			
508	2098·4	318·5			
509	2103·7	318·7	70	2117·3	
Mean	..	318·38	..	..	30·16
536	2251	163·5	73	2216·6	
537	2256·8	164			
538	2262·2	163·8			
539	2267·9	164·2			
Mean	..	163·88	..	..	30·22

TABLE XXXII.—Series II. J 33. September 18, 1892.

Weight of water (*in vacuo*) = 259·500 grms.

Clark cells. $n = 5$. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48), (52, 53, 54).**

Temperature 15°·06 C.

External temperature 241·2 A = 300 E = 19°·260 C. Pressure = ·44 millim.

Calorimeter cooled to 10°·10 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	42·9	..	8	18	
87	48				
88	53·1				
89	58·5	..	10	84·6	
Mean	..	..	..	..	
126	251·7	208·8			
127	257	209			
128	262·2	209·1			
129	267·7	209·2	16	283·8	
Mean	..	209·03	..	..	30·10
176	520·4	268·7			
177	525·7	268·7			
178	531·2	269			
179	536·5	268·8	24	549·2	
Mean	..	268·75	..	..	30·15
†216	737·7	217·3			
217	743	217·3			
218	748·5	217·3			
219	754·1	217·6	31	781·6	
Mean	..	217·38	..	..	30·12
256	959·3	221·6	36	947·4	
257	965	222			
258	970·6	222·1			
259	996·5	222·4	37	980·7	
Mean	..	222·03	..	..	30·14

TABLE XXXII—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1188·9	229·6	43	1180·3	
297	1194·7	229·7			
298	1200·4	229·8			
299	1206·2	229·7	44	1213·6	
Mean	. .	229·7	. .	. .	30·06
346	1482	293·1	52	1479·7	
347	1487·8	293·1			
348	1493·6	293·2			
349	1499·2	293	53	1512·9	
Mean	. .	293·1	. .	. .	30·07
396	1780	298			
397	1786·2	298·4			
398	1792·3	298·7			
399	1798·4	299·2	62	1812	
Mean	. .	298·58	. .	. .	30·10
†446	2088·8	308·8	70	2077·7	
447	2095	308·8			
448	2101·2	308 9			
449	2107·7	309·3	71	2110·9	
Mean	. .	308·95	. .	. .	30·11
506	2472	383·2	81	2441·7	
507	2478·6	383·6			
508	2485·2	384			
509	2491·7	384	83	2507·6	
Mean	. .	383·7	. .	. .	30·25
536	2669·7	197·7	87	2639·8	
537	2676·4	197·8			
538	2682·9	197·7			
539	2689·9	198·2			
Mean	. .	197·85	. .	. .	30·26

TABLE XXXIII.—Series II. J 34. September 18, 1892.

Weight of water (*in vacuo*) = 259·500 grms.

Clark cells. $n = 6$. Nos. **(37, 38, 39)**, **(40, 41, 42)**, **(43, 44, 45)**, **(46, 47, 48)**, **(49, 50, 51)**, **(52, 53, 54)**. Temperature 15° C.

External temperature 241·2 A = 300 E = 19°·260 C. Pressure = ·48 millim.

Calorimeter cooled to 10°·01 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	65·9	. .	0	18·5	
87	69·6				
88	73·7				
89	77·3	. .	2	85·5	
126	217·4	151·5	5	184	
127	221·4	151·8			
128	225·2	151·5			
129	228·9	151·6	7	250	
Mean	. .	151·6	. .	. .	30·23
176	411·5	194·1	11	382·5	
177	415·4	194			
178	419·3	194·1			
179	423·2	194·3	13	448·5	
†Mean	. .	194·13	. .	. .	30·25
216	568	156·5	16	547·5	
217	572	156·6			
218	576	156·7			
219	580	156·8	18	613·5	
Mean	. .	156·65	. .	. .	30·30
256	727·5	159·5	21	712·4	
257	731·5	159·5			
258	735·5	159·5			
259	739·6	159·6	22	745·4	
Mean	. .	159·53	. .	. .	30·23

Table XXXIII.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	891·7	164·2	26	877	
297	895·8	164·3			
298	900	164·5			
299	904·2	164·6	27	910	
Mean	..	164·4	..	..	30·28
346	1100·8	209·1			
347	1104·8	209			
348	1109	209			
349	1113	208·8	34	1140·3	
Mean	..	208·98	..	..	30·40
396	1312·5	211·7	39	1304·7	
397	1317	212·2			
398	1321	212			
399	1325·4	212·4	40	1337·7	
Mean	..	212·08	..	..	30·40
†446	1531·2	218·7			
447	1535·7	218·7			
448	1540	219			
449	1544·6	219·2	47	1567·7	
Mean	..	218·9	..	..	30·43
506	1802·7	271·5	54	1797·8	
507	1807·2	271·5			
508	1811·9	271·9			
509	1816·5	271·9	55	1830·6	
Mean	..	271·7	..	..	30·43
536	1942·3	139·6	58	1929·8	
537	1947	139·8			
538	1951·7	139·8			
539	1956·3	139·8			
Mean	..	139·75	..	..	30·33

For convenience of reference in the succeeding Tables, we repeat here the Table showing the values we have assigned to each range on the rising thermometer.

Table XXXIV.

No. of range.	E_m.	C.	Mean.	$d\theta$.
1	87·5–127·5	13·990–14·963	14·477	·973
2	127·5–177·5	14·963–16·198	15·581	1·235
3	177·5–217·5	16·198–17·185	16·682	·987
4	217·5–257·5	17·185–18·180	17·683	·995
5	257·5–297·5	18·180–19·196	18·688	1·016
6	297·5–347·5	19·196–20·474	19·835	1·278
7	347·5–397·5	20·474–21·756	21·115	1·282
8	397·5–447·5	21·756–23·063	22·409	1·307
9	447·5–507·5	23·063–24·661	23·862	1·598
10	507·5–537·5	24·661–25·471	25·006	·811

We consider it unnecessary to give more of our original observations in full, but have collected in Tables XXXV. and XXXVI. the unreduced experimental results of all the J experiments, with the exception of J 1 to 4. (See p. 475.)

Table XXXV.—Series I. Experiments 5–19.

Group B. W = 188·065 gr.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
Aug. 14 5	16·20 5	Time Rate	168·87 30·21	216·73 30·21	175·85 30·10	179·47 30·06	185·60 30·08	236·90 30·07	— —	250·00 30·05	311·46 30·04	160·12 30·13
Aug. 14 6	16·50 5	Time Rate	— —	— —	175·65 30·13	179·47 30·22	185·41 30·19	236·43 30·35	241·01 30·23	250·20 30·18	311·31 30·17	160·45 30·11
Aug. 16 7	16·70 5	Time Rate	168·97 30·02	217·34 30·00	175·92 30·02	179·70 29·99	186·23 29·92	237·28 30·05	241·18 29·97	250·40 29·97	311·78 30·01	160·40 30·16
Aug. 17 8	15·95 4	Time Rate	249·22 29·87	321·71 29·93	261·36 29·97	267·71 29·97	277·59 30·10	356·73 29·97	364·13 29·94	381·01 30·04	475·88 30·00	246·55 30·00
Aug. 18 9	16·30 2	Time Rate	— —	— —	755·64 29·81	792·83 29·86	844·13 29·90	1116·16 29·90	— —	— —	— —	— —
Aug. 23 10	16·55 4	Time Rate	246·20 31·20	317·62 31·16	258·03 31·14	264·75 31·11	275·23 31·18	351·64 31·22	359·96 31·22	— —	— —	— —
Aug. 24 11	16·60 4	Time Rate	249·84 29·48	322·12 29·66	262·47 29·26	269·52 29·19	279·60 29·15	357·90 29·20	368·69 29·43	378·83 30·36	471·63 30·71	245·38 30·81
Aug. 25 12	16·60 4	Time Rate	248·28 30·20	320·77 30·19	260·30 30·21	267·63 30·24	277·55 30·16	355·27 30·18	364·93 30·04	380·52 30·11	476·35 30·07	246·63 30·06

TABLE XXXV.—(continued).

Group E. W = 277·931 grms.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
Aug. 26 13	16·15 4	Time Rate	330·52 29·77	426·35 29·80	346·63 29·82	355·30 29·87	368·82 29·79	472·58 29·84	484·37 29·78	504·48 29·78	633·02 29·74	327·63 29·71
Aug. 27 14	16·15 4	Time Rate	330·70 29·72	426·22 29·82	347·08 29·84	355·22 29·83	368·75 29·82	473·43 29·61	485·32 29·68	504·85 29·77	632·50 29·78	327·73 29·76
Aug. 28 15	16·05 5	Time Rate	223·60 30·34	287·18 30·38	232·62 30·44	237·23 30·54	245·22 30·52	313·25 30·58	319·28 30·54	330·47 30·61	411·63 30·63	211·52 30·47
Aug. 28 16	16·05 5	Time Rate	224·45 29·87	288·35 29·88	233·45 29·89	238·40 29·90	246·47 29·91	314·65 29·89	320·45 29·85	332·18 29·81	412·50 29·79	213·10 29·77
Aug. 30 17	16·50 4	Time Rate	331·12 30·17	426·88 30·13	346·82 30·02	355·98 30·04	370·02 29·95	473·83 29·92	485·20 30·10	505·12 30·14	633·10 30·11	327·83 30·11
Aug. 31 18	16·10 4	Time Rate	329·57 30·16	427·18 29·68	346·70 30·22	353·90 30·36	367·97 30·39	471·68 30·33	483·25 30·35	504·15 30·31	629·47 30·37	327·50 30·37
Sept. 1 19	16·00 5	Time Rate	224·52 29·73	288·33 29·84	233·50 29·59	238·80 29·79	246·80 29·54	314·42 29·82	320·60 30·03	331·33 29·77	412·60 29·75	211·07 31·52

TABLE XXXVI.—Series II. Experiments 20–34.

Group A. W = 139·776 grms.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1	2	3	4	5	6	7	8	9	10
Sept. 11	15·05	Time	199·20	258·45	209·80	214·70	223·38	286·93	293·75	306·38	384·00	199·38
20	4	Rate	30·54	30·37	30·47	30·53	30·45	30·35	30·41	30·56	30·54	30·42
Sept. 11	15·00	Time	136·60	175·98	142·45	145·63	150·60	191·98	195·38	202·70	252·33	130·20
21	5	Rate	30·36	30·37	30·31	30·13	30·12	30·26	30·37	30·34	30·37	30·40
Sept. 12	15·00	Time	202·40	260·60	211·60	217·00	225·30	289·05	296·72	308·97	387·40	201·25
22	4	Rate	29·48	29·53	29·67	29·64	29·66	29·58	29·69	29·68	29·68	29·63
Sept. 12	15·10	Time	315·03	409·53	336·33	347·45	364·43	470·78	488·35	515·78	—	—
23	3	Rate	30·01	30·06	30·01	30·02	29·95	30·02	30·02	30·02	—	—
Sept. 13	15·00	Time	136·13	175·28	142·05	144·73	149·65	190·60	194·38	201·35	250·75	—
24	5	Rate	30·82	30·89	30·87	30·80	30·96	31·04	30·97	31·06	31·26	—
Sept. 13	15·25	Time	199·80	258·68	210·45	215·68	223·88	287·40	294·58	307·38	384·90	199·68
25	4	Rate	30·25	30·18	30·18	30·21	30·24	30·21	30·23	30·25	30·26	30·27

TABLE XXXVI.—(continued).

Group C. W = 199·674 grms.

Date and number.	Tempe-rature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1	2	3	4	5	6	7	8	9	10
Sept. 14 26	15·10 4	Time Rate	252·40 30·10	326·25 30·12	265·30 30·12	272·03 30·10	282·25 30·12	362·40 30·02	371·25 30·02	387·20 30·05	484·78 30·08	251·40 30·07
Sept. 15 27	15·00 5	Time Rate	173·13 29·88	222·18 30·13	179·95 30·15	183·48 30·22	189·75 30·12	242·83 30·08	246·83 30·11	256·08 30·10	318·38 30·16	163·88 30·22
Sept. 15 28	15·10 4	Time Rate	252·40 30·10	326·00 30·14	265·18 30·15	272·08 30·05	282·10 30·13	361·28 30·33	369·15 30·40	384·95 30·44	482·15 30·47	249·00 30·70
Sept. 16 29	15·00 5	Time Rate	172·70 30·25	221·88 30·36	179·65 30·39	183·25 30·33	189·60 30·27	241·98 30·34	246·00 30·43	255·30 30·39	317·70 30·42	163·73 30·39

Group D. W = 259·500 grms.

Sept. 16 30	15·05 5	Time Rate	209·33 29·93	269·10 29·93	217·80 29·93	222·30 29·85	230·48 29·50	294·45 29·50	300·05 29·47	310·50 29·58	386·40 29·61	199·03 29·70
Sept. 17 31	15·20 5	Time Rate	209·46 30·00	269·23 30·07	217·95 30·05	222·40 29·98	230·00 30·00	293·23 30·07	298·63 30·08	309·63 30·19	384·60 30·18	198·28 30·20
Sept. 18 32	14·80 5	Time Rate	209·13 30·00	268·97 30·11	217·78 30·08	222·38 30·04	229·68 29·92	293·80 29·83	298·95 29·95	309·38 30·15	384·58 30·14	198·25 30·19
Sept. 18 33	14·80 5	Time Rate	209·03 30·10	268·75 30·15	217·38 30·12	222·03 30·14	229·70 30·06	293·10 30·07	298·58 30·10	308·95 30·11	383·70 30·25	197·85 30·26
Sept. 18 34	15·00 6	Time Rate	151·10 30·23	194·13 30·25	156·65 30·30	159·53 30·23	164·40 30·28	208·98 30·40	212·08 30·40	218·90 30·43	271·70 30·43	139·75 30·33

Observations on Tables XXXV. and XXXVI.

Experiments 1 *to* 4.

These have been rejected and the observations are not included in the Tables. The discrepancies over the shorter ranges are absurdly great, and the mean times far too large. During these experiments we had great difficulty in maintaining the potential-balance; this ought to have been sufficient to indicate the source of error, but, at the time, we attributed it to want of practice in the observers.

It must be borne in mind that it was not until many weeks after the close of our observations that the reductions were completed, so that, at the time of experimenting, we were wholly in the dark as to the actual value of our results. Our only means of obtaining any glimpse of the value of a particular experiment was its resemblance to some other one taken under similar conditions, but, as all the conditions were never precisely the same, we could thus obtain little information.

We have now no hesitation in assigning the following cause to the failure of these four experiments. With this weight of water, the depth in the calorimeter, when the stirrer was at rest, was only 2 centims., and as, when rapid stirring took place, a great portion of the contents must have been within the cylinder and in the upper portion of the calorimeter, it is evident that the higher parts of the coil must have been at times uncovered, since it required 1·5 centims. for complete immersion. This would account (1) for the irregular behaviour of the potential-balance, and (2) for the great lengthening of the times shown in these experiments. We at no time anticipated success with so small a mass of water, but the excellent results obtained from our stirring experiments on this weight (see Table XXI.) justified the attempt.

Experiments 5 *to* 12. (*Group B.*)

These experiments are only slightly more satisfactory than those we have rejected. The cause is partly that so frequently alluded to in Section XII., viz., obstructions in the openings at the base of the internal cylinder, the resulting irregularities in the throw of the stirrer affecting not only the non-electrical supply, but also changing the coefficient of increase in the resistance of the wire. That this is the case is indicated by the results of Experiment 9, where the E.M.F. was so reduced that the current was only ·1 ampère, and the changes in the temperature of the wire were less marked. This experiment was performed in order to test the accuracy of our various corrections, rather than as a serious effort to determine the value of J. The observations in this experiment only extended over $\frac{2}{5}$ of our total range, for, had we endeavoured to carry it out through the upper ranges, its duration would have extended over some hours.

In every case (excepting as above stated, Experiments 1 to 4) we have given all the numbers recorded, because, however bad individual observations may appear, we have no sufficient grounds for their rejection. The omissions in the table represent experimental mistakes; in No. 5, Range 7, the storage circuit was broken for a second or two by an accident with the reversing key; in No. 6, the chronograph refused to work throughout the first two ranges; in No. 10, some contact must have taken place between the wires of the coil, as the potential-balance was entirely destroyed after the 7th range. No. 11 was an exceptionally bad experiment, especially in the latter half; the changes in water pressure were so rapid and violent as to pass beyond the control of the motor-regulator, the rate changing from 29·43 to 30·36 in two consecutive observations. Had it not been for our desire to present our results in full, we would certainly have rejected this experiment.

Experiments 13 to 19. (*Group E.*)

In this group, our heaviest weight, the observations although still somewhat irregular are in better agreement. As might be anticipated the throw of the stirrer was less affected by the obstructions at the base when the depth of water was greater. The alterations made in the connections of the stirring rod have been already alluded to.

Our conclusions as to the value of J and the changes in the specific heat of water are based almost entirely on the results of Series II. Although the results deduced from Nos. 13 to 19 are in almost perfect harmony with those from Series II., the agreement, for the reasons given above, must be in part fortuitous. The mean value of J deduced from Nos. 5 to 12 agrees with the most probable value, but the change in the specific heat of water alters from positive to negative according as the group is compared with the results obtained from a lighter or a heavier weight.

Series II.

This series contains but fifteen experiments, which may appear a small number on which to base our conclusions. We would, however, point out that these are only the concluding series of many hundred J experiments conducted by the same observers, who are therefore in the best position to judge of their value. Furthermore, these fifteen experiments are conducted on very different weights of water and with very different currents, and our conclusions would be but little altered if they were based on any two individual experiments rather than on the mean results of all. Want of time alone prevented our increasing their number, but we believe that such increase would have given them small additional weight. The results of Series I. are quite sufficient to corroborate the value of J, although the least trustworthy group of that series gives contradictory values for the change in specific heat of water.

Such irregularities as are observable in the members of these series, are we believe inseparable from the use of mercury thermometers. Although many irregularities which would otherwise be apparent have been eliminated by the method of calibration adopted (Section X.), there yet remain residual ones, not strictly recurrent, whose effect can only be eliminated by taking the mean path of a series of observations over consecutive ranges. The magnitude of such irregularities can be estimated by a comparison of the columns headed "mean" and "from curve" in Table XL.

None of these experiments call for special comment, with the exception of No. 34. This was performed with the object of testing the accuracy of the correction given by the formula $\delta R = \alpha C^2$ (Section VII., Table IX.), and we did not anticipate from it results of much value. The irregularities in ranges 8, 9, 10, of Group D, Column 21, Table XL., are clearly an example of the eccentric behaviour of the thermometer when rising at this rapid rate; their mean however is excellent.

SECTION XIV.—THE CALCULATION OF THE RESULTS.*

The method which we have adopted in our calculations can be put into a general form, thus:—

Suppose a to be the rate of production of heat in the calorimeter at some standard temperature θ, whilst a' denotes the value of a at any other temperature θ_1;

ρ the change in temperature per one second due to radiation, &c., when the difference between the external and internal temperature is 1° C;

M the capacity for heat of the calorimeter and its contents at the standard temperature θ, M′ its value at θ_1;

Then if θ_1 is the temperature of the calorimeter at any time t, the rate (a') of production of heat is

$$M' \frac{d\theta_1}{dt} + M'\rho\,(\theta_1 - \theta_0),$$

where θ_0 is the temperature of the surrounding envelope.

Thus the equation of condition is

$$M' \frac{d\theta_1}{dt} = a' - M'\rho\,(\theta_1 - \theta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now, by JOULE's law,

$$JM \frac{d\theta_1}{dt} = C^2R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

therefore

$$a_\epsilon = M \frac{d\theta_1}{dt} = \frac{C^2R}{J} = \frac{n^2E^2}{JR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where a_ϵ is the rate of production of heat due to the electrical supply, and E is the E.M.F. of a Clark cell, and n is the number of cells used.

But since the rate of production of heat is dependent on the resistance of a platinum wire, and E is kept constant, this rate will diminish as the temperature rises, or,

$$a'_\epsilon = a_\epsilon \{1 - k\,\overline{\theta_1 - \theta + \beta}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

---

* We have changed the notation adopted in Section I. to one more suited for arithmetical operations.

where k is the resistance temperature-coefficient of the wire, and β is the number of degrees that the wire is hotter than the water in contact with it.

Similarly

$$M' = M\{1 + l\,\overline{\theta_1 - \theta}\}$$

where l is the mean coefficient of increase of specific heat of the water and calorimeter.

For the non-electrical supply, we have shown that $tr^3 = K$ (p. 438), where t is the time of rising 1° C. and r is the rate of revolution of the stirrer, and therefore the rise per second is given by

$$\sigma = r^3/K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and

$$\delta\sigma = \frac{3r^2}{K}\,\delta r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

By (6) we can reduce all observations to the standard rate r.

Equation (1) after introducing terms due to stirring becomes

$$M'\frac{d\theta_1}{dt} = a'_{\epsilon.\rho.\sigma.} + (\sigma - \rho\,\overline{\theta_1 - \theta_0})\,M',$$

that is,

$$M\,[1 + l\,(\theta_1 - \theta)]\,\frac{d\theta_1}{dt} = \frac{n^2E^2}{JR}\{1 - k\,\overline{\theta_1 + \beta - \theta}\} + (\sigma - \rho\,\overline{\theta_1 - \theta_0})\,M\,(1 + l\,\overline{\theta_1 - \theta}) \; (7);$$

or

$$\frac{d\theta_1}{dt} = \frac{n^2E^2}{JRM}\cdot\frac{1 - k\,(\theta_1 + \beta - \theta)}{1 + l\,(\theta_1 - \theta)} + \sigma - \rho\,(\theta_1 - \theta_0) \quad . \quad . \quad . \quad . \quad (8^*).$$

If we write

$$\left(\frac{d\theta_1}{dt}\right)_\epsilon \text{ and } \left(\frac{d\theta_1}{dt}\right)_0$$

for the electrical and non-electrical terms in this expression,

$$\frac{d\theta_1}{dt} = \left(\frac{d\theta_1}{dt}\right)_\epsilon + \left(\frac{d\theta_1}{dt}\right)_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

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[*] Equation (8) may be expressed in the following form—

$$d\theta_1/dt = A + \lambda\theta_1,$$

where A is the sum of quantities independent of θ_1, and λ is the sum of a number of small quantities.

The range $d\theta_1$ is reckoned from slightly below θ_1, say from θ', to slightly above θ_1, say θ''; hence if t' and t'' are the times at θ' and θ'', we may assign to $d\theta_1/dt$ the value $(\theta'' - \theta')/(t'' - t')$.

Strictly speaking, the equations ought to be integrated from $t = t'$, when $\theta_1 = \theta'$, to $t = t''$, when $\theta_1 = \theta''$, but the error introduced by the above method may, provided the ranges are sufficiently small, be neglected.

We have adopted throughout our experiments ranges of from 1° to 1°·5 C. It does not appear probable that increased accuracy would have resulted from the adoption of smaller ranges, for the effect of errors in the thermometry is increased as the size of the ranges is diminished.

The experiments on the heat developed by the stirrer give us the values of

$$\left(\frac{d\theta_1}{dt}\right)_0 = \sigma - \rho(\theta_1 - \theta_0).$$

(See Table XXVII.)

1. The first step in the calculation of our observations has been to correct, where necessary, all times for the clock error, which was + ·0004125, up to J 9 inclusive. This correction has been applied in Table XXXVIII.

2. We found the value of $d\theta_1/dt$, *i.e.*, the rise per second produced by the combined effect of all sources of gain or loss (Table XXXVII., col. 4).

3. Since the rate of revolution throughout an experiment was always nearly 30 per second, we eliminated irregularities in the rate by reducing to rate 30 by means of the formula given in (6). Thus we obtained the value $d\theta_1/dt$ would have had, had the rate been uniformly 30 per second. This reduction is shown in Table XXXVII., cols. 5 and 6.

4. In order to be able to more readily compare experiments performed under almost identical circumstances, the next correction applied was that for the temperature of the Clark cells. Lord RAYLEIGH gives the formula

$$E_{15} = E_t \{1 + \cdot 00077\,(t - 15)\}$$

hence

$$E^2_{15} = E^2_t \{1 + \cdot 00154\,(t - 15)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where t is the temperature of the Clark cells.

This correction cannot be applied to $d\theta_1/dt$ as a whole (equation 9) but only to the portion $(d\theta_1/dt)_\epsilon$ the electrical supply. As the correction was always very small, it was sufficient to find the approximate value of $(d\theta_1/dt)_\epsilon$ by subtracting, at any temperature, the value of $(d\theta_1/dt)_0$ as given in Table XXVII., and then find the correction on the value of $(d\theta/dt)_\epsilon$ so obtained by formula (11). The result of this correction gives col. 7, Table XXXVII.

The correction was applied in this order as we could thus compare all the experiments performed with the same E.M.F.'s so long as the values of σ and ρ were the same. The results at this stage of their reduction are given in Tables XXXVIII. and XXXIX.

5. Table XXXVII., col. 8, gives the value of $(d\theta/dt)_0$ taken from Table XXVII. This subtracted from col. 7 gives the value of $(d\theta/dt)_\epsilon$, *i.e.*, the rise per second due to the current only, the difference of potential at the ends of the coil being n Clark cells (col. 9).

Hence equation (8) is reduced to

$$\left(\frac{d\theta_1}{dt}\right)_e = \frac{n^2E^2\{1 - k\,\overline{\theta_1 - \theta + \beta}\}}{\text{JRM}\,(1 + l\,\overline{\theta_1 - \theta})}.$$

6. In the section on resistance in Table XVI. we have collected the values of $\frac{\text{R}}{1 - k\,(\theta_1 - \theta + \beta)}$ at the mean temperature of each range. The value of n, the number of Clark cells used, is given with each experiment. Substituting their values we obtain the time of rising 1° C. with unit resistance and one Clark cell, or

$$\frac{\text{J}}{\text{E}^2}\text{M}\,(1 + l\,\overline{\theta_1 - \theta}) = \text{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

This gives column 10 of Table XXXVII.

7. The results of all the experiments were then meaned and plotted, the columns headed "Mean" and "From curve" in Tables XL. and XLI. giving the results. From these smoothed curves the values of the required constants were calculated thus :—

8. If W_x is the water-equivalent of the calorimeter, and f and g are the temperature coefficients of the specific heat of water and of the calorimeter respectively, and suffixes written to W, l, T, denote the values of these quantities for different weights of water, then from (12) we obtain

and

$$\left.\begin{aligned} \frac{\text{J}}{\text{E}^2}(\text{W}_1 + \text{W}_x)\,(1 + l_1\overline{\theta_1 - \theta}) &= \text{T}_1 \\ \frac{\text{J}}{\text{E}^2}(\text{W}_2 + \text{W}_x)\,(1 + l_2\overline{\theta_1 - \theta}) &= \text{T}_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where

$$l_1 = \frac{f\text{W}_1 + g\text{W}_x}{\text{W}_1 + \text{W}_x} \quad \text{and} \quad l_2 = \frac{f\text{W}_2 + g\text{W}_x}{\text{W}_2 + \text{W}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

Hence, subtracting equations (13), we get

$$\frac{\text{J}}{\text{E}^2}(\text{W}_2 - \text{W}_1)\,(1 + f\,\overline{\theta_1 - \theta}) = \text{T}_2 - \text{T}_1$$

or

and

$$\left.\begin{aligned} \frac{\text{J}}{\text{E}^2}(\text{W}_2 - \text{W}_1)\,(1 + f\,\overline{\theta_0 - \theta}) &= \text{T}_{2,0} - \text{T}_{1,0} \\ \frac{\text{J}}{\text{E}^2}(\text{W}_2 - \text{W}_1)\,(1 + f\,\overline{\theta_2 - \theta}) &= \text{T}_{2,2} - \text{T}_{1,2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (15),$$

where the observations are taken at two temperatures θ_0 and θ_2, the second suffix of the T's referring to these temperatures.

Hence, by division, we obtain the value of f, the temperature-coefficient of the specific heat of water.

And, if θ_0 is our standard temperature, the value of J is found without any knowledge of the water-equivalent of the calorimeter.

Now $W_1 (T_{2,0} - T_{1,0})/(W_2 - W_1)$ is the time that W_1 grms. of water at temperature θ_0 would take to rise 1° C. Subtracting this from $T_{1,0}$ and dividing by $(T_{2,0} - T_{1,0})/(W_2 - W_1)$ we obtain the number of grms. of water at θ_0, to which the calorimeter also at θ_0 is equivalent . . . . (16).

Similarly the expression

$$\frac{T_{1,2} - W_1 (T_{2,2} - T_{1,2})/(W_2 - W_1)}{(T_{2,0} - T_{1,0})/(W_2 - W_1)} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

gives the number of grammes of water at θ_0 to which the calorimeter at θ_2 is equivalent. Thus we obtain both the water-equivalent of the calorimeter and also its temperature-coefficient g.

Finally J is calculated from

$$J = \frac{E^2 . T}{M (1 + l \overline{\theta_1 - \theta})} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18).$$

The following Table gives the complete working out of J 20. This experiment was selected, because it was the first of which we have given the experimental details (See Table XXVIII.).

TABLE XXXVII.—Experiment J 20.

Date, September 11, 1892. W = 139·776 grms. Clark cells, $n = 4$ at 15°·05 C.

$\dot{\sigma} = \cdot 000512$. $\rho = \cdot 000621$. $t r^3 = 5232 \times 10^4$.

$d\theta_1$.	dt.	Rate.	$\frac{d\theta_1}{dt} \times 10^6$.	Rate correction × 10^6.	$\frac{d\theta_1}{dt} \times 10^6$ at rate 30.	$\frac{d\theta_1}{dt} \times 10^6$ corrected for temperature of Clark cells (+ 0·3).	$\left(\frac{d\theta_1}{dt}\right)_0 \times 10^6$.	$\left(\frac{d\theta_1}{dt}\right)_\epsilon \times 10^6$.	T.*	Mean temperature.
·973	199·20	30·54	4884	− 28	4856	4856	809	4047	458·1	14·477
1·235	258·45	30·37	4779	− 19	4760	4761	741	4020	459·4	15·581
·987	209·80	30·47	4704	− 24	4680	4681	672	4009	459·0	16·682
·995	214·70	30·53	4636	− 28	4608	4608	610	3998	458·7	17·683
1·016	223·38	30·45	4548	− 24	4524	4524	548	3976	459·7	18·688
1·278	286·93	30·35	4454	− 18	4436	4437	476	3961	459·6	19·835
1·282	293·75	30·41	4364	− 21	4343	4343	397	3946	459·4	21·115
1·307	306·38	30·56	4266	− 29	4237	4237	316	3921	460·3	22·409
1·598	384·00	30·54	4161	− 28	4133	4134	226	3908	459·7	23·862
·811	199·38	30·42	4067	− 22	4045	4045	155	3890	460·3	25·006
1	2	3	4	5	6	7	8	9	10	11
REFERENCES.										
Table XXXIV.	Table XXVIII.	Table XXVIII.	p. 479, 2	p. 478, eq. 6, p. 479, 3.	p. 479, 3	p. 479, 4, eq. 11	Table XXVII.	p. 479, 5	p. 480, 6 Table XVI.	Table XXXIV.

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* These values can be compared with Table XL., cols. 11–15. Taking the differences between the individual results of Experiment 20, and the mean numbers of all the experiments on this weight, we get by the method of least squares, mean probable error ± ·075 second. This does not include the inevitable errors arising from the thermometry, the values of the ranges being only given to the nearest $\frac{1}{1000}$ of a degree, such errors can only be removed by the smoothed curve.

The two following Tables, XXXVIII. and XXXIX., give the values of $(d\theta_1/dt) \times 10^6$, corrected to rate 30 and for the temperature of the Clark cells. They correspond to Col. 7 of Table XXXVII.:—

Table XXXVIII.—Series I. Values of $(d\theta_1/dt) \times 10^6$.

Group B. W = 188·065.

No. of experiment.	No. of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
9	2	—	—	1314·6	1261·7	1208·9	1150·3	—	—	—	—
8	4	3914	3847	3782	3722	3661	3588	3526	3434	3362	3293
10	4	3916	3853	3801	3725	3655	3597	3523	—	—	—
11	4	3922	3854	3801	3727	3671	3599	3509	3444	3369	3282
12	4	3919	3851	3791	3716	3663	3598	3519	3438	3359	3294
	Mean	3919	3853	3798	3723	3663	3598	3517	3441	3364	3288
5	5	5769	5701	5622	5554	5482	5404	—	5238	5142	5071
6	5	—	—	5629	5551	5488	5406	5326	5232	5143	5064
7	5	5776	5698	5626	5554	5475	5400	5333	5236	5142	5065
	Mean	5773	5699	5626	5553	5482	5403	5329	5235	5142	5067

Table XXXVIII.—(continued).

Group E. W = 277·931.

No. of experiment.	No. of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
13	4	2955	2906	2857	2809	2766	2712	2658	2603	2535	2487
14	4	2955	2907	2852	2811	2764	2714	2655	2600	2535	2484
17	4	2940	2895	2851	2801	2748	2704	2646	2591	2526	2477
18	4	2953	2905	2845	2806	2754	2705	2649	2590	2532	2470
The above four experiments cannot be meaned here, as the values of σ and ρ are different. (See p. 450).											
15	5	4349	4299	4238	4185	4135	4070	4006	3945	3889	3826
16	5	4346	4290	4238	4183	4131	4070	4010	3947	3884	3817
19	5	4347	4291	4245	4178	4135	4075	4004	3956	3884	3804
	Mean	4347	4293	4240	4182	4134	4072	4007	3949	3886	3816

TABLE XXXIX.—Series II. Values of $(d\theta_1/dt) \times 10^6$:

Group A. W = 139·776.

Number of experiment.	Number of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
23	3	3090	3014	2935	2864	2791	2713	2624	2534	(2431)*	(2351)*
20	4	4856	4761	4681	4608	4524	4437	4343	4237	4134	4045
22	4	4834	4763	4681	4604	4526	4442	4336	4246	4141	4049
25	4	4859	4768	4683	4604	4528	4437	4342	4240	4139	4048
	Mean	4850	4764	4682	4605	4526	4439	4340	4241	4138	4047
21	5	7102	6995	6904	6825	6740	6641	6541	6428	6303	6208
24	5	7103	6996	6904	6831	6743	6650	6543	6425	6301	—
	Mean	7103	6995	6904	6828	6741	6646	6542	6426	6302	6208

* Obtained by producing the curve through the preceding numbers. (See Table XXIX.)

Table XXXIX.—(continued).

Group C. W = 199·674.

Number of experiments.	Number of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
26	4	3852	3781	3717	3652	3596	3526	3454	3375	3295	3224
28	4	3852	3783	3717	3654	3596	3523	3457	3377	3296	3226
	Mean	3852	3782	3717	3653	3596	3525	3455	3376	3295	3225
27	5	5625	5554	5479	5414	5349	5264	5188	5101	5013	4939
29	5	5623	5550	5478	5415	5347	5266	5192	5102	5012	4936
	Mean	5624	5552	5478	5415	5348	5265	5190	5102	5013	4937
Group D. W = 259·500.											
30	5	4650	4592	4536	4482	4426	4359	4291	4225	4149	4086
31	5	4646	4585	4529	4476	4419	4357	4290	4222	4151	4082
32	5	4651	4586	4528	4473	4425	4355	4289	4219	4148	4083
33	5	4649	4589	4535	4475	4419	4357	4289	4225	4154	4088
	Mean	4649	4588	4532	4476	4422	4357	4290	4223	4150	4085
34	6	6431	6352	6290	6229	6171	6101	6032	5956	5867	5791

Table XL. gives the values of T (equation 12), *i.e.*, the times of rising 1° C. with unit R, and E = 1 Clark cell at 15° C.

Table XL.—Times of rising 1° C. = T.

Series 1.

	Group B. W = 188·065.					Group E. W = 277·931.			
No. of cells.	2	4	5			4	5		
No. of experiments performed.	1	4	3			4	3		
Temperature.				Mean.	From curve.			Mean.	From curve.
14·477	—	557·3	556·6	557·0	557·1	740·1	740·6	740·4	740·4
15·581	—	557·6	556·5	557·1	557·2	739·7	740·3	740·0	740·5
16·682	558·5	557·0	556·6	557·0	557·3	740·4	739·7	740·1	740·5
17·683	557·0	558·0	557·3	557·6	557·5	740·4	741·2	740·8	740·6
18·688	558·3	558·2	557·8	558·1	557·6	742·2	740·7	741·4	740·6
19·835	558·7	558·0	558·1	558·1	557·7	741·0	741·5	741·3	740·6
21·115	—	558·0	557·0	557·5	557·8	741·2	741·8	741·5	740·7
22·409	—	558·7	558·0	558·4	557·9	740·4	740·4	740·4	740·7
23·862	—	557·3	557·6	557·4	558·0	739·9	738·6	739·3	740·7
25·006	—	558·3	558·2	558·3	558·2	741·1	741·1	741·1	740·8
I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.

TABLE XL.—(continued).

Series II.

	Group A. W = 139·776.					Group C. W = 199·674.				Group D. W = 259·500.			
No of cells.	3	4	5			4	5			5	6		
No. of experiments performed.	1	3	2			2	2			4	1		
Temperature.				Mean.	From curve.			Mean.	From curve.			Mean.	From curve.
14·477	458·7	458·7	458·3	458·7	458·8	580·5	580·8	580·7	580·9	702·8	701·9	702·7	702·9
15·581	458·7	459·0	459·4	459·1	458·9	581·2	580·8	581·0	581·0	703·2	703·5	703·3	703·0
16·682	459·0	458·8	459·4	459·1	459·0	581·2	581·0	581·1	581·1	703·0	703·2	703·0	703·0
17·683	459·2	459·0	458·9	459·0	459·1	581·9	580·8	581·3	581·1	703·0	703·2	703·1	703·0
18·688	459·9	459·4	459·1	459·4	459·2	581·7	581·0	581·3	581·2	703·1	703·0	703·1	703·0
19·835	459·4	459·4	459·1	459·3	459·3	582·0	581·8	581·9	581·3	703·7	703·1	703·6	703·1
21·115	459·4	459·7	459·0	459·4	459·5	581·1	581·1	581·1	581·4	703·5	702·5	703·3	703·1
22·409	459·3	459·9	459·6	459·7	459·6	581·7	581·6	581·6	581·4	703·0	702·5	702·9	703·1
23·862	459·6	459·4	460·0	459·7	459·7	581·0	581·1	581·1	581·5	702·4	702·9	702·7	703·2
25·006	459·9	459·8	460·1	459·9	459·8	581·6	581·4	581·5	581·6	703·2	704·0	703·4	703·2
I.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.	XXII.	XXIII.

Remarks on Table XL.

The column headed "Mean" was obtained by multiplying the numbers in the preceding columns of each section by the number of experiments from which they were deduced, and dividing the sum of the products by the total number of experiments. The "means" were plotted on a scale such that $\frac{1}{10}$ inch of ordinate represented a difference of ·1 second in T, and the smooth curve in each case so drawn that the sum of the positive and negative areas included between it and the experimental curve was zero. The curvature is so slight over our range that by assuming T to be a linear function of θ_1, our "curve numbers" are not affected when written to four figures.

It will be found that our results (obtained from the "smooth curve" values of T) are in much closer agreement than might be expected from a study of Table XL. This indicates that the irregularities are due to some (or all) of the following causes.

(1.) Errors in the comparative values of the ranges. The effect of such errors would be common to all the experiments, and there is evidence that the sixth range is too small, the value of T over that range being nearly always too large; the error, however, does not amount to 0°·001. Again, the ninth range is evidently too great. These errors do not affect the result when the values of T over two or three consecutive ranges are meaned (as is done by the smooth curve), the excess in one case is then compensated by the deficiency in another.

(2.) The *irregular* (as apart from the strictly recurrent) "stickings" of the thermometer; these, as previously mentioned, must mean out when sufficient observations are taken over the same range.

(3.) Irregularities in the temperature of the water ejected through the opening in the cylinder on to the thermometer bulb, water from cooler parts of the calorimeter being followed by a gush of warmer water brought direct from the hot wire. This, no doubt, is a fruitful cause of alternate lag and acceleration, but again, it is evidently an irregularity which would but slightly affect the results of many experiments continued through a sufficient number of intervals of time. We have previously pointed out that the "throw" was irregular throughout Groups B and E, and the effect on the individual experiments is very marked; nevertheless the values of T derived from the fifteen experiments in those groups are in practical agreement with the values deduced from the experiments in Series II.

(4.) Personal errors of observation which, from their very nature, are unlikely to recur at regular intervals or over identical ranges. These, no doubt, are the origin of many discrepancies, the strain on the observer during these experiments being great.

The cumulative effect of all the above causes of irregularity may at times be large. In Series I. differences of 1 in 500 occasionally present themselves, although, in Series II., there is only one case in which the difference between the mean and the curve-numbers exceeds 1 in 1000, and only two others in which it exceeds 1 in 2000.

The values of J deduced from the different groups (using the numbers given by the respective smooth curves) are, however, in much closer agreement, the extreme divergence being less than 1 in 4000 (see Table XLIII.).

It is worth mentioning that in previous years we found the agreement between individual experiments in the same group (having the same E.M.F.) was closer than that shown in the above Table; but, on the other hand, the values of J and of the water equivalent given by different groups (or by members of the same group when the number of cells was changed), differed hopelessly, in some cases by more than 1 per cent., showing that constant causes of error had not been eliminated. We attribute the improvement (1) to the alteration in the form of stirrer, for the mechanical work done in previous years varied capriciously, although no doubt the thermometer bulb was more thoroughly "washed" than before the alteration; (2) to the correction (now introduced for the first time) for the difference between the temperature of the wire and that indicated by the thermometer.

As it is more convenient to examine the values of T for integral values of the temperature, we have read off from the smooth curves the values of T for every two degrees covered by our range, and also the values at 15° C. and 25° C. These are given in the following table.

TABLE XLI.—Values of T from Smoothed Curve.

Temperature.	Series I.		Series II.		
	Group B. W = 188·065.	Group E. W = 277·931.	Group A. W = 139·776.	Group C. W = 199·674.	Group D. W = 259·500.
14·000	557·04	740·44	458·77	580·90	702·88
15	557·14	740·46	458·87	580·95	702·91
16	557·24	740·48	458·97	581·01	702·95
18	557·43	740·55	459·16	581·13	703·00
20	557·62	740·60	459·35	581·25	703·05
22	557·80	740·66	459·53	581·37	703·11
24	557·99	740·72	459·72	581·49	703·17
25	558·09	740·75	459·81	581·55	703·20
26	558·18	740·77	459·92	581·61	703·22
1	2	3	4	5	6

Taking the data given in Columns 4 and 6 of the above table, we give an example of the final calculation of our results.

Using the notation of pp. 478–481, we have the following values

$W_1 = 139{\cdot}776$ $W_2 = 259{\cdot}500$ $\theta_0 = 15°$ C.
$T_{1,0} = 458{\cdot}87$ $T_{2,0} = 702{\cdot}91$ $\theta_1 = 25°$ C.
$T_{1,2} = 459{\cdot}81$ $T_{2,2} = 703{\cdot}20$ $E = 1{\cdot}4344^*$ volts.

Substituting these values in equation (15) and dividing, we obtain

$$\frac{1 + 15f}{1 + 25f} = \frac{243{\cdot}39}{244{\cdot}04} = {\cdot}99734.$$

As the difference in weight of water here amounts to nearly 120 grms., it is probable that the value obtained in this case is the most reliable.

In the same manner using the values of W and T given in Columns 4 and 5, we get

$$\frac{1 + 15f}{1 + 25f} = \frac{121{\cdot}74}{122{\cdot}08} = {\cdot}99722.$$

And again from Columns 5 and 6,

$$\frac{1 + 15f}{1 + 25f} = \frac{121{\cdot}65}{121{\cdot}96} = {\cdot}99746.$$

Hence mean value of $\frac{1 + 15f}{1 + 25f}$ deduced from Series II. $= {\cdot}99734$.[†]

Hence, adopting 15° C. as the standard temperature, the

Specific Heat of Water = 1 − ·000266 (θ_1 − 15).[‡]

Also by means of equation (15) we get the following values of J,

Columns 4 and 6 . . . . . $J = 4{\cdot}1939 \times 10^7$
" 4 " 5 . . . . . $J = 4{\cdot}1940 \times 10^7$
" 5 " 6 . . . . . $J = 4{\cdot}1940 \times 10^7$
Mean . . . . . **$4{\cdot}1940 \times 10^7$.**

This value of J, as pointed out on p. 481, is entirely *independent of the value assigned to the water equivalent of the calorimeter.*

Performing the operations of equation 16, we find that the water equivalent at

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[*] See p. 388, *supra*.

[†] If we mean the results from Columns 4 and 5 and 5 and 6, we must of course obtain the same result as that given by Columns 4 and 6. We have however given the numbers in the above form as they show the nature of the agreement between the different groups.

[‡] Over the range 14° to 26° C.

15° C. in terms of water at 15° C. = 85·340 grms.; and from the operations indicated in equation (17), we get water equivalent at 25° C. in terms of water at 15° C. = 86·174 grms. Hence

Water equivalent = 85·340 $\{1 + \cdot000977\,(\theta_1 - 15)\}$.*

We are now in a position to find the value of $M(1 + l\,\overline{\theta_1 - \theta})$ for any weight of water at any temperature, and the value of J can then be found by equation (18).

The following Table gives the value of $M(1 + l\,\overline{\theta_1 - \theta})$ for each weight of water at temperatures 15°, 20°, and 25° C.

TABLE XLII.—Value of $M(1 + l\,\overline{\theta_1 - \theta})$ at 15°, 20°, and 25°.

Group.	W.	15°.	20°.	25°.
A	139·776	225·116	225·347	225·578
B	188·065	273·405	273·572	273·739
C	199·674	285·014	285·165	285·317
D	259·500	344·840	344·912	344·984
E	277·931	363·271	363·318	363·366

Substituting the values of T (Table XLI.) and $M(1 + l\,\overline{\theta_1 - \theta})$ in equation (18), we get the following values of J. The first column shows the group of experiments from which each value is derived.

TABLE XLIII.—Values of J given by each Group at different Temperatures.

Group.	15°.	20°.	25°.	Mean.
A	4·1940 × 10^7	4·1940 × 10^7	4·1939 × 10^7	4·1940
B	4·1930 ,,	4·1941 ,,	4·1949 ,,	4·1940
C	4·1939 ,,	4·1938 ,,	4·1937 ,,	4·1938
D	4·1940 ,,	4·1939 ,,	4·1940 ,,	4·1940
E	4·1938 ,,	4·1940 ,,	4·1943 ,,	4·1940
				4·1940

We have in the above Table given the values resulting from the calculation at different temperatures since the limit of our experimental errors is thus clearly indicated; for the values of J ought (in the absence of experimental errors) to be identical at all temperatures. The close agreement between the values from different groups, and from the same group at different temperatures, is a satisfactory proof of the accuracy of our determination of the water equivalent of the calorimeter, and or the changes in it and in the capacity for heat of the water. If we reject Group B (and we have already shown that it has little value) the results are practically identical.

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* Over the range 14° to 26° C.

Hence if we assume—

(1.) The unit of resistance as defined in the 'B.A. Report,' 1892.

(2.) That the E.M.F. of the Cavendish standard Clark cell at 15° C. = 1·4342 volts.*

(3.) That the thermal unit = quantity of heat required to raise 1 grm. of water through 1° C. at 15° C.

The most probable value of

$$J = 4{\cdot}1940 \times 10^7.\dagger$$

This, by reduction, gives the following:—

$J = 427{\cdot}45$ kilogrammetres in latitude of Greenwich ($g = 981{\cdot}17$).
$J = 1402{\cdot}2$ ft.-lbs. per thermal unit C in latitude of Greenwich ($g = 981{\cdot}17$).
$J = 778{\cdot}99$,, ,, F ,, ,, ,, ,,

SECTION XV.—DISCUSSION OF THE RESULTS.

As stated in the Introduction, we proposed to determine the value of J in terms of the thermal unit there defined, viz., the quantity of heat required to raise unit mass of water through 1° C. at 15° C. ROWLAND has preferred to tabulate his results by giving the changes in the numerical value of J caused by changes in the capacity for heat of water. We can, however, deduce from his table the expression for the change in the specific heat of water over our range.

Expressed in the same form as above, it becomes

$$1 - {\cdot}000400\,(\theta_1 - 15).$$

The difference between our results on this point is marked, and it is evident that further investigation is required. ROWLAND finds that the minimum value lies between 30° and 35° C.; we hope to carry the investigation beyond that temperature, and some explanation of the difference in our results may then present itself. The whole question is probably one of thermometry, and possibly our revision of this part of the subject may bring our results into closer agreement. ROWLAND himself (p. 198) points out that the whole matter depends on a small difference which he denotes by the letter m. This quantity appears to be of the same nature as that which we term $d\,(=\theta-pt)$ in the platinum-thermometer formulæ. This quantity m had to be determined from a difference of the order of 0°·05 C. between the air- and mercury-thermometers. The method adopted for finding the value of the analogous quantity d would appear much less liable to error.

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* If we assume the E.M.F. of our Clark cells to be the same as that of the Cavendish standard (and we are inclined to think we have over-estimated the difference), we get $J = 4{\cdot}1930 \times 10^7$.

† The value obtained by us in 1891 = $(4{\cdot}192 +) \times 10^7$; *supra*, p. 365.

Rowland himself was evidently anxious to throw more light on this portion of his investigation, for he performed, with great care, a series of experiments by the method of mixtures. These are given on pp. 125 to 130; he has not, however, tabulated his results. We give, with one exception (No. 10), the results of all these experiments which did not involve temperatures above 30° C.

Table XLIV.—Specific Heat of Water by the method of Mixture (Rowland).

Experiment.	Rowland's result.	Mean coefficient of decrease.
1	$\frac{\text{Mean sp. heat } 0^\circ-18^\circ}{\text{,, ,, } 18^\circ-27^\circ}=1\cdot0025$	·000185
2	Above our range	—
3	$\frac{\text{Mean sp. heat } 0^\circ-17^\circ}{\text{,, ,, } 17^\circ-29^\circ}=1\cdot0024$	·000165
4	Rejected by Rowland	—
5	$\frac{\text{Mean sp. heat } 0^\circ-18^\circ}{\text{,, ,, } 18^\circ-30^\circ}=1\cdot0067$	·000447
6	Rejected by Rowland	—
7	$\frac{\text{Mean sp. heat } 0^\circ-16^\circ}{\text{,, ,, } 16^\circ-24^\circ}=1\cdot0010$	·000083
8	$\frac{\text{Mean sp. heat } 0^\circ-17^\circ}{\text{,, ,, } 17^\circ-25^\circ}=1\cdot0027$	·000216
9	$\frac{\text{Mean sp. heat } 0^\circ-21^\circ}{\text{,, ,, } 21^\circ-28^\circ}=1\cdot0045$	·000321
10	Requires explanation, as it gives an increasing specific heat. Also the difference between the mean temperatures (2°·5 C.) is too small to enable us to expect an accurate result. The remaining experiments are above our range	—
	Mean	·000236

It is thus apparent that the results of Rowland's investigation by this method are in closer agreement with our conclusions than they are with his own.* We do not, however, attach much importance to this point, since—as he justly remarks—the method by the mechanical equivalent is of far greater value.

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* Our value ·000266.

The values of J deduced from the two groups of Series I. and the middle group of Series II. are dependent upon the value of the water equivalent deduced from the highest and lowest weights of Series II. The agreement in the results is a sufficient proof of the accuracy of the value assigned to that equivalent.

The coefficient of increase in the specific heat of the calorimeter may appear large. TOMLINSON ('Roy. Soc. Proc.,' 1885), gives the following as the specific heat of copper and zinc:—

$$\begin{aligned}\text{Specific heat of copper}^{*} &= \cdot09008 + \cdot000065\ \theta_1,\\ \text{,,} \quad \text{,,} \quad \text{zinc} &= \cdot09009 + \cdot000075\ \theta_1,\end{aligned}$$

and he quotes the values given by BÈDE as

$$\begin{aligned}\text{Copper} &= \cdot0892 + \cdot000065\ \theta_1,\\ \text{Zinc} &= \cdot0859 + \cdot000084\ \theta_1.\end{aligned}$$

Expressing the mean of these results in the same form as that given to the expression for the water equivalent, we get

$$\begin{aligned}\text{Specific heat of copper} &= \cdot0896\ (1 + \cdot00071\ \theta_1),\\ \text{,,} \quad \text{,,} \quad \text{zinc} &= \cdot0876\ (1 + \cdot00091\ \theta_1).\end{aligned}$$

TOMLINSON states that when reducing his observations, he used REGNAULT's expression for the capacity for heat of water, viz., $1 + \cdot00004\ \theta_1 + \cdot0000009\ \theta_1^2$.

Unfortunately the account of these experiments given in the 'Proceedings' does not supply us with sufficient data to estimate the changes in the coefficient of θ_1 that would be caused by assuming our expression for the specific heat of water. We are, therefore, unable to say how far our rate of increase in the water equivalent agrees with the above determinations. We are, however, enabled to state that the values of the specific heats at 0° C. would be reduced by about 6 parts in 1000, on the assumption that our formula holds over so large a range.

Our calorimeter may be regarded as a mixture of copper, zinc, gold, glass, and mercury, and the coefficient of increase of such a mixture can only be obtained by direct experiment.

ROWLAND, in order to compare his results with those of JOULE, expressed both in kilogrammetres at Baltimore ($g = 980\cdot05$), and gave the numerical value of J in terms of water at differing temperatures, assuming his own value of the specific heat of water. In order to make a similar comparison we give our results in the same form. The following Table is extracted from the Appendix to ROWLAND's paper, with the exception of the last column, which we have added.

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* TOMLINSON's paper (Table II., p. 497) contains an obvious misprint in the coefficient of θ_1.

Table XLV.—Comparative values of J given by Joule, Rowland, and Griffiths.

Date.	Joule's method.	Temperature of water. ° C.	Joule's value in the metric system. Reduced to Rowland's air-thermometer.	Rowland, 1880.	Griffiths, 1892.
1847	Friction of water . . .	15	442·8	427·4	427·9
1850	" " . . .	14	426·8	427·7	428·1
1850	Friction of mercury . .	9	427·5	428·8	428·6
1850	" " . .	9	428·7	428·8	428·6
1850	Friction of iron	9	429·1	428·8	428·6
1850	" "	9	428·0	428·8	428·6
1867	Electric heating . . .	18·6	428·0[*]	426·7	427·6
1878	Friction of water . . .	14·7	425·8	427·6	427·9
1878	" " . . .	12·7	427·1	428·0	428·2
1878	" " . . .	15·5	426·0	427·3	427·9
1878	" " . . .	14·5	422·7	427·5	428·0
1878	" " . . .	17·3	426·3	426·9	427·7

Rowland remarks as follows : "Joule rejected quite a number of his results, but I have thought it best to include them, giving them small weight however. In this way we obtain a value of Joule's experiments of 426·75 at 14°·6 C., my value at this point being 427·52 ; the difference amounts to 1 in 550 only."

Adopting this method of comparison we get (at 14°·6 C., and expressed in kilogrammetres at Baltimore) :

Joule.	Rowland.	Griffiths.
426·75	427·52	427·98

Thus our difference from Joule amounts to 1 in 350,[†] and our difference from Rowland amounts to 1 in 930.[‡]

The difference between Rowland's determinations of the changes in the specific heat of water and ours, would cause the values of J to be identical if expressed in terms of a thermal unit at 11°·5 C., and below that temperature Rowland's value would be the greater.

[Notes by E. H. G., added April, 1893.

1. No change in the value of the various units, or constants, involved in our

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[*] Value deduced by Rowland by assuming his own value of the ohm.

[†] If we attach equal weight to the different values given by Joule (Rowland in obtaining the above numbers attached arbitrary values to different experiments and methods) we obtain J = 428·23 as the mean value at 13°·2 C., *i.e.*, J = 428·08 at 14°·6 C. This *exceeds* our value by 1 in 4281.

[‡] If we assume the E.M.F. of our cells to be that of the Cavendish standard, our value would be 427·86 and these differences would become 1 in 386 and 1 in 1260 respectively.

investigation could bring our results into absolute agreement with those obtained by ROWLAND, since, owing to the difference in the expressions for the temperature coefficients of the specific heat of water, it is inevitable that our conclusions should agree at some one temperature, but must necessarily differ when expressed in terms of a thermal unit at any other temperature, and, thus, changes in the values of the units would only alter the temperature of agreement. For example, Dr. GUILLAUME has pointed out to me that the experiments of Commandant DEFFORGES lead to the conclusion that the value of g, at Greenwich, should be increased from 981·17 centims. to 981·24 centims. A similar correction would slightly increase ROWLAND's value of J, and thus cause our point of agreement to be about 12° C. instead of 11°·5 C. Again, the value of g, assumed by Lord RAYLEIGH ('Phil. Trans.,' A., 1884, p. 427), would have to be slightly increased, and the resulting values of the electro-chemical equivalent of silver, and of the absolute electromotive force of a Clark cell, would require modification, but the only result of any such change would, as before, be to shift the temperature of agreement.

It is, therefore, evident that the difference is chiefly due to errors in thermometry. An error of 0°·01 C., in the value of the range 14° to 25° C., would suffice to explain the discrepancy between the results, and the close agreement in the range value, as obtained from our standard and the Paris nitrogen standard (see p. 430, *supra*), appears to me to warrant a brief criticism of ROWLAND's methods and conclusions.

The science of exact thermometry has made great strides during the past fourteen years, and, no doubt, much of this progress is due to ROWLAND's initiative, for his work undoubtedly marks a distinct advance in this subject. Great, however, as his services have been, he would, I feel sure, be the first to admit that his investigations of 1878-79, by no means exhaust the possibilities of further progress in the science of thermometry. In his paper, he constantly indicates possible causes of error, and, in some cases, makes valuable suggestions as to methods by which they may be eliminated.

In a previous paper ('Phil. Trans.,' A., 1891, p. 155) it was pointed out by Mr. CALLENDAR and myself that "the indications of the air-thermometer cannot be relied on beyond 0°·01 C.... the limit of accuracy of a single reading of the barometer."

On p. 95 of his paper, ROWLAND gives the data on which the following statements are based. A difference of ·03 millim. in the reading of his barometer gave a difference of 0°·01 C. in the air-temperature, and an error of 0°·3 C. in the estimation of the temperature of his barometer-column would produce the same effect. From his remarks, it is evident that he regarded his possible error from this cause alone as ± 0°·01 C.

Again, his temperatures depended on the direct observation of the height of a mercury manometer column by means of a cathetometer, and everyone who has worked with such an instrument knows the difficulty of obtaining measurements accurate to ± ·02 millim. (It is true that *small* differences can be read to a much closer degree of accuracy by means of a microscope fitted with a micrometer scale, but this method is not applicable when large differences have to be read, nor does it appear to have been adopted by ROWLAND.) On p. 94, ROWLAND remarks that on repeating the observations upon an accurate scale, they "agreed with the scale to within two or three hundredths of a millimetre, which was as near as I could read on such an object," and 0·03 millim. appears to have been equivalent to 0°·01 C. Also, the temperature of the manometer column was only ascertained by means of a thermometer placed alongside it, and the uncertainty thus introduced might nearly double the resulting error.

A study of ROWLAND'S Tables, XI. to XV., leads to the conclusion that the discrepancy between individual observations, due to the above and other causes, in some cases amounted to as much as 0°·03 to 0°·04 C.

The following extract will, I think, show that ROWLAND did not rely upon his observations to ± 0°·01 C. In his concluding remarks he says: "I now believe that, with the improvement to the air-thermometer of an artificial atmosphere of constant pressure, we could be reasonably certain of obtaining the temperature at any point up to 50° C., within 0°·01 C. from the mean of two or three observations. I believe that my own thermometers scarcely differ much more than that from the absolute scale at any point up to 40° C., but they represent the mean of eight observations. However, there is an uncertainty of 0°·01 C. at the 20° C. point, owing to uncertainty in the value of m. But, taking $m = \cdot 00015$, I hardly think that the point is uncertain to more than that amount for the thermometers Nos. 6163, 6165, and 6166."

The causes of error above enumerated do not affect the platinum thermometer determinations to the same extent. It is true that the values of d (which corresponds to ROWLAND'S quantity, m), are dependent upon the boiling-points of water and sulphur, which are again dependent on barometer readings, but, on the other hand, an error of as much as 0°·1 C., at those points, would produce so little change in d as to leave the value of our range unaltered, although it would slightly affect the actual elevation; more especially, it should be noticed that any small changes of pressure which may take place *during the comparison of the platinum thermometer with other thermometers* do not affect the results.

The above considerations appear to me to justify the conclusion that the value of ROWLAND'S temperature range, 14° to 25° C., may be in error by as much as 0°·011 C., and, if this is the case, the values of the C.G.S. unit of heat, and of the temperature coefficients of the specific heat of water, as deduced from his experiments, may be identical with ours.

2. I have, since the publication of an abstract of this paper, received several communications criticising our statement (see p. 496, *supra*), that we differ from the results obtained by JOULE, by "1 part in 350," and my correspondents have pointed out that the ratio, 772·55 to 778·99, is very different from that of 349 to 350. It would, therefore, appear necessary that I should indicate, more fully than I have

done, the ground upon which we based the above statement. Joule himself, in 1879, made a careful comparison of his thermometer with one which had been standardized by Rowland (Baudin's Therm., No. 6166). The results were published as an appendix to Rowland's paper in the 'Proceedings of the American Academy' for March, 1880. A considerable correction in the values obtained by Joule was then found to be necessary. Again, Joule had assumed Regnault's expression for the specific heat of water, and a correction for the capacity for heat of the calorimeter was also rendered necessary by the changes in the thermometry. As the appendix above referred to appears to be but little known in this country, I give one example (p. 44, *ibid.*) to indicate the nature of the various corrections.

Joule's result in 1878 (friction of water)		774·6
Correction for thermometer	+	3·7
,, ,, capacity for heat	+	·2
,, ,, latitude (to Baltimore)	+	·9
,, ,, vacuum	−	·9
Corrected value (at 12°·7 C.) . . .		778·5

As these corrections were published in 1880, it appears probable that, had they not met with Joule's approval, he would have signified his dissent, but I can find no record of any such expression of opinion on his part.

As previously stated, Rowland attached what appear to me to be curiously arbitrary values to certain of Joule's experiments. If we accept his verdict as to their respective merits, the mean of the selected experiments, as corrected by Rowland (but reduced by me to temperature 15°, and to latitude Greenwich), is 776·75 ft.-lbs. ; but if we assign equal value to all Joule's published results, the corrected mean becomes 779·17 ft.-lbs., and it was on these numbers that our statement was based.

It is extraordinary that, although these results (or rather the corrections by which they are deduced) were published as far back as 1880, we still find that our modern books of reference continue to give 772·55 as the number resulting from Joule's own experiments.]

APPENDIX I.

Determination of the "Null Point" and an alternative Method of calculating the Results.

As stated in the introduction we append a full analysis of the "null point," *i.e.*, the point at which the radiation is self-eliminated.

Using a similar notation to that on p. 478, we have as the equation of condition

$$\mathrm{M}\frac{d\theta}{dt} = a - \rho\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where the temperatures are measured from that of the surrounding envelope.

For the sake of simplicity we can assume that the values of a and M remain constant.

Integrating and putting $\lambda = \rho/\mathrm{M}$, $\mu = a/\mathrm{M}$, and determining the constant from the fact that when $t = 0$, $\theta = -\theta_0$, we obtain the equation

$$e^{\lambda t} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda} - \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

If there had been no radiation, $\rho = 0$, and the equation of condition would have been $d\theta/dt = \mu$.

Integrating, and using the same constant as before

$$\theta = \mu t - \theta_0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

If we find the points of intersection of (2) and (3) one point $(-\theta_0, 0)$ is that at which the experiment commenced, the other (Θ, T) is the point on (2) at which the radiation is eliminated.

It is more convenient, for experimental work, to obtain an expression involving T rather than Θ; substituting therefore the value of θ given by (3) in (2), we obtain

$$e^{\lambda \mathrm{T}} - \frac{\frac{\mu}{\lambda} + \theta_0}{\mu} \, \frac{e^{\lambda \mathrm{T}} - 1}{\mathrm{T}} = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

This equation can be solved for T when the values of λ, μ, θ_0 are known.

In order to obtain λ, μ, we can take the following observations: Commence an experiment at $-\theta_0$, note the time t when $\theta = 0$, and again note the time t_2, when $\theta = +\theta_0$.

Then equation (2) gives the relations

$$e^{\lambda t_1} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and

$$e^{\lambda t_2} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda} - \theta_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).^*$$

From this pair of equations we can eliminate $\lambda\theta_0/\mu$ and obtain

$$2\{e^{\lambda(t_2 - t_1)} - 1\} - \{e^{\lambda t_2} - 1\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

which is the equation from which to determine λ.

So far as we are aware, the solution of these equations cannot be expressed as an algebraical expansion, except on the assumption that λt_1, λt_2, λT are small. Making this assumption, and writing $x = t_2 - 2t_1$, successive approximation leads to

$$T = 2t_1 + \tfrac{1}{3}x - \tfrac{2}{9}\frac{x^2}{t_1} + \frac{787}{200}\frac{x^3}{t_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The method which we have found best adapted to the numerical solution of such equations amounts practically to tracing their graph, using, for instance, in equation (7), λ and $f(\lambda)$ as abscissa and ordinate. A rough approximation to the value of λ is soon obtained, and afterwards by means of a table of logarithms as close an approximation as is necessary can be found.

Since μ does not occur as an exponential, its value is at once found from (5) or (6).

Equation (4) can be solved for T in a similar manner to equation (7), using T and $f(T)$ as abscissa and ordinate. A rough approximation to the value of T is given by equation (8),

$$T = 2t_1 + \tfrac{1}{3}(t_2 - 2t_1).$$

The forms given to equations (4) and (7) are those which we have found most convenient from which to calculate the values of $f(\lambda)$ and $f(T)$.

Besides being thus able to eliminate radiation, we can at the same time eliminate the change in all those quantities whose rate of variation can be expressed as a linear function of the temperature. Thus, on p. 477, we have shown that when the variations in M and a are considered, their full values are

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* Equation (6) shows that θ_0 is not altogether at our disposal, but must be less than μ/λ. For, otherwise, the calorimeter would reach its final state, where the loss by radiation balances the supply of heat, and could never attain the temperature $+\theta_0$.

$$a' = a\{1 - k(\theta + \beta)\}$$
$$\mathrm{M}' = \mathrm{M}\{1 + l\theta\},$$

and, hence, equation (1) becomes

$$\mathrm{M}(1 + l\theta)\frac{d\theta}{dt} = a\{1 - k(\theta + \beta)\} - \rho\theta. \quad . \quad . \quad . \quad . \quad . \quad (9).^*$$

By neglecting squares and products of small quantities (9) can be written in the same form as (1), thus,

$$\mathrm{M}\frac{d\theta}{dt} = a(1 - k\beta) - \{\rho + a(k + l)\}\theta,$$

and the values of λ and μ become

$$\lambda = \frac{l + a(k + l)}{\mathrm{M}}, \quad \mu = \frac{a(1 - k\beta)}{\mathrm{M}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

where $a = n^2\mathrm{E}^2/\mathrm{JR}$.

Let λ_{11}, μ_{11} denote the values of λ and μ when a weight of water W_1 and an E.M.F. of n_1 Clark cells are used, the first suffix referring to the weight, the second to the number of Clark cells; further, as before, let W_x be the water equivalent.

Then

$$\mu_{11} = \frac{n_1^2\mathrm{E}^2\{1 - k\beta_1\}}{\mathrm{JR}(\mathrm{W}_1 + \mathrm{W}_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(β_1 is written for β as its value is dependent on n).

Similarly

$$\mu_{21} = \frac{n_1^2\mathrm{E}^2\{1 - k\beta_1\}}{\mathrm{JR}(\mathrm{W}_2 + \mathrm{W}_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence, by division of (11) and (12),

$$\frac{\mu_{11}}{\mu_{21}} = \frac{\mathrm{W}_2 + \mathrm{W}_x}{\mathrm{W}_1 + \mathrm{W}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Equation (13) gives the means of calculating the water equivalent.

Writing M_1, M_2 for $\mathrm{W}_1 + \mathrm{W}_x$ and $\mathrm{W}_2 + \mathrm{W}_x$ respectively, we have

$$\mu_{12} = \frac{n_2^2\mathrm{E}^2\{1 - k\beta_2\}}{\mathrm{J.R.M}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

which, with (11), gives

$$\frac{\mu_{12}}{\mu_{11}} = \frac{n_2^2}{n_1^2}\cdot\frac{1 - k\beta_2}{1 - k\beta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

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* This does not introduce σ, the heat developed by the stirrer, but this can be found by preliminary experiments and due allowance made for it.

If k, the temperature coefficient of the resistance of the wire, is determined by preliminary observations, and we assume $\beta = p\text{E}^2$ where p is a constant (p. 400), we have with (15) sufficient data to determine the values of β_1, β_2. Equation (11) then gives

$$\text{J} = \frac{n_1{}^2\text{E}^2 \{1 - k\beta_1\}}{\text{M}_1 . \text{R} . \mu_{11}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16).$$

Again equation (10) gives

$$\lambda_{11} = \frac{\rho}{\text{M}_1} + \frac{n_1{}^2\text{E}^2 (k + l_1)}{\text{J} . \text{R} . \text{M}_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17),$$

and

$$\lambda_{12} = \frac{\rho}{\text{M}_2} + \frac{n_2{}^2\text{E}^2 (k + l_1)}{\text{J} . \text{R} . \text{M}_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18),$$

which are two equations to give ρ and l_1.

Similarly, by performing the operations λ_{21} and λ_{22}, we could determine the value of l_2; and since (p. 480)

$$l_1 = \frac{f\text{W}_1 + g\text{W}_x}{\text{W}_1 + \text{W}_x} \quad \text{and} \quad l_2 = \frac{f\text{W}_2 + g\text{W}_x}{\text{W}_1 + \text{W}_x},$$

we can find both f and g, the coefficients of increase in specific heat of water and of the calorimeter.

APPENDIX II.

The Regulator for Maintaining the Laboratory at a Constant Temperature.

A narrow glass tube, several feet in length, was fixed in a horizontal position on one of the walls.* This tube contained chloride of silver, and a stream of dry ammonia gas at a low temperature was passed through it until the compound AgCl,3NH_3 was formed.† This compound, as pointed out by ISAMBERT, dissociates at ordinary temperatures, the pressure of the vapour at about 15° C. altering by more than 12 millims. per degree. Care had to be taken to completely saturate the AgCl, for the pressure of the vapour AgCl,2NH_3 changes at ordinary temperatures by a much smaller amount. The horizontal tube communicated with a gas regulator of the ordinary pattern, except that the diameter of the regulator tube could be made of any size, since we are dealing with the pressure of a saturated vapour, not with that of a gas, and thus the changes in *volume* caused by a movement of the mercury column could be disregarded, the diameter of the smallest pipe in our regulator exceeding 1 centim. In order to diminish the effect of changes in the atmospheric pressure, some air was allowed to remain in the horizontal tube. As the column of mercury communicating with the regulator was about 800 millims. in height, this residual air was under a pressure of two atmospheres, and, its volume being very small, and the diameter of the mercury tube large, a small movement sufficed to compensate changes in atmospheric pressure. Thus, an alteration of 20 millims. in the barometer produced a change of only a fraction of a degree in the resulting laboratory temperature. The gas, after passing through the regulator, communicated with a large Doulton ware stove, in which small pilot lights were always kept burning by means of an independent supply. An entrance and an exit tube, closed by glass taps, were so fixed in the vertical tube containing the mercury that, by their means, the instrument could be adjusted.

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* The exterior of this tube was blackened.

† This substance was suggested by Mr. C. T. HEYCOCK.

The action of the apparatus is, in many ways, interesting. On a day when the gas would be burning, if the room was unoccupied, the presence of one person in the room would diminish the supply, whereas the presence of two or three would cause it to be entirely cut off. We were surprised to learn, by this means, the amount of heat communicated to a room by the presence of a human being.

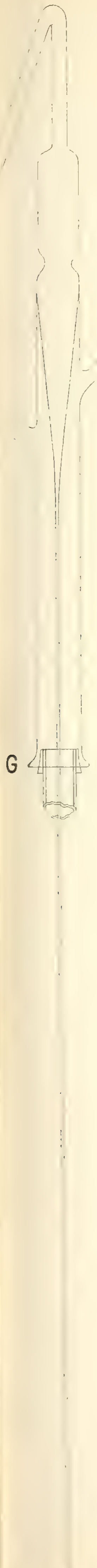
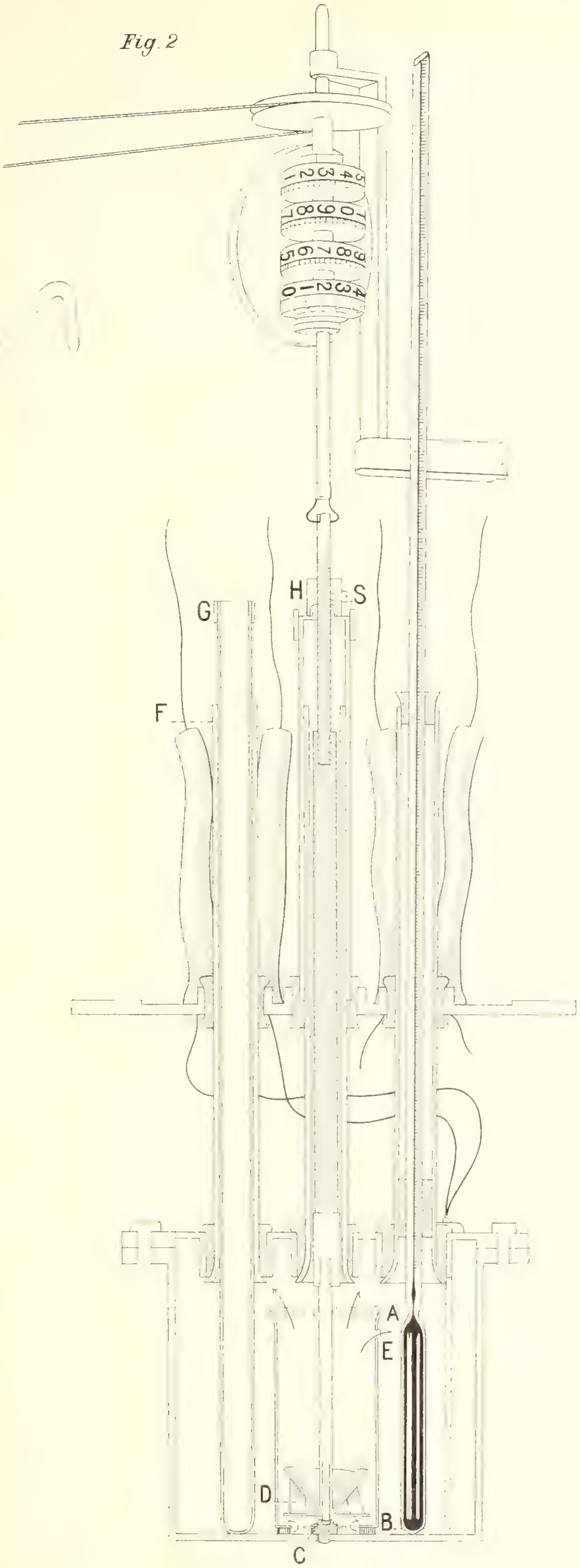


Fig. 2



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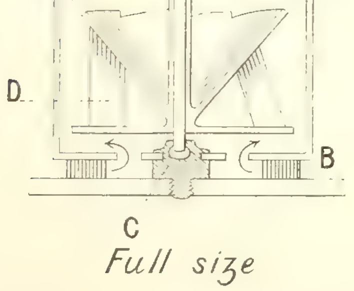
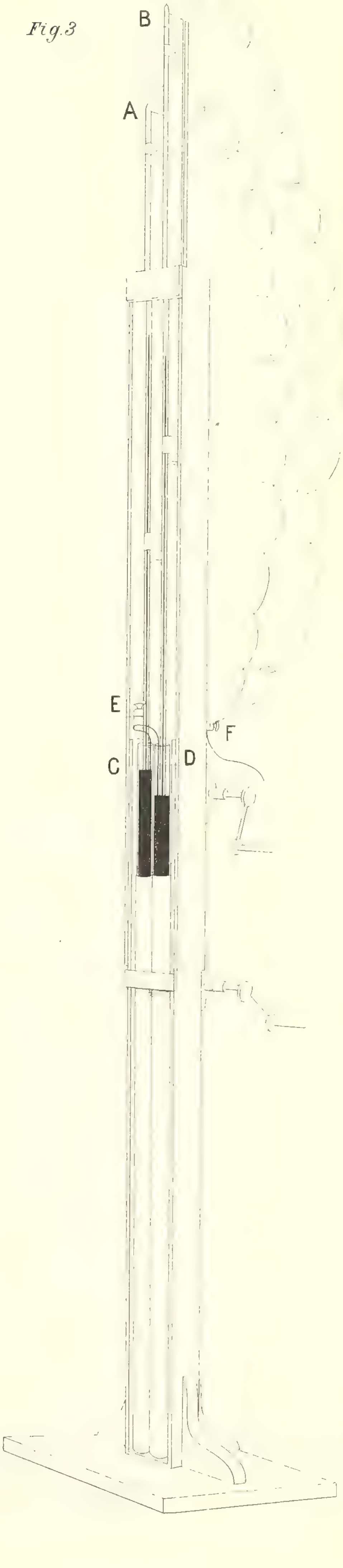


Full size

Fig. 3



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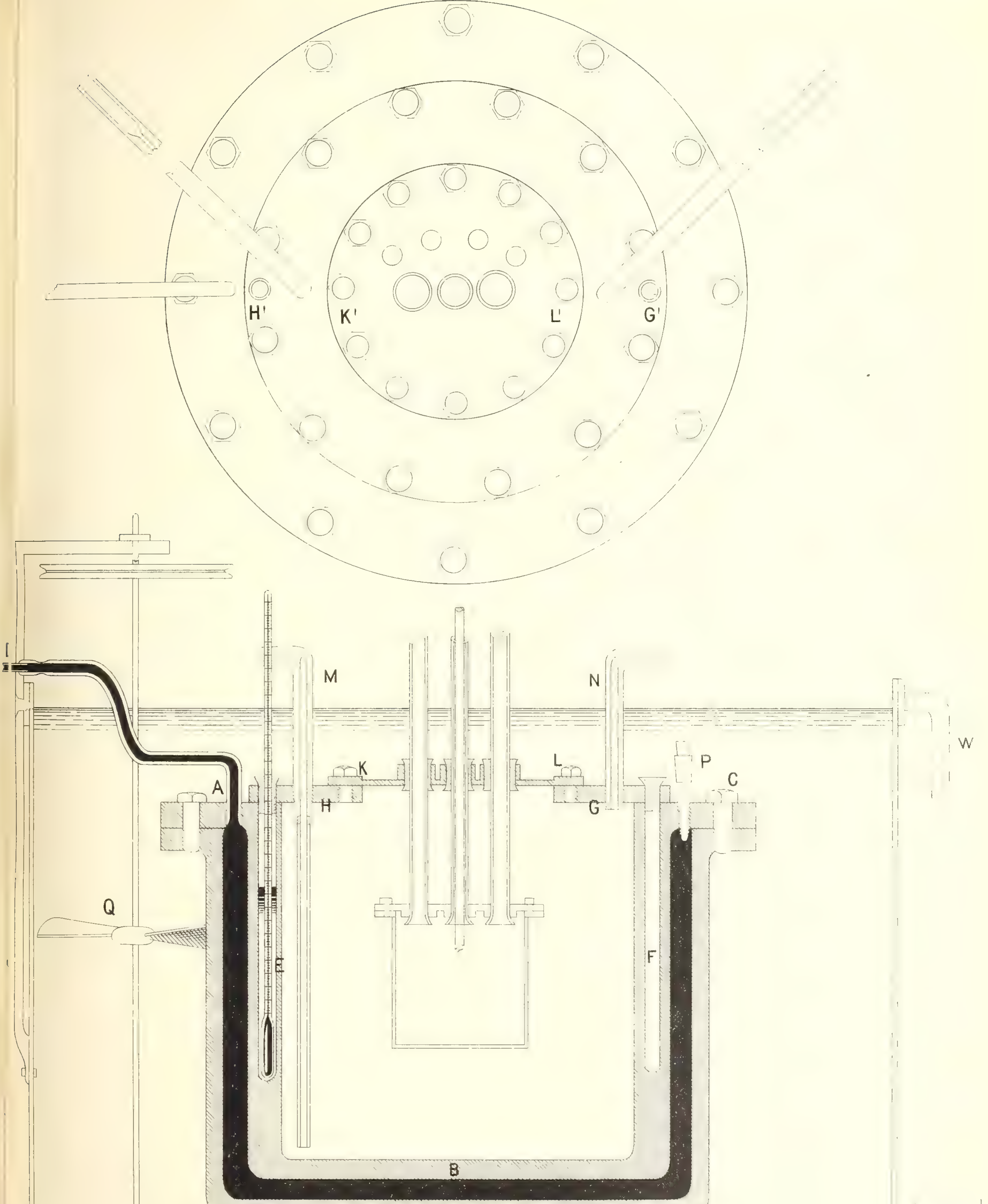

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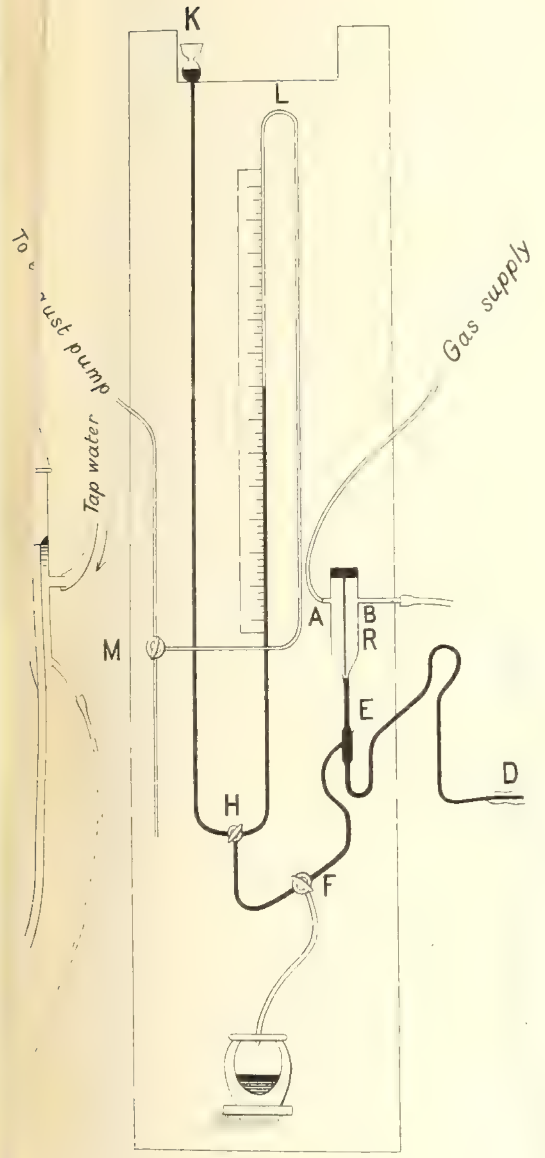
NATURAL HISTORY

Fig 1.



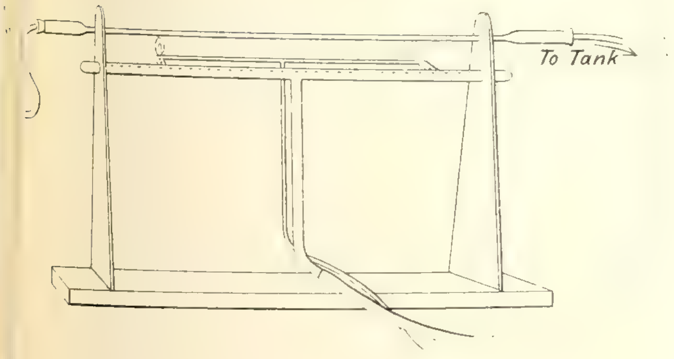
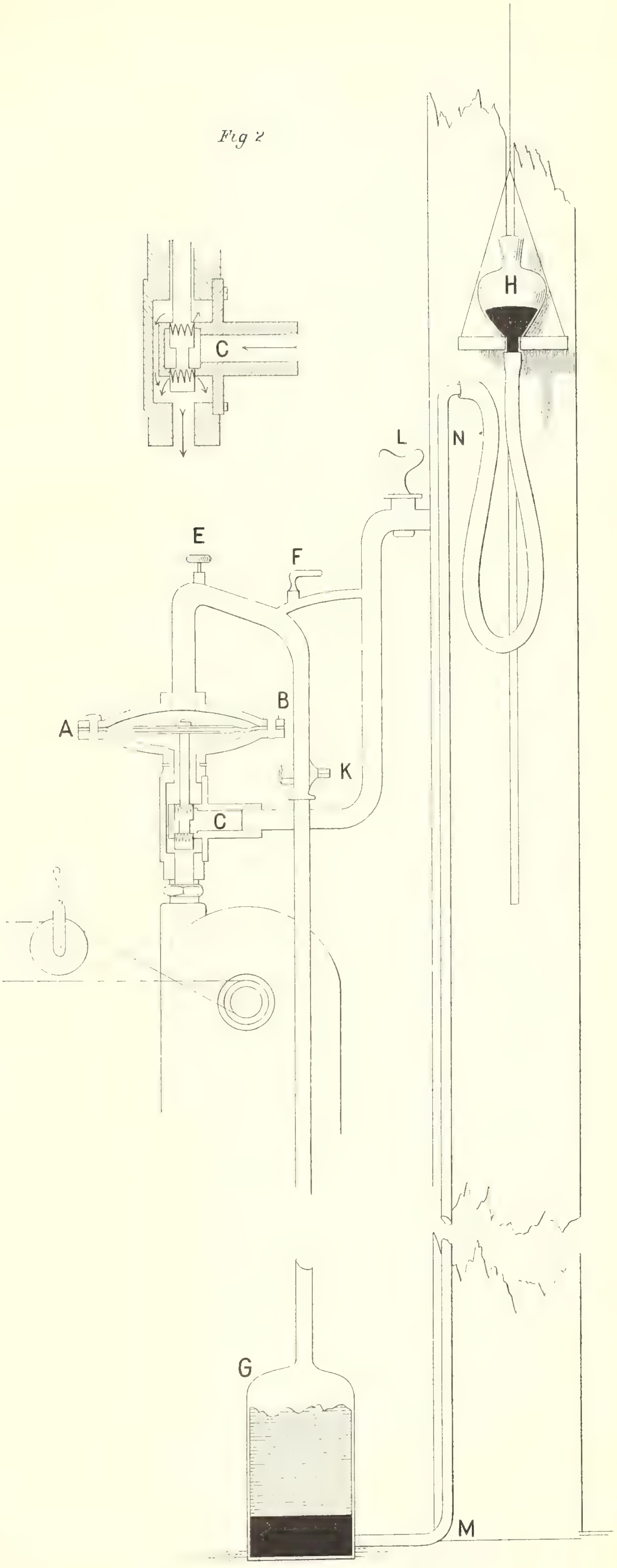


Fig 2



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VII. *On the Thickness and Electrical Resistance of Thin Liquid Films.*

By A. W. Reinold, *M.A., F.R.S., Professor of Physics in the Royal Naval College, Greenwich, and* A. W. Rücker, *M.A., F.R.S., Professor of Physics in the Royal College of Science, London.*

Received March 10,—Read May 4, 1893.

In a paper on the "Limiting Thickness of Liquid Films," presented to the Royal Society in 1883,* we proved that in the case of a large number of films the mean thickness of that part which was thin enough to display the black of the first order of Newton's rings was 11·6 $\mu.\mu.$ (micromillimetres).

The validity of the methods employed was assured by the agreement between the results obtained. The mean thickness of a number of black films was measured optically by a method which involved the assumption that the refractive index of the substance of the film was the same as that of a large mass of the same liquid. The thickness of black films was also determined by measuring their electrical resistances, and assuming that the specific resistance was the same as that of the liquid in mass. The means of the results obtained by the two methods were 11·8 $\mu.\mu.$ and 11·3 $\mu.\mu.$ respectively, which were sufficiently in accord to justify the assumptions.

While, however, there was this close agreement in the mean results, the numbers obtained by different experiments differed rather widely. We then expressed the opinion that "the differences between the numbers given by the optical method are, perhaps, not much in excess of the probable error of experiment, but in the case of the electrical observations they far exceeded it."† We then proceeded, in a passage which it is perhaps unnecessary to quote at length, to give reasons for our belief that "the thickness of the black portions is really different in different films," and added that "into the cause of the variations in the thickness of different black films we do not now propose to enquire." Since that date we have made a large number of new experiments in which the thicknesses of black films of different composition have been determined both by electrical and optical methods. Some of the observations were described to the Society in a paper "On Black Soap Films," which was read on March 1, 1890. The main object then before us was to enquire whether the rate of formation of the black depends on its thickness. Before the paper was published, fresh facts came to our knowledge which appeared to show that however this might be, there was an undoubted and much more clearly marked relation between the composition of the film and the thickness of the black which it produces. We therefore requested that the publication of our paper might be delayed, and we now include the results then obtained in an account of further experiments which have been carried out as opportunity offered during the last three years.

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* 'Phil. Trans.,' 1883, Part 2, p. 645.

† *Loc. cit.*, p. 660.

We think that it will be most convenient to state our results in a series of propositions, and to describe in each case the experimental evidence on which the conclusions are based. It is necessary to state, by way of preface, that the optical and electrical methods of measuring the thickness of the films are in all essentials identical with those described in our paper "On the Limiting Thickness of Liquid Films," already referred to, and that the precautions therein detailed have been carefully attended to.

Proposition I.—In the case of soap films formed with a solution of oleate of soda (one part of oleate in about 70 *of water) containing about* 3 *per cent. of potassium nitrate, the mean thickness when they are thin enough to appear black, is the same whether measured optically or electrically, and is about* 12·1 $\mu.\mu.$

The proof of this proposition is contained in our paper on the limiting thickness of liquid films already referred to.

It will, perhaps, be remembered that a similar statement is also true when the solution is mixed with two-thirds of its volume of PRICE's glycerine.

In the next two propositions we deal exclusively with results obtained by the optical method of measuring the thickness of the films.

Proposition II.—If the proportion of potassium nitrate mingled with the solution be diminished the thickness of the film as measured by the optical method increases.

This is proved by the following Table. In all cases oleate of soda was used, the proportion being 1 part of soap to 50 of water in the case of the first two liquids, and 1 of soap to 40 of water in the last two. These variations in the proportion of soap to water were made from time to time in consequence of the great difficulty of getting black films at all. Often an entire day was devoted to the work without a single result being obtained. It thus became necessary to try a new solution, which was purposely made a little different from the old one, in the hope of securing some black films on which measurements could be made. Even had we been aware, at the time when many of these experiments were carried out, of the effect on the thickness of the film of a change in the proportion of soap, we could hardly have made all the experiments under precisely similar conditions. The question as to why a particular solution sometimes does and sometimes does not readily thin to the black cannot at present be answered, and it is only by varying the conditions that any results can be obtained.

In the Table the results of the individual experiments are given, each thickness being the mean of the thicknesses of about 100 films.

TABLE I.

Percentage of KNO_3.	Thickness of black films in $\mu.\mu.$, measured optically.	Mean.
3·0	12·6, 12·2, 12·3	12·4
1·0	15·7, 11·4, 13·7, 13·2	13·5
0·5	13·4, 15·7	14·5
0·0	23·2, 22·3, 20·3, 22·8	22·1

It is satisfactory to note the close agreement of the above thickness for the 3 per cent. solution with that previously obtained by us ('Limiting Thickness,' *loc. cit.*, p. 661), the numbers being 12·4 and 12·1 $\mu.\mu.$ respectively. We may note, however, that the solution previously used contained less soap (viz., 1 in 70 instead of 1 in 50), a point to which we shall refer again.

The number of observations is hardly sufficient, and the discrepancies between the individual measures on 1 per cent. and 0·5 per cent. solutions are too great to warrant our insisting on the gradual increase in the thickness at these percentages. There can, however, be no possible doubt that when no KNO_3 is present, the numbers are much larger than when the solution contains 3 per cent. of the salt.

This result is confirmed by the following measurements in which soft soap (oleate of potash) was used instead of oleate of soda, and sodium sulphate instead of potassium nitrate. The solution became thick when a larger proportion than one per cent. of Na_2SO_4 was mingled with it.

TABLE II.—Soft soap; 1 part in 40 of water.

Percentage of Na_2SO_4.	Thickness of black films in $\mu.\mu.$, measured optically.	Mean.
1·0	14·1, 16·3, 14·3	14·9
0·0	20·1, 18·2, 19·8, 21·3, 17·8, 19·7	19·5

A further confirmation is obtained by the use of *liquide glycérique.* It is practically impossible to employ a soap solution in which more than 3 per cent. of potassium nitrate has been dissolved, as the liquid becomes "ropy" and does not thin to the black. With a *liquide glycérique*, however, containing two parts by volume of glycerine to three of soap solution, the proportion of potassium nitrate can be largely increased. The following Table (Table III.), exhibits the results obtained. We follow the plan which we have adopted in previous papers of calculating the percentage of KNO_3 with reference to the soap solution only. Thus, if equal volumes of the two liquids were compared, a 10 per cent. *liquide glycérique* would contain the same amount of KNO_3 as a 6 per cent. solution of soap in water only, the proportion of salt to soap being the same in each case.

TABLE III.—*Liquide Glycérique;* 1 part of soap to 40 of water in all cases.

Date.	Percentage of KNO_3.	Thickness in $\mu.\mu.$, measured optically.	Mean.
1891	8·66	8·3, 9·7, 9·4, 10·5	9·5
,,	5·2	10·7, 11·0, 10·6 11·8, 11·2, 11·2	11·1
1883	3·0	10·6, 10·2, 12·5, 10·7 10·2, 11·0, 9·4	10·7
1891	0·0	23·9, 25·4	24·7

These figures are, we think, sufficient to prove the truth of the proposition.

We now pass from a study of the effect of a change in the proportion of salt to that of an alteration in the quantity of soap dissolved, the optical method of measuring the thickness being still adhered to.

Proposition III.—If the proportion of oleate of soda in a solution containing no potassium nitrate be diminished, the thickness of the film, as measured by the optical method, increases.

TABLE IV.—Hard Soap. No dissolved Salt.

Proportion of soap to water.	Thickness of black film in $\mu.\mu.$, measured optically.	Mean.
1 to 30	21·8, 21·7, 21·0, 22·0	21·6
1 ,, 40	23·2, 22·3, 20·3, 22·8	22·1
1 ,, 60	28·1, 27·4, 27·6	27·7
1 ,, 80	28·8, 29·4, 29·8	29·3

The first point to notice is that the results given in this Table strengthen the argument in favour of Proposition II. One part of soap was used with 50 of water in the two solutions first referred to in Table I., but in the case of the last two the proportion was 1 to 40. The above Table proves that, had the proportion of soap been the same in all cases, the relative thickness of the films formed with a solution containing no KNO_3 would have been even greater than appears from Table I. It is, therefore, not too much to say that the average thickness, measured optically, of a black soap film formed of a solution of 1 part soap in 50 of water, but containing no KNO_3, is double that of a similar film made of a solution of the same composition in which 3 per cent. of KNO_3 has been dissolved.

The results obtained with the plain hard soap solution are confirmed by experiments on *liquide glycérique.*

TABLE V.—*Liquide glycérique.* No Salt dissolved.

Proportion of soap to water.	Thickness of black film in $\mu.\mu.$, measured optically.	Mean.
1 to 40	23·9, 25·4	24·6
1 ,, 60	28·6, 30·0, 32·7, 27·9	29·8

In this case, again, a decrease in the proportion of soap produces an increase in the apparent thickness of the black.

When the solution contains 3 per cent. of KNO_3, variation in the proportion of dissolved soap has comparatively little influence on the thickness of a black film. This statement is justified by the figures given in Table VI.

TABLE VI.—Hard Soap. 3 per cent. of KNO_3.

Proportion of soap to water.	Mean thickness of black in $\mu.\mu.$, measured optically.	Greatest and least values.	Date.
1 in 40	13·0	13·5–12·7	April, 1891
1 ,, 50	12·1	12·9–11·6	,, ,,
1 ,, 60	11·55	Single observation	July, 1892
1 ,, 70	12·1	14·4–10·3	1883

The question now arises whether the large increase in the thickness of black films, formed from what we may, perhaps, for convenience, call an *unsalted* solution is real, or whether it is due to some incorrect assumption. The only point where error is possible is in the hypothesis that the refractive index is the same as in the liquid in mass. The thickness of the film varies inversely as $n - 1$ (n being the refractive index), and as the refractive index of the soap solution is about 1·34, it would have to be reduced to about 1·17 in order that the calculated thickness might be doubled. It appears, therefore, *à priori,* extremely improbable that the mere addition of 3 per cent. of KNO_3 should so completely change the optical properties of the liquid that, whereas, if the salt be added, the refractive index is practically the same in thin films and in the liquid in mass, as is proved both by our experiments of 1883 and by many made subsequently (see Tables I. and VI.); yet, without the salt, the refractive index of the film should be as much as 13 per cent. less than that of the liquid in mass.

To this *à priori* argument the following may be added. During the progress of our work a paper has appeared by Herr DRUDE ('Wied. Ann.,' vol. 43, 1891, p. 169), in which the author, using an experimental method totally different from that employed by us, compares the refractive indices of black and coloured soap films, of which the latter may unquestionably be taken as nearly the same as if not absolutely identical with that of the liquid in mass. Using solutions both with and without the addition of a salt (viz., Na_2SO_4), the author finds that the refractive indices of the black and coloured films do not differ by more than about 1 part in 140. Such a variation, if satisfactorily established, would only affect the apparent thickness of the films as measured by the optical method by 2·6 per cent., whereas, according to the measurements given above, the presence or absence of the salt alters the apparent thickness by 100 per cent.

On the whole, then, we certainly think that the differences of thickness indicated by the optical method of measurement may be relied upon as being not merely apparent but real, and we shall hereafter treat this point as established.

Having thus studied the effect of the composition of the films on their thickness, we proceed to describe the results of observations in which the apparent thickness was measured by the electrical method.

Proposition IV.—If the proportion of potassium nitrate mingled with the solution be diminished, the thickness of a black film, as measured by the electrical method, increases, but in a far larger ratio than would be inferred from the optical method. If the proportion of salt be diminished to zero the thicknesses thus calculated are greater than the greatest thickness at which a film can appear black. In such cases, therefore, the electrical method does not give the true thickness of the film, and the hypothesis that the specific conductivities of the film and of the liquid in mass are identical is untenable.

This statement is proved by the following table. A very large number of observations have been made with unsalted solutions, but we give the results of the latest experiments only which were made with all the precautions which experience has shown to be necessary.

TABLE VII.—Hard Soap; 1 part to 70 of water.

Percentage of KNO_3.	Thickness in $\mu.\mu.$, measured electrically.	Mean.
3	9·6, 11·6, 12·3, 7·7, 10·0, 12·5	10·6
2	12·3, 20·0, 8·3, 10·7, 10·5, 11·3, 11·1, 17·3	12·7
1	20·1, 28·0, 22·1, 30·3, 21·6	24·4
0·5	29·8, 29·8, 25·2, 25·8, 22·0, 26·4, 26·4	26·5
0·0	146, 103, 139, 215, 197, 240, 134. 98, 149, 155, 129, 79, 118, 167	148
0·0*	156, 84, 213, 240, 159, 250	184

The first thing to notice with regard to this table is that when the solution contained 3 per cent. of KNO_3, both the variations in the results of the individual experiments and the mean value of the thickness are in close agreement with the corresponding values obtained by us in 1883 ("Limiting Thickness," &c., *loc. cit.*, p. 661). This fact is proved by the following table, in which the thicknesses are as usual expressed in $\mu.\mu.$

TABLE VIII.

Date.	Largest value.	Smallest value.	Mean of all values.
1882 . . .	14·5	7·2	11·7
1891 . . .	12·5	7·7	10·6

Next we note that in Table VII. the indications of an increase in the calculated thickness are unmistakable when the quantity of dissolved salt is reduced to 1 per cent. When no salt is used, and the proportion of soap to water is as 1 to 40, the mean thickness is in one case no less than 184 $\mu.\mu.$ The colour corresponding approximately to this would be the beginning of the yellow or of the red of the first order, according as the incidence of the light was grazing or normal.

Under no possible circumstances could a film of this thickness appear black. The beginning of the black, according to NEWTON, occurs when the thickness is 36 $\mu.\mu.$, which is only about one-fourth or one-fifth of the mean value obtained for unsalted solutions. We are therefore driven to the conclusion that the close agreement between the results of the optical and electrical measures of the thickness of black films obtained both in 1883 and in 1890–92, when the solution contained 3 per cent. of KNO_3, does not hold in the case of unsalted solutions. We have already stated that we believe the optically measured thicknesses to be correct mean values in all cases, but, whether this is so or not, the electrically measured values cannot for the reasons just given be real thicknesses. There must, therefore, when no metallic salt is present, be a difference between the specific conductivities of a film and the liquid of which it is formed.

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* One part of soap to 40 of water.

This conclusion is borne out by the results of experiments made with soft soap.

The following table contains the results of measurements on 23 films made in the summer of 1889.

TABLE IX.—Soft soap. No salt dissolved.

Thickness in $\mu.\mu.$ as given by the electrical method.				
169,	166,	185,	171,	189,
173,	120,	126,	150,	151,
150,	145,	142,	143,	148,
150,	188,	80,	157,	165,
179,	252,	232,		
Mean 162.				

Proposition V.—The apparent thickness, as measured by the electrical method, of different black films formed of an unsalted solution is very variable. In some cases, at all events, this is due to a real variation in thickness.

The results given above are sufficient proof of the apparent variability of the thickness of the film.

The best proof that this variability is sometimes real, is the fact that we have distinctly seen two different shades of black in the same film. NEWTON recognised two different intensities of black in soap bubbles, and Lord KELVIN has on more than one occasion asked us whether we have noticed the same phenomenon. So long as our experiments were made with a solution containing a relatively large proportion of salt our answer was always in the negative. Since we have been working with solutions which contain but little salt, we have occasionally noticed traces of two different black tints in films containing not more than one per cent. of KNO_3. In films containing no salt the phenomenon is of frequent occurrence. The difference of tint is but slight and may easily be overlooked, but in many cases a definite line of demarcation can be observed between the two tints, so that we can certainly state that the upper part of the film showed one tint and the lower part the other, which may be distinguished as β_1 and β_2 respectively. The line of separation is very irregular in form. That which divides the black and coloured parts of the cylindrical films we employ is almost always a horizontal circle. This is by no means the case with the boundary between the two black tints. Sometimes one completely encloses a patch of the other, sometimes the line of separation is sinuous or stands higher at one part of the film than at another. In some cases when no difference of hue was observed, the passage of an electrical current through the film either created the difference or enabled us to notice a boundary line which we had not before perceived. It is a very curious fact that under such circumstances tiny white flecks appear in the black along the boundary line between the two tints. Sometimes this line has been observed before the passage of the current, sometimes it has not been noticed until the flecks have drawn attention to its existence.

These eye observations are confirmed by electrical measurements. It is indeed only on one fortunate occasion that we have been able to measure electrically the apparent thicknesses corresponding to the two black tints when exhibited by the same film. Both hues were then seen and the line of demarcation was above the upper needle. The thickness (measured electrically) of the part between the needles was 200 $\mu.\mu.$ When the line of separation was about half-way between the two needles, the thickness (assumed uniform throughout, and, therefore, really the mean thickness of the two parts) was 157 $\mu.\mu.$ When the line had reached the lower needle, so that both needles had passed through the upper black region, the thickness was 116 $\mu.\mu.$ In this film the curve of separation was approximately a horizontal circle, so that when the needles were in the upper or lower black region the whole of the corresponding horizontal ring on the cylinder was of the same hue. In a few other cases we have seen the line of demarcation pass the upper needle, and as it sank, the apparent thickness of the film, as measured electrically, invariably diminished. In one case we obtained a measurement when the two needles were in β_1 the apparent thickness was 110 $\mu.\mu.$, which, as will be seen by comparing the results for unsalted solutions in Table VII. and IX. is relatively small, but agrees with the value 116 $\mu.\mu.$ obtained on the occasion already described. Unfortunately on this occasion no measure could be made on β_2.

The value of the confirmation—given by the variation of the electrical measures made upon the same film—of the fact disclosed by eye observation that, in an unsalted film there are discontinuous thicknesses, both of which are black, is enhanced by comparison with the extreme constancy of electrical measures which we have previously obtained with salted solutions, provided only that they were made upon the same film.

When experimenting with salted solutions we have failed ("Limiting Thickness, &c.," *loc. cit.*, p. 652) to detect any change in the thickness of the black part of a film which lasted for more than two hours.

These observations certainly suggest that even in cases where we have not noticed a line of separation, which is the only optical indication of the discontinuity of the black part of the film which our methods allow, the extreme variability in the electrically measured thicknesses is to be accounted for by the gradual substitution of one black tint for the other. The line, as we have said, is difficult to see. It is quite probable that the white flecks which draw attention to it are more numerous on some occasions than on others, or may sometimes be altogether absent, and thus the fact that no line has been seen does not prove that the black film was not really divided into the two parts of different thicknesses. It will be observed, however, that we have so framed the proposition under discussion that the possible agency of other causes in bringing about changes in the electrically measured thickness is not denied.

Proposition VI.—Whereas the specific conductivity of black films formed of solutions containing about 3 per cent. of KNO_3 is approximately the same as that of the liquid in mass, the specific conductivity of similar films formed of solutions containing no KNO_3, but consisting of a solution of oleate of soda or oleate of potash in water is very different from that of the liquid in mass.

This is merely a deduction from our previous work and from Propositions III. and IV.

If the optical and electrical methods give the same result, it may be inferred that the specific resistance is not affected by the tenuity of the film. Hence, since this is the case for films which contain considerable proportions of salt, the first part of the proposition is true. If the results obtained by the optical and electrical methods are at variance, one of the assumptions made (viz., that the refractive index and the specific conductivity are independent of the thickness) is incorrect. We have already shown that the change is too great to be accounted for by a variation in the refractive index. We are, therefore, driven to attribute the anomalous, or rather impossible, values of the thickness of unsalted films, obtained by the electrical method, to a change in the specific conductivity.

Proposition VII.—The change in specific conductivity—

(1.) *Is independent of moderate changes of temperature;*

(2.) *Is not due to the absorption or evaporation of water by the film;*

(3.) *Is not due to a change in the composition of the liquid by electrolytic decomposition produced by the current used to measure the resistance of the film;*

(4.) *Is not affected by a very large change in the quantity of carbonic acid in the air around the film.*

(1.) The first of these statements is proved by such facts as the following. The value of β (the apparent thickness of the black) was, on one occasion, 139 $\mu.\mu.$ when the temperature was 22°·2 C. The same liquid on the same day gave $\beta = 103$ $\mu.\mu.$ at 18°, and, in the case of another film, $\beta = 147$ $\mu.\mu.$ at 18°. In this case the variations of β, when the temperature was constant, were greater than the change which might otherwise have been attributed to a rise of 4° in temperature. Sometimes the larger value was found at the lower temperature. Thus, in one case, when

$t = 26°·3$ $\beta = 135$ $\mu.\mu.$
$t = 21°·0$ $\beta = 237$ $\mu.\mu.$

Sometimes precisely the reverse result was obtained, as

$t = 21°·1$ $\beta = 201$ $\mu.\mu.$
$t = 18°·7$ $\beta = 171$ $\mu.\mu.$

Instances such as these might be multiplied indefinitely, and suffice to show that the relatively small variations of temperature which occurred during our experiments are not the cause of the fact that a film, of which the true thickness is about 22 $\mu.\mu.$, appears when measured electrically to have a thickness of 184 $\mu.\mu.$

(2.) The precautions taken to prevent absorption or evaporation by the films were precisely the same as those which our previously published results have shown to be efficient in the case of *liquide glycérique*, which is much more sensitive to changes of hygrometric state than solutions containing no glycerine.

The bottom of the vessel in which the films were formed was flooded with the liquid. A continuous roll of linen dipped into the liquid, and could be moved so that all portions in turn were moistened, thus exposing a large evaporating surface. A hair hygrometer was enclosed in the vessel. No observations were made until the index of the hygrometer ceased to move. A constant hygrometric state was thereafter maintained. In the case of liquids not containing glycerine, we found that the layer of liquid at the bottom of the vessel was sufficient for this purpose, as the results were the same, whether the roll of linen was or was not used. The vessel in which the films were made was surrounded by a tank of water, the object of which was to check variations of temperature. To assure ourselves that the measurements were not affected by the diffusion of water vapour, into the interior we made experiments when the tank was empty as well as when it was full, and obtained precisely the same results in the two cases. Hence, considerable variations in the hygrometric state of the air close to the vessel, and in the means taken to secure a constant hygrometric state within the vessel produced no effect on the results.

(3.) As, in order to assist in reducing the thickness of a film, we frequently use electric currents which, although they are very weak, pass for some time through the film and are relatively strong when it is thick, we thought it very important to prove that the change in specific conductivity is not due to the passage of the current. Comparative observations were therefore made, in some of which a current from a battery of from fourteen to twenty-eight Leclanché cells was passed through the film from the moment of its formation, in others no current was used until the time for measurement arrived, and then it was produced by seven Leclanché cells only, and was applied intermittently so that the circuit was not completed unless an observation was actually in progress.

The following results were obtained with an unsalted oleate of potash solution. Each of the values of β was derived from a different film, and the number of cells indicated is in each case that employed to pass a continuous current from the first formation of the film. Of course in all cases a current was used at the moment of measurement.

TABLE X.

Number of cells.	β (apparent thickness of black film measured electrically).
0	150, 171, 148, 150
14	151, 145, 142
28	150, 157, 179

These numbers prove that the results were not affected by the strength or duration of the current which was passed through the film before the measurements were actually made.

These observations were supplemented by others in which the resistance of films could be compared by means of currents which were practically instantaneous. For this purpose the needles were connected with the terminals of the secondary of a Ruhmkorff coil, a ballistic galvanometer being included in the circuit.

The primary circuit could be completed or broken by an independent key. In experiments on different films the circuit was always completed for the first time when the black had descended to the same distance below the lower needle, so that the throw would be nearly inversely proportional to the resistance of the black film.

The deflections of the galvanometer were identical in two cases, in one of which a current from twenty cells had been passed through the film (formed of an unsalted hard soap solution) from the time of its formation, while in the other a film formed of the same liquid had not been subjected to any electrical action until the transient current used in the measurement was employed. It might possibly be urged in answer to this that we have no evidence that the thickness of the black was the same in each case—that we may have been dealing with β_1 in the one case and β_2 in the other. If this were so the equality of the throws could only be explained by the hypothesis that the product of the thickness and specific conductivity is the same for β_1 and β_2, and this is inconsistent with the large change in resistance which the substitution of one for the other has been proved above to produce.

In fact the observations are perhaps open to criticism of a precisely opposite kind, viz., that in view of the irregularities in the value of β obtained by our ordinary method of measurement, it is strange that on this occasion we should have obtained the same result in the case of two consecutive films. Such coincidences were not uncommon ; but the objection is, perhaps, best answered indirectly by comparing the throws in the case of salted and unsalted solutions, and thus determining whether the ratio of the computed apparent thicknesses is about the same when transient and continuous currents are used. This experiment we have performed. With a solution containing 3 per cent. of KNO_3 the throw was 19 divisions of the scale. With an unsalted solution when the black had descended to the same level the throw was 9·5 divisions. The specific resistances of the two liquids were 36 and 890 ohms per cubic centim. respectively. If r_1 and t_1 are the resistance and calculated thickness of the unsalted film, and ρ_1 is the specific resistance of the liquid of which it is formed ; and if r_2, t_2, ρ_2 have similar meanings with respect to the salted film, we have the equation

$$\frac{t_1}{t_2} = \frac{\rho_1 r_2}{\rho_2 r_1},$$

or, remembering that the throws are inversely proportional to the resistances,

$$\frac{t_1}{t_2} = \frac{890}{36} \times \frac{9{\cdot}5}{19} = 12{\cdot}4.$$

But t_2, the thickness of a salted film, is about 12 $\mu.\mu.$ Hence t_1, the calculated apparent thickness of the unsalted film, = 149 $\mu.\mu.$ This number is intermediate between the largest and smallest of those obtained by the continuous current method and exhibited in Table VII, and does not differ very widely from the mean, which is about 160.

On the whole then the evidence clearly points to the conclusion that the passage of the current does not produce the change of specific conductivity under discussion.

(4.) Passing now to the question as to whether absorption of CO_2 from the air is the efficient cause, it must be remembered—and this remark applies also to the possibility of the results having been affected by evaporation or change of temperature—that all our observations are based on a comparison between two solutions which differ only in the addition to one of them of not more than 3 per cent. of KNO_3, or 1 per cent. of Na_2SO_4. If, therefore, the change in specific conductivity were ascribed to any one of these causes, it is not only necessary to assume that the cause was competent to produce the change, but that its efficiency was very greatly modified by the addition of the salt.

As it is difficult to argue from the results of experiments conducted on the liquid in mass to the behaviour of a film, and as it is probable that, if sufficient time were allowed, the amount of CO_2 absorbed by a liquid containing a substance such as oleate of soda, upon which it acts, might be very considerable, we have tested the matter by direct experiment. A series of observations were made when the air which surrounded, and that which was used to blow the films had been carefully freed from CO_2 by passage through three tubes containing caustic alkali. As it would have been extremely inconvenient to have made the apparatus air-tight, a current of the purified air was passed through it continuously during the whole of the experiment. Although, therefore, the proportion of CO_2 in the air around the films was—if not absolutely *nil*—very much reduced below its normal value, the results as regards the electrical thickness were practically identical with those obtained when no such precautions were taken.

TABLE XI.

Date.	Proportion of oleate to water.	Electrical thickness of black in $\mu.\mu.$			
		Highest.	Lowest.	Mean of all.	Number of films.
June and July, 1890	$\frac{1}{70}$	240	79	148	14
		Taken without any special precaution. Linen band not used.			
November and December, 1890 . .	$\frac{1}{70}$	243	73	154	7
		Linen band used. Films blown with air freed from CO_2, but no continuous flow of purified air through the film box.			
July, 1891	$\frac{1}{40}$	250	84	184	6
		See Table VII. Every known precaution used. Continuous flow of air freed from CO_2 through the film box.			

Although the mean values differ somewhat, it will be observed that the highest and lowest values in each set are about the same. There is thus no indication that the presence or absence of CO_2 in the air surrounding the film has any marked influence on its electrical thickness. This argument against CO_2 being the cause of the high conductivity of the thin films is much strengthened if we consider the ratio of the electrical to the optical thicknesses, *i.e.*, the ratio of the specific conductivity of the film to that of the liquid in bulk. Dividing each of the electrical thicknesses in the above table by the corresponding optical thickness given in Table IV., we get the three values, 5·2, 5·4, 8·3. Hence the specific conductivity of the film was about 50 per cent. greater when every precaution was taken to exclude CO_2 than when no such precaution was taken, which is certainly opposed to the theory that CO_2 is the efficient cause.

Proposition VIII.—The change in specific conductivity is practically unaltered if the films are formed in an atmosphere of oxygen.

When the films were formed in an atmosphere of either oxygen or nitrogen they did not thin in the normal way. The black was either not formed at all, or, if formed, was streaked with lines and flecks of other colours. An electrical measurement therefore only gave the mean apparent thickness of a non-uniform mass, from which the thickness of the black could not be deduced. When the film case was filled, and the film itself blown with purified oxygen, from which all traces of chlorine had been carefully removed, it was, however, possible to make observations when the thickness corresponded to the white of the first order. This colour changes so slowly as the film thins that no accurate estimate of the thickness can be based upon it. Alternate series of measurements were, however, made when the box was filled with air and oxygen respectively with the following results :—

TABLE XII.

Air purified from CO_2.		Oxygen.		Air purified from CO_2.	
Colour of the first order.	Electrical thickness in $\mu.\mu.$	Colour of the first order.	Electrical thickness in $\mu.\mu.$	Colour of the first order.	Electrical thickness in $\mu.\mu.$
Yellow	407	Yellowish-white . .	403	White	465 to 383
White	387	Grey to white . .	326		
White, full of black spots	346	Grey (with black specks) to grey	296	Grey to white . .	316
Grey	286	Grey (mixed with black spots) to white	245	Grey to white . .	296
,, ...	261				
,, ...	236				

It must be clearly understood that the estimates of colours of films as thin as these are very untrustworthy, but they are, we think, sufficient to prove that the phenomena are substantially identical in air and in oxygen.

The thickness corresponding to the white of the first order, as given in NEWTON'S table, is 97 $\mu.\mu.$ Hence, in all cases, the electrical very much exceed the true thicknesses. The first two numbers in the case of oxygen (in Table XII.) correspond to colours given by thicknesses a little greater and less respectively than that which corresponds to the white. Taking the mean of these, and also the mean of the extreme thicknesses for which the film appeared white in the second air experiment, we get the following Table :—

TABLE XIII.

	Electrical thickness of white.	Ratio to true thickness.
Air	387	3·99
Oxygen	364	3·75
Air	424	4·37
Air (later experiments; see p. 522	434	4·47

The ratio obtained when oxygen was used is less than in the other cases, but the difference is small when the uncertainty of the colour estimates is allowed for; and the agreement is, at all events, sufficiently close to prove that a very great increase in the quantity of oxygen in the neighbourhood of this film produces little or no change in the specific resistance.

The nitrogen used was produced by heating a mixture of ammonium chloride and potassium nitrite, and was passed through dilute sulphuric acid and caustic potash. Whatever the cause may have been, we found that in this gas the film did not thin regularly, so as to exhibit pure and unmixed colours. Estimates of the thickness were therefore impossible.

Before proceeding to discuss these results further, it may be well to ask whether the large changes in specific conductivity affect the black films only, or whether similar phenomena can be detected when the thickness is greater. On this point we have made numerous observations, which have led us to the following conclusion :—

Proposition IX.—The specific conductivity of a soap film increases as the thickness decreases, but this increase is less in the case of a film formed of a solution to which a salt has been added, and cannot be detected in the case of a "liquide glycérique" such as that we have employed.

To justify this statement we adduce the following facts which are only samples of the results of a large number of experiments. Perhaps the most convenient way of testing the constancy of the specific conductivity of a film is to calculate the thickness from observations on the electrical resistance, assuming that the specific resistance is the same as that of the liquid in mass, and to compare the number so obtained with the thickness given by a direct optical measurement based on the colour. If the ratio of the electrical thickness to the optical thickness is unity there is no reason to suspect a change in the specific conductivity. If it differs much from unity the presumption is in favour of such a change. One film, formed of a solution containing one part of oleate of soda to 60 of water without the addition of any metallic salt, which was observed on July 7, 1892, lasted two hours and a half. During this time it thinned to the white of the first order, was thickened by sending a current up it, and was again allowed to become thin. The following table gives the optical thickness and the ratio of the electrical to the optical thickness. It will be observed that the latter differs considerably from unity, and increases as the film becomes thinner.

TABLE XIV.

Optical thickness in $\mu.\mu.$	Ratio of electrical to optical, *i.e.*, to true thickness.
641	1·66
484	1·69
388	1·60
332	1·81
320	1·78
314	2·07
,,	1·90
307	1·84
296	1·98

The electrical thickness was also measured on each occasion when the film displayed the white of the first order. Another film was then formed, and after its thickness had been very rapidly reduced by means of a current, the electrical thickness of the white was again determined. The numbers thus obtained were 407, 430, and 465 $\mu.\mu.$ respectively, the mean being 434 $\mu.\mu.$ As the true thickness corresponding to this colour is 97 $\mu.\mu.$, it will be seen that the ratio is 4·47.

Summing up we have the results given in the following Table.

TABLE XV.—Solution of oleate of soda containing 1 part of oleate in 60 of water, but no metallic salt.

Colour.	Optical, *i.e.*, true thickness, in $\mu.\mu.$	Ratio of electrical to optical thickness.
Green of fourth order G (4, 1)	641	1·66
Green of second order G (2, 1)	296	1·98
White of first order	97	4·47
Black	27·7	5·8

The optical thickness assumed for the black is the mean of those obtained from films of similar constitution (see Tables IV. and VII.), the electrical thickness, viz., 160 $\mu.\mu.$, is deduced from the value obtained for films containing 1 part of soap to 40 and 70 of water respectively.

It is important to remember that after the white had been exhibited, the film was thickened and then allowed to thin to the white again, and that the ratio of the electrical and optical thicknesses changed from a large to a relatively small, and then again to a large value. This is shown below.

TABLE XVI.

Time.	Colour.	Optical thickness.	Ratio.
h. m. s.			
3 16 0	Yellow of the second order Y (2, 0). .	320	1·78
3 17 45	Green of the second order G (2, 3) . .	301	1·84
Later	White of the first order	97	4·19
4 23 15	Red of the second order after the film had been thickened	388	1·60
4 49 0	White of the first order	97	4·47

The numbers given in this Table, combined with the fact that we always get a much larger ratio (of which the mean is 5·8), when the film is thin enough to display the black, are certainly consistent with the view than when the same colour is reproduced, the ratio is, at all events, approximately the same, and that, therefore, the change in specific conductivity indicated by the value of the ratio is a function of the thickness. The connection is further illustrated by the curve on p. 526.

We now proceed to consider films formed of liquid, to which a metallic salt has been added.

The following Table gives the values of the ratios of the electrical and optical thickness as obtained from a number of different films. The numbers are arranged in such a way as to bring out the various points which are worth attention.

TABLE XVII.

Solutions containing 3 per cent. of KNO_3 but no glycerine.			
Number of film.	Date.	Optical, *i.e.*, true thickness.	Ratio of electrical to optical thickness.
	1892.		
Film 2	July 11	662	1·06
		563	1·07
,, 2	,, 12	600	1·03
		545	1·00
		528	0·94
		496	0·93
		429	1·08
		412	1·15
		337	1·24
		305	1·21
		253	1·16
,, 3	,, 11	320	1·28
		300	1·14
		289	1·12
,, 4	,, 11	401	1·22
		307	1·25
		305	1·27
		215	1·25

In the first place it will be noticed that in all cases the ratio is much less than when no salt is added to the solution. At optical thicknesses above 450 $\mu.\mu.$, the number varies on each side of unity, giving no evidence of a change in specific conductivity. This is proved by the films numbered 2, on July 11 and 12. At thicknesses between 200 and 450 $\mu.\mu.$, the ratio is generally above unity, being in some cases as large as 1·27 or 1·28. It is, however, to be observed that there is no clear indication of an increase in the ratio as the film thins. The value corresponding to the smallest optical thickness is not the greatest in the cases of film 2, July 12, and film 4, July 11, and is actually the least of those furnished by measurements on film 3, July 11.

The optical and electrical measurements of the thickness of black films formed of a 3 per cent. solution are in as close accord as the nature of the experiments will permit, the electrical being as a matter of fact rather the smaller of the two. Thus, from Tables VII. and I., we get 10·6 $\mu.\mu.$, and 12·4 $\mu.\mu.$, for the mean electrical and optical thicknesses respectively. The discrepancies between the individual experiments are too great to enable us to insist on the accuracy of this difference, but it is evident that there is no such striking divergence between the results of the two methods as is remarked when no salt is present.

We have tried from time to time to obtain some measures of the electrical thickness of films formed of a solution containing 3 per cent. KNO_3, at thicknesses intermediate between that corresponding to the black and 200 $\mu.\mu.$ Such observations would be very valuable in helping us to decide whether there is or is not a real maximum of specific conductivity at about 100 $\mu.\mu.$, which is suggested by the fact that (in spite of the discrepancies above noted) the ratio of the electrical to the optical thickness is less for thick films and for black films than for those whose thicknesses lie between 200 $\mu.\mu.$, and 400 $\mu.\mu.$ Unfortunately, though we have exhausted every device known to us, sometimes leaving the film to thin unaided, sometimes passing weak and sometimes strong currents through it, we have never succeeded in getting the white of the first order. This connecting link cannot, therefore, be at present supplied, but the facts are sufficient to justify the proposition that the increase in specific conductivity observed in the case of an unsalted solution is largely diminished by the addition of a salt. It is indeed open to question whether the evidence would not support the statement that there is no appreciable change in the specific conductivity of a film containing 3 per cent. of salt, but although we hardly feel justified in drawing this general conclusion, there can be no doubt whatever that it is true for a *liquide glycérique* for the black on the one hand, and for thicknesses greater than 374 $\mu.\mu.$ on the other, and therefore, in all probability (though not quite certainly) for intermediate thicknesses. This conclusion rests upon our previous work. In 1881 we proved that there is no change in the specific resistance of a film of salted *liquide glycérique* down to a thickness of 374 $\mu.\mu.$ ('Phil. Trans.,' Part 2, 1881, p. 447, *et seq.*) while the optical and electrical thicknesses of black films are 10·7 and 11·9 $\mu.\mu.$, respectively ('Phil. Trans.,' 1883, Part 2, p. 661). Both of these conclusions have been verified by more recent experiments. In the following table we give the ratios of the electrical to the optical thicknesses found in the case of two films observed on Jan. 9, 1892, which are in entire agreement with our 1881 observations on thick films.

TABLE XVIII.

Optical thickness in $\mu.\mu.$	Ratio of electrical to optical thickness.	
	Film I.	Film II.
789	0·996	
735	1·012	
732		1·064
711		1·022
708	0·989	
672	1·012	
661		1·045
631		1·052
627	1·014	
618		1·023
611		1·005
585	0·995	

Taking the means of these values and making the proper corrections for temperature we get

Specific resistance of liquid in mass	142·6	ohms. per cub. centim.
,, ,, from Film I.	142·0	,, ,,
,, ,, from Film II.	138·0	,, ,,

On referring to page 479 of our paper of 1881 it will be seen that the agreement here obtained is as good as that on which our former conclusions are based.

The thickness of black films formed of salted *liquide glycérique* has also been measured again and again, with the result that the accuracy of our 1883 measurements has in every case been confirmed.

Preliminary Discussion of the Results.

Up to this point we have as far as possible confined our attention to experiment. We now propose to give our reasons for postponing any full discussion of the possible causes of the phenomena we have observed.

The most probable directions in which to look for an explanation are in the formation of a pellicle of different conductivity from the interior, or in a change in the conductivity due to the direct action of the surface forces as the film becomes thin, or in a combination of both.

If we assume that the film consists of two surface layers and an interior, which can be sharply distinguished from each other, and whose properties are different, it is easy to express the relation between the optical and electrical thicknesses by a formula.

Let T and x be the total thickness and the thickness of a surface layer of the film respectively.

Let K, k, and κ be the conductivities of the liquid in bulk, of the interior of the film, and of the surface layer respectively.

Then if T_ϵ be the thickness deduced from a measurement of the resistance we have, if

$$T > 2x,$$

$$r = \frac{T_\epsilon}{T} = \frac{(T - 2x)\,k + 2x\kappa}{TK}.$$

Hence

$$T_\epsilon = \frac{Tk}{K} + \frac{2x\,(\kappa - k)}{K}.$$

If, now, the results of the experiments be plotted with T_ϵ and T for ordinate and abscissa respectively, the formula shows that—

(1.) If the film be of constant conductivity equal to that of the liquid in bulk, the results will be represented by a straight line through the origin inclined at 45° to the axes. This follows from the assumption $K = k = \kappa$.

(2.) If the conductivity be constant throughout the film, but different from that of the liquid in bulk, the line will pass through the origin, but will not be inclined at 45° to the axes.

In this case $K \neq k$ and $k = \kappa$.

(3.) If the conductivity of the surface layers be different from that of the interior $\kappa \neq k$, and the line will not pass through the origin.

(4.) If the conductivity of the interior be variable, and if k can be expressed by a formula of the type

$$k = k_0 + \frac{A}{T^n} + \&c.,$$

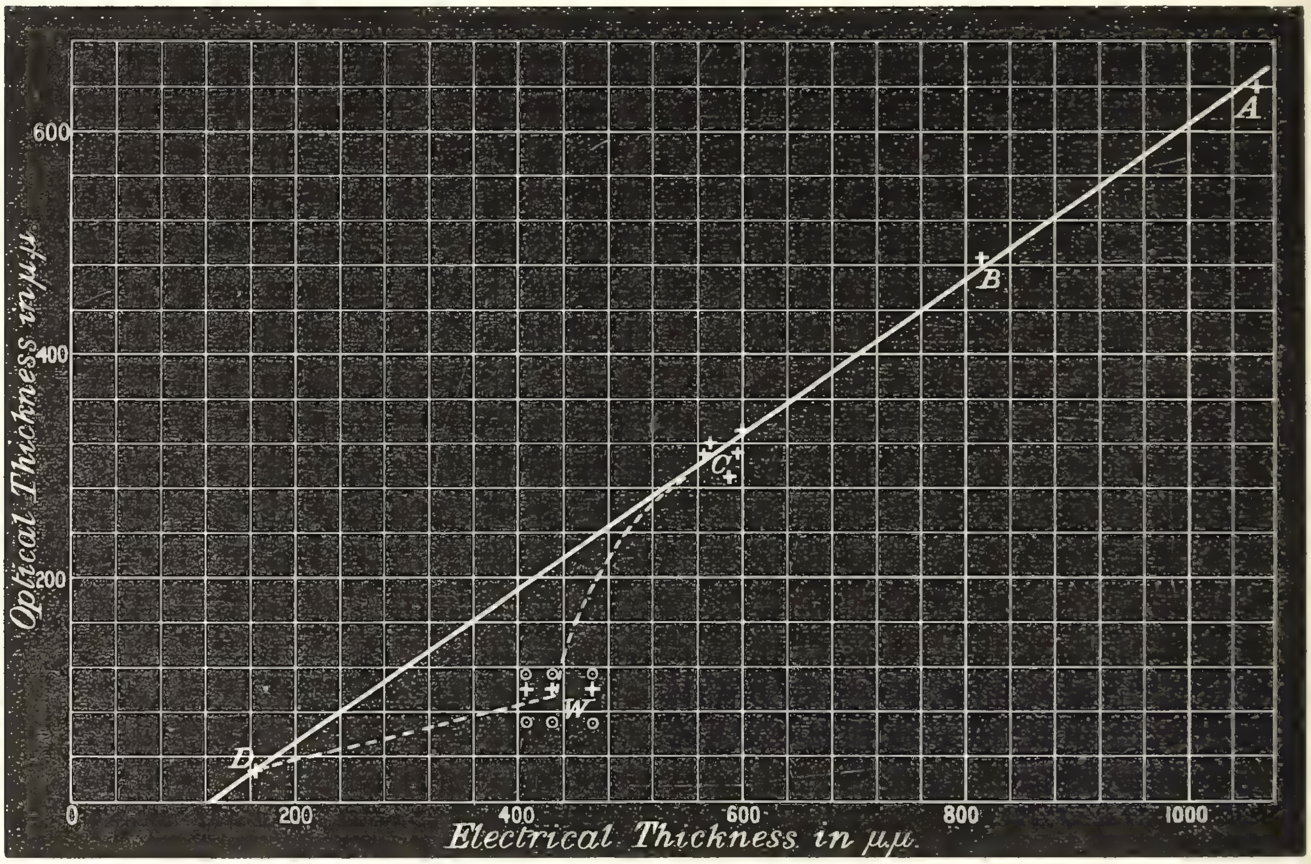
the expression becomes

$$T_\epsilon = \frac{Tk_0}{K} + 2x\frac{(\kappa - k_0)}{K} + \frac{T - 2x}{K}\frac{A}{T^n} + \&c.$$

All terms similar to that in which A occurs vanish when $T = 2x$. Hence if n be large, the term in T^n will be small when T is large, and will vanish when $T = 2x$. Thus a straight line will pass through the points corresponding to large values of T and through the point corresponding to $T = 2x$. For intermediate values the points given by the observations will not lie on the line in question.

The accompanying diagram shows how far our observations can be made to accord with these theories.

The point D is determined from the mean optical and electrical thicknesses of the black. It must, however, be remembered that we have every reason to believe that there are two thicknesses corresponding to this tint, and that unless the conductivity and the thickness are connected by a linear relation, the mean thickness of a large number of films will not necessarily correspond to their mean conductivity.



The three crosses at W correspond to the three electrical thicknesses of the white obtained by different observations. The optical thickness is assumed to be 100 $\mu.\mu.$ The circles indicate the extreme errors which could be due to errors in colour estimation. The other observations represented by A and B, and the group represented by C, do not require special remark. The principal deductions which can be made from these observations are as follows :—

(1.) The points do not lie on a straight line inclined at 45° to the axes. Hence the conductivity of the interior of the film is different from that of the liquid in mass.

(2.) The points A, B, D, and the centre of the group C, lie nearly on a straight line. If, therefore, we waive the objection above pointed out with regard to D, and note the fact that the group W certainly does not lie on this line, the facts are consistent with the theory that the film consists of an interior of different conductivity from the liquid in mass, bounded by pellicles of different conductivity from itself, and that when it becomes thin (about 300 $\mu.\mu.$), new forces (presumably surface forces) modify the conductivity in a way which can be approximately represented by supposing that the conductivity of the interior alone alters.

We do not, however, at present wish to draw from these observations any but two conclusions :—

(1) The theory of the formation of a pellicle of different conductivity from the rest of the film will not by itself explain all the facts.

(2) Something in the nature of a discontinuity appears to occur at a thickness of about 100 $\mu.\mu.$, which is but little greater than that which we have already fixed as the upper limit of the range of unstable thicknesses to which we attribute the discontinuity at the edge of the black part of a film.

On the whole, then, we think it most likely that the changes observed at the smaller thickness are in part at all events due to the direct action of the surface forces, but that their action is complicated by some cause which at greater thicknesses forms a pellicle and also affects the interior of the film.

There is indeed a great deal of evidence for the fact that if the surface of a solution of soap in water has been long exposed a pellicle is formed upon it. The surface tension of such a solution is about one-third of that of pure water, but Lord RAYLEIGH has shown that there is no measurable difference if the surface of the solution has only been formed for a very small portion of a second. It is obvious, therefore, that the surface of a solution of soap in water is modified when exposed to the air.

We, on the other hand, have proved ('Phil. Trans.,' 177, Part 2, 1886, p. 659) that every disturbance of the surface of a soap film increases its surface tension, and that the normal value proper to what may be called an *aged* surface is not regained for several minutes.

In this respect the variations of surface tension and of electrical conductivity behave similarly. Both are affected when the film is disturbed by the passage of an electrical current.

On referring to Table XIV. we observe that two of the values of the ratio of the electrical to the true thickness (r), viz., those which correspond to the thicknesses 388 and 314 $\mu.\mu.$ respectively, are inconsistent with the others.

If the table is rearranged in the order in which the experiments were performed, these facts gain significance. The exact sequence of events was as follows :—

Current from 20 cells sent down the film before the measurement was made :

(1) T (true thickness) $= 641\ \mu.\mu.$, $r = 1{\cdot}66$.

Current from 20 cells sent up the film :

(2) $T = 484\ \mu.\mu.$ . . . . . . $r = 1{\cdot}69$,
332 ,, ... 1·81,
314 ,, ... 1·90,
White . . . . . . 4·19.

Current sent up and down the film in turn :

(3) $T = 320\ \mu.\mu.$ . . . . . . $r = 1{\cdot}78$,
307 ,, ... 1·84,
296 ,, ... 1·98.

Film began to thicken spontaneously :

(4) $T = 314\ \mu.\mu.$ . . . . . . $r = 2{\cdot}07$.

An upward current was next applied and the film allowed to thin with the following results :

(5) $T = 388\ \mu.\mu.$ . . . . . . $r = 1{\cdot}60$.
White . . . . $r = 4{\cdot}47$.

It will be noticed that every time the film was disturbed the value of r fell, *i.e.*, the specific conductivity of the film was reduced more nearly to that of the liquid in mass, nor was this merely due to the fact that the first effect of the currents was in every case to thicken the film, the value of r corresponding to a given thickness was always less in a more recently disturbed film. This is proved by comparing the values of r corresponding to 320 $\mu.\mu.$ in groups (3) and (2), and those corresponding to 388 $\mu.\mu.$ in groups (5) and (2).

We take it that this is sufficient proof that though the phenomenon we are studying depends on the thickness, it is not merely an explicit function of the thickness of the film, *i.e.*, is not due exclusively to a peculiar molecular constitution depending on the thickness. If this were so, the molecular conditions peculiar to a particular thickness would be immediately re-established as soon as the film regained that thickness after disturbance.

We think that the next step should be to try to separate the effects due to the pellicle and to the capillary forces if both co-exist. The pellicle is possibly due to chemical actions between the liquid and the atmosphere. An apparatus is in course of construction which will enable us to experiment in other gases more satisfactorily than hitherto. For the moment we venture to lay our results before the Society without a full explanation.

We believe that the main fact, viz., the change in the conductivity of an unsalted soap film is entirely new. We hope that the observations which we have made on it will be of interest.

It is difficult to assign a reason why the addition of salt to the liquid should produce so great a change in the results. In part, the better conducting salt probably masks effects which when soap alone is used become predominant, but we think it likely that in part at all events it actually prevents the changes to which the change of conductivity is due.

However this may be, we venture to think that there can be no doubt as to the facts that whereas the conductivity of a thin unsalted soap film is considerably greater than that of the liquid in bulk, the difference is so reduced in the case of a salted film with or without the addition of glycerine that the mean thickness of a number of black films whether measured electrically or optically is the same to within the limits of the error of experiment.

VIII. *On the Measurement of the Magnetic Properties of Iron.*

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[PLATES 5–16.]

THIS paper contains some of the results and a description of the methods employed in a series of experiments on the rate of change of an electric current, immediately after the application or reversal of a constant electromotive force, in a circuit containing the magnetizing coil of a large electromagnet or the primary coil of a transformer. The experiments have been carried out in the Electrical Laboratories of the Rose Polytechnic Institute, Terre Haute, Ind. The results seem to be of considerable interest, and the method of studying the magnetic properties of iron here proposed has, I believe, some important advantages in many practical cases where the magnetic circuit is closed and of large section.

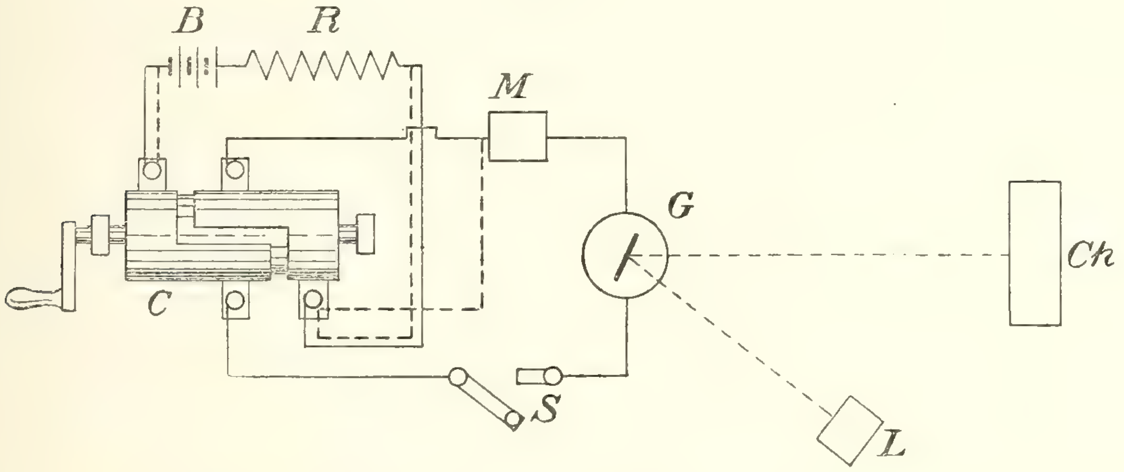
In the experimental determination of the magnetic properties of iron it has been usual to determine by means of a series of successive experiments the value of the total magnetization produced by different magnetizing forces. From these results the magnetic permeability of the iron, the self-induction of the circuit, and so forth, can, of course be calculated. Several methods are well known, by means of which reliable results can be obtained in this way, but they are, in many cases, inconvenient. For closed magnetic circuits, for example, the method commonly employed has been to measure, by means of the current induced in a coil of wire surrounding the iron, and in circuit with a ballistic galvanometer, the changes of magnetization produced by different changes of the current in a magnetizing coil. By this method, the value of the integral $\int_{c_1}^{c_2} \mathrm{L}\, dc$, or its equivalent $\int_{t_1}^{t_2} e\, dt$, can be measured. In the first form of the integral, c_1 and c_2 are the initial and final values of the current in, and L the coefficient of induction of, the magnetizing coil. In the second form, $t_2 - t_1$ is the interval of time required for the current to change from the value c_1 to the value c_2, and e is the back electromotive force induced by the rate of change of the current at any instant between the times t_1 and t_2. When the masses of iron experimented on are large, the interval of time $t_2 - t_1$ becomes too great for the value of the integral to be accurately measured by means of any ordinary form of ballistic galvanometer joined in circuit with a second, or induction, coil surrounding the iron.

It occurred to me, somewhat more than a year ago, that the cause of the difficulty in the ordinary method of measuring magnetic quality in large masses of iron might be made an advantage if the method of experiment were changed, that, in fact, the desired information might be more easily obtained by recording, not the total value of the induction for each of several particular values of the magnetizing force, but the curve showing the rise of current with time, immediately after the circuit is closed. Preliminary experiments showed that, for many of the cases most difficult by the ordinary method, this could be done both accurately and easily. In these preliminary experiments the coils of a large electromagnet, having laminated cores and pole pieces, were joined in circuit with a storage battery and a non-inductive resistance. An automatic arrangement was devised, by means of which the circuit could be closed and the instant of closing recorded by a chronograph. The ends of the non-inductive resistance were at the same time connected to the quadrants of an electrometer. At any desired interval after the closing of the circuit the electrometer could be disconnected from the resistance, and the instant of disconnecting recorded by the chronograph. The deflection of the electrometer needle then showed the difference of potential between the ends of the non-inductive resistance at the instant when it was disconnected. By this means a sufficient number of points on the curve of rise of current could be observed to enable it to be drawn. The results obtained in these early experiments agree with those given in this paper, and the method may be found useful in cases where the rise of current is too rapid to allow of the successful operation of the method since used. It is evident that a modification of this method with a ballistic galvanometer substituted for the electrometer could be easily arranged to give satisfactory results. The ordinary ballistic galvanometer and induction coil method can also be used with a break circuit arrangement, allowing the total time integral to be taken by steps. This method of experiment was not used after the preliminary results for which it was devised had been obtained. It was abandoned, partly because of its considerable complication, but mainly because a considerable number of separate experiments are required to obtain the complete curve, thus involving a corresponding number of independent magnetizations of the magnet.

The arrangement of the apparatus used in the experiments, the results of which are here illustrated, is shown in the annexed diagram (fig. 1). In this diagram *M* is the electromagnet experimented on; *C*, a commutator for reversing the battery, or, when the connection to it was as shown by the dotted lines, for simultaneously cutting out the battery and short-circuiting the magnet coils; *B* is the battery, and *R* a resistance of about one ohm in its circuit to prevent excessive current, when, by the operation of the commutator, it is for an instant short-circuited; *G* is a mirror galvanometer, nearly "dead beat" and of short period; *L*, the galvanometer lamp; *Ch*, a chronograph, and *S*, a key for closing the circuit. The drum of the chronograph is

14 inches long and 20 inches in circumference, and it can be run at a rate of one turn per minute or of six turns per minute, the speed being in each case almost perfectly uniform. The galvanometer lamp stand was provided with cross-wires, the galvanometer with a light plane mirror, and the cross-wires were clearly focussed on the chronograph sheet by means of a lens placed between the lamp and the mirror. The chronograph sheets were, for convenience of experiment and subsequent reduction, formed of "cross-section" paper, the cross-lines on which were one tenth of an inch apart.

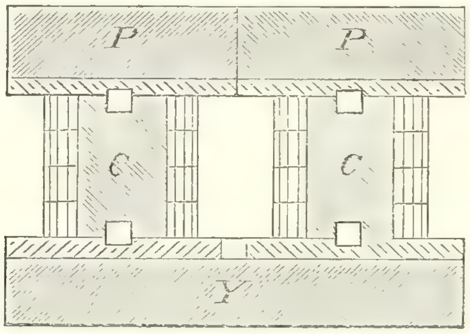
Fig. 1.



The method of experiment is simply to hold the finger on the switch *S* and observe the motion of the chronograph until the image of the cross-wires comes to a particular line on the paper sheet; at this instant close the switch, and then follow the image of the cross-wires with a soft sharp pencil, dotting the line occasionally, but taking care not to influence the motion of the chronograph. When the deflection has reached a maximum, the commutator is suddenly turned as the image of the wires passes a known position and the cycle of reversal marked out on the paper. The chronograph is now stopped, and the complete curve lightly marked. The experiment is then repeated, the image of the wires being simply observed as it passes over the curve, and any disagreement noted. Any disagreement being corrected and tested, the curve is ready for reduction. The time required to take a complete set of curves, check, and correct them, only requires a few minutes, after a little dexterity has been acquired in dotting out the curves. The curves obtained by this method have the advantage of being drawn to a large scale, the galvanometer sensibility being varied by means of shunts, so as to make the double deflection nearly cover the whole breadth of the chronograph sheet, while for many of the experiments made the length of the diagram extended several times round the drum. I am at present constructing an automatic recording apparatus on the principle of the Thomson recorder for the purpose of drawing diagrams for smaller masses of iron, where the interval of time is so short as to make the initial motion of the spot of light too rapid to be followed with certainty. Should this apparatus prove successful, I hope to apply it to investigate the effect of cyclic speed on the dissipation of energy, using a varying impressed electromotive force.

The results here quoted were obtained from experiments on a large electromagnet belonging to the Electrical Laboratory of the Rose Polytechnic Institute. I have, however, found it easy to trace curves for an ordinary 6000-watt transformer, although the time is in this case considerably shorter than that required for the magnet. I hope to supplement the results here given by results for various transformers, as soon as they can be reduced. The electromagnet was constructed in the shops of the Rose Polytechnic Institute, and was designed by my colleague, Dr. C. L. Mees, and myself for convenient use in the instruction of students in magnetic measurements. It consists of two cores *C C* (fig. 2), each surrounded by four coils of insulated wire, 2·5 millims. diameter. Each coil is divided into four concentric coils of five layers each, and there are 24 turns in each layer. The total number of turns in the magnetizing coils, when all are joined in series, is thus 3840, while the arrangement allows a considerable variation to be made, if desired, in the distribution, position, power, or resistance of the coil. For example, the inside set of concentric coils may be compared with the outside set to illustrate the law that the magnetizing effect is proportional to the number of turns, and for closed magnetic circuits practically independent of the diameter of the turns. Again, one set of alternate coils may be used for a primary, while the others, joined in any way, may be used for the secondary of a transformer. The magnet, when joined in this way, is capable of transforming from 30,000 to 40,000 watts. For transformer purposes this arrangement has the advantage for purposes of instruction and experiment that the action of an open and a closed circuit transformer can be directly compared. Various other modifications of the arrangement of the magnet coils and cores will readily be seen, with which we are not concerned in this paper.

Fig. 2.



The cores *C C* rest on a yoke *Y*, and on the top of the cores the pole pieces *P P* rest. The length of the iron circuit can be varied by moving the cores further apart and pushing forward the pole pieces. To facilitate this operation, a screwed rod is run from end to end of the yoke, sunk just beneath the surface; one half has a right, and the other half a left-handed screw, which work in half nuts fixed in the cores. Thus, by turning the screw, the cores can be moved apart or brought closer together. A similar arrangement serves to move the pole pieces and to pull them apart after they have been strongly magnetized on closed circuit. When an air space is left between the pole faces, the magnetic attraction is resisted by distance pieces, placed between the poles. The necessity for the somewhat elaborate arrangement just described arises from the great weight of the pieces, the cross section of the iron circuit being about 50 square inches, and the length of the yoke 40 inches. The cores and pole pieces were constructed of soft homogeneous iron, laminated in sheets one-twentieth of an inch thick. Each part is held together by plates three-eighths of an inch thick, bolted together through the sheets. One plate of each piece forms a common nut for all the bolts, while the heads of the bolts are countersunk into the other plate. The insulation between the sheets and between the bolts and the holes is simply the oxidized surface of the rolled metal coated with shellac varnish.

Let the impressed electromotive force be E, the resistance of the circuit R and the current at any instant C. Then $E - RC = e$, the back electromotive force on the magnetizing coil due to induction. This is also equal to $L\,(dC/dt)$, and hence we have $e = E - RC = L\,(dC/dt)$ or $e\,dt = L\,dC$. Consider the curve *Obd* in the diagram,

Fig. 3.

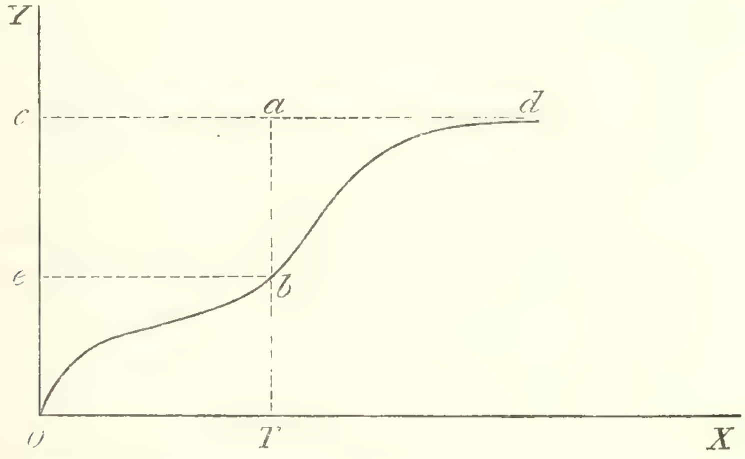


fig. 3, to be a curve showing the relation between current and time immediately after the magnetizing circuit is closed. Let the distance along *OX* to any point represent the interval of time $t_2 - t_1$ and the vertical ordinates the successive values of RC. Then the maximum vertical ordinate $Oc = E$, and the line *ab* corresponding to any time T is the value of *e* at that time. Hence the area *Obac* is the value of the integral $\int_0^T e\,dt$ or of $\int_0^{C_T} L\,dC$, the total change of induction from the time of closing the circuit to the time T; the current being supposed zero when T is zero. By means of this curve the total induction produced by any current between zero and the maximum that the impressed E.M.F. is capable of producing can thus be studied. Again, by drawing tangents to the curve at points corresponding to different values of the currents flowing through the circuit, the value of dC/dt and, consequently, of

$e/(d\mathrm{C}/dt) = \mathrm{L}$, the coefficient of induction at any instant, can be obtained for different values of the magnetomotive force applied, or by comparison with the curve of total induction, for different degrees of magnetization. Similarly, if the curve of variation of current from maximum in one direction to maximum in the other direction, due to a reversal of the impressed E.M.F. be traced, the various changes of total induction, and of L through a complete cycle can be studied. The curve of total induction for the cycle forms generally, as is well known, a closed loop. The area of this loop is the difference of the values of $\int e\mathrm{C}\,dt$ for the two halves of the cycle, and, consequently, is proportional to the energy dissipated in a cycle of magnetization. This area includes the dissipation due to magnetic retentiveness, and also that due to currents set up in the iron or in conducting currents surrounding the iron. It is interesting to study these curves, as is done for a few cases below, for different conditions of the magnetic circuit.

This method of experiment applied to electrical transformers has the advantage of giving directly the total dissipation of energy in the iron, and also of furnishing data from which the magnetic quality of the iron can be studied. The electromagnetic arrangement used for the experiments quoted in this paper was, as has already been stated, of such a form that it could be used either for a simple electromagnet or for an open or closed circuit transformer. Diagrams are given illustrating the action in these different cases. It should perhaps be stated that these curves are copies on a reduced scale of the original curves from which the calculations were made, and may not be sufficiently accurate to bear exact comparison. The values of the dissipation of energy, total induction, &c., calculated from the original curves, are therefore marked on the diagrams. The method usually adopted in reducing the curves was to integrate by means of the cross-section paper the successive values of $e\,dt$ for successive values of t, say, for one, two, three, &c., seconds, and plot a new curve with these areas as ordinates and the corresponding value of the current through the magnetizing coil as abscissæ. When the proper scale is adopted this gives the magnetization curve with magnetomotive force as abscissæ and density of magnetic induction as ordinates. From this curve, by a repetition of the integration of areas, the dissipation of energy was deduced. The values of L used in the curves for coefficients of induction were obtained by drawing tangents to the current curve at a sufficient number of points, and from these tangents reading the values of $d\mathrm{C}/dt$. The corresponding value of e, for example, ab in fig. 3, was then divided by $d\mathrm{C}/dt$, and the quotients used as ordinates in curves having current through the coil, that is, with proper change of scale, magnetomotive forces, as abscissæ. Curves of permeability can, of course, be also obtained by taking the ratio of the ordinates to the abscissæ of the induction or magnetization curves, that is $\left(\int_0^c \mathrm{L}\,d\mathrm{C}\right)\Big/4\pi n\mathrm{C}$ as ordinates, and the corresponding values of $4\pi n\mathrm{C}$ as abscissæ. These curves have not been drawn, as the curves on the diagrams seem to be already sufficiently numerous, and should the permeability be wanted at any time it can be readily obtained from those already given. The lines of impressed

E.M.F. have also been omitted from the diagrams as unnecessary. In the results, as shown by the derived curves, it should also be stated that no correction has been made for the induction through the coils which did not pass through the iron.

The curves in figs. 1–6 show the rise of current through the circuit of the electromagnet with different impressed E.M.F.'s. The full line curves were obtained after the magnet had been several times magnetized in the same direction as that produced by the current, while the dotted line curves show the rise of current when the magnet had been previously magnetized in the opposite direction by a current of the same maximum strength. The iron circuit was closed during these experiments, and had a mean length of about 265 centims. It will be observed from these figures that the time required to complete the magnetization is nearly inversely proportional to the E.M.F. applied to the poles of the magnet, and, also, that the shape of the curve changes considerably with change of impressed E.M.F. The time required for the current to reach different percentages of its maximum value is shown in figs. 7 and 8. The ordinates of these curves are proportional to the time required, and the abscissæ to the impressed E.M.F. It appears from these curves that for any particular percentage of the maximum current there is always a particular voltage which requires greatest time. The curves also indicate that above a certain voltage, higher than the percentage considered, the time required to complete the magnetization will become nearly constant. Figs. 9 and 10 show the values of the coefficient of induction for different values of the current flowing through the magnetizing coil, and the effect on this coefficient of the degree of magnetization ultimately reached. The most noticeable feature of these curves is the gradual change in the value and position of the maximum. This depends largely on the amount of residual magnetism previously existing in the magnet, and consequently, on the maximum value of the current which had previously flowed through the circuit. The exact value of the maximum coefficient is somewhat difficult to obtain by this method when the current produces a reversal of magnetization, because of the extreme smallness of dC/dt at the point of the curve where this maximum occurs. No particular importance is attached, therefore, to the apparent maximum of maxima shown in fig. 10. That there is a maximum of maxima, as shown in fig. 10, is, however, possible when we consider the cause of the change in position and magnitude of the maximum value of L from the point of view of the magnetization curve. It is well-known that this curve begins to rise somewhat slowly, and has a point of zero curvature corresponding to a comparatively small value of the magnetizing force. When the residual magnetism is great, and in the same direction as that which the current produces, the results show that the curve never attains to so great a steepness, and attains the greatest steepness later than if no residual magnetism existed. When the residual magnetism is of opposite kind from that produced by the current, the maximum steepness seems at first to increase with increase of residual magnetism, and afterwards to diminish; the position of the point of greatest steepness corresponding to greater and greater magnetizing forces as the residual magnetism is increased. That there is a particular value of the residual magnetism, which gives greatest steepness of the reversal of the magnetization curve, is possible, but it is not here considered established. The interpretation of the results, with reference to this point, is somewhat complicated by the fact that Foucault currents no doubt produce considerable shifting of the position of the point of maximum, as indicated by the apparent magnetizing current. These currents may also modify the value of the maximum coefficient. This point will be more readily understood in connection with the effect of a secondary coil referred to below.

In fig. 11, a set of curves are shown which illustrate the rise of current when the previous residual magnetism is in the same and in the opposite direction, together with a continuation of the curve showing the rate of diminution of current when the battery is suddenly cut out, the magnet circuit being left closed. The dotted-line curves show the rise and fall of the magnetic induction for the two cases, the curve of fall of current being, of course, the same for both. It will be observed, from the total induction or magnetization curves here shown, that the residual magnetism has amounted to about 60 per cent. of the whole magnetization produced. These, and similar curves shown further on, also illustrate the points referred to in connection with figs. 9 and 10, as to the variation and great difference in the value of the coefficient of induction, as depending on previous magnetization. Fig. 12 shows the current and derived curves for a complete cycle of magnetization. The full line shows the variation of the current after reversal of the impressed E.M.F., while the dotted line shows the corresponding variation of the coefficient of induction. The dot-dash line shows the induction cycle, and the area enclosed by this curve gives the dissipation of energy due to the combined effects of magnetic retentiveness and Foucault currents.

The curves in figs. 13–16 are similar to those in figs. 11 and 12. The impressed E.M.F. was 23 volts for all three sets, but the iron circuit in the magnet contained half a centimetre of air-space for figs. 13 and 14, and two centimetres of air-space for figs. 15 and 16. Comparing the latter two sets of curves with those obtained when the iron circuit was closed, we find as one of the most noticeable features the evidence they afford of the great effect a small air-space has on the magnetic retentiveness of the magnet. Curves (1) and (2) of figs. 13 and 15 are now nearly the same, and the sharp rise in the coefficient of induction has disappeared. The maximum value of the coefficient of induction is now nearly the same, whether the current repeats or reverses previous magnetization, but, somewhat curiously, seems to be somewhat less for reversal than for repetition. The fall of current curves show slower fall at first, due, apparently, to the less firm hold of the residual magnetism and consequent larger initial value of the coefficient of induction just after reversal. This effect would probably be still more marked, were the total induction not considerably diminished by the air-space. Figs. 14 and 16 show the complete cycle, and thus correspond to fig. 12. The sharp inflection in the current curve does not appear in these curves, and there is evidence that, for a given impressed E.M.F., there will be a particular air-space, which will give the maximum time for the current to pass through zero, and reverse after the reversal of the impressed E.M.F. The curves of coefficient of induction are now comparatively flat, and the curve of total induction is nowhere nearly so steep as it is for the closed magnetic circuit. There is an important difference also in the amount of energy dissipated in the cycle. Of course, a direct comparison of the amount of energy dissipated in the cycle cannot be made from these curves alone, but a separate set of experiments, illustrated in figs. 22–27, bring out the result that considerably less energy is dissipated on the open circuit for the same amount of total induction than is the case with the closed circuit. From the curve (fig. 26) it appears that the energy dissipated per cycle on closed magnetic circuit for total density of induction of 13,200 C.G.S. units is 92×10^7 ergs, while the actual experiment on open circuit only gives 74×10^7 ergs, the air-space being only half a centimetre. Again, from the same curve, it appears that, for closed magnetic circuit and a total density of induction of 8610 C.G.S. units, the energy dissipated per cycle is 4×10^8 ergs, while for the actual experiment, with two centimetres of air-space in the circuit, it is only 3×10^8 ergs per cycle. This is equivalent to a saving of about two and a half foot-pounds of energy per cycle. That the dissipation of energy due to magnetic retentiveness becomes comparatively small when the distance between the poles is as much as half a centimetre, was proved by static experiments made for the purpose. These experiments showed that, for a total induction of 14,000 C.G.S. units per square centimetre across the air-space, the dissipation, as measured by the variation of the field in the air-space, was only 590×10^6 ergs. The intensity of magnetization here quoted was obtained with about 130 volts on the terminals of the magnet, and approaches near to saturation of the cores. When the circuit is open, the dissipation of energy is largely due to Foucault currents in the iron, and, as will be seen from the values calculated and marked on the curves, it approaches towards being proportional to the square of the induction through the iron. This is a result which we should expect so long as the law of the variation of the induction is not very different in the cases compared. The practical bearing of the result here arrived at on the question of open *versus* closed circuit transformers is evident.

The group of figs. 17–21 illustrate results obtained with the magnet used as a transformer with equal numbers of turns on the primary and secondary coils. The coils were joined alternately so as to form two sets of four coils each, each pair of primaries containing between them one secondary and *vice versâ*. Fig. 17 is similar to fig. 12, except that the current is greater and the number of turns in the magnetizing coil less. The different curves in this set will be at once recognized. In fig. 18, the effect of closing the secondary coil through a total resistance, consisting of the coil and incandescent lamps, is shown. The curves of primary and secondary current are clearly marked in the figure, and can be better studied from it than from a description.

It may be remarked that the curve of secondary current is slightly affected by the heating of the lamps, which became a dull red towards the end of each reversal of current in the primary. The effect is slight, however, and the equality of the back E.M.F. on the primary, and the forward E.M.F. on the secondary, is as perfect as the experiment can show. The heavy dot-dash line encloses an area which represents the total dissipation of energy in the iron and in the secondary coil and circuit. We have here the means of illustrating the effect of Foucault currents on the *L* curve. It will be noticed, by comparing figs. 17 and 18, that the high values of the *L* curve are displaced to right and left by the existence of the secondary currents, which, by their demagnetizing action, allowed the primary current to become stronger before taking the inflection due to high permeability. The magnetizing force being now the difference of the magnetizing forces of the two coils, the curve of coefficient of induction should be plotted to this difference as abscissæ. This is done in figs. 20 and 21, which show that the maximum value of the coefficient comes earlier the stronger the current in the secondary. Fig. 19 is the same as fig. 18, except that the resistance of the secondary is now simply that of its own coil and the galvanometer circuit, namely, 5·62 ohms. There is now, of course, a greater dissipation area and a greater displacement of the *L* curve. There are one or two points of interest in connection with the *L* curves and the curves shown by fine dotted lines which may be noted. The *L* curve, when calculated from the primary current, besides being displaced, has a higher maximum value the greater the current in the secondary. The dissipation of energy in the magnet itself is less the greater the dissipation in the secondary circuit. When there was no current in the secondary, the energy dissipated per cycle was 1950×10^6 ergs; when the resistance in the secondary was 12·7 ohms, this fell to 1440×10^6 ergs, and when the secondary was short-circuited, it fell to 946×10^6, or less than half the first value. When the differences of the magnetizing forces are considered, the *L* curve has its maximum nearer to zero current, the greater current in the secondary, and has nearly a constant value.

The effect of varying the impressed E.M.F. on the dissipation of energy is illustrated in figs. 22 to 25, the abscissæ of which are proportional to the impressed E.M.F., and the ordinates to the induction as calculated from the reversal of current curves which were similar to that shown in fig. 12. Although the number of experiments here shown are not sufficient to allow any law to be accurately deduced from them, the results are shown by the black dots in the curves figs. 26 and 27. Fig. 26 is drawn from an equation of the form $e = \mathrm{AE} + \mathrm{BI}$, where e is the energy dissipated, E the impressed E.M.F., I the total induction, A and B constants. For this particular curve $\mathrm{A} = \frac{1}{2}$ and $\mathrm{B} = 25300$. The curve shown in fig. 27 has impressed E.M.F. for abscissæ and dissipation for ordinates, and is drawn to show that for the range of experiment taken the dissipation was practically proportional to the impressed E.M.F. Independent experiments indicate that the amount of energy dissipated in the secondary circuit when the magnet is used as a transformer, in the way above described, is nearly proportional to the impressed E.M.F., and hence this was assumed to be the case for the Foucault currents also. The other constants were then obtained by solving for them by means of simultaneous equations of the form $e = \mathrm{AE} + \mathrm{BI}^x$, the necessary data to form the equations being taken from the experimental results. It was found in this way that x was practically unity, not differing by more than one per cent. for the cases taken, and hence it is taken unity in the curve. The results of static experiments would indicate that the term BI^x, which represents the energy dissipated due to the retentiveness of the iron, should vary more rapidly than the induction, that is, that x should be greater than unity, about 1·6 for the experiments of EWING and others. This result does not seem to be reached by the kinetic method here adopted, and an examination of recent work on the subject seems to confirm the results here arrived at. In an elaborate series of experiments recently published by Mr. C. P. STEINMETZ an attempt is made to show that the exponent x is always 1·6. Unfortunately his results have most of them been reduced on this assumption, together with another based partly on calculation for Foucault currents. When the value of x given by the results is deduced from a series of equations by the methods indicated above the exponent unity seems to agree just as well, sometimes better, than 1·6. An excellent series of experiments for this purpose have recently been made by Mr. ALEXANDER SIEMENS and the results communicated to the Institute of Electrical Engineers.* In this paper also the losses from Foucault currents are calculated on an assumption of size of wire, specific resistance, &c. The estimate of the energy dissipated seems to be somewhat too small, and with a proper correction on it, about 25 per cent., the dissipation given under hysteresis becomes simply proportional to the induction. The results of these experiments treated by the simultaneous equation method for the determination of the law of variation of the different elements of the dissipation give for Foucault currents AI^2 and for hysteresis BI^1 with almost perfect exactness. The fact that a sufficient number of fairly accurate experiments furnishes data for the mathematical determination of the law of variation of the different elements entering into the dissipation of energy in cases like that here considered seems to have been very generally overlooked.

[Note added June 20, 1893.—Subsequent experiments, the results of which I hope soon to place before the Royal Society, show that the constant A in the above equation should have been zero, as there is practically no Foucault current loss. The exponent x is also shown by these experiments to agree closely with the number obtained by Mr. STEINMETZ from Professor EWING'S experiments and, since this paper was written, more fully established by his own experiments.

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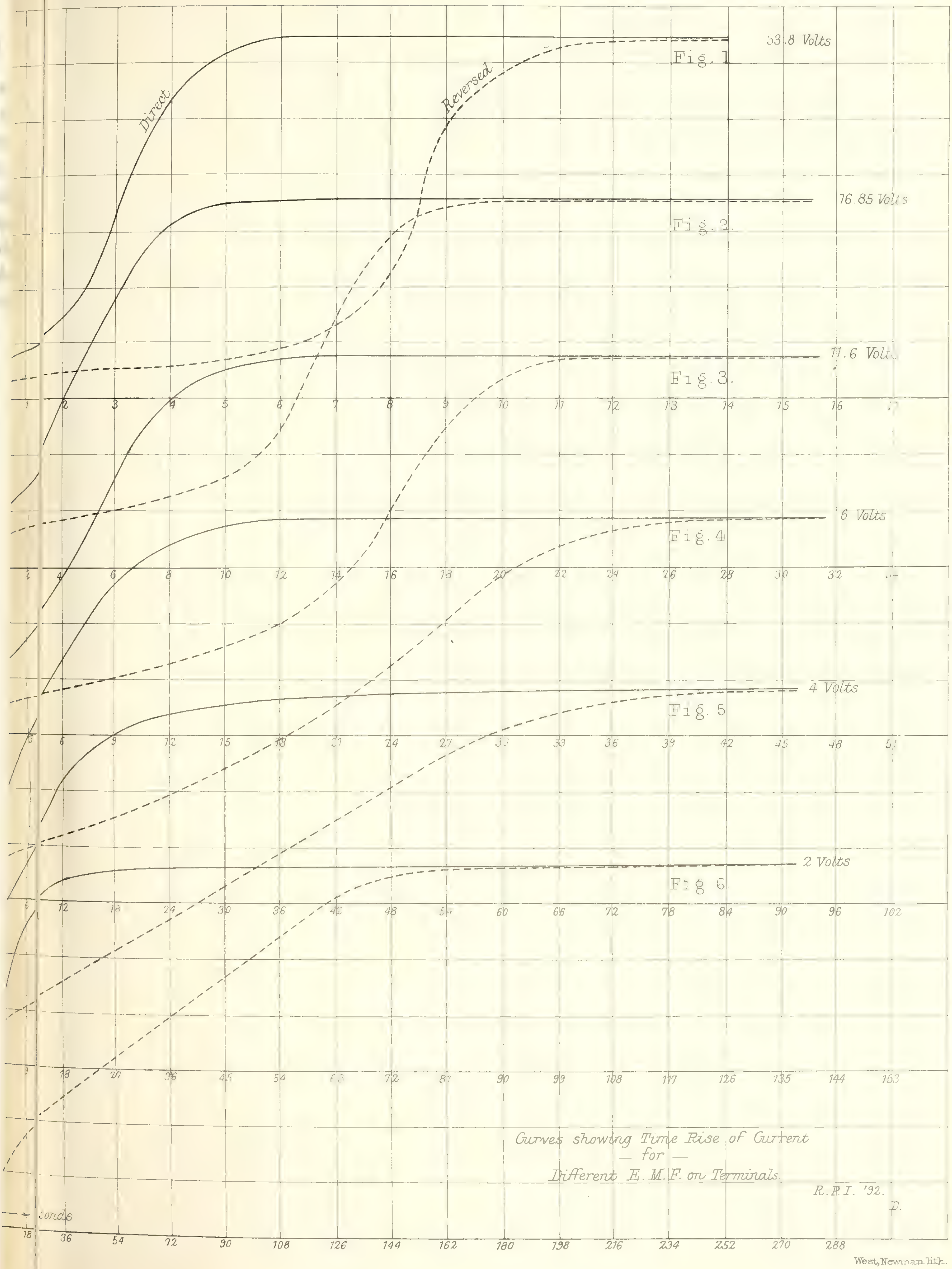
* "Some Experimental Investigations on Alternate Currents." See 'Electrical Engineer,' February 19th and 26th, 1892.

The simultaneous equation method referred to above is more generally stated as follows:—Let the dissipation of energy e be expressed by an equation of the form $e = AI^{\alpha} + BI^{\beta} + \&c.$ Then, if, by proper experiments, the curve expressing e in terms of I be determined, the values of the constants A, B, &c., and α, β, &c., can be determined from the curve by taking a sufficient number of corresponding particular values of e and I from the curve to form the requisite number of equations. Experience shows, however, that it is difficult to obtain the experimental curve with sufficient accuracy. In the particular case given above, the results were neither sufficiently numerous nor sufficiently accurate for the purpose, and the conclusion arrived at is undoubtedly wrong.—T. G.]



Curves showing Time Rise of Current
— *for* —
Different E. M. F. on Terminals.

R.P.I. '92.

D.

West, Newman lith.

MUSEUM

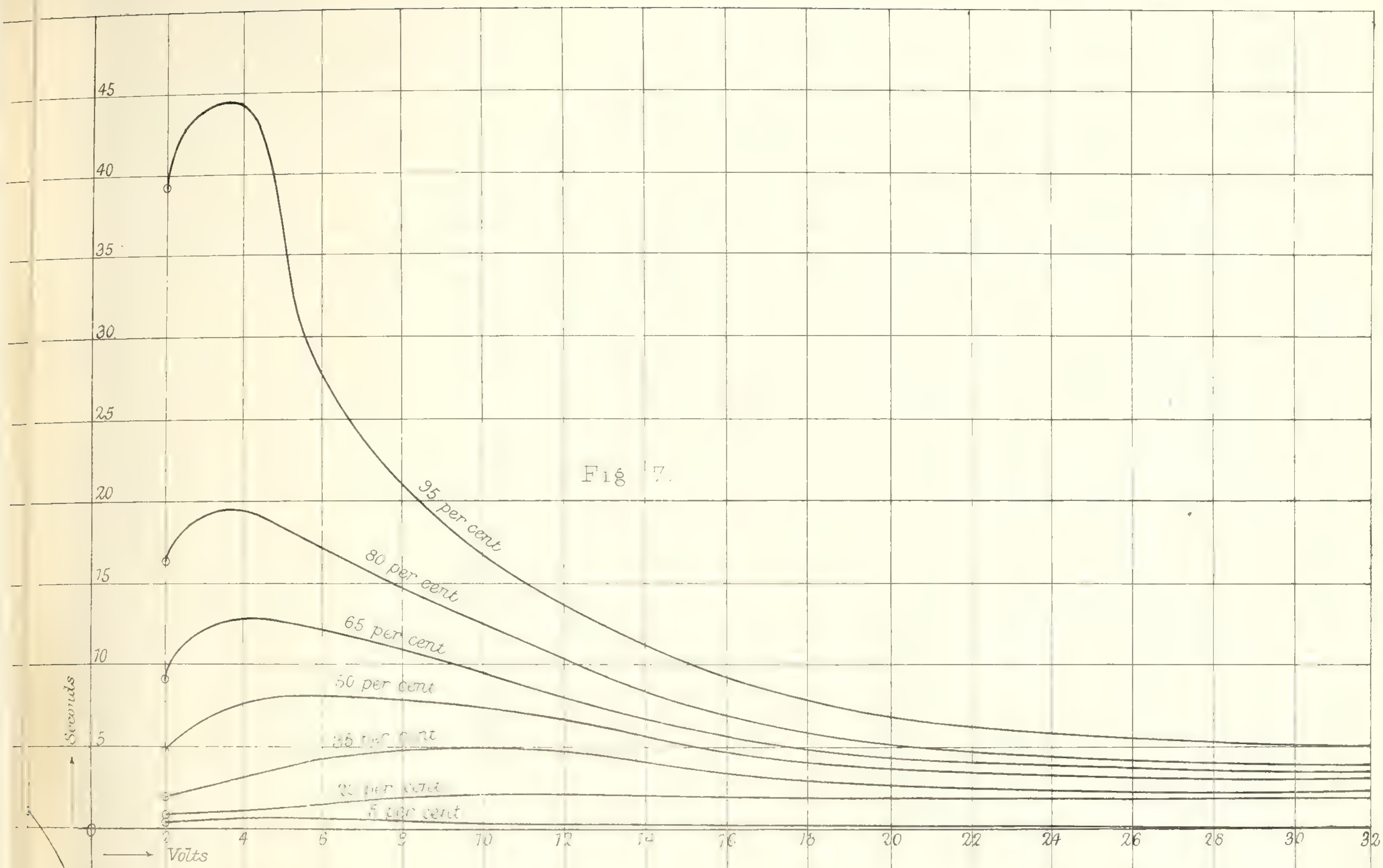


Fig 7.

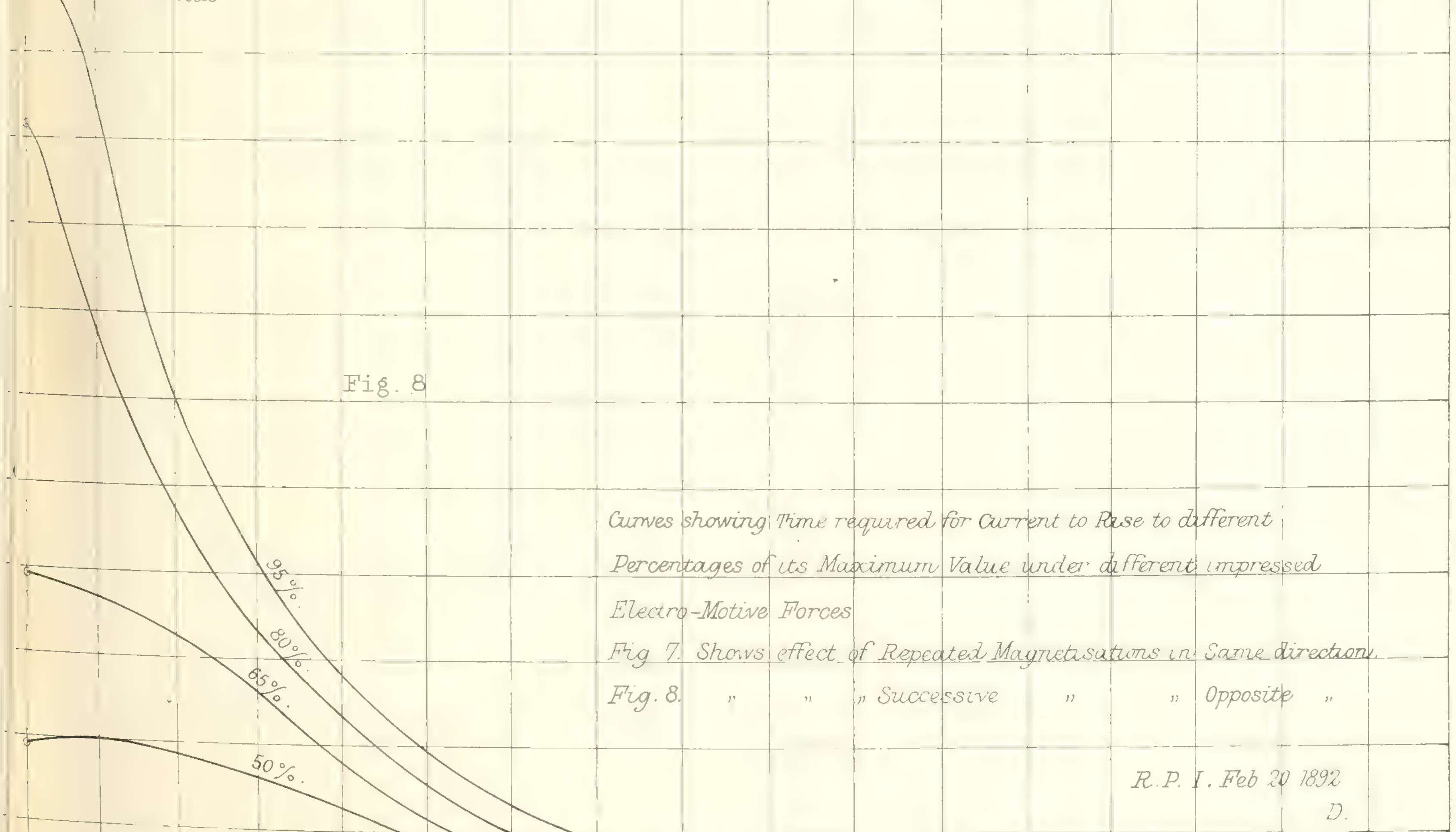


Fig. 8

Curves showing Time required for Current to Rise to different Percentages of its Maximum Value under different impressed Electro-Motive Forces

Fig 7. Shows effect of Repeated Magnetisations in Same direction.

Fig. 8. " " " Successive " " Opposite "

R. P. I. Feb 20 1892

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NATURAL HISTORY

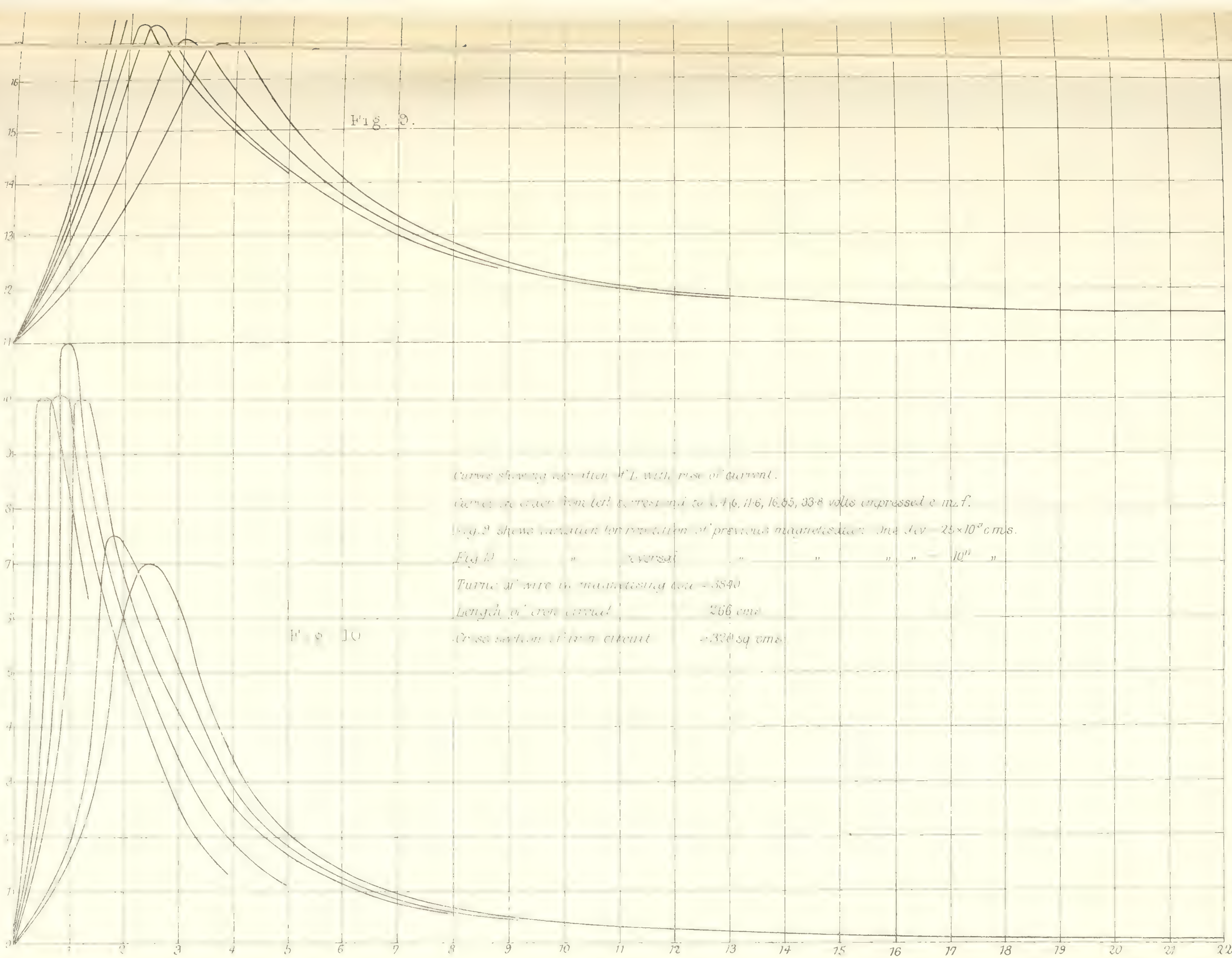
Fig. 9.

Fig. 10.

West, Newman lith.

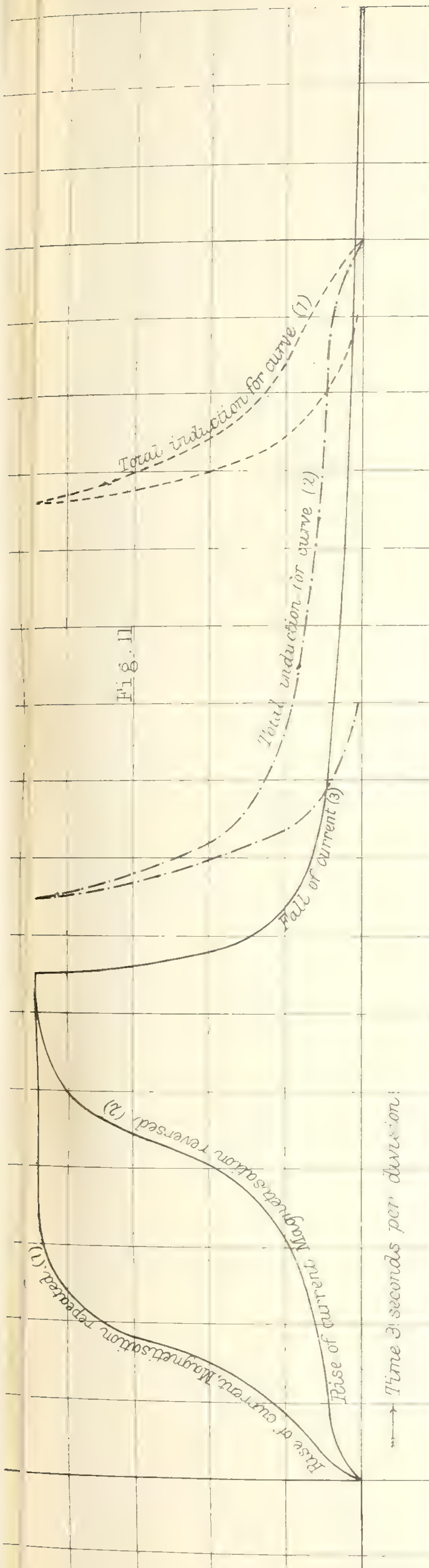
Fig. 11

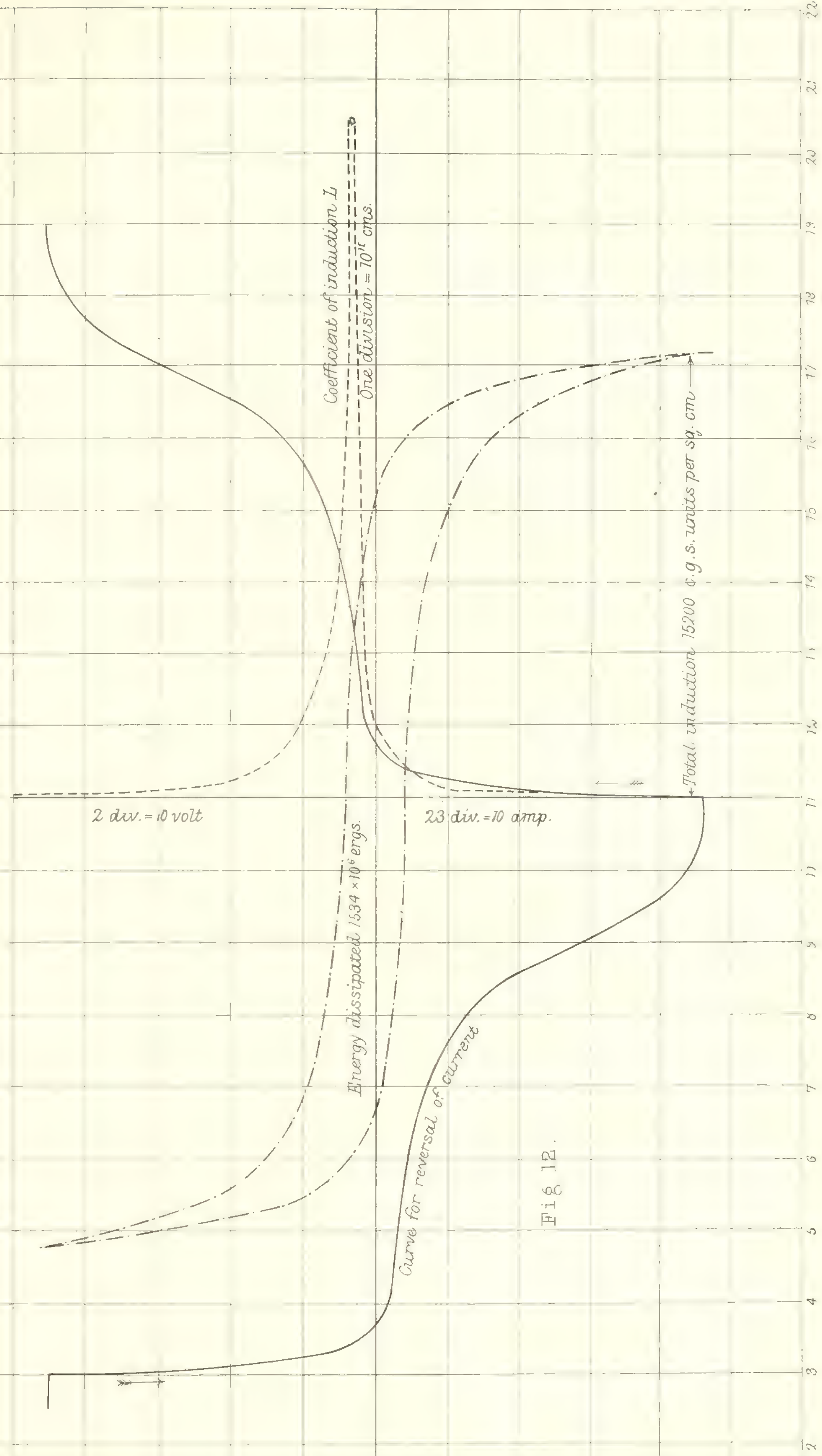
Fig 12.

West, Newman lith.

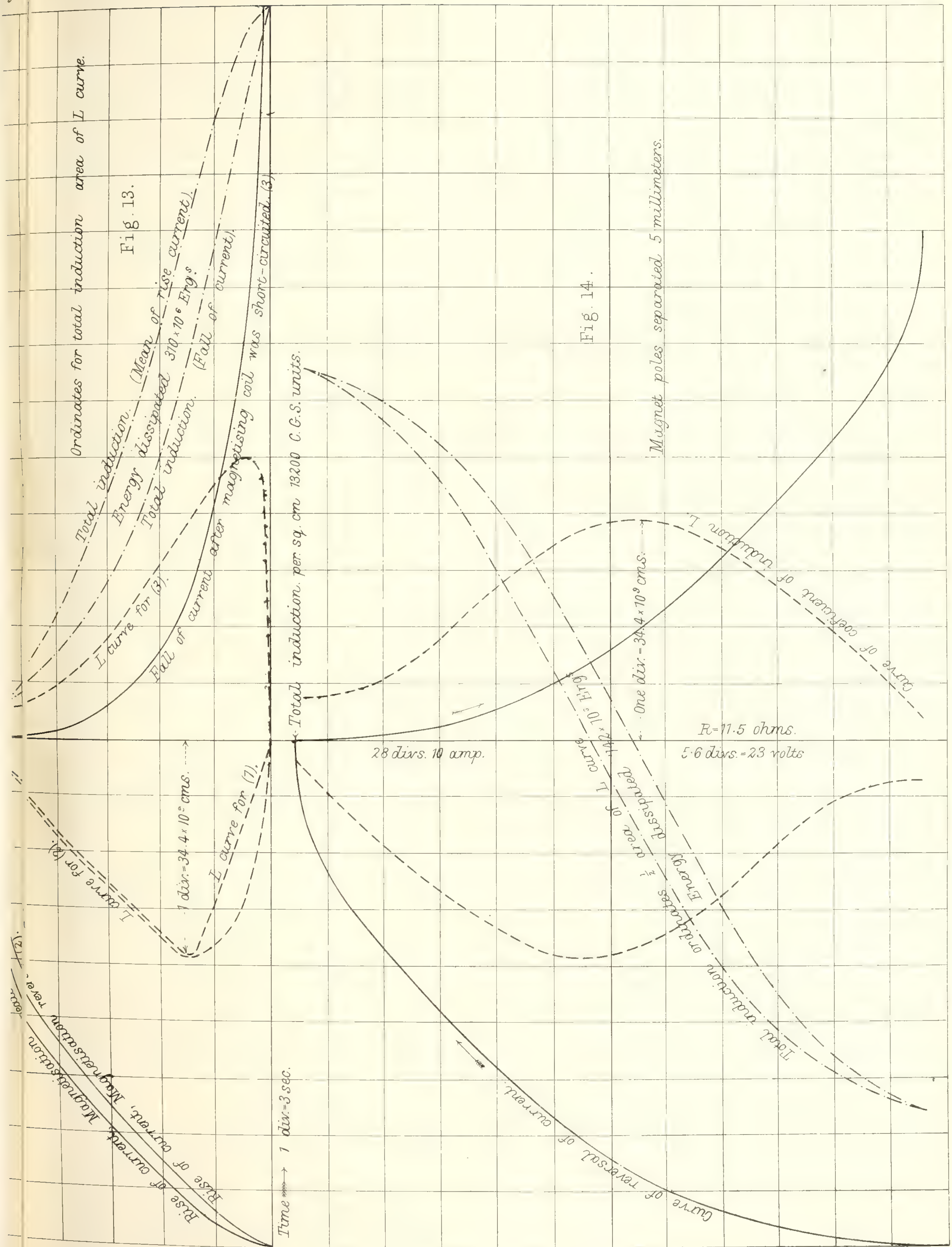

Fig. 13.

Fig. 14.

West, Newman, lith

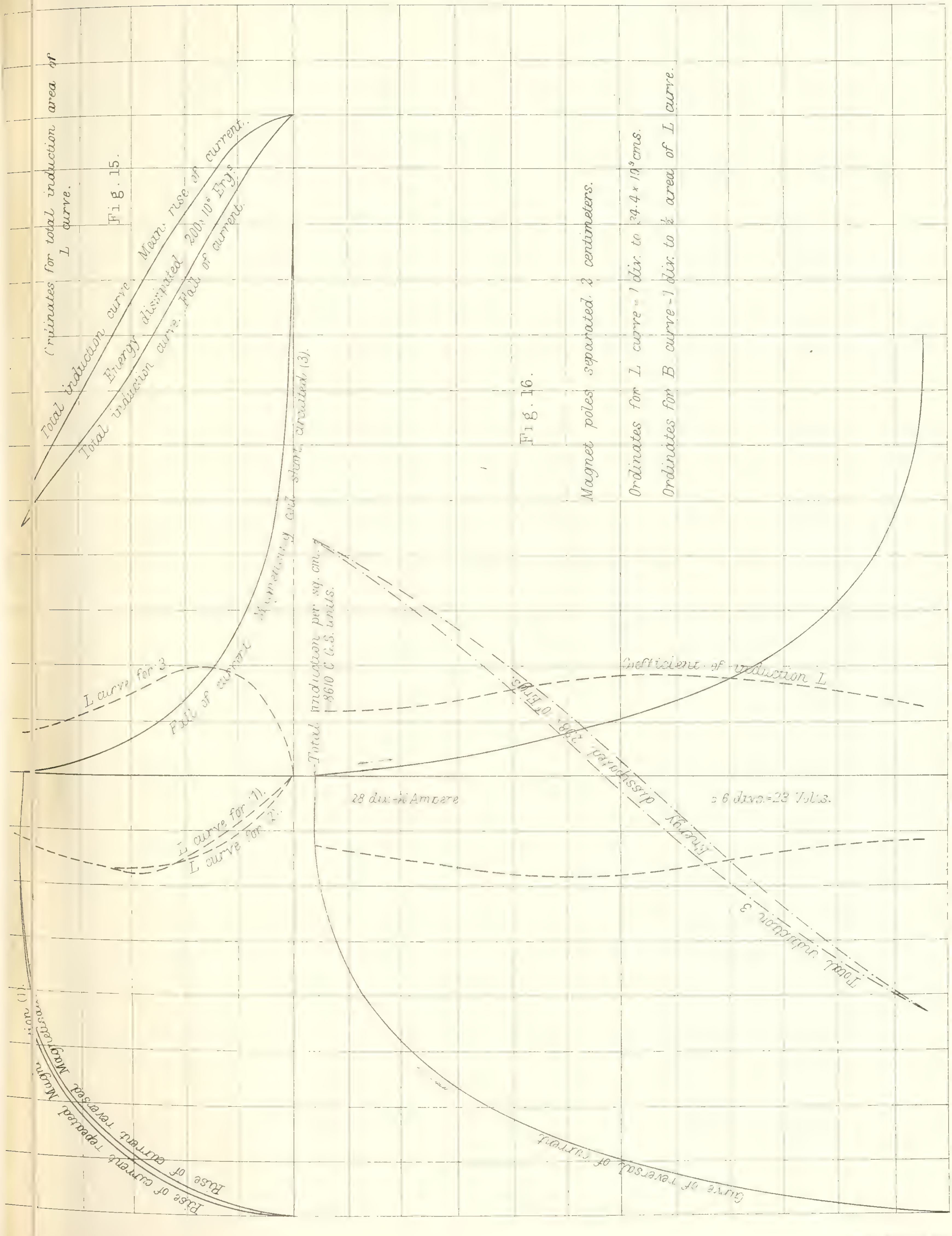

Fig. 15.

Fig. 16.

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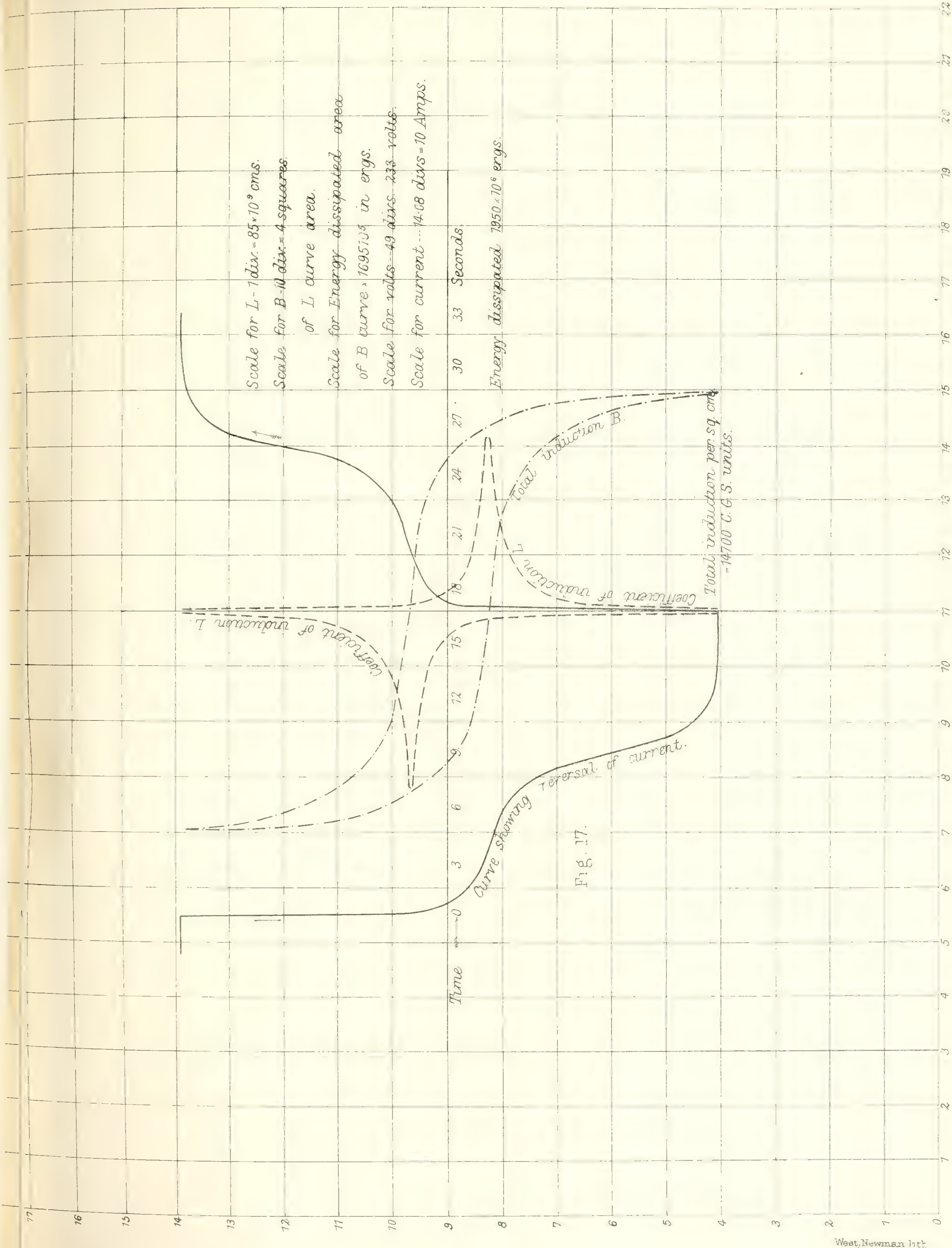

Fig. 17.

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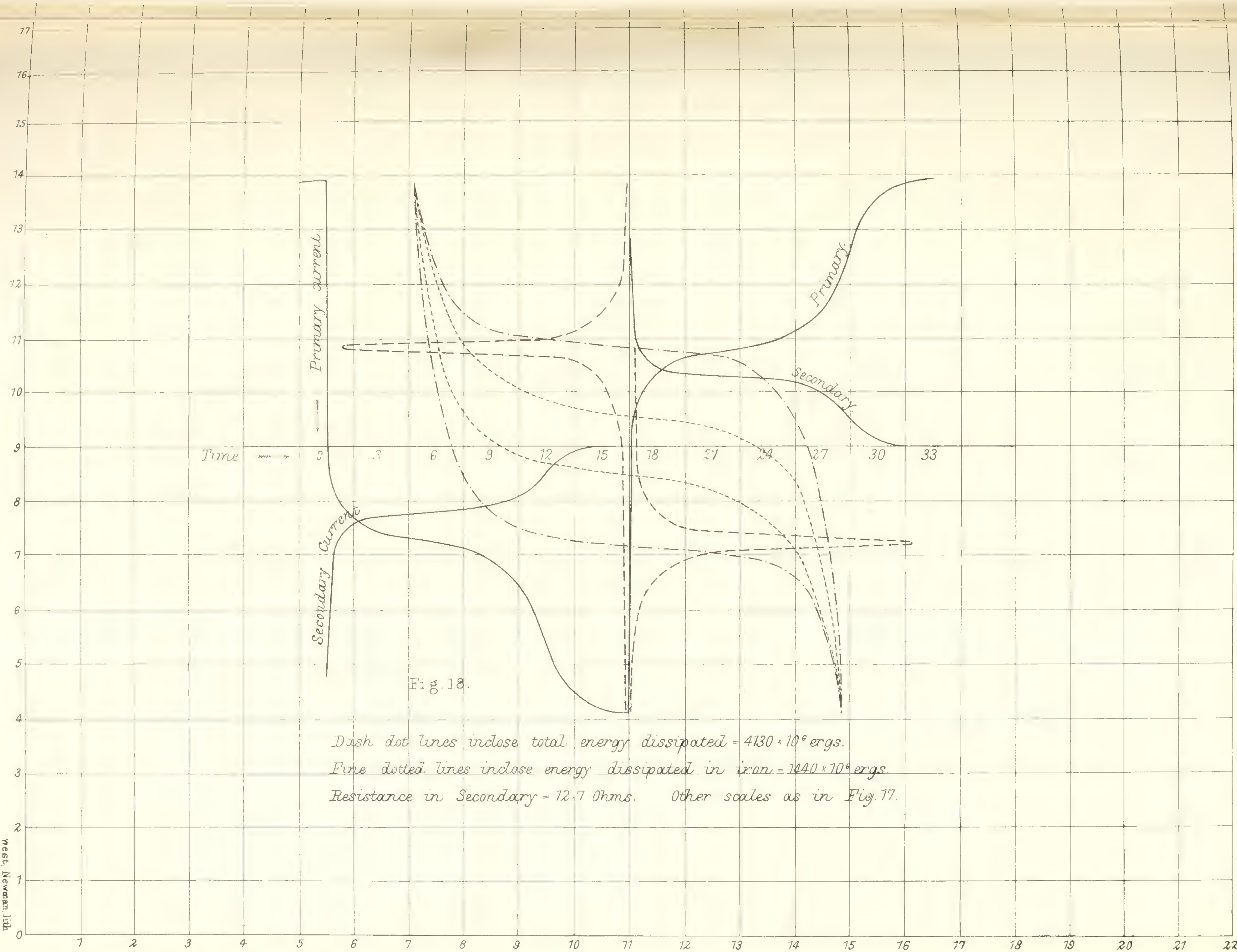


Fig. 18.

Dash dot lines inclose total energy dissipated = 4130 × 10^6 ergs.
Fine dotted lines inclose energy dissipated in iron = 1440 × 10^6 ergs.
Resistance in Secondary = 12·7 Ohms. Other scales as in Fig. 17.

West, Newman, lith.

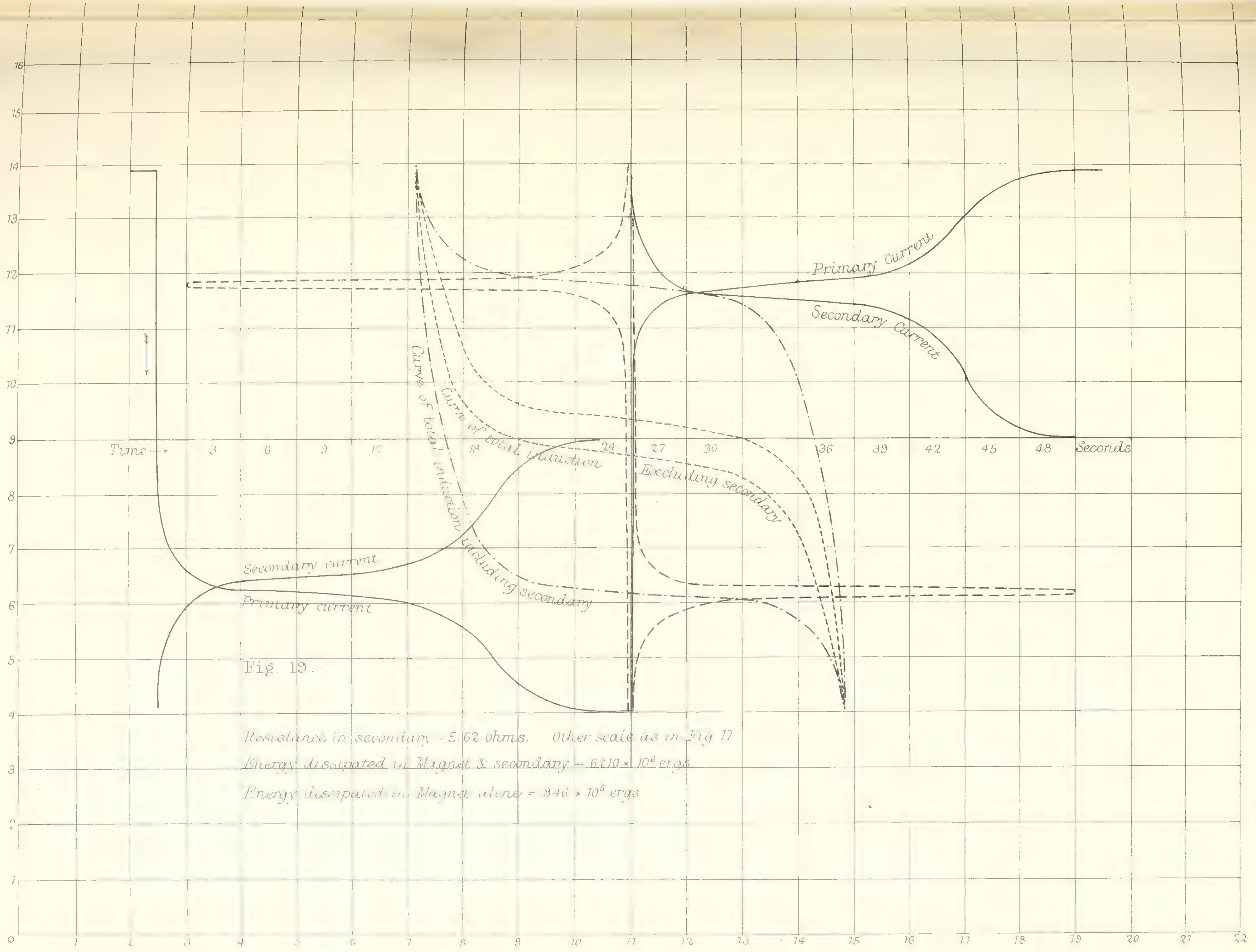


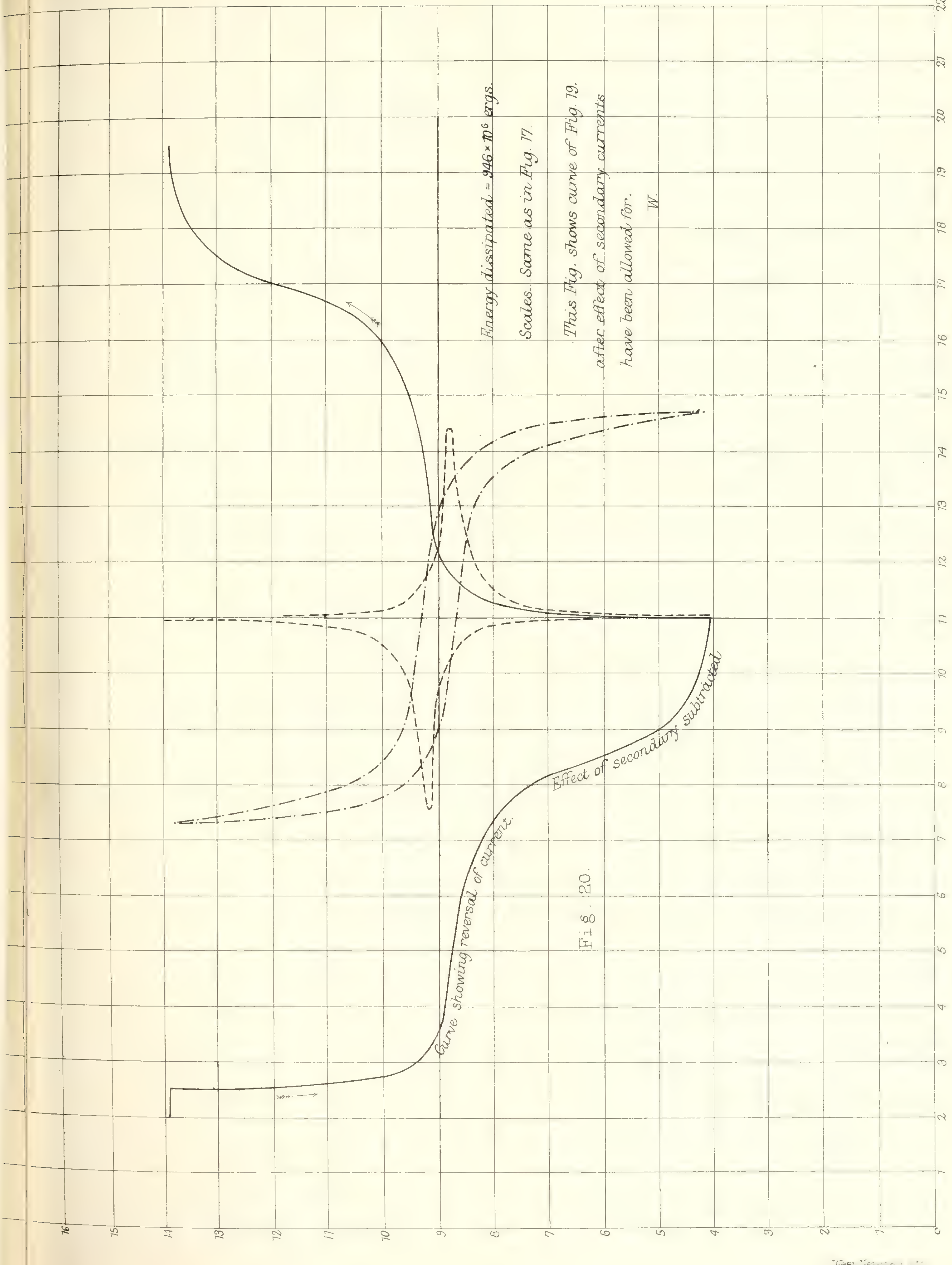
Fig. 19.

Resistance in secondary = 5.62 ohms. Other scale as in Fig 17

Energy dissipated in Magnet & secondary = 6210 × 10^6 ergs.

Energy dissipated in Magnet alone = 946 × 10^6 ergs

West, Newman lith.



Energy dissipated = 946×10^6 *ergs.*

Scales.–Same as in Fig. 17.

This Fig. shows curve of Fig. 19.
after effect of secondary currents
have been allowed for.

W.

Fig. 20.

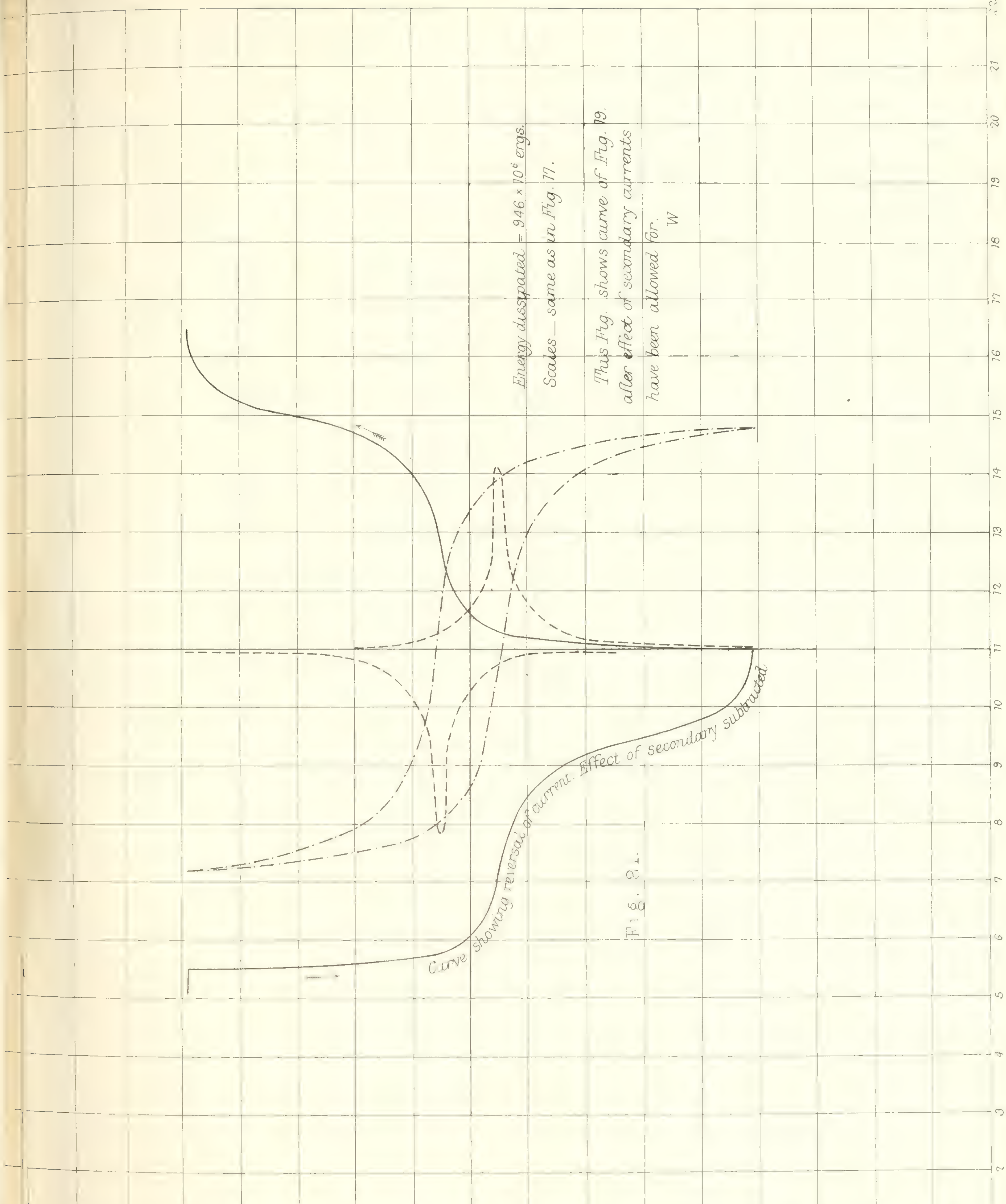
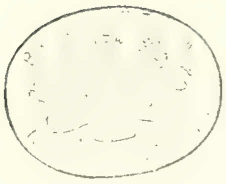
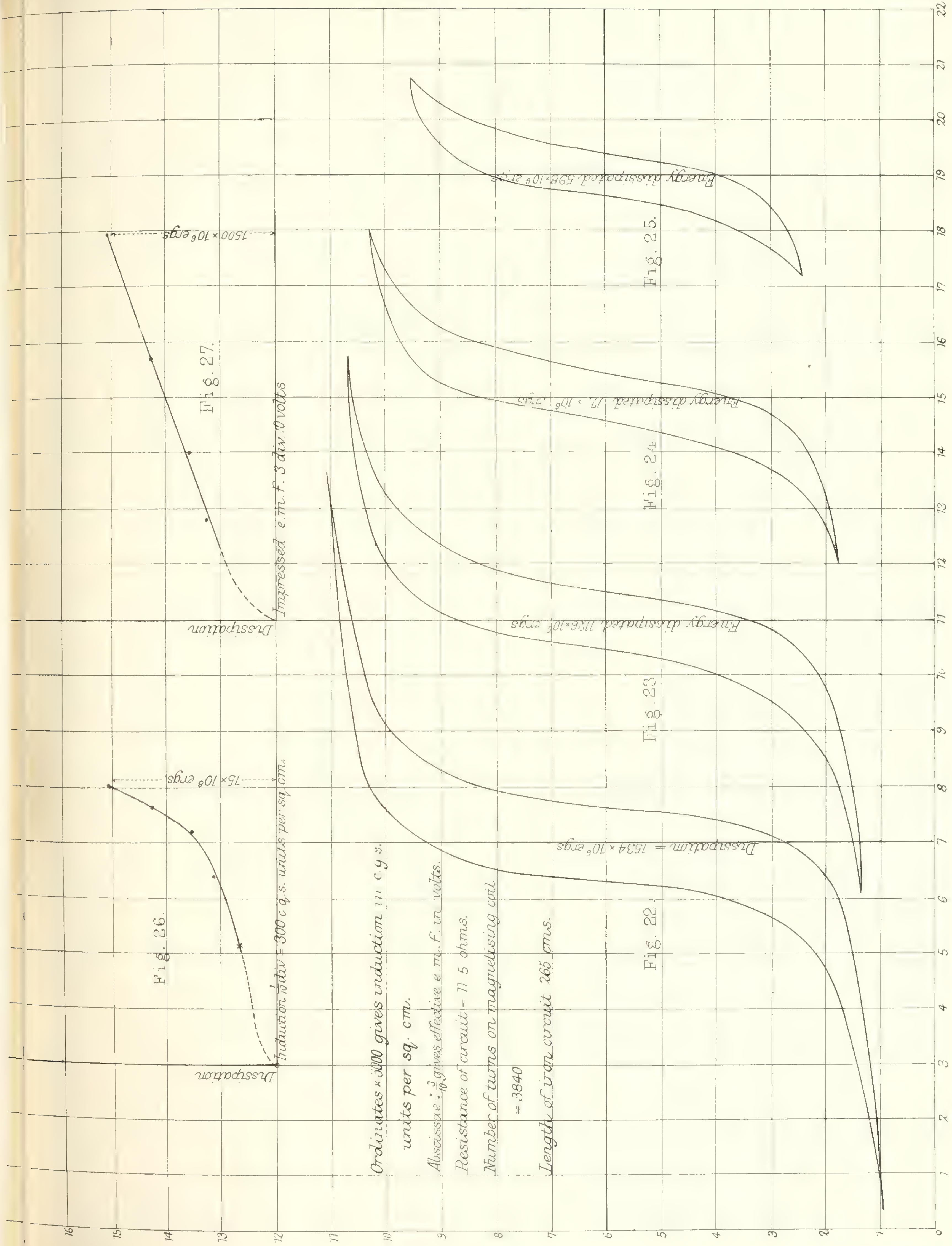


Fig. 21.

West, Newman lith.





West, Newman lith.

IX. *On the Composition of Water by Volume.*

By Alexander Scott, *M.A., D.Sc., Jacksonian Demonstrator in the University of Cambridge.*

Communicated by Lord Rayleigh, *Sec. R.S.*

Received March 4,—Read March 23, 1893.

In a preliminary note, "On the Composition of Water by Volume," presented to the Society in June, 1887, I remarked that up to that time we had no direct measurements of this important constant, except those of Gay-Lussac and Humboldt. The value they deduced from their experiments corresponded so far with the number I then gave as the most probable one, in that it required rather less than two volumes of hydrogen to one volume of oxygen. The results of my experiments, in spite of this coincidence, astonished me, for it was to be expected that as oxygen is more compressible than hydrogen (to say nothing of its coefficient of expansion), it would contain more molecules per unit volume than hydrogen, and, therefore, necessarily require more than two volumes of hydrogen for one volume of oxygen.

In a short note, published in "Nature," of March 8th, 1888, I announced that I had detected a most important, and hitherto unsuspected, source of error which had led to the consumption of oxygen, and this naturally accounted for the relatively small number for the hydrogen. This source of error lay in the use of a lubricant of a combustible nature, not because of its volatility, but because of its tending to form a greasy layer on the surface of the eudiometer, and which was, to a certain extent, burnt with each explosion in the eudiometer.

In the 'American Journal of Science,' for March and April, 1891, E. W. Morley gives an account of twenty experiments made with a very elaborate apparatus for preparing his gases, and with a measuring apparatus practically the same as I described in my former paper, except that he explodes his gases in his measuring vessel, which is a wide cylindrical tube, thus necessitating microscopical readings of levels and heights, which he claims he *can* make with accuracy to $\frac{1}{500}$th of a millimetre, and that they actually were made to $\frac{1}{200}$th in his experiments. How this can be done satisfactorily after the explosion and the accumulation of water as the result of it, must astonish all who have tried to measure heights of mercury in glass tubes with accuracy. No mention is made as to where the water was expelled to, or how it could be got rid of.

Throughout I have endeavoured to use the simplest apparatus possible and to prepare the gases themselves from only the purest materials, and those of the simplest composition that I could find, so that no purification should be required and all unnecessary contact with other chemical substances avoided, to work with an apparatus of glass throughout, so that no diffusion could take place, and, finally, so to work, that with a given amount of materials, I might compare the gas given off in the first fractions continuously to the last fractions, and thus endeavour to detect any possible impurity, either by variation of the ratio, or by actual observation from the residual gas.

My new experiments are divided primarily into two series, as two different forms of measuring vessel were used, one of the type described in my former paper in which both pressure and volume had to be measured, in the later form the pressure only required measurement, the volume being kept constant. This tended greatly to increase the accuracy of measurements, both because it lessened the number required, and, also, even if the total volume were as much as one cubic centimetre wrong, it would make no appreciable error in the ratio found, as all were measured in the same vessel. This will be obvious on slight consideration of the conditions.

The hydrogen required was obtained from the following sources:—

1. Electrolysis of dilute sulphuric acid
2. ,, ,, hydrochloric acid

} Using in both cases a platinum electrode for the hydrogen to be evolved from, and amalgamated zinc for the absorption of the oxygen.

3. Action of steam on sodium.
4. Palladium hydride, formed by passing hydrogen from sources 2 and 3 over palladium at the ordinary temperature.

The oxygen was prepared from—

1. Re-crystallized potassium chlorate, sometimes with the addition of a little pure sodium hydrate (from sodium).
2. Mercuric oxide.
3. Silver oxide.
4. Silver oxide and barium sulphate.

Before using either the sulphuric or hydrochloric acid, after the necessary dilution it was boiled with some zinc amalgam for an hour or an hour and a-half to get rid of all dissolved gases, and then added to the generating vessel while hot.

The sodium used was such as had never been at any time in contact with paraffin or naphtha, and was obtained direct from the manufacturers. It was, before use, fused in a clean iron ladle and cast in an iron mould. It was always heated up till the anhydrous oxide fused so that the metal could be poured quite free from all scum.

The palladium was used in the form of thin sheets and was ignited in a current of pure air for some time to a full red heat, to remove all traces of greasy matter which it might have acquired during the rolling.

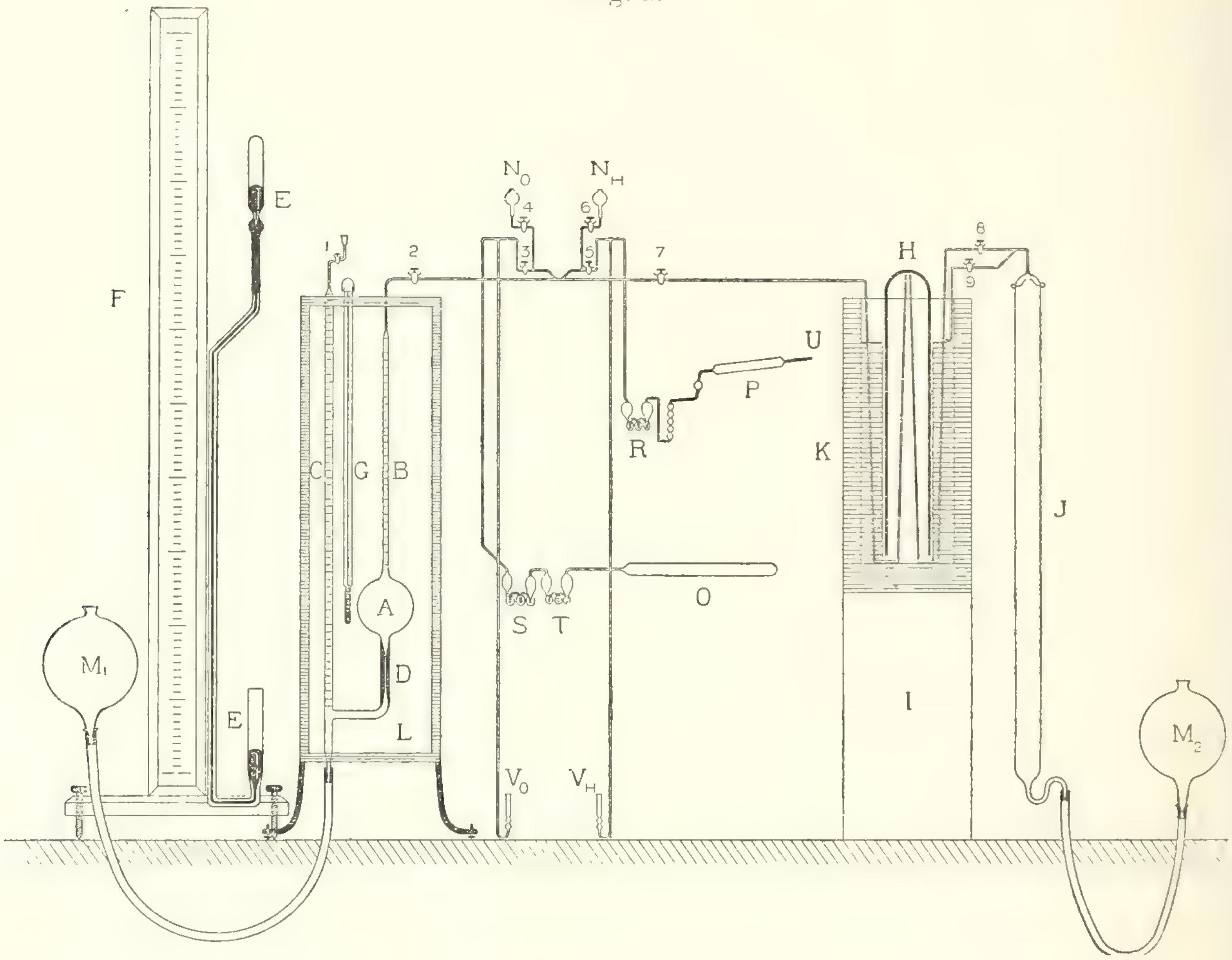
The potassium chlorate was very carefully recrystallized and fused in a platinum dish before use, and in the later experiments a small quantity of pure sodium hydrate made from the metal was added, to prevent, if possible, the evolution of small quantities of chlorine.

Very few experiments were made with mercuric oxide as the temperature required for its decomposition was rather high for soft German glass to stand with safety. Two samples were used, the first of which at first gave off some carbon dioxide, so a sample was prepared with great care from the metal, by first converting it into mercurous chloride, then into mercuric chloride, and finally into the oxide which, after thorough washing, was moistened with pure soda solution and dried. The soda effectually retained any carbon dioxide in this sample, but the repeated fractures of the tubes containing the mercuric oxide led me to abandon its use. Silver oxide seems to be by far the most convenient source of pure oxygen, and has been used in all the later experiments. It was made at first by precipitating silver nitrate solution by means of pure barium hydrate, washing free from barium nitrate, and then drying rapidly on the water bath. The first sample prepared had absorbed so much carbon dioxide from the air during its drying that it rendered worthless several experiments. The second sample was, therefore, prepared by the same reaction, but was very carefully washed with thoroughly boiled water in closed flasks, and dried, *in vacuo*, in vessels with ground glass stoppers, free from all grease. Thinking that nitrates might still adhere to the precipitate as STAS showed they were so prone to do, a quantity of silver sulphate was precipitated with barium hydrate, so that a mixture of silver oxide and barium sulphate was obtained and used after drying as above, and worked extremely well. The silver oxide used in the last experiments, and which has given gas in which no impurity, however small, can be detected, was prepared from silver sulphate and excess of pure potassium hydrate, and after washing, giving another treatment with some fairly dilute potassium hydrate solution by boiling for half an hour or so, again washing with water well boiled and just coloured with potassium permanganate, till free from sulphates, then allowed to settle, and the clear liquid decanted and a little pure soda added (as was done always with the silver oxide) and the oxide dried, *in vacuo*, at 110° C.

Fig. 1 shows the apparatus as used in the later series of experiments, and is drawn to scale as regards the size of the principal parts; the tubes are merely elongated so as to show their connection clearly. The scale F, which is 1 metre in length, indicates the size of the various parts.

The constant volume is contained between a mark on the narrow tube above B, and a similar mark about the middle of the tube D. The internal diameter of these tubes, which are of strong Sprengel tubing, is almost exactly 2 millims., as they

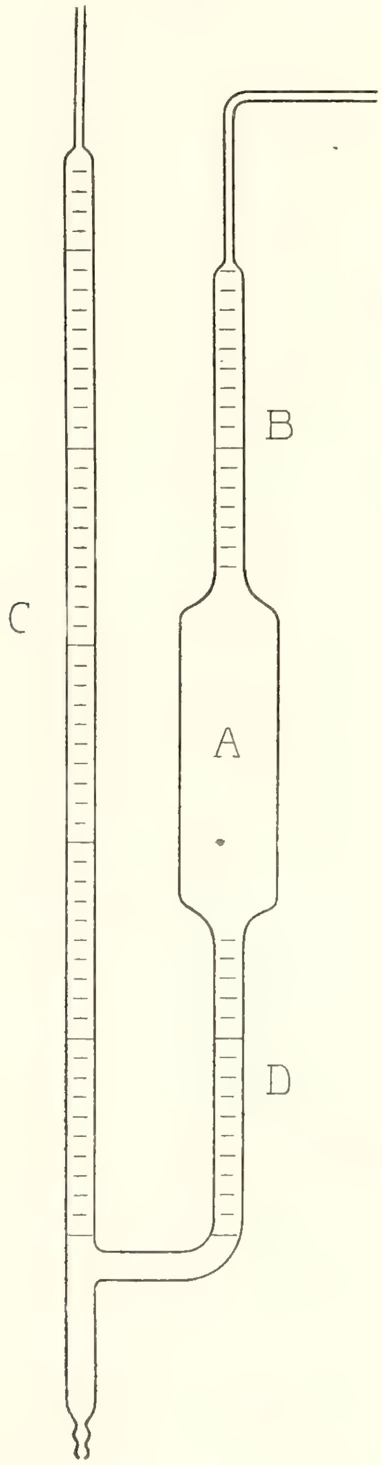
Fig. 1.



A,	Bulb for measuring volume of gas.
B,	Graduated tube calibrated for measuring residual volumes.
C,	Manometer tube.
D,	Narrow tube with mark.
EE,	Barometer.
F,	Standard metre scale divided into millimetres.
G,	Thermometer graduated in $\frac{1}{50}$° C.
H,	Jar for mixing gases before explosion.
I,	Wooden block to support mercury trough.
J,	Tube for exploding gases under diminished pressure.
K,	Mercury trough of wood, varnished.
L,	Water tank, for maintaining temperature steady, with stirrer and levelling screws (contents = about 14 litres).
M_1, M_2,	Mercury reservoirs.
N_O, N_H,	Small mercury reservoirs for driving gases out of tubes into A and B.
O,	Tube, with silver oxide.
P,	Tube, with palladium.
R, S,	Geissler's bulbs, with pure water.
T,	Geissler's bulbs with pure sodium hydrate (from sodium and water).
U,	Tube for hydrogen to pass in at from generating vessel, but sealed up when palladium hydride is used.
V_O, V_H,	Gauges and safety valves with mercury covered with water for oxygen and hydrogen.
	1, 2, 3, 4, 5, 6, 7, 8, and 9, safety taps with syrupy phosphoric acid.

contain ·0395 grm. of mercury per millim. length. The capillarity correction, due to the difference in diameter between the manometer tube C and the tube D, amounted to 3·9 millims., and between B and C to ·1 millim. Temperatures were read to

Fig. 2.

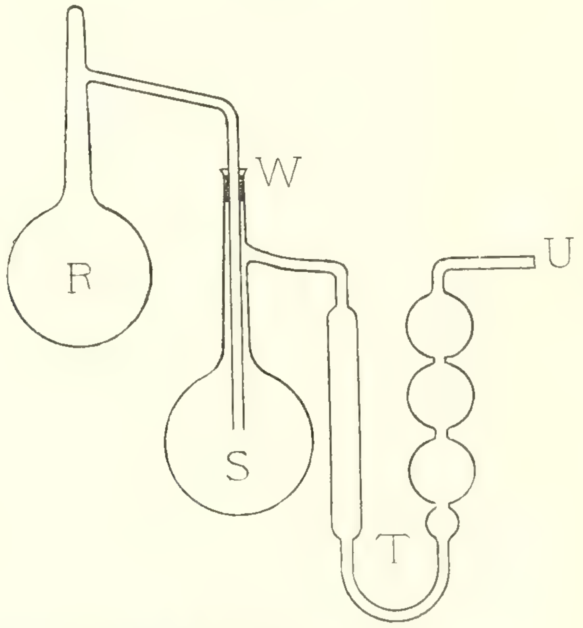


A, Bulb for measuring gases.
B, Tube for measuring residues.
C, Manometer tube.
D, Calibrated tube.

$\frac{1}{100}$th of a degree centigrade in the later series of experiments, but only to $\frac{1}{10}$th previously, by means of the thermometer G, after thorough mixture by means of a stirrer not shown in the figure. All heights were referred to the standard scale F, both those of the barometer EE, as well as those in C and D, by means of a cathetometer Z (fig. 4) constructed specially for the purpose. This consisted essentially of a vertical steel axis carrying two horizontal telescopes with cross wires, which were set one at each level, and then the axis rotated to the scale F, and the heights registered to ·1 millim. This method was necessitated from the want of steadiness in the floor of the room in Durham School Laboratory, where the majority of the experiments were carried out. The same arrangement (which is very convenient as well as accurate) was also used in the later experiments which were carried out under more favourable conditions in the gas analysis room of the Cambridge University Chemical Laboratory.

Fig. 2 shows the earlier form of measuring vessel used when the varying pressures and volumes were both measured. The tubes B, C, and D were all of the same diameter, so that no capillarity corrections were required.

Fig. 3.

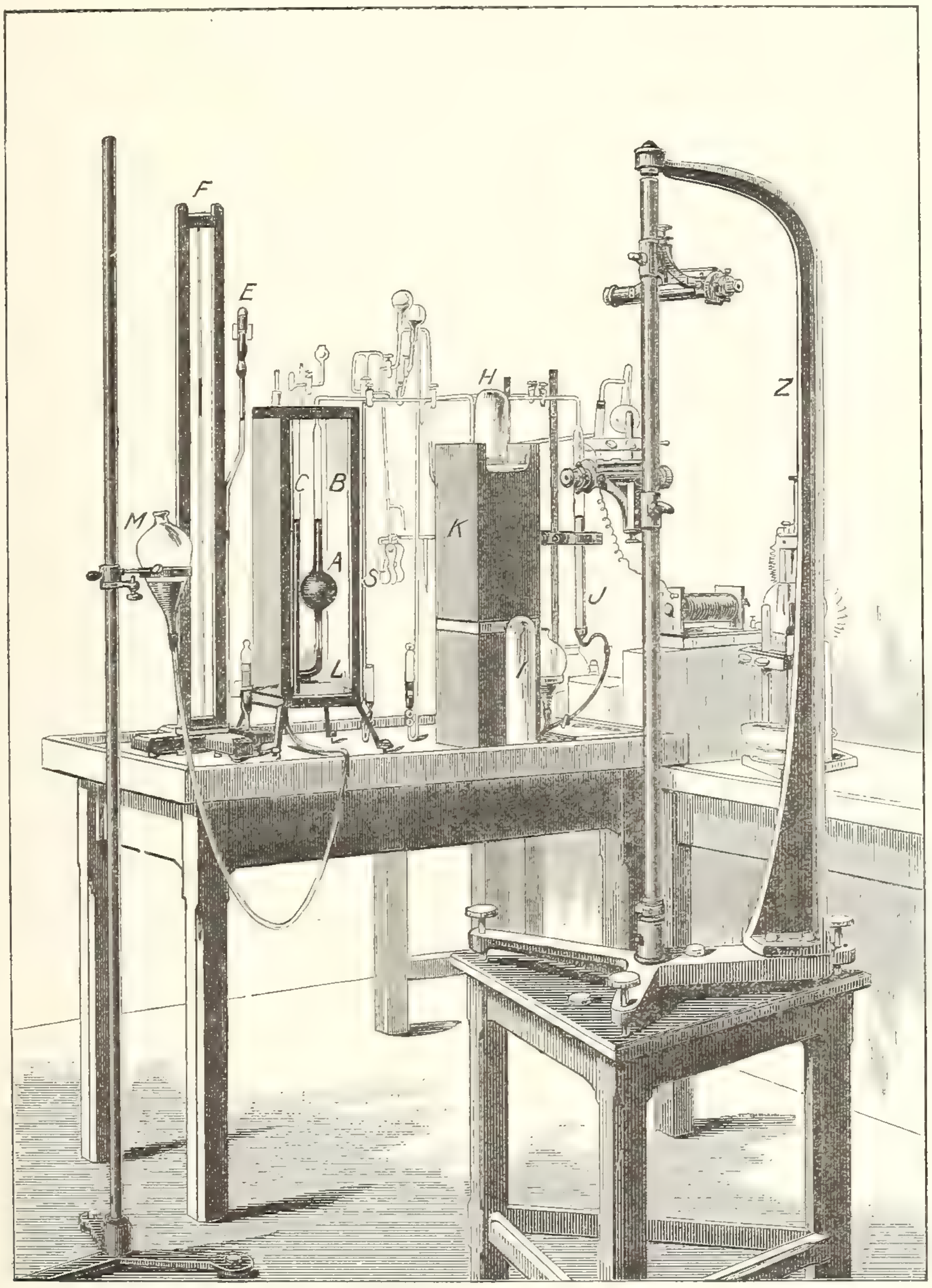


R, Flask for water to be boiled.
S, Flask for sodium.
T, Washing tube, with *very dilute* hydrochloric acid.
U, Exit tube for hydrogen, sealed to U (fig. 1) when in use.
W, Small india-rubber stopper.

Fig. 3 shows the simple apparatus employed for the preparation of hydrogen from sodium by acting on it with steam. When it was used the end of the tube U was sealed to the tube U (fig. 1), which contained palladium in its wider part, P. The only india-rubber connection is at W, where there is a stopper of that material, but diffusion through this was practically impossible to any injurious extent, as the space between the tubes was very narrow and the stopper was 6 to 8 millims. long, and after setting up the apparatus it was sealed over with paraffin containing enough vaseline to prevent its cracking, whilst it allowed of its being easily shaken.

Before beginning a series of experiments the apparatus was carefully exhausted by filling A, B, and C with mercury, as well as the tubes leading to the bulbs N_0 and N_H, by having the taps open, and by raising the reservoir M_1, closing all the taps 1, 3, 4, 5, 6, and 7, then lowering M_1 and opening tap 5, which opened communication with the hydrogen generating apparatus, M_1 being lowered until the mercury fell to D. Then tap 5 was closed and M_1 raised till the air was under more than atmospheric pressure, tap 6 opened and the air inside the apparatus expelled by forcing mercury up into the bulb N_H. This was repeated till no more air could be thus pumped out. A little hydrogen was now generated and pumped out, and the process repeated twice,

Fig. 4.



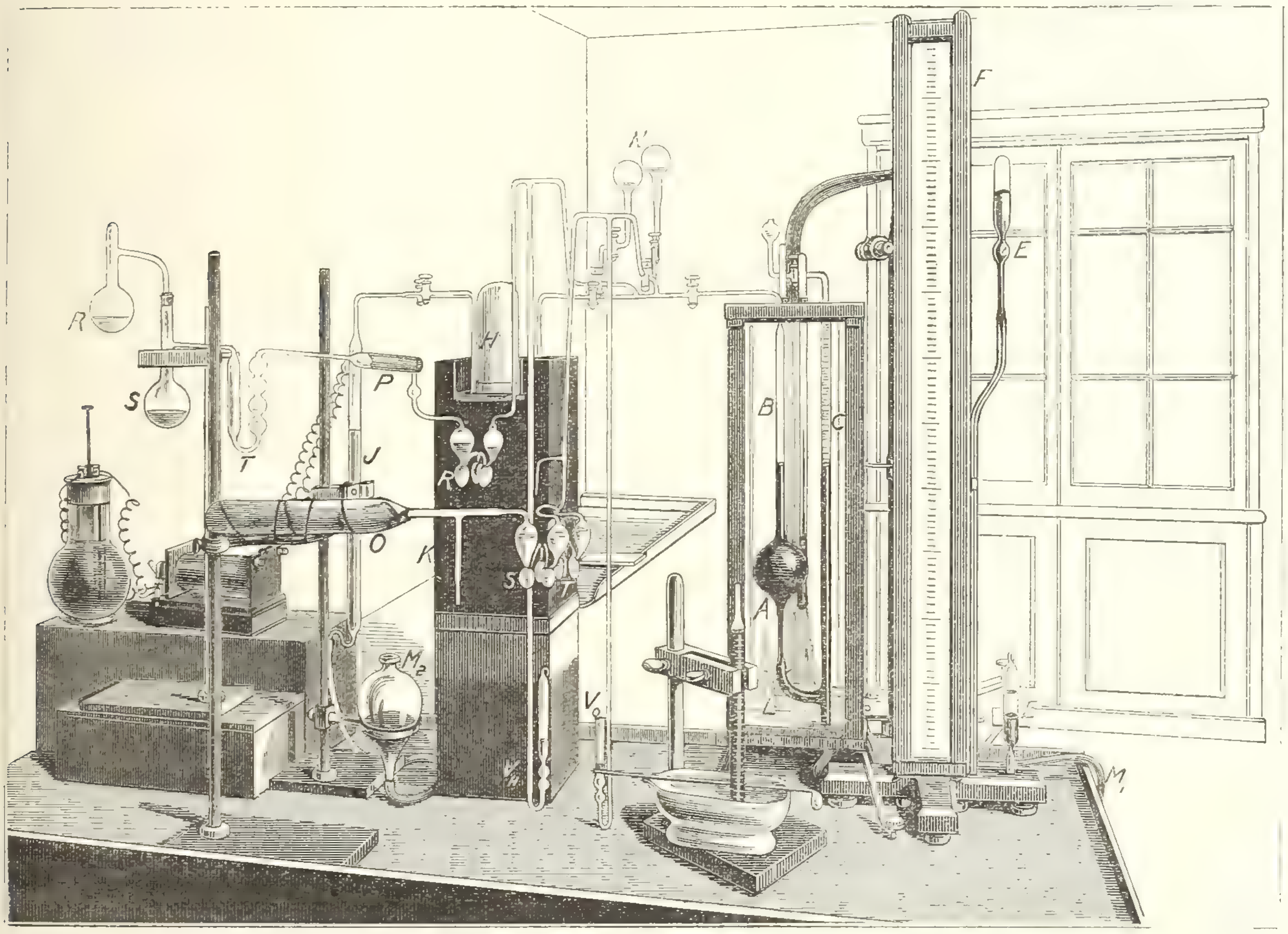
General view of apparatus from front, showing cathetometer.

when the gas now generated might be regarded as absolutely free from admixed air. The same was repeated with the oxygen apparatus. It was easy to arrange before raising the reservoir M_1, so that there was no "untraversed space" by filling the tubes with mercury from the bulbs N_H and N_O.

The mode of using the apparatus will be best understood by the description of an actual experiment.

Oxygen was evolved from the oxide of silver in O by heating till it was almost equal in pressure to that of the external atmosphere, as indicated by the gauge and safety valve V_O, when taps 3 and 2 were opened, M_1 being about the level indicated, and the heating continued till B and A were almost filled to the mark on D, when the flame was extinguished and tap 2 shut, and then mercury was run into the fine tubes from N_O, so that it passed the tap 4 a short distance; that tap was then closed, and 2 opened carefully, so that the gas in the tube was driven into the calibrated space, and the mercury was stopped at the mark by closing taps 2 and 4. Tap 1 on the manometer was opened generally before filling the volume with gas, as one could judge better of the exact pressure by so doing. All being now ready for the final adjustments the mercury reservoir M_1 was raised or lowered by a fine screw on its holder till the mercury came exactly to the mark on D, and, after thorough stirring of the water in L, left for a quarter of an hour, and the adjustments looked to, and then repeatedly stirred, and readings taken in about half an hour after filling if the volume was quite steady. The temperature and barometric pressure were then read. Closing tap 1 on the manometer, and opening taps 2 and 7, the oxygen was next expelled into the jar H (which had been completely filled with mercury) by raising M_1 sufficiently. When all the gas has been thus driven over into H, tap 7 is closed, M_1 is returned to its former position, tap 1 opened again, and the same operation repeated twice for the hydrogen. By opening tap 8 quantities of the mixed gases are drawn over into the explosion-tube J, and there caused to combine under diminished pressure by passing a spark, and this continued till nearly all has combined, when what remains in the explosion-tube is passed back into the jar H, and mixed and drawn back again, and exploded as before. A small residue of one or other gas in excess now remains. This was now passed into H, but the water formed was expelled by closing tap 8 and opening tap 9 when the water flowed out on to the surface of the mercury; it was easy to arrange so that nothing but the gaseous residue was driven into H, from which it was now drawn into B, and its volume accurately determined. After measurement it was again expelled into H, and, if hydrogen, some oxygen in excess was drawn into B, and measured as before, passed into H, and the mixture exploded, and the new residue, measured as before, again passed back into H, and then into J, and expelled into an absorption-tube, measured and treated first with pure potassium hydrate solution to estimate any carbon dioxide, and then with solution of pyrogallol to absorb the oxygen and determine the amount of impurity, if any, present. The final residue gave thus a *maximum* value for any impurity which existed as permanent gas, and might be either nitrogen, carbon monoxide, or hydrogen which had escaped combination in the relatively large excess of oxygen which usually was present in the final explosion. In all probability this is the real explanation of the permanent gas found, as the apparent impurity was almost always larger when the excess of oxygen was large than when it was small and was roughly in proportion to it. This will be easily seen on inspection of the table of results. Taking a few examples in Series IIc, we have

Fig. 5.



General view of apparatus from back, with barometer turned round.

Residue.	Impurity.
10·2	·3
28·9	·5
41·1	·8

There are, no doubt, exceptions to this, but it seems to hold generally that the larger the oxygen residue the larger the impurity found as shown above. The other explanation which might account for the same thing is, that carbon monoxide might be given off easily (from the pyrogallol) to the small extent usually found,

·3 corresponding to an impurity of about $\frac{1}{34,000}$.
1·0 ,, ,, ,, $\frac{1}{10,000}$.

The impurity was called 0 when the impurity was less than ·1 = about $\frac{1}{140}$ of a cubic centimetre, or = $\frac{1}{100,000}$ of the volume of the gases used.

The gases were measured moist, and the surface of the measuring vessel was kept thoroughly moist by washing the apparatus with distilled water in the evening and expelling the water, and then leaving it full of mercury and expelling what had risen through the night, so that no appreciable error was introduced from water present as such in the vessel. One advantage of working with moist gases and vessels with moist surfaces is that the gases can be so completely got rid of, none remaining as a film. When both manometer and measuring vessel were completely filled with mercury and the reservoir lowered so as to produce a Torricellian vacuum in both, it was usual for the mercury to remain for some time 300 to 400 millims. higher in the measuring vessel than in the manometer, the mercury not leaving the moist glass till a trace of gas from the stopcock 2 set it off.

The following experiments taken at random show the modes of measurement and calculation adopted.

August 24th, 1889. Experiment XXXIV., Series IF. Apparatus with Variable Volume.

Oxygen from mercuric oxide—

Volume 50·0 = 3881·0. Temperature = 15·4° C.

	millims.		millims.
Height of mercury in manometer tube on scale =	71·3	on cathetometer =	10·0
,, ,, volume tube on scale . =	71·6	,, =	10·3
Correction . . . . . . =	−0·3	=	−0·3

	millims.
Barometer 777·1 − 26·0 . . . . . . =	751·1
Pressure of aqueous vapour . . . . =	13·0

∴ Pressure of oxygen = 751·1 − ·3 − 13·0 . . = 737·8

Hydrogen from electrolysis of dilute hydrochloric acid—

Volume 45·3 = 3819·6. Temperature = 15·35° C.

	millims.
Height of mercury in manometer tube on scale	118·5
,, ,, volume tube on scale	118·5
Correction . . . . =	·0

Barometer 777·4 − 25·8 . . . . = 751·6

Pressure of aqueous vapour . . . . = 13·0

∴ Pressure of hydrogen = 751·6 − 13·0 . . . . = 738·6

Hydrogen as before—

Volume 45·6 = 3823·5 Temperature = 15·4° C.

	millims.		millims.
Height of mercury in manometer tube on scale	= 115·8	on cathetometer	= 7·1
,, ,, volume tube on scale .	= 115·5	,,	= 6·8
Correction	= + 0·3		+ 0·3

	millims.
Barometer 777·6 − 25·7	= 751·9
Pressure of aqueous vapour	= 13·0
Pressure of hydrogen = 751·9 + ·3 − 13·0 .	= 739·2

Residue (of oxygen)—

Volume 8·6 = 56·5 Temperature = 15·4° C.

	millims.		millims.
Height of mercury in manometer tube on scale	= 491·9	on cathetometer	= 10·0
,, ,, volume tube on scale .	= 484·7	,,	= 2·8
Correction	+ 7·2		+ 7·2

	millims.
Barometer = 777·7 − 25·4	= 752·3
Pressure of aqueous vapour	= 13·0
Pressure of gas = 752·3 + 7·2 − 13·0 . . .	= 746·5

On analysis of residue ·6 of unabsorbable gas remained after treatment with potassium hydrate and pyrogallol; no carbon dioxide being detected.

Total oxygen volume reduced to 0° C. and 760 millims. pressure, becomes

millims.

$3881{\cdot}0 \times \frac{737{\cdot}8}{760{\cdot}0} \times \frac{273{\cdot}0}{288{\cdot}4}$ . . . . . = 3566·4

Residual volume

$56{\cdot}5 \times \frac{746{\cdot}5}{760{\cdot}0} \times \frac{273{\cdot}0}{288{\cdot}4}$ . . . . . = 52·5

·6

Oxygen in residue . . . = 51·9

Hydrogen volumes become

$3819{\cdot}6 \times \frac{738{\cdot}6}{760{\cdot}0} \times \frac{273{\cdot}0}{288{\cdot}35}$ . . . . . = 3514·4

$3823{\cdot}5 \times \frac{739{\cdot}2}{760{\cdot}0} \times \frac{273{\cdot}0}{288{\cdot}4}$ . . . . . = 3520·2

Total hydrogen volume . = 7034·6

Assuming the impurity to be equally distributed between the two gases we have

7034·2 of hydrogen unite with 3514·3 of oxygen,

or

2·0016 ,, ,, 1 ,,

If all impurity be in oxygen, we get the ratio

2·0019 : 1,

and if all in the hydrogen,

2·0014 : 1.

For the experiment with the apparatus arranged as in fig. 1, I have chosen an experiment in which the hydrogen measured is in excess of the oxygen. The constant volume of the apparatus = 3826·3 grms. of mercury.

April 7th, 1892. Experiment XXXIII., Series IIE.

Oxygen from silver oxide (from silver sulphate and potassium hydrate)—

millims.

Height of mercury in manometer tube . . . . = 154·5
,, ,, volume tube . . . . = 145·4

Correction . . . . + 9·1

Pressure of aqueous vapour . . . . = 13·4
Capillarity correction . . . . . = 3·9

− 17·3

Barometer 790·6 − 29·1 . . . . = 761·5
Pressure of oxygen = 761·5 + 9·1 − 17·3 . . . . . . = 753·3
Temperature = 15·92° C.

Hydrogen from palladium hydride—

millims.

Height of mercury in manometer tube . . . . = 154·6
,, ,, volume ,, ... = 145·3

Correction . . . . + 9·3

Pressure of aqueous vapour . . . . 13·5
Capillarity correction . . . . . . 3·9

− 17·4

Barometer, 790·7 − 29·0 . . . . = 761·7
Pressure of hydrogen = 761·7 + 9·3 − 17·4 . . . . . = 753·6
Temperature = 15·97° C.

Hydrogen, second volume—

millims.

Height of mercury in manometer tube . . . . = 155·0

,, ,, volume ,, ... = 145·5

Correction . . . . + 9·5

Pressure of aqueous vapour . . . . = 13·2

Capillarity correction . . . . . = 3·9

− 17·1

Barometer, 791·7 − 28·0 . . . . = 763·7

Pressure of hydrogen = 763·7 + 9·5 − 17·1 . . . . . = 756·1

Temperature, 15·70° C.

The oxygen volume and the first hydrogen volume were measured in the afternoon, the second hydrogen volume in the evening. This is why there is such a difference in the temperature and barometric pressure in the latter case.

Volume of residue, 6·5 = 15·3.

millims.

Height of mercury in manometer tube . . . . = 127·1

,, ,, volume ,, ... = 562·7

Correction . . . . − 435·6

Pressure of aqueous vapour . . . . = 13·3

Capillarity correction . . . . . = ·1

= − 13·4

Barometer, 791·4 − 28·2 . . . . = 763·2

Pressure of hydrogen = 763·2 − 435·6 − 13·4 . . . . = 314·2

Temperature, 15·80° C.

These volumes, when reduced to 0° C. and 760 millims. pressure, become—

For the oxygen . . . . . . $3826{\cdot}3 \times \frac{753{\cdot}3}{760} \times \frac{273{\cdot}0}{288{\cdot}92} = 3396{\cdot}4$

,, hydrogen . . . . . $3826{\cdot}3 \times \frac{753{\cdot}6}{760} \times \frac{273{\cdot}0}{288{\cdot}97} = 3397{\cdot}0$

,, ,, ... $3826{\cdot}3 \times \frac{756{\cdot}1}{760} \times \frac{273{\cdot}0}{288{\cdot}70} = 3411{\cdot}5$

Total hydrogen volume . . . . 6808·5

For the residue (hydrogen) $15{\cdot}3 \times \frac{314{\cdot}2}{760} \times \frac{273{\cdot}0}{288{\cdot}80} = 5{\cdot}98 = 6{\cdot}0$

Hydrogen used . . . . = 6802·5

In order to analyse the residue it was passed into the jar H, and a quantity of oxygen drawn from the oxygen apparatus and measured in B, in exactly the same way as the residue; then both were mixed and again measured, giving the corrected volume 14·8 for the volume before explosion. After explosion 5·9 remained, and on treatment as in the other experiment described above, first with potassium hydrate and then with pyrogallol, no carbon dioxide could be detected, and all the gas was absorbed with the exception of a minute bubble of as nearly as possible 1 millim. in diameter. The residue of 5·98 was therefore pure hydrogen, being confirmed by the analysis $\frac{2}{3}(14{\cdot}8 - 5{\cdot}9) = 5{\cdot}93$.

The combining volumes are therefore—

6808·5 − 6·0 = 6802·5 of hydrogen with 3395·8 of oxygen,

or—

2·0028 ,, ,, 1·0 ,,

In order to determine the degree of accuracy with which the volumes of the gases might be determined, the same volume of gas was repeatedly measured during a fortnight, so as to have it under much greater variations of temperature and pressure than ever occurred during any experiment. The following corrected values were found :—

3279·4
3280·3
3280·6
3280·1

3279·7
3279·4
3279·8

giving as a mean 3279·9, with the probable error on a single observation of ± ·3.

These are highly satisfactory when one remembers that ·1 millim. in pressure corresponds to about ·5 in volume, and ·01° C. to rather more than ·1.

In the following experiments the volumes of the gases measured and the residues are given reduced to 0° C. and 760 millims. pressure. They are divided into two primary series, corresponding to the two forms of measuring vessel employed; and these are again divided into sub-series, according to the sources of the oxygen and hydrogen.

All experiments are given which were completed, except a very few, which were contaminated by such large quantities of impurity, such as carbon dioxide (as in the first experiments with oxygen from mercuric and argentic oxide), which rendered the determination of the ratio impossible or very unsatisfactory. The first seven experiments of Series Iᴀ were also omitted, as they were rendered practically valueless, as were all those formerly published, owing to the use of vaseline as a lubricant for the glass taps. Experiments VIII. and IX. show how this source of error was detected, and on using syrupy phosphoric acid as the lubricant, and safety taps with cups, carbon dioxide vanished as an ordinary impurity, and the total impurity fell away to a very small amount indeed.

In the following Tables the following data are given :—

Column A gives the number of the experiment.
,, B ,, ,, date on which it was performed.
,, C ,, ,, hydrogen volume as measured and corrected to 0° C + 760 millims.
,, D ,, ,, oxygen ,, ,, ,, ,, ,,
,, H ,, ,, hydrogen in residue.
,, O ,, ,, oxygen ,, ,,
,, N ,, ,, impurity, determined by potash and pyrogallol.
,, R ,, ,, ratio of combining volume of hydrogen to that of oxygen when the impurity was assumed to be equally distributed in both gases.
,, S ,, ,, ratio of combining volume of hydrogen to that of oxygen when the impurity was supposed to be all in the hydrogen.
,, T ,, ,, source of the hydrogen.
,, W ,, ,, ,, ,, oxygen.

Series IA.

A.	B.	C.	D.	H.	O.	N.	R.	S.	T.	W.
	1887									
VIII.	August 30 . .	6909·4	3451·2	38·9	. .	7·9 CO_2 7·2 CO ·7 N_2	. .	. .	Electrolysis of H_2SO_4	$KClO_3$
IX.	„ 30 . .	6882·2	3414·2	74·9	. .	5·3 CO_2 2·6 CO 3·6 N_2	. .	. .	Electrolysis of H_2SO_4	$KClO_3$
	Series IB.									
	1888									
X.	January 11 . .	Lost some hydrogen after exploding some.								
XI.	„ 16 . .	Burst eudiometer								
XII.	„ 17 . .	7657·2	3798·7	63·3	. .	·8	1·9991	1·9989	Electrolysis of H_2SO_4	$KClO_3$
XIII.	„ 17 . .	7561·4	3777·9	19·9	. .	·7	1·9962	1·9961		
XIV.	„ 18 . .	Burst eudiometer								
	Series IC.									
XV.	April 30 . . .	Oxygen flask cracked and much air in oxygen.								
XVI.	„ 30 . . .	7177·4	3609·7	. .	19·0	·8	1·9989	1·9987	Electrolysis of H_2SO_4	$KClO_3$
XIX.	May 2	7042·7	3605·8	. .	83·0	1·2	1·9992	1·9989		
	Series ID.									
XX.	May 2	6946·8	3582·9	. .	111·8	1·2	2·0013	2·0010	. .	Old $KClO_3$
XXI.	„ 5	7144·0	3657·1	. .	83·4	2·4	1·9990	1·9984	Electrolysis of HCl	Now $KClO_3$
XXII.	„ 5	7247·2	3654·1	. .	27·1	·6	1·9981	1·9980		
XXIII.	August 31 . .	7171·1	3613·0	. .	19·4	·5	1·9958	1·9956		

Series IE.

A.	B.	C.	D.	H	O.	N.	R.	S.	T.	W.
XXV.	1889. August 8 . .	6875·5	3533·0	..	93·5	1·7	1·9990	1·9985	..	$KClO_3$
XXVI.	,, 13 . .	7014·7	3558·0	..	49·6	3·2	1·9994	1·9985	Electrolysis of HCl	$KClO_3+NaHO$
XXVII.	,, 14 . .	6964·9	3535·6	..	56·8	1·7	2·0021	2·0016		
XXVIII.	,, 14 . .	6972·3	3544·6	..	61·6	3·2	2·0018	2·0009		
XXIX.	,, 15 . .	7028·0	3562·9	..	53·8	·8	2·0028	2·0025		
XXX.	,, 16 . .	6989·7	3528·1	..	35·9	·4	2·0015	2·0013		
Series IF.										
XXXI.	August 22 . .	6963·1	3490·9	28·9	..	16·3 CO_2 19·9 CO 3·2 N_2	2·0014	2·0005	Electrolysis of HCl	HgO
XXXII.	,, 23 . .	7004·3	3569·7	..	70·2	·8	2·0015	2·0013		
XXXIII.	,, 23 . .	6981·8	3538·8	..	48·8	·8	2·0006	2·0003		
XXXIV.	,, 24 . .	7034·6	3566·4	..	51·9	·6	2·0016	2·0014		
XXXV.	,, 24 . .	7041·6	3603·5	..	86·2	·8	2·0019	2·0017		
XXXVI.	,, 27 . .	7070·1	3609·7	..	74·0	·9	1·9996	1·9994	Pd_2H	HgO
XXXIX.	,, 30 . .	6968·2	3622·9	..	144·3	1·1	2·0032	2·0028	Pd_2H	Ag_2O

Series IIA.

A.	B.	C.	D.	H.	O.	N.	R.	S.	T.	W.
	1890.									
I.	January 20 . .	6883·7	3461·4	..	25·3	·3	2·0033	2·0033	Pd_2H	Ag_2O
II.	,, 20 . .	6860·4	3513·6	..	88·6	·4	2·0030	2·0029		
III.	,, 21 . .	6940·1	3523·4	..	54·5	·3	2·0007	2·0006		
IV.	,, 21 . .	6741·6	3433·8	..	66·6	·2	2·0021	2·0021		
VI.	May 26 . . .	6821·9	3499·5	..	93·9	·9	2·0031	2·0028	$Na+H_2O$	New Ag_2O
VII.	June 3	6750·8	3431·7	..	58·3	1·2	2·0012	2·0008	,,	Ag_2O
VIII.	,, 5	6760·6	3393·2	..	20·3	·3	2·0043	2·0043	,,	
IX.	,, 7	6833·6	3449·2	..	35·8	·2	2·0019	2·0019	Pd_2H	
X.	,, 9	6734·6	3399·4	..	38·2	·5	2·0036	2·0035		
	1891.									
XII.	August 11 . .	6745·3	3388·7	..	16·4	·5	2·0003	2·0001	Pd_2H	Ag_2O
XIII.	,, 11 . .	6793·9	3425·3	..	25·7	·5	1·9985	1·9983		
XIV.	,, 13 . .	6792·8	3429·2	..	36·1	·3	2·0019	2·0019	$Na+H_2O$	
XV.	,, 13 . .	6726·2	3430·8	..	70·45	·35	2·0017	2·0016		
XVI.	,, 14 . .	6746·4	3404·1	..	35·8	·3	2·0029	2·0028		
XVII.	,, 14 . .	6724·4	3378·4	..	23·7	·3	2·0044	2·0044		
XVIII.	,, 15 . .	6669·0	3342·8	..	13·9	·1	2·0033	2·0033	Pd_2H	
XIX.	,, 15 . .	6660·0	3330·8	..	3·3	·0	2·0015	2·0015		
XX.	,, 17 . .	6835·0	3421·4	..	4·3	·1	2·0002	2·0002		
XXI.	,, 17 . .	6821·4	3414·2	..	3·5	·3	2·0000	2·0000		

Series IIB.

A.	B.	C.	D.	H.	O.	N.	R.	S.	T.	W.
I.	November 21 .	6691·5	3371·1	..	36·5	·9	2·0067	2·0064	$Na+H_2O$	$Ag_2O+BaSO_4$
II.	,, 21 .	6643·8	3335·2	..	26·0	1·2	2·0076	2·0073		
III.	,, 24 .	6721·1	3378·7	..	22·9	·7	2·0028	2·0026		
IV.	,, 26 .	6717·2	3369·1	..	13·0	·4	2·0015	2·0014		
V.	,, 26 .	6677·1	3365·0	..	32·2	·7	2·0034	2·0032		
VI.	,, 28 .	6715·1	3373·6	..	17·9	·4	2·0011	2·0010		
VII.	December 5 . .	6665·5	3328·7	..	2·0	·2	2·0036	2·0036		

A.	B.	C.	D.	H.	O.	N.	R.	S.	T.	W.
Series IIc.										
VIII.	1891. December 7 . .	6475·8	3247·0	..	7·6	·5	1·9991	1·9989	Pd_2H	$Ag_2O+BaSO_4$
X.	,, 8 . .	6714·2	3367·0	..	2·5	6·3	1·9993*	..		
XII.	1892. February 23 . .	6647·4	3329·2	..	4·7	·3	1·9995	1·9994	Pd_2H	$Ag_2O+BaSO_4$
XIII.	,, 24 . .	6715·0	3379·6	..	28·9	·5	2·0040	2·0039		
XIV.	,, 27 . .	6805·3	3409·7	..	10·2	·3	2·0019	2·0018		
XV.	March 1 . . .	6784·7	3418·1	..	30·6	·3	2·0029	2·0028		
XVI.	,, 2 . . .	6803·3	3420·5	..	24·4	·4	2·0032	2·0031		
XVII.	,, 3 . . .	6873·5	3453·2	..	18·0	·4	2·0009	2·0008		
XVIII.	,, 4 . . .	6848·2	3462·0	.	41·1	·8	2·0019	2·0016		
XIX.	,, 10 . . .	6853·5	3429·0	..	5·9	·7	2·0021	2·0019		
Series IId.										
XX.	March 28 . . .	6655·0	3344·4	..	23·2	·3	2·0038	2·0037	$Na+H_2O$	New Ag_2O
XXI.	,, 30 . . .	6905·2	3461·7	..	15·7	·1	2·0038	2·0038	New $Na+H_2O$	
XXII.	,, 30 . . .	6890·5	3455·9	..	15·0	·0	2·0025			
XXIII.	,, 31 . . .	6871·4	3452·2	..	20·8	·0	2·0025			
XXIV.	,, 31 . . .	6855·7	3430·2	..	6·5	..	2·0024			
Series IIe.										
XXV.	April 1 . . .	6863·8	3443·8	..	15·4	·3	2·0020	2·0019	Pd_2H	Ag_2O
XXVI.	,, 1 . . .	6870·0	3432·9	..	2·1	·0	2·0024			
XXVII.	,, 2 . . .	6870·1	3439·7	..	9·2	·0	2·0026			
XXVIII.	,, 4 . . .	6848·7	3422·1	..	2·9	·0	2·0030			
XXIX.	,, 4 . . .	6792·5	3386·6	13·5	..	·0	2·0022			
XXX.	,, 5 . .	6809·2	3399·5	1·5	..	·0	2·0025			
XXXI.	,, 5 . . .	6793·9	3399·6	..	7·7	·0	2·0029			
XXXII.	,, 6 . . .	6789·6	3389·5	2·9	..	·0	2·0023			
XXXIII.	,, 7 . . .	6808·5	3396·4	6·0	..	·0	2·0028			
XXXIV.	,, 8 . . .	6793·1	3395·8	..	2·1	·0	2·0017			
XXXV.	,, 8 . . .	6786·5	3395·0	..	5·4	·0	2·0022			
XXXVI.	,, 9 . . .	6814·8	3411·9	..	9·3	·0	2·0028			

* Impurity all in oxygen.

The experiments in Series I. were all made in the apparatus with variable volume.

Series IA.

Only two experiments are given of this series, and these two the last that were done with taps lubricated with a very stiff mixture of vaseline and paraffin. The ratios of the volumes of carbon dioxide and carbon monoxide to one another are interesting, being almost 1 : 1, when the excess of hydrogen is rather small, and 2 : 1 when the excess of hydrogen is much larger, recalling the results of BUNSEN's experiments on limited oxidation of carbon monoxide.

Series IB.

These show how experiments failed frequently. They also led to the discovery that the fracture of the eudiometer was generally due to its being allowed to become too hot from the number of explosions following one another too rapidly, and then the cold mercury cracked the glass. The temperature at which they were performed was about 4° C. The ordinary glass taps were lubricated with syrupy phosphoric acid.

Series IC.

These experiments were made with the apparatus fitted with safety taps, and lubricated with phosphoric acid.

Series ID.

These experiments were the first made with dilute hydrochloric acid instead of dilute sulphuric acid, and it will be noticed that the ratio gradually falls from the first to the last of the four experiments, pointing out clearly that either the hydrogen is not the same or that the oxygen is not the same throughout, and that the impurity tends to increase or to diminish as the materials get used up. The first experiment of this series is the first in which the ratio exceeds 2 : 1.

Series IE.

The experiments in this series, especially the later ones, seem to indicate clearly that the low values found for the ratio were due to traces of chlorine escaping with the oxygen, even after passing through GEISSLER's bulbs filled with pure soda, as the value rises except in the first case, after which potassium chlorate mixed with a little pure soda was used as the source of the oxygen.

Series IF.

Oxide of mercury was used instead of potassium chlorate as the source of oxygen when the value found is practically the same as in the later experiments of the last sub-series. It is plain that the source of the hydrogen ought to be varied. This was done by passing the hydrogen over palladium, and getting the hydrogen from the hydride formed by heating it.

Finally silver oxide was substituted for mercuric oxide, as it decomposes at a more convenient temperature.

The fracture of the measuring vessel rendered a new apparatus necessary, and one or two improvements were now introduced, the chief being the use of a constant volume, so that pressures only required to be measured for the large volumes, and the use of a narrower tube for measuring the residues.

One interesting point about the fracture of the measuring vessel is that, while surrounded by water and full of mercury, it broke in exactly the same way that a spare one which was kept ready for fear of accidents broke *at the same time.* The one in use was found fractured on a Wednesday morning, and the spare one had been handled and put on a shelf on the Monday afternoon just preceding. On going on the Wednesday to take it down to replace the other which had broken, it was found to have broken in such a similar manner that the pieces had to be fitted to each other to find out to which apparatus they belonged. The spare one had never been used, and not even calibrated.

In Series II. the oxygen was always got from oxide of silver, and the hydrogen from sodium and water or from palladium hydride, the hydrogen for which was furnished by sodium and water, with the exception of the first four experiments in Series IIA., when the hydrogen charge in the palladium was what was left in it from Series IF., the hydrogen for which was obtained by the electrolysis of dilute hydrochloric acid. The 93 grms. of palladium absorbed usually somewhat over 6 litres of hydrogen, enabling twelve experiments to be done consecutively from one charge.

In Series IIA. the only two experiments which perhaps ought to be rejected are Nos. XII. and XIII., which were the first two performed after the apparatus had stood unused for over a year. Why these results should be so low I can offer no suggestion, but they differ notably from those of the same charge both before and after them.

Series IIB.

The results in this series vary remarkably, two of the ratios being very high and the mean of the whole being high, but with a large probable error. The oxygen from the mixture of silver oxide and barium sulphate seemed to be very pure.

Series IIC.

The hydrogen from the palladium hydride at first behaved in a peculiar way, and some air seemed somehow to have got in with the oxygen, and in two of the experiments the residues were very large, so large that the results were absolutely valueless, the non-absorbable residue amounting to many cubic centimetres in Experiments IX. and XI. It is therefore advisable to neglect also Experiments VIII., X., and XII., the results of which are somewhat abnormal. No reason can be given for the abnormal behaviour of all five experiments, but being all consecutive there is no doubt that some common source of impurity infected all five.

Series IID.

The first experiment was made with the last lot of hydrogen from a charge of sodium; fresh sodium was added, and the second experiment was made with the hydrogen at first given off. Both these results are high. The next three agree wonderfully with one another and are, as it were, made with the purest hydrogen obtainable from sodium, any impurity which would come off first being used in Experiment XXI., and any which tended to remain being left behind, and was rejected as not quite enough sodium remained to give hydrogen for another experiment. The gases were in the last three experiments also perfectly pure as far as residual gas was concerned. The silver oxide in this series and the next was the same sample and prepared from silver sulphate and potassium hydrate.

Series IIE.

This series as a whole is the best which has been done, the gases being throughout pure, with the exception of very small impurity in the first experiment. They agree remarkably with one another, whether the hydrogen or the oxygen was in excess. The whole twelve experiments were successful without exception, and were all that could be performed with one charging of the palladium.

The values deduced from the various series are given below and together so that they may readily be compared. The equations employed in their reduction are the following well known ones :—

1. For the probable error of the arithmetical mean

$$\text{probable error} = \pm \cdot 6745 \sqrt{\left(\frac{s}{n(n-1)}\right)}$$

where n = number of observations,

s = sum of squares of the variations of the individual results from the mean.

2. For the probable value of the result from several series of experiments

$$M = \frac{\frac{A}{a^2} + \frac{B}{b^2} + \frac{C}{c^2} \;\&c.}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \;\&c.}.$$

M = general mean.
A, B, C, &c., being the mean results as above.
a, b, c, &c., being the probable error of each.

3. For the probable error of this general mean $= m$

$$m = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \;\&c.\right)}}.$$

These mean values are for

Series I.

	Number of experiments in each sub-series.	Impurity equally in both gases.	Number of experiments.	Impurity all in hydrogen.
IB.	(2)	1·9976 ±·001	(2)	1·9975 ±·001
IC.	(2)	1·99905 ±·0001	(2)	1·9988 ±·00006
ID.	(4)	1·9985 ±·0007	(4)	1·9982 ±·0007
IE.	(6)	2·00011 ±·0004	(6)	2·00055 ±·0005
IF.	(5)	2·0014 ±·00005	(5)	2·00104 ±·00018
		General mean for Series I.		
	(19)	2·000903 ±·00004	(19)	1·99925 ±·00005

Series II.

	Number of experiments in each sub-series.	Impurity equally in both cases.	Number of experiments.	Impurity all in hydrogen.
IIA.	(19)	2·0020 ±·00024	(19)	2·0019 ±·00024
IIB.	(7)	2·0038 ±·0062	(7)	2·0037 ±·0060
IIC.	(10)	2·0015 ±·00036	(9)	2·0016 ±·00036
IID.	(5)	2·0030 ±·00022	(5)	2·0030 ±·00022
IIE.	(12)	2·00245 ±·00007	(12)	2·00245 ±·00007
		General mean for Series II.		
	(53)	2·002435 ±·00006	(52)	2·002431 ±·00006

If we omit experiment XIII., Series IIA.
,, ,, VIII., X., XII., Series IIC.
,, ,, XX., XXI., Series IID.

we get for

	Number of experiments in each sub-series.	Impurity equally in both gases.	Number of experiments.	Impurity all in hydrogen.
IIA.	(18)	2·0022 ±·00022	..	2·0021 ±·00022
IIC.	(7)	2·0024 ±·00024	..	2·0023 ±·00026
IID.	(3)	2·00247 ±·0000033	..	2·00247 ±·0000033
	which with the values above for IIB., IIE., gives the general mean for Series II.			
	(47)	2·002466±·000003	(46)	2·002466±·000003

Neglecting altogether Series I., as in all probablity (as pointed out above) the oxygen was contaminated by chlorine, we have the values—

2·002433,
2·002466,

as the extreme values for the number of volumes of hydrogen uniting with 1 volume of oxygen to form water. As Series D was a small one, especially when the two first experiments in it are omitted, the second value is unduly weighted by the extremely small probable error in the remaining three, I think that the value—

2·00245,

may be taken as the true value for the ratio sought.

It is highly satisfactory to note that this is exactly the value found in the last series of twelve experiments made with the purest gases, and the advantage of prolonged experience in the manipulation of these difficult and tedious experiments. Although the last of the experiments was performed in April of last year, I have refrained from publishing the results till I could feel certain that I was unable to detect any further source of error, and until I felt assured that all the possible sources of error which have been suggested to me by various friends who have taken an active interest in my work, had been successfully overcome or did not apply to my methods of manipulation and measurement.

With regard to the results published by MORLEY, I think I may fairly claim that the simplicity of my apparatus for measuring and preparing my gases, the fact that they were never transferred from one mercury trough to another, that they were always saturated with water in the same way, and that the water formed by the union was never admitted to my measuring vessel, amongst other advantages, render my results more worthy of confidence than his. The means adopted by him for detecting any impurity are unable to detect such small amounts of foreign gas as were not only easily detected but measured by my method. As long as the bubble could be seen it could be measured, and this, as measured, was always a maximum. His excess over 2 volumes is just one-tenth part of what I find.

The value for the ratio found by LEDUC[*] from his determinations of the densities of hydrogen, oxygen, and the detonating gas from the electrolysis of strong potassium hydrate solution, agrees far more nearly with my value, which it exceeds somewhat, being—

$$2{\cdot}0037.$$

I do not think, however, that this method can compare in accuracy with a direct method, even if it be proved that the detonating gas so prepared unites completely without any residue whatever of either oxygen, hydrogen, or foreign gas.

There is but little doubt that the carbon in MORLEY's hydrogen is due to acetates in his potassium hydrate giving ethane on electrolysis, some of which, although evolved at the opposite pole, being soluble to a certain extent in water, will find its way into the hydrogen. It is well known that potassium hydrate, purified by alcohol, is useless for gas analysis.[†]

The values for the ratio of the volumes are, therefore,

MORLEY	2·00023 : 1,
LEDUC	2·0037 : 1,
SCOTT	2·00245 : 1,

and for densities—

Lord RAYLEIGH	15·882,
LEDUC	15·905.

Combining my value with that of Lord RAYLEIGH, we get for the atomic weight of oxygen—

$$15{\cdot}862,$$

a number almost identical with that of—

DITTMAR . . . . = 15·866,

and with

COOKE and RICHARDS . . . = 15·869.

Combining LEDUC's two values, we get for the same—

$$15{\cdot}876.$$

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[*] 'Compt. Rendus,' August 8, 1892.

[†] HEMPEL's 'Gas Analysis,' DENNIS' Translation, p. 115.

The value 2·00245 is for temperatures about 14° to 18° C. If, however, the values for the coefficients of expansion of oxygen and hydrogen be taken as $\frac{1}{272{\cdot}16}$ and $\frac{1}{273{\cdot}13}$ respectively, we have for the ratio at 0° C., 2·00285.

In conclusion, I have only to say that *every measurement in all the experiments* was made by me, and that all the solutions required were made up, and cleansings of the apparatus, &c., were carried out by my own hands, so that no ambiguity could arise without my being in the position to elucidate any possible error at any time.

X. *The Absolute Thermal Conductivities of Iron and Copper.*

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THE experiments described in this paper were undertaken at the suggestion of Professor A. GRAY, M.A., University College, Bangor, with the object of contributing something to the results still necessary to establish the experimental work bearing on the absolute thermal conductivity of metals on a more satisfactory basis. They were also intended to furnish a determination of the absolute conductivity of *pure* copper at different temperatures.

The data already accumulated* on the thermal conductivities of iron and copper are due chiefly to ÅNGSTRÖM, FORBES, NEUMANN, TAIT and MITCHELL, and more recently to KIRCHHOFF and HANSEMANN and LORENZ.

FORBES first published,† in 1861 and 1865, the details of his determination of the conductivity of iron, and showed that its value diminished with rise of temperature. Some years later TAIT confirmed‡ this result, and in addition gave determinations of the conductivities of copper, lead, &c. His results for copper showed that, unlike iron, its conductivity increased with increase of temperature. ÅNGSTRÖM, however, in 1861, published§ results for iron and copper, evidently obtained with great care and by a very reliable method, which showed that the thermal conductivities of both metals decreased with increase of temperature. As pointed out by TAIT, both ÅNGSTRÖM and FORBES omitted to correct their results for change of specific heat with temperature, but the application of this correction does not materially affect the nature of their results. NEUMANN‖ also determined the absolute conductivities of

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[*] See table given below, p. 588.

[†] 'Trans. Roy. Soc. Edinb.,' vol. 23, pp. 133–146, and vol. 24, pp. 73–110. A preliminary communication had previously been made to the British Association at Belfast in 1852.

[‡] 'Trans. Roy. Soc. Edinb.,' 1878.

[§] Communicated to the Royal Swedish Academy in January, 1861. German and English translations of his paper are to be found in POGGENDORFF'S 'Annalen' for 1863 and in the 'Phil. Mag.,' 1863 (first half-year).

[‖] 'Ann. de Chim.,' vol. 66, pp. 183–185.

several metals, but gave no results bearing on the change of conductivity with temperature. More recently KIRCHHOFF and HANSEMANN* and LORENZ† have published results for iron and copper, but their methods were not so reliable as their work, and the results for copper are exceptionally low.

Although there can be no doubt as to the substantial accuracy of most of the results obtained by the experimenters mentioned above, yet it is evident that those bearing on the change of conductivity with temperature are very unsatisfactory, and that a good deal of work has yet to be done on the subject.

The method of determination adopted in the work here described was that due to FORBES, with the exception that a single thermo-electric couple was employed for the determination of temperatures instead of a number of thermometers. In addition to the uniformity secured by measuring all temperatures on the same scale, there is the advantage that by this method the holes in the bar for the insertion of one junction of the couple may be very much smaller than those required for thermometers, and may therefore be drilled at shorter distances apart, thus admitting of a more accurate determination of the distribution of temperature along the bar. Further, as no mercury is put in the holes, no lining of iron is necessary in the copper bar. Another advantage in the use of a thermo-electric couple is, that in observing the cooling of the short bar used to determine the rate of cooling at different temperatures, the junction inserted in the bar cools at practically the same rate as the bar, whereas, in the case of a thermometer, the cooling of the thermometer always lags behind that of the bar ;‡ an experiment showed that at temperatures between 150° C. and 200° C. the temperature of the thermometer may be nearly a degree higher than that of the bar in which it is inserted even when this bar is sufficiently massive to cool rather slowly.

The bars used in the experiments were of iron and copper. The iron bar was a $\frac{3}{4}$-inch square bar, about $4\frac{1}{2}$ feet long, of ordinary commercial wrought-iron, having its surface filed up and very lightly polished with black-lead to secure uniformity of radiating power.

The copper bar was a round hard-drawn bar half-an-inch in diameter and about 7 feet long, of practically pure electrolytically deposited copper, prepared by Messrs. BOLTON and Co., Oakamoor Mill, Staffordshire. The sectional area of the bar was perfectly uniform and its surface was smooth and polished.

For the reception of the thermo-electric junction, holes about a millimetre in diameter were drilled in these bars at distances apart which preliminary experiment had shown to be suitable. The depth of the holes was about equal to three-quarters of the diameter of the bar, and care was taken in drilling that their axes were at right angles to the bar, and passed approximately through its central axis.

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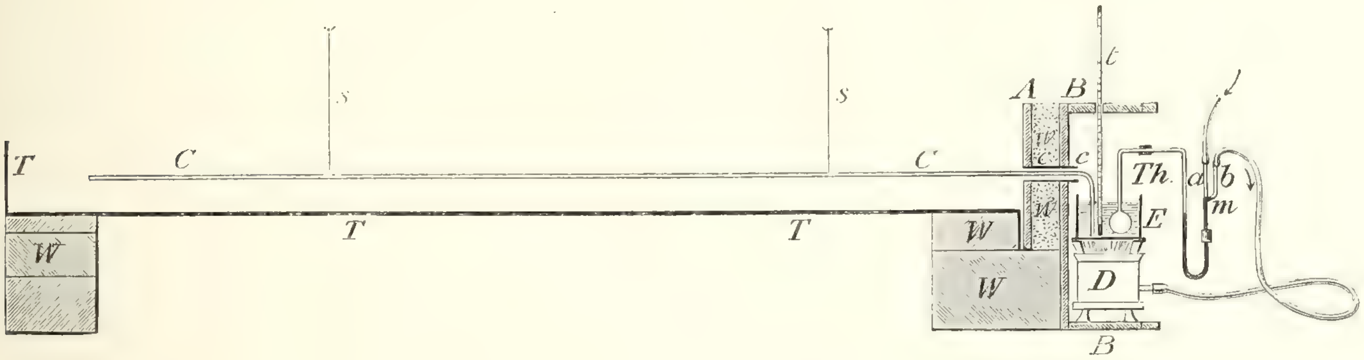
* WIEDEMANN'S 'Annalen,' 9, p. 1 ; 13, p. 406.

† WIEDEMANN'S 'Annalen,' 13, p. 422.

‡ See FOURIER'S 'Theory of Heat,' §§ 298–300.

The arrangement of apparatus finally employed was that shown in fig. 1. One end of the experimental bar was immersed in a vessel of melted lead maintained at a constant temperature, and the rest of its length was exposed to the air, but protected from draughts and external radiation by a wide trough of sheet zinc, *T T T*, arranged as shown in the figure. The molten lead was maintained at a constant temperature by a thermostat, *Th.*, which consisted of an air thermometer having its bulb immersed in the lead, and controlling, by means of a column of mercury in the U-shaped bend of its stem, the passage of the gas at *m*, the junction of the entrance and exit tubes *a* and *b*. By means of this arrangement the lead could be maintained at a temperature constant within a range of about half a degree for any length of time.

Fig. 1.



A. Wooden screen.
B. Wooden box lined with sheet iron.
C C. Metal bar.
D. Gas burner.
E. Pan containing melted lead.
T T T. Trough (of sheet zinc) protecting bar.
Th. Thermostat (air thermometer with mercury cut-off at *m*).
c c. Thin copper tube.
w w. Cotton wool.
s s. Strings supporting bar.
W W. Blocks of wood supporting *T*.

The bar was protected from the radiation from the heating apparatus by enclosing this apparatus in a box B, lined with sheet iron, and by screening off this box by a wooden screen A, and lightly packing the space between the box and screen with cotton wool. The tube, *c.c.*, shown in the figure was intended to shield the part of the bar within the box from irregular heating by the flame of the burner, and served also to prevent contact between the cotton wool and hot bar. The space between this tube and the bar was packed at one end with asbestos to prevent a current of hot air passing out of the box along the bar.

The thermo-electric couple employed was one of a good soft iron wire and German silver. The wires were about ·5 millim. in diameter, and the junctions were made by fusing the wires together under borax and filing down to the required size. Repeated calibration showed that with this couple the deflection on the galvanometer scale was, between 0° C. and 200° C., strictly proportional to the difference between the temperatures of its junctions. The galvanometer in use with the couple was a low resistance reflecting galvanometer, adjusted so as to give a deflection of about four scale-divisions for a difference of 1° C. between the temperatures of the junctions. The scale deflections could be easily read to half a division, so that the difference between the temperatures of the junctions could be estimated to about one-tenth of a degree.

The distribution of temperature along the bars in the stationary state was determined by inserting one junction successively into the small holes drilled in the bar, while the other junction was maintained at a constant temperature* by inserting it in a hole in a short iron bar resting on the bottom of the zinc trough *T*. The deflections obtained were, in this way, proportional to the difference of temperature between the several points on the bar and the temperature of the surrounding air. It was found necessary to carefully insulate the ends of the wires of the couple from one another just above the junctions, and from the bars in which they were inserted; this was done by filing down the wires, inserting a thin slip of silk between them, and then whipping them round with a fine silk thread. The outer surface of the thread charred slightly at the higher temperatures, but not sufficiently to destroy its efficiency.

In taking a series of observations of the distribution of temperature along the bar, the heating was allowed to go on for five or six hours, and then the deflections corresponding to each hole along the bar were observed, up and down the bar, time after time, until a set of perfectly constant and concordant readings were obtained. It was not always possible to attain this end; a change in the temperature of the room, or any cause tending to produce draught in the room, destroyed the constancy of the results, but by repeated observations under favourable circumstances, and by shielding the bar from draughts by means of the trough *TT*, it was found possible to obtain perfectly constant data, except in the case of the copper bar, for the holes nearest the heated end. It was found that the temperatures of these holes when above 180° C. to 200° C. varied continually, through a range having a maximum value of about 1° for the hole nearest the end, and quickly diminishing to zero at the third or fourth hole from the end. This fluctuation in temperature was probably due, partly to the want of massiveness of the bar and partly to the formation of a film of oxide on the surface of this part of the bar; but by keeping the temperature from going much above 200° C. and by taking the mean of ten or twelve successive readings the mean temperature of these holes was determined with certainty to within a quarter of a degree. A much greater error than this would have no effect on the results given below, which are for temperatures below 200° C.

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* This temperature was indicated by a thermometer placed with its bulb in a hole drilled in the bar close to the small hole in which the junction was placed. In no case was the variation of temperature more than a fifth of a degree. The bar was a short massive iron bar, not the one used for cooling experiments.

The rates of cooling of the bars at different temperatures were determined by heating a piece of each bar, about a foot long, in a sand-bath to a temperature higher than required, and observing its rate of cooling under the same conditions as those to which the experimental bar was exposed. For this purpose it was suspended in the position originally occupied by the long bar, the thermo-electric junction inserted in a hole drilled at its middle point, and the deflection of the galvanometer observed at regular intervals during the cooling. It was found that the observations of the rate of cooling were not trustworthy for the first few minutes after it had commenced, so that it was necessary to heat the bar to a temperature considerably higher than the highest at which the rate of cooling was required. With this precaution it was found that the data obtained for rates of cooling under similar conditions agreed well together—for example, the data given in columns *a* and *b* of Table I. 2 were obtained on different days but under similar conditions. Experiments showed that the presence of a thin film of oxide on the surface of the copper bar had no perceptible effect on its rate of cooling, for it was found that the film did not form unless the heating was continued for a long time, so that by varying the time of heating in the sand bath it was possible to observe the cooling with the surface in the different initial stages of oxidation, and the rate of cooling was found to be practically the same in all cases.

In reducing the results, the graphical method was largely used—the distribution of temperature along the bars and the cooling observations were plotted in the usual way, and from the curves so obtained differential curves were plotted showing, in the one case, the gradient of temperature at any point on the bars, and, in the other, the rate of cooling at any temperature. In the case of the curves showing the stationary state of the bars, the scale adopted was such that abscissæ represented actual distances along the bar, and the lengths of the ordinates in millimetres gave the corresponding temperatures in scale divisions. In the differential curves, however, showing the gradient of temperature at any point on the bar, the ordinates were drawn on ten times this scale, so that one millimetre represented about $\frac{1}{40}$th of a degree Centigrade.

The cooling curves were drawn on a somewhat larger scale in overlapping sections. The scale varied, according to the data to be plotted, from one in which 4 centims. represented one minute and 5 millims. one scale division (1 millim. to $\frac{1}{20}$th of a degree Centigrade) to one in which 1 centim. represented one minute and 4 millims. one scale division. Owing to the size of the scale, it was necessary to draw the curves in sections, and experience showed that the sections must overlap considerably, in order to secure continuity. The differential curves showing the rate of cooling at any temperature were drawn with abscissæ on the same scale as the observational curves, but the ordinates were in each case on a scale ten times as great. As it is impossible to reproduce these curves on the scale to which they were drawn, two specimen curves are given on a reduced scale in figs. I and II (p. 590). Fig. I shows the upper portion of the temperature curve for the copper bar, and fig. II gives the upper part of the cooling curve for the iron bar.

The experimental results obtained and the details of their reduction are set out in the Tables given below, which, taken in connection with the following general explanation of the notation adopted, indicate the successive steps of the reductions.

Let k denote the absolute thermal conductivity at any normal section of the bar at a distance x from the heated end, and where $d\theta/dx$ is the gradient of temperature. Then, if a denote the area of cross-section of the bar, $ka\,(d\theta/dx)$ denotes the heat which crosses the section considered in unit time. This heat is lost by radiation from the surface of the bar beyond this section, hence, if l denote the length of a short element in this part of the bar, θ its mean temperature, and $d\theta/dt$ its rate of cooling at this temperature, then the heat lost per unit of time from the surface of this element is $lasd\,(d\theta/dt)$, where s denotes the specific heat, and d the density of the material of the bar at the temperature θ. From this it follows that, if all the units involved be consistent, we have

$$ka\,(d\theta/dx) = \Sigma\, lasd\,(d\theta/dt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Introducing corrections for temperature this relation becomes

$$k\,(1 + 2\alpha\theta)\,(d\theta/dx) = d_0 \Sigma l_0 s\,(d\theta/dt) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where l_0, a_0, d_0 are the values of l, a, d at 0° C., θ the temperature of the cross section of the bar, and α the coefficient of linear expansion of the metal. The value of k for any cross-section can evidently be found directly from (2) by calculating the corresponding value of $\Sigma l_0 s\,(d\theta/dt)$, where s has its proper value for each element of the bar;* the reduction can however be more conveniently made by taking $\Sigma l_0 s\,(d\theta/dt)$ as equal to $\bar{s}\,\Sigma l_0\,(d\theta/dt)$, where $\bar{s}$ is the specific heat at a mean temperature $\bar{\theta}$, which is given by

$$\bar{\theta} = [\Sigma l_0\,(d\theta/dt)\,\theta]/[\Sigma l_0\,(d\theta/dt)].$$

This gives us

$$k\,(1 + 2\alpha\theta)\,(d\theta/dx) = d_0 \bar{s}\,\Sigma l_0\,(d\theta/dt),$$

or

$$k = \frac{d_0 \bar{s}}{1 + 2\alpha\theta}\,\frac{\Sigma l_0\,(d\theta/dt)}{(d\theta/dx)} = \mathrm{M}\,\frac{\Sigma l_0\,(d\theta/dt)}{(d\theta/dx)},$$

where M denotes the factor involving $\bar{s}$.

In the reductions given below it will be seen that the absolute conductivity, k, first calculated from the values of M, $d\theta/dx$, and $\Sigma\, l_0\, (d\theta/dt)$, and the diffusivity κ, afterwards calculated by dividing by the proper values of (sd), the thermal capacity of unit volume of the material of the bar.

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* [An error connected with this point, in the results first communicated to the Royal Society, w pointed out to me and has now been corrected.—R. W. S., 13.7.93.]

To obtain the values of s and d the densities of the iron and copper were carefully determined in the usual way, and, as the iron was not pure its specific heat at different temperatures could not be obtained from previous results ; it was therefore determined by BUNSEN'S calorimeter, as described below. For the specific heat of the copper the results given by BÈDE [*] were adopted.

It will be evident that if $d\theta/dx$ and $d\theta/dt$ are expressed in the same units, the absolute value of these units need not be known, hence, in what follows, temperatures are given in scale divisions (galvanometer deflections), and the absolute value of a scale division in degrees Centigrade has but little bearing on the results obtained.

IRON.

TABLE I. 1.—Table showing distribution of temperature along iron bar during steady state.

Number of hole.	Distance in centims. from fixed point at heated end. [x.]	Excess of temperature in scale divisions. [θ.]
1	1·00	539
2	3·00	458
3	5·55	375
4	8·55	283
5	12·30	221
6	20·00	130
7	27·65	81
8	35·35	50
9	45·60	26
10	55·90	14
11	66·20	7
12	81·55	4
13	96·95	2
(end of bar)	103·20	—

Temperature of air . . . . 83 scale divisions.

Between 0° C. and 100° C., 1° C. = 3·75 ,, ,,

,, 100° C. ,, 200° C., 1° C. = 3·74 ,, ,,

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[*] 'Mémoires couronnés de l'Acad. de Bruxelles,' vol. 27, p. 1.

TABLE I. 2.—Table giving Cooling Data for Short Iron Bar.

[Temperature of the air 21° C.]

Time (t).	Temperature in scale divisions (θ).		
	a.	b.	Mean.
m. s.			
0 0	482	483	482·5
0 15	474	476	475
0 30	468	470	469
0 45	464	461	462·5
1 0	457	453	455
1 15	450	446	448
1 30	443	440	441·5
1 45	435	433	434
2 0	428	428	428
2 15	421	422	421·5
2 30	414	415	414·5
2 45	407	407	407
3 0	400	400	400
3 15	395	394	394·5
3 30	389	390	389·5
3 45	384	385	384·5
4 0	377	378	377·5
4 15	373	372	372·5
4 30	368	366	367
4 45	362	361	361·5
5 0	357	356	356·5
5 15	353	350	351·5
5 30	347	346	346·5
5 45	341	340	340·5
6 0	335	334	334·5
6 15	330	329	329·5
6 30	325	325	325
6 45	321	320	320·5
7 0	315	315	315
7 15	311	310	310·5
7 30	308	306	307
7 45	304	301	302·5
8 0	298	297	297·5
8 15	293	294	293·5
8 30	288	290	289
8 45	284	285	284·5
9 0	281	. .	281
9 15	278	278	278
9 30	274	. .	274
9 45	270	269	269·5
10 0	. .	266	266
10 30	259	262	260·5
11 0	252	255	253·5
11 30	246	247	246·5
12 0	239	240	239·5
12 30	232	233	232·5
13 0	226	227	226·5
13 30	222	222	222

TABLE I. 2 (continued).

Time (t).	Temperature in scale divisions (θ).		
	a.	b.	Mean.
m. s.			
14 0	216	216	216
14 30	210	211	210·5
15 0	204	205	204·5
15 30	199	199	199
16 0	194	194	194
16 30	189	189	189
17 0	184	184	184
17 30	181	180	180·5
18 0	177	176	176·5
18 30	172	172	172
19 0	168	168	168
19 30	164		164
20 0	159		159
21 0	150		150
22 0	145		145
23 0	139		139
24 0	132		132
25 0	125		125
26 0	119		119
27 0	112	114	113
28 0	109		109
29 0	103	103	103
30 0	100		100
31 0	96		96
32 0	91		91
33 0	87		87
34 0	83		83
35 0	79		79
36 0	76		76
37 0	73		73
38 0	70		70
39 0	67		67
40 0	64		64
42 0	..	59	59
43 0	57	57	57
46 0	49		49
49 0	44		44
52 0	40		40
55 0	35		35
60 0	29		29
65 0	26		26
70 0	21		21
73 0	18		18
84 0	12		12

TABLE I. 3.—Table giving Loss of Heat from Surface of Iron Bar.

[For Notation, see p. 574.]

l_0.	θ (scale divisions).	$d\theta/dt$ (scale divisions per minute).	$l_0\,(d\theta/dt)$.	$\Sigma\,[l_0\,(d\theta/dt)]$.
centims.				
19·2	2·5	0·09	1·73	1·73
10	4	·15	1·50	3·23
10	6	·23	2·30	5·53
5	10	·39	1·95	7·48
5	14	·55	2·75	10·23
5	18	·63	3·15	13·38
5	25	·90	4·50	17·88
2	30	1·10	2·20	20·08
2	35	1·35	2·70	22·78
2	40	1·55	3·10	25·88
2	46	1·85	3·70	29·58
2	52	2·15	4·30	33·88
2	58	2·40	4·80	38·68
2	66	2·80	5·60	44·28
2	74	3·20	6·40	50·68
2	84	3·70	7·40	58·08
2	95	4·25	8·50	66·58
1	103	4·65	4·65	71·23
1	110	5·05	5·05	76·28
1	118	5·5	5·50	81·78
1	126	5·9	5·90	87·68
1	136	6·5	6·50	94·18
1	144	6·9	6·90	101·08
1	154	7·5	7·50	108·58
1	166	8·3	8·30	116·88
1	178	9·0	9·00	125·88
1	190	9·7	9·70	135·58
1	204	10·6	10·60	146·18
1	218	11·5	11·50	157·68
1	232	12·4	12·40	170·08
1	248	13·4	13·40	183·48
1	266	14·6	14·60	198·08
1	286	15·9	15·90	213·98
0·5	298	16·7	8·35	222·33
·5	310	17·5	8·75	231·08
·5	320	18·3	9·15	240·23
·5	332	19·1	9·55	249·78
·5	344	20·0	10·00	259·78
·5	356	20·9	10·45	270·23
·5	370	22·0	11·00	281·23
·5	385	23·0	11·50	292·73
·5	400	24·2	12·10	304·83
·5	415	25·4	12·70	317·53
·5	432	26·8	13·40	330·93
·5	450	28·2	14·10	345·03
·5	468	29·7	14·85	359·88
2·5	..	..	..	..

TABLE I. 4.

Table showing values of k, the absolute thermal conductivity of iron, at temperatures between 60° C. and 150° C.

T.	x.	$d\theta/dx$ (per centim.).	$\Sigma\,[l_0\,(d\theta/dt)]$. (per minute).	$\bar{\theta}$.	M.	k [C.G.S. units].
°C.	centims.			°C.		
60	19·90	9·0	93·0	41	0·856	[·147]
80	13·70	14·2	146·7	51	·862	[·148]
100	9·20	20·1	206·6	63	·868	·148
110	7·50	23·4	235·3	68	·871	·146
120	5·90	26·9	267·0	73	·873	·144
130	4·60	30·6	297·6	79	·877	·141
140	3·45	34·5	325·7	84	·880	·138
150	2·50	38·5	352·7	88	·882	·135

From these values (between 100° C. and 150° C.), if we assume the conductivity at t° C. to be given by

$$k_t = a - bt,$$

we have, on reducing by method of least squares,

$$k_t = \cdot175 - \cdot00026t,$$

or

$$k_t = \cdot175\,(1 - \cdot0015t).$$

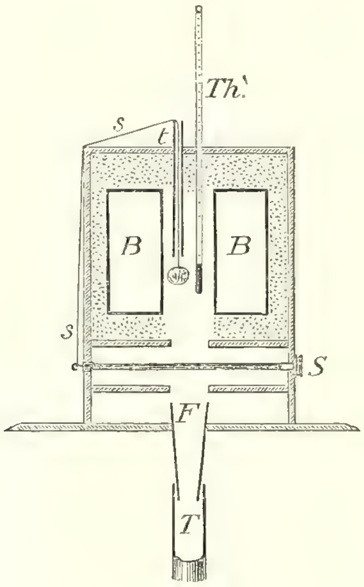
Determination of Specific Heat of Iron.

Advantage was taken of the snowfall in January to use BUNSEN's calorimeter for this determination. The calorimeter was prepared for use by the method described by BUNSEN in his paper given in the 'Phil. Mag.,' 1871, p. 161, and all the precautions there detailed were observed. The tube used for determining the change of volume attendant on the melting of ice was of very fine bore, but was perfectly clean, so that there was no trouble with the sticking of the mercury thread. This tube was carefully calibrated, and a calibration curve drawn from which a length measured in any part of the tube could be converted into a length corresponding to the uniform bore taken as the basis of the calibration.

The heater used was made specially for the work. It consisted of a hollow cylindrical oil bath shown in section at BB (fig. 2). This bath was heated up to a temperature of about 250° C., and then quickly placed in the wooden case shown in the figure and packed round on all sides with cotton wool. Thus packed, the bath took nearly a day to cool, and could, therefore, be used for a complete series of determinations at temperatures below 200° C.

The piece of iron to be heated was suspended by a silk fibre in the cylindrical space in the centre of the heater in close proximity to the bulb of the thermometer, *Th.*, which passes through a cork in the lid of the case. The silk thread passed through the glass tube, *t*, and was attached at the other end to a small hook in the end of the slide, *S*. The tube, *t*, was just long enough to be in contact with the piece of iron, so that when the slide, *S*, was withdrawn quickly the thread broke and the iron was free to fall through the holes in the double bottom of the case, between which the slide, previous to withdrawal, acted as a screen. When it was desired to make a determination the heater was lifted on to the lid of the calorimeter case and placed in a marked position, so that the holes in the bottom were directly over the mouth of a paper funnel, *F*, acting as a guide to the calorimeter tube, *T*. When placed in position, the thermometer was watched and the slide, *S*, withdrawn, when the required temperature was indicated. The iron at once fell straight into the calorimeter tube and the heater was then removed, the mouth of the funnel, F, closed with cotton wool, and the observation of the mercury-thread in the indicating-tube proceeded with. In the meanwhile, a second piece of iron was suspended in the heater so as to be ready for another determination. The cooling of the heater was so slow that there was no doubt about the temperature of the iron when dropped into the calorimeter, and the time of fall into the tube, T, was too small to admit of any appreciable cooling. Experiment also showed that placing the heater on the lid of the calorimeter case, even for a long time, had no effect on the calorimeter.

Fig. 2.



In the reduction of the results obtained the latent heat of water and the specific heat of ice were assumed to be correct as given by BUNSEN.

The rapid melting of the snow and some accidents with the calorimeter limited the time of working and prevented any extension of the results given.

The principal data of two sets of experiments are given in the following Table.

I.

TABLE showing the Data for Determination of Bore of Tube.

Temperature.	Weight of mercury.	Length of thread. [Corrected by calibration curve.]	Volume of 1 centim. of bore.
° C.	grms.	centims.	
13	0·9092	99·80	·000672
13	0·8700	95·15	·000674
			·000673 (mean)

II.

TABLE giving Data for Specific Heat Determinations.

Mass of iron.	Temperature of iron (t).	Deflection of mercury thread.	Heat required to raise unit mass of iron from 0° C. to t° C. [h_t.]
grms.	° C.	centims.	
2·1025	41	16·0	4·5172
2·1025	55	21·7	6·1264
2·1025	98	39·3	11·0950
2·1025	102	40·6	11·4360
1·3440	150	39·7	17·5340
1·3440	170	44·7	19·7420

Assuming

$$h_t = at + bt^2,$$

and reducing by method of least squares, we have

$$h_t = \cdot 1095t + \cdot 0000401t^2,$$

and, therefore,

$$dh_t/dt = s_t = a + 2bt = \cdot 1095 + \cdot 00008t,$$

or

$$s_t = \cdot 1095\,(1 + \cdot 00073t)$$

where s_t denotes the true specific heat at t° C.

Determination of Density of Iron, and the Value of (sd) the Thermal Capacity of Unit Volume.

The density of the iron was found to be 7·556 at 0° C. For present purposes this may be taken as 7·56, and the density at t° C. will be given by

$$7\cdot 56\,(1 - \cdot 000038t).$$

Hence the value of (sd) at $t°$ C. is given by

$$7{\cdot}56\,(1 - {\cdot}000038t) \times {\cdot}1095\,(1 + {\cdot}00073t) = {\cdot}828\,(1 + {\cdot}000692t).$$

Value of κ, the Diffusivity of Iron, in Absolute Units.

From the results obtained above the value of κ at $t°$ C. will be given by

$$\kappa_t = (k/sd)_t = {\cdot}175\,(1 - {\cdot}0015t) \div {\cdot}828\,(1 + {\cdot}000692t).$$

That is,

$$\kappa_t = {\cdot}211\,(1 - {\cdot}0015t)\,(1 - {\cdot}000692t),$$

or,

$$\kappa_t = {\cdot}211\,(1 - {\cdot}00219t + {\cdot}00000105t^2).$$

This result is probably represented with sufficient accuracy by

$$\kappa_t = {\cdot}211\,(1 - {\cdot}0022t).$$

Copper.

Table C. 1.—Table showing Distribution of Temperature along Copper Bar for Two Independent Determinations, I. and II.

Number of hole.	Distance from first hole.	Excess of temperature (θ) (in scale divisions).	
		I.	II.
	centims.		
1	·0	764	889
2	2·65	697	807
3	5·35	637	735
4	8·00	584	671
5	10·67	537	615
6	13·28	493	565
7	15·81	454	520
8	19·63	405	463
9	23·40	362	413
10	27·20	326	369
11	31·06	293	329
12	36·20	254	284
13	41·30	219	245
14	46·41	189	212
15	51·55	166	184
16	56·70	144	159
17	61·80	124	139
18	69·48	102	115
19	77·07	85	95
20	84·70	70	79
21	92·34	58	66
22	102·47	45	52
23	112·59	35·5	41
24	122·73	28	33
25	132·84	22	27
26	142·89	17·5	22
27	152·95	14	18
28	168·14	10	14
29	183·29	8	12
30	198·49	7	10·5

Temperature of room, I., 15° C.; II., 17° C.
From calibration of couple before experiments,

1° C., between 0° C. and 100° C. = 4·24 scale divisions.

1° C., between 100° C. and 200° C. = 4·25 scale divisions.

From calibration of couple after experiments,

1° C., between 0° C. and 200° C. = 4·25 scale divisions.

In determinations I. the end of the copper bar was inserted in a socket in a thick piece of copper standing in the melted lead, which was kept at a constant temperature of 345° C.

For II. the whole apparatus was taken down and the end of the copper bar bent round so that it could be directly inserted in the melted lead in the way shown in fig. 1. By a coincidence the constant temperature of the lead during this determination was 345° C. as in I.

TABLE C. 2.—Table giving Cooling Data for Short Copper Bar.

[Temperature of the room constant at 15° C.]

Time (t).			Temperature (θ) (in scale divisions).	Time (t).			Temperature (θ) (in scale divisions).	Time (t.)			Temperature (θ) (in scale divisions).
h.	m.	s.		h.	m.	s.		h.	m.	s.	
7	34	0	829	7	41	30	473	7	51	0	247
		15	814			45	464			30	239
		30	799	7	42	0	455	7	52	0	232
		45	785			15	446	7	53	0	219
7	35	0	771			30	439	7	54	0	205
		15	756			45	432	7	55	0	193
		30	742	7	43	0	424	7	56	0	181
		45	729			15	415	7	57	0	170
7	36	0	715			30	409	7	58	0	161
		15	702			45	403	7	59	0	152
		30	688	7	44	0	397	8	0	0	144
		45	675			15	389	8	1	0	137
7	37	0	663			30	382	8	2	0	129
		15	650			45	376	8	4	0	114
		30	638	7	45	0	369	8	6	0	102
		45	627			15	364	8	8	0	92
7	38	0	614			30	357	8	10	0	83
		15	602			45	351	8	12	0	75
		30	589	7	46	0	345	8	14	0	67
		45	579			15	339	8	17	0	57
7	39	0	568			30	333	8	20	0	49
		15	557			45	328	8	23	0	43
		30	547	7	47	0	323	8	26	0	38
		45	536			30	313	8	29	0	33
7	40	0	527	7	48	0	302	8	34	0	27
		15	515			30	292	8	39	0	22
		30	507	7	49	0	282	8	49	0	15
		45	497			30	273	9	9	0	7
7	41	0	489	7	50	0	265	9	29	0	3
		15	480			30	256				

TABLE C. 3.—Table showing Loss of Heat from Surface of Copper Bar for Determinations I. and II.

$[l_0]$.	$[\theta]$.		$[d\theta/dt]$.		$[l_0\,(d\theta/dt)]$.		$\Sigma\,[l_0\,(d\theta/dt)]$.	
Centims.	I.	II.	I.	II.	I.	II.	I.	II.
10	7	10·4	0·25	0·40	2·5	4·0	2·5	4·0
10	7·5	11·5	·28	·45	2·8	4·5	5·3	8·5
10	8·5	13·0	·30	·56	3·0	5·6	8·3	14·1
10	10·5	15·0	·40	·68	4·0	6·8	12·3	20·9
10	13·2	17·5	·50	·75	5·0	7·5	17·3	28·4
10	16·5	21·0	·70	·85	7·0	8·5	24·3	36·9
10	20·7	25·5	·85	1·00	8·5	10·0	32·8	46·9
10	26·2	31·0	1·05	1·30	10·5	13·0	43·3	59·9
10	33·6	39·0	1·41	1·75	14·1	17·5	57·4	77·4
10	42·5	49·0	1·94	2·35	19·4	23·5	76·8	100·9
5	50·5	58·5	2·37	2·88	11·85	14·4	88·65	115·3
5	57·3	65·5	2·78	3·30	13·90	16·5	102·55	131·8
5	65·0	73·8	3·21	3·82	16·05	19·1	118·60	150·9
5	73·5	83·0	3·75	4·35	18·75	21·75	137·35	172·65
5	83·5	94·0	4·38	5·05	21·90	25·25	159·25	197·90
5	94·5	105·5	5·05	5·85	25·25	29·25	184·50	227·15
5	107·3	119·2	5·88	6·85	29·40	34·25	213·90	261·40
5	122·0	136·0	6·90	7·85	34·50	39·25	248·40	300·65
5	140·0	155·3	8·00	9·25	40·00	46·25	288·40	346·90
5	160·0	178·5	9·38	10·75	46·90	53·75	335·30	400·65
2	176·5	197·0	10·62	12·20	21·24	24·4	356·54	425·05
2	187	208·0	11·45	13·00	22·90	26·0	379·44	451·05
2	197	220·5	12·10	13·90	24·20	27·8	403·64	478·85
2	209	233	13·00	14·90	26·00	29·8	429·64	508·65
2	221	247	13·95	16·00	27·90	32·0	457·54	540·65
2	234	261	14·90	17·10	29·80	34·2	487·34	574·85
2	248	277	16·00	18·2	32·00	36·4	519·34	611·25
2	262	293	17·04	19·4	34·08	38·8	553·42	650·05
2	277	311	18·20	20·7	36·4	41·4	589·82	691·45
2	293	329	19·40	22·1	38·8	44·2	628·62	735·65
2	309	349	20·70	23·6	41·4	47·2	670·02	782·85
2	328	370	22·10	25·4	44·2	50·8	714·22	833·65
2	347	393	23·50	27·3	47·0	54·6	761·22	888·25
2	367	418	25·05	29·5	50·1	59·0	811·32	947·25
2	389	444	26·9	31·9	53·8	63·8	865·12	1011·05
2	413	472	29·0	34·5	58·0	69·0	923·12	1080·05
2	439	502	31·3	37·1	62·6	74·2	985·72	1154·25
2	467	534	33·8	39·9	67·6	79·8	1053·32	1234·05
2	498	569	36·6	42·9	73·2	85·8	1126·52	1319·85
2	530	608	39·3	46·1	78·6	92·2	1205·12	1412·05
1	557	638	41·5	48·6	41·5	48·6	1246·62	1460·65
1	575	660	43·0	50·2	43·0	50·2	1289·62	1510·85
1	594	682	44·5	52·1	44·5	52·1	1334·12	1562·95
1	614	706	46·1	54·0	46·1	54·0	1380·22	1616·95
1	634	731	47·6	55·9	47·6	55·9	1427·82	1672·85
1	655	757	49·2	57·9	49·2	57·9	1477·02	1730·75
1	677	784	50·8	60·0	50·8	60·0	1527·82	1790·75
1	700	812	52·6	62·2	52·6	62·2	1580·42	1852·95
1	725	842	54·4	64·5	54·4	64·5	1634·82	1917·45
1	751	873	56·3	67·0	56·3	67·0	1691·12	1984·45

The values given in the last column of this Table are subject to a correction (a reduction of about 2 per cent.), due to the fact that, in the case of the short bar used in determining the rates of cooling, the cooling surface includes the terminal faces of the bar.

Table C. 4.—Table showing the values of k, the absolute thermal conductivity of Copper, at temperatures between 40° C. and 200° C.

I.

T.	x.	$d\theta/dx$ (per centim.).	$\Sigma\,[l_0\,(d\theta/dt)]$ (per minute).	$\overline{\theta}$.	M.	k (C.G.S. units).
° C.	centims.			°C.		
40	68	2·70	192·50	28	0·816	[·956]
60	46·2	5·20	369·74	39	·821	[·973]
80	33·1	7·65	558·66	50	·828	1·007
100	23·5	10·55	758·50	59	·833	·999
120	16·5	13·60	950·70	71	·840	·979
140	11·0	16·80	1142·50	84	·846	·958
160	6·4	20·20	1334·50	93	·851	·937
180	2·5	23·90	1523·70	103	·856	·909

II.

T.	x.	$d\theta/dx$ (per centim.).	$\Sigma\,[l_0\,(d\theta/dt)]$ (per minute).	$\overline{\theta}$.	M.	k (C.G.S. units).
° C.	centims.			°C.		
40	75·50	2·40	191·5	29	0·816	1·085
60	51·65	5·00	375·2	40	·822	[1·027]
80	38·10	7·60	561·9	49	·826	[1·018]
100	28·60	10·30	753·6	60	·833	1·015
120	21·45	13·25	945·8	69	·838	·995
140	15·65	16·50	1145·7	80	·844	·976
160	11·00	19·93	1339·1	89	·850	·953
180	7·00	23·53	1532·3	100	·854	·927
200	3·70	27·20	1714·5	109	·860	·903

Note 1.—On comparing the results here tabulated, it will be noticed that the data of columns 3 and 4 are very similar in I. and II.; but reference to column 2 shows that the data of column 4 extend over a different length of the bar in the two cases.

Note 2.—In I. we find the same falling-off in the value of k at temperatures below 100° C. as occurs in the case of the iron bar.

Reducing the results given in these tables by the method of least squares, and assuming that the conductivity at t° C. is given by

$$k_t = a - bt,$$

we have

$$\text{I.}\quad k_t = 1{\cdot}113 - {\cdot}00112t = 1{\cdot}113\,(1 - {\cdot}001t)$$

$$\text{II.}\quad k_t = 1{\cdot}130 - {\cdot}00112t = 1{\cdot}130\,(1 - {\cdot}001t).$$

The mean value of these results gives

$$\mathbf{k_t = 1{\cdot}12\,(1 - {\cdot}001t).}$$

Determination of Density and Value of (sd), the Thermal Capacity of Unit Volume of Copper.

The density of the copper was found to be 8·907 at 0° C. Taking this as 8·9, we may take the density of copper at t° C., as given by $8{\cdot}9\,(1 - {\cdot}000056t)$.

The specific heat of copper at t° C. is given by BÈDE* as $\cdot0892 + {\cdot}000065t$. This result may be expressed as $\cdot0892\,(1 + {\cdot}00073t)$, hence the value of (sd) at t° C. is

$$8{\cdot}9\,(1 - {\cdot}000056t) \times {\cdot}0892\,(1 + {\cdot}00073t),$$

or

$$\cdot794\,(1 + {\cdot}000674t).$$

Value of κ, the Diffusivity of Copper, in Absolute C.G.S. Units.

From the results obtained above the value of κ at t° C will be given by

$$\kappa_t = (\kappa/sd)_t = 1{\cdot}12\,(1 - {\cdot}001t) \div {\cdot}794\,(1 + {\cdot}000674t).$$

That is,

$$\kappa_t = 1{\cdot}41\,(1 - {\cdot}001t)\,(1 - {\cdot}000674t),$$

or,

$$\kappa_t = 1{\cdot}41\,(1 - {\cdot}0017t + {\cdot}0000007t^2).$$

This result is probably represented with sufficient accuracy by the formula

$$\boldsymbol{\kappa_t = 1{\cdot}41\,(1 - {\cdot}0017t).}$$

It will be seen that the results of the experiments here described go to show that for both iron and copper the conductivity *decreases* with rise of temperature.

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* [Recent determinations of the specific heat of pure copper (probably identical with that used in these experiments) give the specific heat at 100° C. as ·0928. BÈDE's formula gives ·0957, a value about 3 per cent. too high. The determinations here referred to were made by J. JOLY, Esq, F.R.S., with his steam calorimeter, and were kindly communicated to me by MR. JAMES H. GRAY, M.A., B.Sc., Glasgow University.—12.4.93.]

For convenience of comparison these results are given below, followed by a Table showing previous results.

Values obtained for Absolute Thermal Conductivities of Iron and Copper from Experiments detailed in this Paper.

[C.G.S. units.]

IRON . . $k_t = 0·175\ (1 - ·0015t)$.

COPPER . $k_t = 1·12\ (1 - ·001t)$.

TABLE showing the results of previous Determinations of the Absolute Thermal Conductivity of Iron and Copper.

[C.G.S. units.]

Iron.	Copper.	Authority.
$0·209\ (1-·00147t)$	..	FORBES
·164	1·108	NEUMANN
$0·199\ (1-·002874t)$	$\left\{\begin{array}{l} 1·027\ (1-·00214t) \\ ·983\ (1-·00152t) \end{array}\right\}$	ÅNGSTRÖM
$0·197\ (1-·00002t)$	$\left\{\begin{array}{l} 1·08\ (1+·0013t) \\ ·71\ (1+·0014t) \end{array}\right\}$	TAIT
$0·17\ (1-·00002t)$	$0·72\ (1+·00004t)$	LORENZ
$0·17\ (1-·002t)$	$0·51\ (1+·0057t)$	KIRCHHOFF and HANSEMANN

The Emissive Powers of the Surfaces of the Iron and Copper Bars.

From the data obtained in the cooling experiments involved in the conductivity research, it is evidently possible to calculate the emissivities of the surfaces of the bars at different temperatures.

As it may be of interest to place these on record, they are given in the following table.

The iron bar was a $\frac{3}{4}$-inch square bar, and its surface was blackened and lightly polished with blacklead.

The copper bar was a round bar of $\frac{1}{2}$-inch diameter, and its surface was smooth and polished.

TABLE showing the emissive powers of the surfaces of the Iron and Copper Bars.

The emissivities are given in gramme degrees of heat lost per square centim. of surface for each degree Centigrade difference of temperature between the surface and the surrounding air. The results are corrected for variation of specific heat with temperature.

Difference of Temperature.	Emissive Power.	
	Blackened surface of Iron Bar.	Polished surface of Copper Bar.
°C.		
20	0·000275	0·000220
40	313	253
60	348	280
80	378	297
100	405	315
120	429	333
140	454	346
160	476	360
180	..	374

On comparison of these results with those published by Mr. D. MACFARLANE[*] and by Professor TAIT,[†] for somewhat similar surfaces, it will be found that the agreement is fairly close.

The experience gathered in the conduct of the experiments described above goes to show that in order to obtain a thoroughly satisfactory determination of thermal conductivity and its variation with temperature, the bars employed should be somewhat massive—not less than an inch in diameter—and their surface should be plated (to prevent oxidation) and well polished. Further, to avoid the use of very long bars, and to diminish as far as possible the importance of the loss of heat from the surface of the bar as an element in the determination, ÅNGSTRÖM's method should be adopted in preference to FORBES'.

It is noteworthy that with FORBES' method the influence of experimental errors seems to cause an apparent decrease of conductivity with fall of temperature. A trace of this effect is seen in the tables giving the values of κ for iron and copper, but the reduction of an unsatisfactory set of observations made with the copper bar before the use of the trough TT (fig. 1) gave results which were not only much too low, but which decreased rapidly with fall of temperature.

In conclusion the author has to express his thanks to Professor GRAY for constant help and advice during the progress of the work.

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[*] 'Proc. Roy. Soc.,' 1872, p. 93, or EVERETT's 'Physical Constants,' p. 107.

[†] 'Proc. Roy. Soc. Edinb.,' 1869–70, p. 207.

Fig. I.

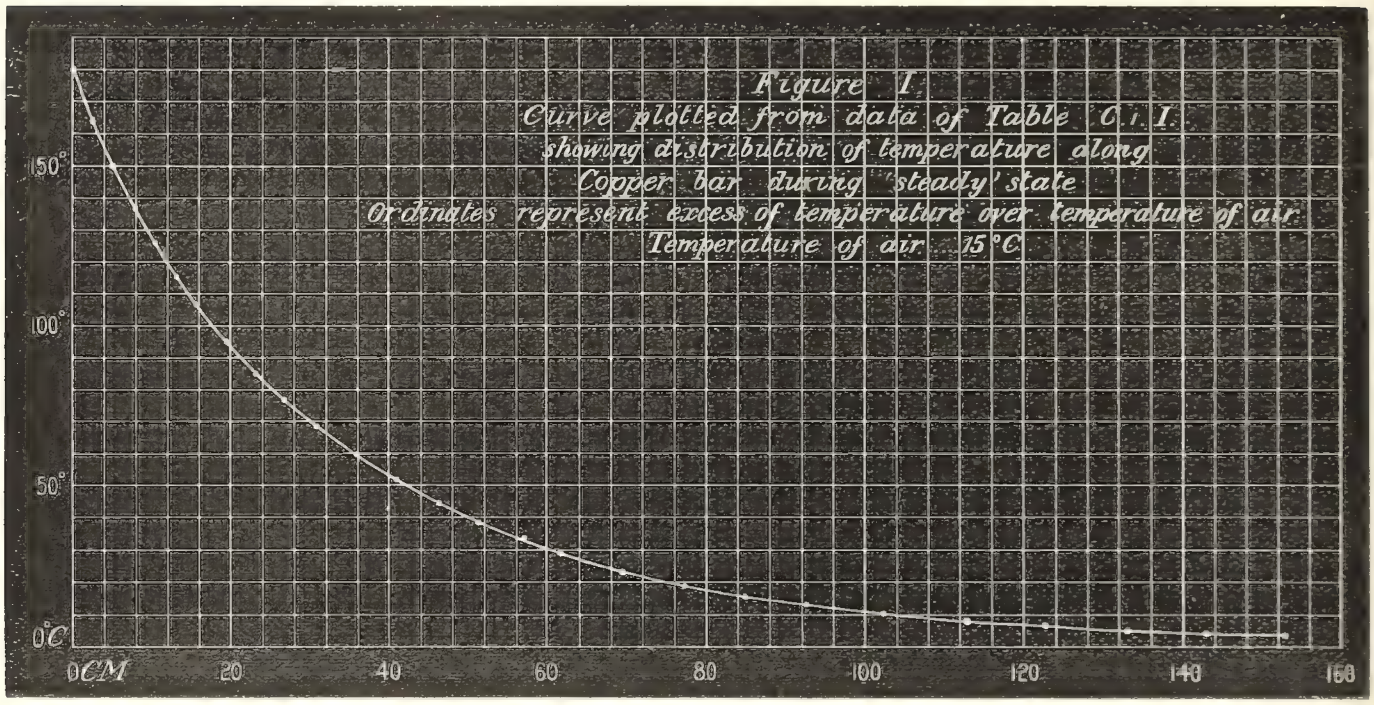
Figure I.
Curve plotted from data of Table C.1.I.
showing distribution of temperature along
Copper bar during 'steady' state
Ordinates represent excess of temperature over temperature of air.
Temperature of air 15°C.
150°
100°
50°
0°C
0 CM
20
40
60
80
100
120
140
160

Fig. II.

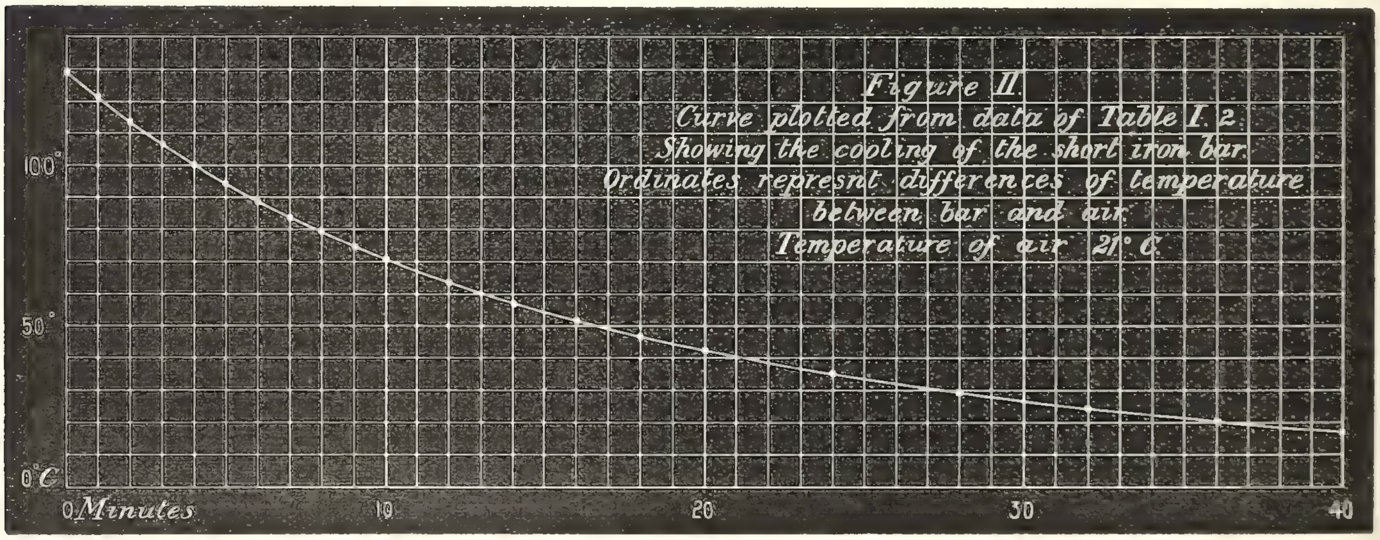
Figure II.
Curve plotted from data of Table I. 2
Showing the cooling of the short iron bar.
Ordinates represnt differences of temperature
between bar and air.
Temperature of air 21° C.
100°
50°
0°C
0 Minutes
10
20
30
40

XI. *On Thermal Radiation in Absolute Measure.**

By J. T. Bottomley, *M.A., D.Sc., F.R.S.*

Received and Read June 16, 1892.

[Plates 17, 18.]

In 1884† and in 1887‡ I had the honour of submitting to the Royal Society the results of certain experimental determinations of thermal emission in absolute measure. In these experiments metallic wires heated by an electric current were used. Much remains to be done by the methods and with the apparatus then employed; and I have made some progress, particularly in the determination of radiation from incandescent filaments of carbon. The loss of heat from a heated body, however, depends to some extent on the form and dimensions of the body, and it seemed very important to experiment on the emission of heat from bodies differing in form from the wires I had been using, and larger in dimensions. I desire now to give an account of some experiments in this direction.*

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* [Since this paper was written, and since it was accepted by the Royal Society, a paper has appeared in the 'Philosophical Transactions,' 1893, A., vol. 183, by Professor W. E. Ayrton, F.R.S., and Mr. H. Kilgour, "On Thermal Emissivity of Thin Wires." In their preliminary remarks the authors, referring to my previous papers (among others), say: "But it was not part of these investigations to determine the change in the emissivity that is produced by change in the shape and size of the cooling body."

I cannot avoid pointing out that this remark is wholly erroneous. My work on thermal emission, which was commenced on November 9, 1883, had altogether for its object, in the first instance, the finding of the effect of dimensions and form of the cooling body on the thermal emission; and the question of dimensions has never been lost sight of by me, and has, I think, been referred to in every paper I have published on this subject.

In the 'Electrician' for April 19, 1884, there appeared a short account of my experiments, which were commenced as an experimental counterpart to a paper on "The Efficiency of Clothing," read by Lord Kelvin to the Royal Society of Edinburgh, on March 3, 1883. I found, however, that Professor G. Forbes had already (British Association, 1882, and 'Electrician,' September, 1882) given important conclusions, derived theoretically, as to the effect of dimensions on the temperature of wires carrying the electric current. The theoretical conclusions of Professor G. Forbes were completely borne out by the results of my experiments; and those now given by Professor Ayrton and Mr. Kilgour agree.

On June 17, 1884, I had the honour of reading a paper, which was wholly occupied with the effect of dimensions, before the Royal Society ('Proc. Roy. Soc.,' No. 232, 1884). In the autumn of the same year I contributed a paper to the British Association on the same subject. These papers (of which the latter was published in 'Nature,' September, 1884) contain a practical experimental investigation of the loss of heat from thin bare wires in air, so far as dimensions are concerned, as well as experimental results as to the effect of coverings of various kinds and different thicknesses.

I have still a large number of results on these subjects hitherto unpublished; and as I have received, from the £4000 Scientific Fund, grants in aid of these researches, I am unwilling that there should be any misunderstanding in this matter.—March 2, 1893.]

† 'Proceedings Royal Society, No. 232, 1884.

‡ 'Phil. Trans.,' 1887.

In the autumn of 1889 I commenced to experiment on loss of heat from the copper globes used in the experiments of Mr. D. MACFARLANE, 'Roy. Soc. Proc.,' 1872, p. 93. These globes are two exactly equal solid copper spheres, 4 centims. in diameter; one has a smooth copper surface, the other is now thoroughly well electroplated with silver. A considerable number of preliminary experiments were tried, using apparatus similar to that which MACFARLANE employed, and in particular the same enclosure; and then, as my object was to experiment first with full air pressure and afterwards with vacuum more or less complete up to the highest attainable, new and suitable arrangements were made for this purpose.

It is unnecessary here to describe in detail the apparatus used and the mode of experimenting in these preliminary investigations.† It was similar in every respect to that of MACFARLANE. The thermo-junctions were new, however, and were standardized for the purpose of these experiments; the galvanometer arrangements were also fresh and were very carefully made to secure sensitiveness and permanence of the thermometric value of the deflections. The globes had their surfaces prepared in the ways shown in the columns of the table on p. 593; and the metallic enclosure used by MACFARLANE was employed. The headings of the table will, it is hoped, make it self-explanatory.

---

* The experiments described in what follows were concluded some time ago. I have delayed the giving an account of them in the hopes of finding time and opportunity for obtaining further results; but I have been unable to do so up to the present, and judge it best to publish now the results which I have obtained.

† In connection with these preliminary experiments I wish to acknowledge very cordially the great assistance I received from Mr. JAMES H. GRAY, M.A., B.Sc., at the time of the experiments a young student in the Laboratory, now the holder of the first 1851 Exhibition Scholarship for Glasgow University. To his care and assiduity I am greatly indebted.

Table I.—Showing Loss of Heat per square centim., per second, per 1° C. of Excess of Temperature of Cooling Globe above Temperature of Surrounding Enclosure; Macfarlane's Enclosure being used, and the Surface of Globe being prepared in various ways as shown in the Headings of the Columns.

Copper globe polished bright. (Mean of six experiments.) Temperature of enclosure approximately 14°·5 C.		Copper globe thinly coated with lamp-black. (Mean of six experiments.) Temperature of enclosure approximately 14°·5 C.		Copper globe polished bright and thinly lacquered. (Mean of six experiments.) Temperature of enclosure approximately 14°·5 C.		Copper globe plated with silver. (Mean of six experiments.) Temperature of enclosure approximately 14°·5 C.	
Temperature of globe.*	Loss of heat per square centim., per second, per 1° C. of excess.	Temperature of globe.	Loss of heat per square centim., per second, per 1° C. of excess.	Temperature of globe.	Loss of heat per square centim., per second, per 1° C. of excess.	Temperature of globe.	Loss of heat per square centim., per second, per 1° C. of excess.
°		°		°		°	
		21	·000278	21	·000246		
22	·000165	22	·000281	22	·000250		
23	·000170	23	·000284	23	·000254	23	·000174
24	·000174	24	·000287	24	·000257	24	·000178
25	·000178	25	·000290	25	·000260	25	·000182
26	·000181	26	·000293	26	·000263	26	·000185
27	·000184	27	·000296	27	·000265	27	·0001875
28	·000187	28	·000299	28	·000268	28	·000190
29	·000190	29	·0003015	29	·000271	29	·000192
30	·000192	30	·0003045	30	·000273	30	·000194
31	·000194	31	·000307	31	·000276	31	·000196
32	·000196	32	·000310	32	·000278	32	·000198
33	·000198	33	·000313	33	·000280	33	·000200
34	·0001995	34	·000316	34	·0002825	34	·000201
35	·000201	35	·000320	35	·000285	35	·000203
36	·000202	36	·000323	36	·000287	36	·000205
37	·0002035	37	·000326	37	·000290	37	·0002065
38	·000205	38	·000329	38	·000292	38	·000208
39	·0002065	39	·000332	39	·000294	39	·0002095
40	·000207	40	·000335	40	·000297	40	·000211
41	·000208	41	·000338	41	·000299	41	·0002125
		42	·000341	42	·0003015	42	·000214
				43	·000304	43	·000215
				44	·000307	44	·000216
				45	·000309	45	·000218

* N.B.—It is the temperature of the cooling body which is stated in this column, not the excess temperature.

In two cases, the highest polished copper and the sooted globe, the surfaces are similar to those used by MACFARLANE. In the other two cases, "copper polished and finely lacquered" and "copper silvered," the surfaces are new. It is interesting to note that the results for lamp-black differ very appreciably from those obtained by MACFARLANE. All my experiments show that there is difference in radiation from different kinds of lamp-black; though I doubt, considering the way in which "lamp-black radiation" is often referred to, whether the ideas commonly held quite correspond with this view.

On the other hand, the column showing loss of heat from the "copper ball polished bright," agrees, with almost perfect exactness, with the numbers obtained by MACFARLANE under the same circumstances. This confirms, in a very important way, both sets of results. It is quite natural that if both are accurate the two should agree perfectly, because to obtain a well-polished surface of copper again and again in much the same condition is comparatively easy. It was found very difficult, however, to keep the polished surface of copper unoxidized, even for a few hours. Hence, I tried a thinly-lacquered copper surface. The lacquer, of course, alters the condition entirely from that in which the surface is highly polished and unlacquered, but the condition was permanent, and, indeed, it is only in this condition that bright copper is ever met with in the arts or in the laboratory. It might be interesting to try a dull surface of copper lacquered, just because it is the most common of all. The silvered surface was the first I had tried of the kind. It was bright; but no special pains were taken (such as I afterwards took) to get the very highest possible polish. It is worth observing that the bright fresh copper shows a slightly lower rate of loss of heat, probably lower radiation, than the silver-coated globe, while the increase with temperature of the rate of loss is almost the same in these two cases, but very different in the other cases.

In all these cases the air was at the natural pressure of the day. It was not artificially moistened, as was the case in MACFARLANE'S experiments, nor artificially dried, as in the case of my own experiments which followed.

The experiments of MACFARLANE, and those just referred to, were made with the copper globes suspended in an enclosure of large dimensions, and at ordinary atmospheric pressure. To make use of a Sprengel vacuum, however, and to compare results obtained with vacuum of various degrees of pressure and at full air pressure, it is necessary, according to the theory of CROOKES' results, to have an enclosure of moderate dimensions, so that the condition may be reached in which the length of free path of the gas molecules becomes comparable with distances, from point to point, of portions of the enclosure.

For the reasons stated in my former paper,* I also considered it essential that the enclosure should be metallic. I considered whether a glass globe lined inside with metal, by first silvering and subsequently depositing copper on the silver, would be satisfactory; but finally determined, in spite of the difficulty of joining a metal enclosure to the Sprengel pump, to use a hollow copper sphere.

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* 'Phil. Trans.,' 1887, p. 444. Uncertainty, with a substance so badly conducting and so little diathermanous as glass, as to the temperature of the inner skin of such an enclosure.

At the centre of the hollow sphere, the heated globe is suspended and allowed to cool, the enclosure being kept immersed in water at a known temperature. The temperature of the cooling globe is read off at equal intervals of time by means of a thermo-electric junction, and from these readings the rate of cooling, and the absolute loss of heat per unit of cooling surface, per unit difference of temperatures of surface and surroundings, per unit of time, are calculated.

The details of the arrangement will be understood from the following description, by reference to figs. 1 and 2, Plate 17. In fig. 1, *aa* is the spherical enclosure. It is a copper spherical shell, 10 centims. in internal diameter, constructed like the Magdebourg hemispheres, but surmounted by a brass tube, *tt*, which is brazed to the upper opening of the copper shell. At the top of the brass tube is a groove, made by brazing on a small piece of wider brass tube; and into this groove the extremity of the glass tube, *gg*, is inserted. The extremity of the glass tube is prepared by grinding to fit the socket; and it is joined in, air-tight, at the last moment, when all the other preparations are complete, by means of *siegelwachs*, a soft German cement which I have found invaluable for these purposes, and have already mentioned in my former paper.* By means of a narrow glass tube, *ee*, the enclosure is attached to the Sprengel pump, being hermetically sealed to it in the usual way. An outer tube of brass, TT, forms a water-jacket to the tube, *tt*; and, by means of a suitably-arranged tube, water is kept flowing round the *siegelwachs* joint. The copper sphere is suspended in the enclosure by one of the wires of the thermo-junction.

For the thermo-junction I employ platinoid and platinum. The latter is used because it is to be sealed into the glass tube *gg*, as is shown at *s*; and no wire is known at present, except platinum, or an alloy of platinum, which, when sealed into glass, will make a joint sufficiently tight for a Sprengel vacuum. I have found platinum and platinoid to give a very satisfactory thermo-junction; and, along with Mr. A. TANAKADATE, have published a paper on the thermo-electric position of platinoid alloy.†

The platinoid wire is passed through the brass tube *tt* at *p*, and soldered in its place. It is, therefore, not insulated; and it bears the weight of the copper sphere. The platinum wire bears no weight. It is insulated all the way up through the brass tube by being encased in a very thin tube or sleeve of glass, which is slipped over it. The platinum wire has a small length of spiral at the top to take up any slack that may exist (as it is not desirable, for glass-blowing reasons, to cut off the platinum wire to the exact length), and to avoid any pull on the glass at the place where the wire is sealed into the wall of the tube. The platinoid wire is cut by trial to the proper length, so that, when the weight of the copper globe is on it, the globe shall hang at the centre of the spherical enclosure : the ends of the platinum and platinoid are then brought together and inserted into a hole drilled in a copper plug, and are silver-soldered to each other and into the hole. This copper plug is firmly fitted by screwing into a hole, which is carefully cut in each of the copper globes.

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* 'Phil. Trans.,' A., 1887, p. 433.

† 'Roy. Soc. Proc.,' vol. 46, 1889, p. 286.

The operations just described are, of course, carried out at the commencement of a series of experiments, the copper plug being once for all attached to the thermo-junction. The globes can be attached or detached at pleasure for the purpose of having the surfaces polished, sooted, or otherwise treated. When a globe has been hung in its place for experiment, the lower half of the spherical enclosure is brought up, pressed firmly to the upper half, and soldered there.

It has already been mentioned that, during the cooling of the heated globe, the enclosure is immersed in a bath of cold water, which is kept at constant temperature. An ordinary chemical water-bath of copper is used for this purpose. It is shown surrounding the enclosure *aa*. One of the thermo-junctions being, as has been described, at or near the centre of the cooling globe, the second junction is soldered to a small lump of copper, which is tied to the bulb of a thermometer and immersed in the water-bath. The platinum wire is made sufficiently long to pass from the globe up through the brass tube, out through the side of the glass tube at *s*, and down to the water-bath, where its extremity joins the second platinoid wire at *m*, the small lump of copper just mentioned. The two platinoid wires, *pd*, one coming from the centre of the cooling globe, and the other from the junction *m*, are carried forward to *o*, where they are soldered to the copper wire electrodes of a reflecting galvanometer to be spoken of immediately. It is necessary that no electromotive force, due to differences of temperature, should arise at these joinings. These joinings are, therefore, wrapped in tissue paper for insulation ; and then a large quantity of cotton wool, *o*, is wrapped round the whole and along the wires for some distance. The protection and equable distribution of temperature is so perfect that the cotton wool bundle may be handled freely with warmed hands without the galvanometer being in any way disturbed.

The galvanometer used in these experiments is a THOMSON's reflecting galvanometer. It has a very excellent coil, 0·9 ohm resistance, specially wound for me for use with thermo-junctions, by JAMES WHITE, Glasgow. I use a Steinheil plane parallel mirror, hung by a spider line instead of silk fibre, in a THOMSON's dead-beat plug. The comfort and satisfaction of a spider line for purposes where the galvanometer deflection is to be read, and not merely a zero noted, or where a steady unchanging zero is desired, are almost indescribable.*

Placed opposite the mirror is a Steinheil telescope and a scale. Every part of the galvanometer apparatus is screwed down on a wooden bracket fixed to the stone wall of my experimenting room, and the field magnets are fixed down firmly with cement. I use a small weak "zero magnet," with its magnetic axis perpendicular to

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[*] J. T. BOTTOMLEY and A. TANAKADATE. 'Roy. Soc. Proc.,' vol. 46, 1889, p. 286.

the plane of the mirror, for the purpose of bringing the zero of the scale to the cross wire of the telescope. This is not often required, but occasionally, when the sun shines strongly on the iron frames of the Glasgow University windows, the zero may be pulled round by a few divisions; and, occasionally, also a movement of some iron in the laboratory slightly displaces the zero. The "zero magnet" being moved slightly, with its axis along a line perpendicular to the plane of the mirror, adjusts the zero without altering the field of force. I find it an excellent plan to put a little "soft wax"* on the zero magnet, and press it on to the table in front of the galvanometer shelf. By cautiously pressing on the magnet it may be made to slide very slowly, with viscous yielding of the wax, for the purpose of setting the zero of the galvanometer.

I have mentioned already that I have made, with the assistance of my friend, Mr. A. TANAKADATE, a determination of the thermo-electric value of a platinum-platinoid couple. We considered it necessary, however, to make special determinations of the values of the actual couples used in these experiments, and this was done each time a fresh soldering of the platinoid and platinum was made. Platinum wire is so variable in its qualities that it is not safe to assume that the physical properties of different portions of the same hank even are the same.†

The standardizing of the thermo-junction is performed with the circuit precisely as it is to be used during the radiation experiments, with every joint soldered, and with the field of force of the galvanometer suitably adjusted and rendered unchangeable, as described above. Along with the thermo-electric wires, the leads, and the galvanometer, a platinoid resistance of about 12 ohms was introduced into the circuit. The sensitiveness of the arrangement was quite sufficient to permit of this being done; and it was of great advantage to introduce this added resistance, as it rendered insensible small variations of resistance (due to temperature) in the long platinum thermo-electric wire, which in early experiments gave rise to curious and embarrassing variations in the galvanometer readings.

From the standardizing, or calibration, as it is often called, of the thermo-electric circuit in use, a formula is obtained, from which is calculated, from the "*corrected* deflection"‡ of the galvanometer, the difference of temperatures of the two junctions in terms of centigrade temperatures on the air thermometer.§ Every temperature referred to in this paper is thus temperature on REGNAULT's "normal constant volume" air thermometer. All the mercury-in-glass thermometers employed for subsidiary purposes were also compared directly with the air thermometer, and the suitable corrections applied wherever they were sensible.

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* The material used by chemists for making diachylon sticking plaster answers admirably.

† The best makers will not, I believe, guarantee to supply *uniform* platinum, in the physical sense of the word uniform, knowing that it cannot be done, at present at all events.

‡ The deflection is read off on a plane scale, and is *corrected* for the fact that the angle turned through by the light ray is twice the angle of deflection of the magnet and mirror. In order to apply this correction with facility, a table of numbers was constructed. The correction may now, however, be easily applied by the use of the recently published "Reductions-Tabellen zur GAUSS-POGGENDORFFSCHEN Spiegelablesung," by Dr. PAUL CZERMAK. J. SPRINGER, Berlin.

§ Sir WILLIAM THOMSON'S article "Heat," 'Encyc. Brit.,' and J. T. BOTTOMLEY, 'Proc. Roy. Soc., Edinburgh,' 1888, and 'Phil. Mag.,' August, 1888.

One part of the arrangement of the thermo-electric circuit still remains to be mentioned. By far the most satisfactory way of experimenting for purposes like those under consideration is to take readings (in the way to be described immediately) of the galvanometer deflection on each side of the middle, or zero, position, at equal intervals of time, instead of reading only the deflection on one side, as is often done. Accordingly, I inserted in the circuit a mercury reverser, constructed with care to avoid the introduction of small uncertainties as to resistance. The use of this piece of apparatus also enabled me from time to time to apply to the galvanometer and its leading wires a very small known current from a standard cell, in order to assure myself of the constancy of the magnetic field.

I may now explain the mode of experimenting. The value of the thermo-junction having been determined, and the copper globe screwed on to the plug, and the apparatus, which is supported by clips on a retort stand, having been adjusted so that the globe shall hang in the centre of the enclosure, the lower half of the enclosure is brought up and held pressed against the upper half by means of temporary clips. Owing to the edges of the flange having been well prepared by grinding, and by having a very thin layer of solder or tinning applied to them, the parts fit together very exactly; and in this position they are now soldered together, with the help of a soldering bolt and a Bunsen burner. Resin is used for the soldering, except just at the end of the operation, when a little soldering fluid is applied round the outer edges to make sure of perfect filling up of the minutest holes. The soldering up of these edges in such a way as to be tight for the Sprengel vacuum is a very troublesome operation, and one which requires the greatest care.

When the soldering is finished, the joint between the brass tube and the glass tube is made complete by means of *siegelwachs;* and finally the apparatus is connected to the SPRENGEL pump by means of a portable blowpipe. The intention, generally speaking, in these experiments, was to determine the loss of heat from surfaces prepared in various ways, both with air present (and later with other gases), and also in more or less complete vacuum. Sometimes the cooling in vacuum was tried first, and the cooling in air afterwards; sometimes the order was reversed. Sometimes the cooling in air was determined, then in vacuum, and then in air again.

This being understood, I may say that everything being prepared as described above, water was now poured into the water bath; a flame was applied beneath, and the boiling water communicated heat to the enclosure, and gradually to the suspended ball within. The galvanometer was watched and the time roughly noted to show the progress. I have not, however, made use hitherto of these observations on speed of heating.

With full air present, the process of heating was not inconveniently slow. With vacuum it was excessively tedious, several hours being required for the heating. The process was carried on till the thermo-junction showed that the boiling-point of water had been very nearly reached by the suspended globe.

Then, preparations having previously been made, the lamp was extinguished; the boiling water was very quickly run out with the help of siphons, and cold water run in, in place of it. A supply of cold water, at very nearly the temperature of the laboratory, was kept in readiness for this process, and a plentiful supply was run through the water-bath until all the surroundings had come to a steady temperature. Neighbouring parts of the table, retort stands, &c., were often cooled by wetting to obtain subsequent steadiness of temperature. In my experiments I had the advantage of the able assistance of Mr. A. TANAKADATE. I cannot sufficiently express my indebtedness to him, either for his dexterous manual aid, or for his many ingenious and valuable suggestions during the progress of the work. Single-handed, some of the work would have been impossible; and there are few, even of trained experimenters, who could render such help as he gave me.

As soon as the galvanometer could be read, after the cold water had been applied, regular observation commenced, and the method of observation was as follows:—It was found that an observation every 5 minutes was sufficient. (In experiments at higher temperatures, described below, the intervals were $2\frac{1}{2}$ minutes.) A chronometer, beating half-seconds, stood close by; and by listening to the beats, as in astronomical observations, the galvanometer reading was taken at 10 seconds before the exact minute. The reverser was instantly shifted, and the first reading then recorded. By the time that 4 or 5 seconds after the exact minute was reached, the observer was once more ready, looking through the telescope and counting the seconds, and the vibrations of the galvanometer (well damped*) had long before completely subsided; and, at exactly 10 seconds after the minute, the observer was able to take the second reading. The sum of these readings, to right and left of the galvanometer zero, was taken to represent the temperature at the exact minute noted; and the mean of the pairs of readings, if it agreed with the known zero of the galvanometer, afforded a check, and showed (1) that no mistake had been made in noting; (2) that the galvanometer needle was *not* on the swing at the second reading; and (3) that there was no disturbance from the distant dynamo or from the moving about of magnets in the laboratory.

The readings were at once plotted down on squared paper; and at first this plotting revealed little faults in observation, shakiness of apparatus, change of resistance in the leads, &c., all of which faults were soon eliminated. The curves then became so perfectly regular as to leave nothing to be desired.

Before giving an account of the results obtained, I will briefly describe a modified apparatus which I found it necessary to use for temperatures higher than the boiling point of water.

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* Damped by the air damping of a Thomson dead-beat galvanometer plug. It seems strange that the Thomson dead-beat plug is so little known to instrument makers, and to users of the reflecting galvanometer. It has been in use in all the galvanometers supplied by JAMES WHITE for about twenty-five years, and no instrument of the kind which is not dead-beat would be tolerated in the Glasgow laboratory.

When I had finished a considerable number of determinations, carried on precisely as described above, with boiling water for the superior temperature, I attempted to get to a temperature of 200° C. or 250° C. I found, however, that the soldered joint would not bear a temperature of anything like 200° when the Sprengel vacuum was applied. The solder became crystalline and brittle and leaky below 150°. The solder used was good ordinary solder; and I think that slight leakiness (very perceptible to the Sprengel) manifested itself at a temperature at which the solder was for ordinary purposes quite good and sound. I am not sure, however; and it is quite possible that by trials I might have got a better solder by using less tin in the composition. I did not delay with such trials, but preferred to alter the enclosure to the form shown in fig. 2. *aa* is a copper shell nearly spherical, but with a wide mouth and neck. The shell is beaten into shape. The neck is carefully brazed on, and the parts round the brazing finished by beating. The neck has a flange with the upper edge ground flat. The whole is very thick and solid, and an outer shell of copper was subsequently added, squeezed into shape and fitting loosely round the other for the purpose of equalizing the heating (and cooling) of the enclosure. The brass tube *tt* and outer water-jacket are also shown in the diagram; and it will be seen that, to close the copper shell, the whole of this piece is brought down on the flange at the neck of the shell; and the soldering is made at this joint. This construction was found to be in every way an improvement on that used at first. The only little trouble being the lowering in of the copper globe without touching the sides, and the making the measurements necessary for bringing the copper globe to the centre of the enclosure. These troubles are scarcely worth mentioning, and the new arrangement gave much greater freedom for working at the globe while it was hanging on the wire before being lowered into its place.

In the arrangement just described it will be seen that the soldering is kept cool by the proximity of the cold water in the water-jacket; and this can be done even although the temperature of the shell enclosure is very high. In practice a stream of cold water was kept running through the water-jacket during the heating process described above, and for heating the enclosure a Bunsen burner or, in the latest experiments, one of FLETCHER'S high-power burners was applied direct to the bottom of the copper shell.* When the temperature of the globe inside was found, by means of the galvanometer, to have risen to the desired point, the flame was withdrawn, and the cold water, at the temperature of the laboratory, applied as before. It was also found advisable for steadiness of temperature to stop the current of cold water through the water-jacket, and to warm up the water in the water-jacket (which, while the flow continued, was kept at the temperature of the public supply) to be roughly the same as the air temperature of the laboratory.

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* A curious phenomenon, which requires farther examination, was noticed during the heating of the globe described in the text. I found it impossible to exhaust the enclosure with the Sprengel pump while the flame was applied; but as soon as the flame was removed I could exhaust the enclosure perfectly. Whether the copper shell became porous at the high temperature (just visible redness) and thus admitted gas enough to prevent my obtaining a vacuum; or whether some minute crack or opening was made at some of the joints owing to the great variation of temperature from point to point of the shell, I am unable to say at present. The leaking ceased, however, as soon as the heating ceased. This leak-

The Sprengel pump used for exhausting the enclosure and the duplicated MacLeod gauge used for determining the residual pressure are fully described, and certain difficulties and uncertainties are pointed out, in my former paper,* and need not be further spoken of here.

For the purpose of calculation two pieces of numerical information are still required. One is the capacity for heat of the copper globes; and the other, the cooling surface.

The capacity for heat was determined very carefully for these particular globes by Mr. D. MACFARLANE in 1872; and my determination has not sensibly varied from his result. I propose, however, to make a fresh determination of this constant as soon as possible.

The surface of the globe is obtained by careful measurement with calipers. But it will be observed that, besides loss of heat by radiation, there is also a loss by conduction along the two thermo-electric wires. When a sooted globe is cooling in full air pressure the amount of heat conducted along the thin wires used is very small in comparison with that lost by radiation and convection. But when the globe is cooling in vacuum or, at any rate, when a highly polished silvered globe is cooling in vacuum, the heat lost by the wires is by no means negligible in calculating the value of the surface emissivity.

In making use of the results obtained up to the present time I have used for calculation an equation of the form

$$-c\frac{dv}{dt} = e\text{S}(v - v_0),$$

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ing gave rise to a most disturbing difficulty at the commencement of the corresponding vacuum series; because it was desirable to commence the observations on cooling as soon as the cooling commenced, but it was not till ten or fifteen minutes later that a proper vacuum could be secured, even with the most vigorous use of the pump.

* 'Phil. Trans.,' 1887, pp. 444–446. There is just one remark which it seems worth while to make, on account of repeated misunderstanding, namely, that the MacLeod gauge does not measure (nor profess to measure) pressure in the pump of any gas collapsible by pressure, for any reason whatever, at the existing temperature of the gauge. Thus it does not measure the pressure of mercury vapour which may be present in the so-called vacuum, nor the pressure of vapour of water, for example. Nor does it even measure the pressure of any gas which may with the help of a small pressure be caused to collapse against the walls of the tube, or by absorption into, or on, the mercury used in the gauge. I have shown the possibility of this last-named occurrence in a paper on an "Improved Mercury Pump," British Association, 1886.

where v is the temperature of the cooling globe, v_0 that of the enclosure, c the capacity for heat of the globe, and S its surface, while e is a co-efficient which has been called the *emissivity*. It is easy to show by an application of FOURIER's equations that, to a first approximation, sufficient for my present purpose, the loss by conduction by the wires may be allowed for by adding to S, or I should rather say to eS, a small quantity depending on the diameter, conductivity, and emissivity of the wire, the length of the wires being so great that no heat passes away by conduction at the ends. The numerical value of the quantity to be added to eS I have not yet determined exactly; but I have made an estimate of it and applied it as a correction in obtaining the results given below.

Another interesting point requiring consideration is the following:—The arrangements explained above give approximately the temperature at or near the centre of the copper globes. Those of Mr. D. MACFARLANE did the same in the case of his experiments. In a criticism of MACFARLANE's results, by M. CORNU, 'Journal de Physique,' December, 1873, an objection was taken to the thermo-electric method as compared with the method used by DULONG and PETIT, where the bulb of a thermometer was the cooling body, on the ground that the temperature of the centre may be different from that of the surface. This objection was completely answered by numerical calculations founded on the dimensions and conductivities of the globes in question; and, in an interesting appendix the whole question is discussed in the 'Royal Society Proceedings,' June 10, 1875.*

With these explanatory remarks I proceed to a brief statement of the results obtained, but I wish it to be clearly understood that I do not yet consider myself entitled to draw general conclusions as to laws of radiation from the observations completed and calculated down to the present date. Theories of radiation have, in recent years, been discussed by STEFAN,† H. F. WEBER,‡ FERREL,§ and others; but it appears to me that the main difficulty of making any satisfactory theory is the want of experimental data on which to found a theory, and by which to test it. I venture to think that the data required are still far from sufficient; and, indeed, this is admitted and pointed out as a difficulty by the most recent of these writers, the late Mr. FERREL.

I have experimented solely on the copper globes with their surfaces prepared by sooting and silvering. The sooting was done very carefully with a gas flame or with paper or cotton waste impregnated with powdered resin, or powdered camphor, or moistened with turpentine. The endeavour was to put on a thoroughly continuous but thin coating of lamp-black, and thus to obtain what I presume we may take to be a surface giving maximum of radiation. On the other hand the silvered globe was polished to the highest degree possible. The polishing was done very frequently, and with special care, by the kindness of Mr. THOMAS SMITH, Electroplater, Glasgow. After polishing the globe was wrapped in tissue paper and kept, surrounded by a very large quantity of cotton wool, in a box until the last moment before it was screwed on to the plug by which it was suspended. It was never touched by the bare hands, being held in its tissue-paper cover while the operation of screwing on was performed. The surface was so smooth and slippery that it was difficult to keep fast hold of the globe while it was being screwed on; and when suspended it presented the most beautiful mirror surface I have ever seen.

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* Republished in 'Lord KELVIN's Collected Papers,' vol. 3, p. 245.

† STEFAN, 'Berichte der Wiener Akademie,' vol. 79, 1879.

‡ Professor H. F. WEBER, 'Akademie der Wissenschaften zu Berlin.' Berlin, 1888.

§ W. FERREL, 'American Journal of Science,' July, 1889, and February, 1890.

I am supposing that the minimum of radiation is had from such a silver surface as has just been described; and it was extremely interesting to find that a marked effect, in reducing the amount of radiation in vacuum from the silvered globe, was produced by efforts to attain the highest possible degree of polish.

In calculating the emissivity from the recorded deflections of the galvanometer, the first process was to obtain the "corrected deflection"* of the galvanometer. Secondly, a correction is applied to the numbers for the fact that the deflection produced by the thermo-junction is not in simple proportion to the difference of temperatures (defined by the air thermometer) between the two junctions.† Thus a series of numbers is obtained corresponding to the successive times noted, the intervals being 5 minutes or $2\frac{1}{2}$ minutes, as has already been mentioned. From these numbers I have calculated the value of the "emissivity," or quantity of heat lost per second, per square centimetre of surface, per degree centigrade of difference of temperatures of surface and surroundings.

Let c be the capacity for heat of the cooling globe, S the surface, and $(v - v_0)$ the difference of temperatures of source and surroundings at any time t. Then in the formula

$$- c \frac{dv}{dt} = e\mathrm{S}(v - v_0),$$

the coefficient e corresponds with the definition of "emissivity" just given; and this formula is commonly taken to be a representation of the "law of cooling," whether in air or in any other gas, or in vacuum, the range of temperatures dealt with being moderate. The numerical value of e, however, depends on the circumstances under which the cooling takes place; and, when air is present, on the dimensions and shape of the cooling body. Important theories, to which I have already alluded, have been from time to time put forward as to the form, practically speaking, of e as a function of v for the case of pure radiation.

One way, however, of dealing with the matter, from an experimental point of view,

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* See note p. 599.

† For example, one of the formulas, with numerical coefficients actually used, was as follows:—

$$t = \frac{1}{3{\cdot}38}\delta - {\cdot}000086\,\delta^2,$$

where t is the difference of temperatures, and δ the "corrected deflection" of the galvanometer.

is to solve the equation above as though e were constant—which is, of course, approximately true if the difference of temperatures is small—and then to determine, numerically, the value of e at different places in the temperature scale, taking an exact account of the circumstances. When a sufficiently large number of such values have been obtained a basis for a proper theory will have been laid in a logical way. This is practically what I am endeavouring to do.

Taking, then, the equation

$$-c\frac{dv}{dt} = e\mathrm{S}\,(v - v_0),$$

and its solution

$$\log_\epsilon \frac{v' - v_0}{v - v_0} = \frac{e\mathrm{S}}{c}\,t,$$

where v' is the temperature of the cooling globe when $t = 0$, we have for the numerical calculation of e

$$e = \frac{c}{\mathrm{S}t}\,\{\log_\epsilon (v' - v_0) - \log_\epsilon (v - v_0)\},$$

or, using common logarithms,

$$e = \frac{\mathrm{M}c}{\mathrm{S}.t}\,\{\log (v' - v_0) - \log (v - v_0)\}.$$

Here $t = 300$, the intervals of time used in my experiments being 5 minutes; $c = 28{\cdot}31$ and $\mathrm{S} = 50{\cdot}26$, with an addition which, for the present, I roughly estimate at about 0·6 per cent.—the correction applied for the carrying away of heat by the conducting wires. This I calculate, assuming that the emissivity of the surface of the conducting wires is much the same as that of a *tolerably clean* silvered surface.

The following tables and curves (Plate 18) show most of the numerical results obtained. The tables are made self-explanatory, as far as possible, by means of suitable headings; and the curves which I have drawn corresponding to most of the series are dated, and the multiplier is given for reducing the ordinates to C.G.S. measure. The curves were drawn chiefly for my own satisfaction; but there are one or two matters to which I may be permitted to call attention.

In the first place, it will be seen from the curves that in each series the results are in very exact accordance. Certainly they are more so than I ventured to hope when I commenced putting them down. The difficulties surrounding experiments of this kind are really very great, if anything more than rough approximation is aimed at; and, unfortunately, uncertainties connected with the temperature measurement easily creep in, and are difficult to detect till it is too late to remedy them, or even to determine the limits of possible error.

The different series of determinations represented by the several curves are also in agreement, and some of them, as, for instance, those of March 7 and 10 (silvered globe in air), and those of October 29, 1889, and April 2 and 8 (sooted globe in vacuum), are quite unexpectedly close to each other. There are differences between the absolute positions of the curves, but most of these depend on some differences of circumstances. The only case which I cannot explain is that of the silvered globe in air, November 6. I do not understand why the loss of heat was so small in an experiment which bears every sign of being thoroughly good. I have no note (such as would certainly have been made) of any change of galvanometer zero (indicating possible alteration of magnetic field); nor was there alteration of the apparatus of any kind.* Extreme dryness of the air is a possible explanation.

A feature, which seems to me remarkable, about the curves representing these late experiments is that none of them show any sign of emissivity, as it were, accelerated with increase of temperature, that is to say, having an increasing rate of increase of emissivity with increasing temperature, but rather the reverse. The curves are all slightly concave towards the axis of temperatures. It is, however, well ascertained, particularly from the experiments of SCHLEIERMACHER and myself, that, at all events, in the case of *wires* losing heat there is a very rapid acceleration or increase of rate of increase of *radiation* with increase of temperature. This is most easily seen by looking at the curves of my former paper, 'Phil. Trans.,' 1887, A, Plate 25. Looking at the matter in this light also, it will be seen from those curves that the curvature (proportionate acceleration) is greatly masked when more than $\frac{1}{4}$ millim. or $\frac{1}{2}$ millim. of air pressure is present, and that the lower the air pressure the more striking is the convexity towards the axis of temperatures in that series of curves.

Referring now particularly to the tables accompanying this paper. Table I. has been already explained (pp. 591, 592, 593). Table II. shows the results of four nearly successive days of experimenting in air and in vacuum on a globe with two different coatings of lamp-black. In the first pair the coating was much more dense than in the second pair. In the latter case the globe was passed quickly a few times through the gas flame of my blowpipe table. In this way a coating of the finest texture was obtained; the colour of the copper globe was distinguishable through it. I had a reason for wishing to keep the coating very thin in connection with experiments on incandescent wires which I am making.

Table III. shows the results of similar experiments on a silvered globe.

Both these sets of experiments were made with the apparatus of fig. 1; and I believe that in neither was the interior of the enclosure sooted in the way which was afterwards done.

Tables IV. and V. show the results of experiments on the sooted globe in air and in vacuum, and on the silvered globe in air at very high temperatures. These experiments were made in apparatus of fig. 2, Plate 17, and were most satisfactory in every way.

I come lastly to Table VI. It was in the hope of finding time (three months at least) for repeating and extending these experiments that I have delayed the publication of this paper. The operation of soldering up the enclosure so thoroughly as to withstand the high temperature required for heating up the hanging globe and to remain perfectly air-tight for the Sprengel pump was at all times difficult. But when the enclosure contained the silvered globe, in the highest state of polish, it became almost impossible to perform the operation without tarnishing the globe. The experiment of April 14 was satisfactory, except for the somewhat variable state of the vacuum; which was, however, very high throughout. The numbers quoted show the effect of this variableness. A new experiment was tried on April 17; but the result was less satisfactory; the globe turned out to be tarnished when the enclosure was opened; and the rapidity of loss of heat corresponds.

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* To imagine a cause for unexpected excessive loss would be easy; but it is not easy to account for the globe retaining its heat.

Unfortunately, the time at which these experiments were tried was the very end of the Glasgow winter session. Indeed at this time I had already become involved in the heavy University work belonging to this period of the year. Before we could resume experimenting, my friend Mr. TANAKADATE was obliged to leave Glasgow for Berlin; and I have not since then found the time and circumstances necessary for repeating these last experiments with an enclosure slightly modified to make the soldering more easy and more sure.

TABLE II.—Copper globe sooted. (Apparatus fig. 1.)

October 22, 1889. Vacuum. Pressure $\frac{1}{1\cdot 3}$ M.* Temperature of bath surrounding enclosure, 17°·5 C.		October 24, 1889. Air. Pressure 75 centims. Temperature of bath surrounding enclosure, 16°·4 C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.
79°·6	$1\cdot 37 \times 10^{-4}$	72·1	$2\cdot 76 \times 10^{-4}$
74·0	1·36 ,,	62·3	2·69 ,,
68·9	1·34 ,,	54·1	2·62 ,,
64·2	1·32	47·2	2·55
59·9	1·30	41·3	2·48
55·9	1·29	36·2	2·41
52·2	1·27	31·9	2·34
48·8	1·25	28·2	2·27
45·7	1·23	25·1	2·20
42·8	1·22	22·4	2·13
40·1	1·20	20·0	2·06
37·7	1·18	18·0	1·99
35·4	1·16	16·2	1·92
31·3	1·13		
27·8	1·09		
24·8	1·06		
22·2	1·02		
20·0	0·989		
18·0	0·954		

---

* M denotes a pressure equal to one-millionth of an atmosphere. The measurement, however, is not of the TOTAL pressure in the enclosure; but only of the pressure of the gas *not collapsible* in the MacLeod gauge.

TABLE II. (continued).—Copper globe very finely sooted.

October 25, 1889. Air. Pressure 76 centims. Temperature of bath surrounding enclosure, 17° C.		October 29, 1889. Vacuum. Pressure $\frac{1}{2\cdot2}$ M. Temperature of bath surrounding enclosure, 15°·3 C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.
76°·1	$2{\cdot}80 \times 10^{-4}$	83°·6	$1{\cdot}25 \times 10^{-4}$
68·25	2·73	78·2	1·23
59·1	2·65	73·3	1·21
51·6	2·57	68·7	1·19
45·1	2·50	64·6	1·17
39·75	2·42	60·7	1·15
34·8	2·34	57·1	1·14
30·7	2·26	53·9	1·12
27·2	2·19	50·7	1·10
24·2	2·11	47·8	1·08
21·6	2·03	45·15	1·06
19·6	1·95	42·7	1·05
18·1	1·88	40·4	1·03
		36·3	0·99
		32·7	0·95

It will be of interest, in order to observe the way in which the cooling proceeds, to remark that the excess temperatures set down in the temperature columns are practically those observed at the ends of successive intervals of five minutes.

TABLE III.—Copper globe, silvered and polished. (Apparatus, fig. 1.)

November 4, 1889. Vacuum. Pressure $\frac{1}{3·04}$ M. Temperature of bath surrounding enclosure, 14°·4 C.		November 5, 1889. Vacuum, and most thorough drying since yesterday. Pressure $\frac{1}{2·5}$ M. Temperature of bath surrounding enclosure, 15°·6 C.		November 6, 1889. Air, very dry, admitted in full pressure to enclosure of yesterday. Barometer not noted. Temperature of bath surrounding enclosure, 15°·7 C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.
67°·4	4·53 × 10^{-5} *	72°·1	3·54 × 10^{-5}	66°·8	1·75 × 10^{-4}
65·6	4·50 *	70·8	3·52	61·2	1·72
64·0	4·50 *	69·5	3·50	55·9	1·68
62·5	4·44	68·2	3·47	51·2	1·65
59·7	4·38	66·9	3·45	46·9	1·61
57·0	4·32	65·7	3·44	43·1	1·58
54·2	4·25	64·5	3·42	39·7	1·54
52·0	4·20	63·4	3·40	36·6	1·51
49·1	4·12	62·2	3·38	33·8	1·47
46·6	4·05	61·1	3·36	31·25	1·44
45·6	4·02	59·0	3·32	29·0	1·41
43·7	3·96	58·0	3·30	26·9	1·37
41·9	3·90	56·0	3·26	25·0	1·34
40·2	3·84	54·1	3·22		
36·4	3·69	53·2	3·20		
34·4	3·60	51·4	3·16		
		49·7	3·12		
		48·1	3·08		
		46·6	3·04		
		45·1	3·00		
		43·7	2·96		

---

* Vacuum not satisfactory at commencement. Improved to amount stated, $\frac{1}{3·04}$ M, by working pump; but somewhat variable all through. Probable cause, want of perfect drying. This would not be indicated as pressure by the MacLeod gauge.

TABLE IV.—Copper globe sooted. (Apparatus, fig. 2.)

March 29, 1890. Globe freshly sooted, very thin coat, but very complete and uniform. Air. Pressure 75·6 centims. Temperature of bath surrounding enclosure 17°·5 C.		April 8, 1890. Vacuum. Pressure $\frac{1}{5\cdot1}$ M. Temperature of bath surrounding enclosure 15° C.		April 10, 1890. Dry air admitted to enclosure of April 8. Pressure 75·5 centims. Temperature of bath surrounding enclosure 15°·5 C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Average loss of heat per sq. centim. per second per 1° C. of excess, between the temperatures indicated in preceding column.	Excess of temperature of globe above temperature of surroundings.	Average loss of heat per sq. centim. per second per 1° C. of excess, between the temperatures indicated in preceding column.
224°·0	3·92 × 10^{-4}	230°·4		250°·3	
189·9	3·82		2·23 × 10^{-4}		4·44 × 10^{-4}
155·5	3·68	204·6		217·3	
128·2	3·55		2·00		4·26
106·5	3·42	184·0		189·6	
89·1	3·29		1·78		4·12
75·0	3·16	152·2		173·7	
63·6	3·03		1·63		3·81
54·3	2·90	139·5		157·0	
46·7	2·77		1·52		3·70
40·4	2·64	128·7		142·2	
35·3	2·51		1·44		3·70
31·0	2·38	119·2		126·4	
			1·42		3·63
		110·5		116·9	
			1·36		3·55
		102·8		106·4	
			1·27		3·48
		96·1		89·25	
			1·29		3·23
		89·7		81·9	
			1·20		3 02
		84·1		58·5	
			1·22		2·98
		78·8		49·9	
			1·20		2·86
		74·0		42·9	
			1·26		
		69·2			
			1·15		
		65·0			
			1·07		
		61·4			

Table V.—Copper globe silvered and highly polished. (Apparatus, fig. 2.)

March 7, 1890. Air. Pressure 75·8 centims. Temperature of bath surrounding enclosure, 12° C.		March 10, 1890. Air. Pressure 76 centims. Temperature of bath surrounding enclosure, 13° C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.
°		°	
214·9	$2·53 \times 10^{-4}$	231·3	$2·55 \times 10^{-4}$
188·7	2·49	221·1	2·52
165·5	2·45	193·4	2·48
145·4	2·41	169·5	2·44
128·0	2·37	148·9	2·40
113·0	2·33	131·2	2·35
99·9	2·29	118·5	2·31
88·5	2·25	102·4	2·27
78·6	2·21	90·8	2·23
69·8	2·17	80·7	2·18
62·3	2·13	71·9	2·14
55·8	2·09	64·2	2·10
50·0	2·05	57·4	2·06
44·9	2·00	51·5	2·01
40·4	1·96	46·3	1·97
36·4	1·92	41·7	1·93
32·8	1·88	37·5	1·89

Table VI.—Copper globe silvered and very highly polished for experiment of April 14; believed to be tarnished during experiment of April 17. (Apparatus, fig. 2.)

April 14, 1890. Vacuum. Pressure at commencement about 10 M or 12 M, coming down in 25 minutes to less than $\frac{1}{10}$ M, and kept so, though somewhat variable, by vigorous working of Sprengel. Temperature of bath surrounding enclosure, 13°·3 C.		April 17. Vacuum. Pressure 2 M. Considerably more at commencement. Temperature of bath surrounding enclosure, 12° C.	
Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.	Excess of temperature of globe above temperature of surroundings.	Loss of heat per sq. centim. per second per 1° C. of excess.
°		°	
240·5	5·58 × 10^{-5}	254·5	8·39 × 10^{-5}*
233·4	5·10	246·6	8·21 *
227·2	4·06	235·1	7·35
222·3	3·07	214·7	7·28
218·7	2·68	199·8	7·18
211·4	2·64	184·3	6·98
208·4	2·55	171·2	6·87
205·7	2·59	159·3	6·66
202·9	2·59	146·0	6·57
198·2	2·16	136·1	6·31
195·9	2·25		
193·6	1·95		
191·5	1·64		
187·7	1·73		
183·9	1·86		
179·9	1·77		
176·6	1·90		
174·9	1·90		
167·9	1·92		
164·5	1·94		
161·2	1·94		

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* Vacuum unsatisfactory, improving by working pump. Surface of globe found to be tarnished at conclusion of this experiment.

APPENDIX.

Added March, 1893.

The method of calculation of the results of my experiments is explained in the body of my paper; but it has been suggested that a more detailed account of the calculations, with, perhaps, a numerical example, might be of value. Accordingly, though I fear it cannot be done without considerable repetition, I beg to supply the want in this appendix.

The numerical results of my experiments are obtained, as has been explained, in the form of readings of the galvanometer deflection taken at equal intervals of time, five minutes being the interval most commonly adopted. The arrangements for making the observations at the precise instant are explained on p. 599 of the paper. These arrangements were perfectly successful. Thanks largely to the skill of Mr. TANAKADATE in this kind of observation, it was very rarely that a reading was missed; and when by any chance a reading was missed the failure was known and the exact time of the next observation was noted. There was on no occasion whatever any *uncertainty*.

At suitable times also the temperature of the surrounding vessel, and the state of the vacuum and other particulars, were noted.

While an experiment on the cooling of a globe was in progress the galvanometer readings were plotted on to a large sheet of squared paper. This was done because it was impossible to see into the enclosure; and some check was wanted by which it could be known that everything was proceeding satisfactorily and had been carried out properly in the preparations for the experiment. Usually each experiment had some special feature for which special preparation was made; and it was quite possible that some contact of the wires of the thermo-junction, or some flaw in the last soldering of the halves of the enclosure, should make the result of the long course of waiting and observing valueless. Fortunately no such mischance did occur; but a check was necessary, and the plotting of the curve, which could be quickly compared by eye with others of the same kind, afforded a test which was particularly easy and satisfactory.

The following is a copy of one of the pages of my note-book. It explains itself; as, indeed, each page ought to do :—

April 2, 1890. Observed—J. T. Bottomley, A. Tanakadate. Sooted globe in vacuum; thin coating.

Time.	Deflection.	Temperature of bath.	Vacuum, &c.
h. m.		C	
3 37 P.M.	1171·9	16·0 16·2	Good vacuum.*
40	1034·5	15·7	Vacuum improving.
45	864·6	15·9	Volume 10 Pressure 13 Readings of volume and pressure on the MacLeod gauge.
50	756·3	..	Volume 10 Pressure 12
55	669·9		
4 00	597·0	15·9	
5	535·9		
10	483·1	..	Volume 10. Pressure 12.
15	436·7	15·9	
20	399·0		
25	365·2		
30	335·4	..	Volume 10. Pressure 12.
35	309·4	15·6 15·9	
40	286·5	15·9	
45	266·5	..	Volume 10. Pressure 12.
50	248·4		
55			
5 00	218·9	15·9	

* Neglect first two readings, vacuum not good enough.

It will be seen that the galvanometer readings are noted opposite particular instants of time. The galvanometer readings, properly corrected, give the temperature of the globe at those instants of time. The differences of the logarithms of these readings give the cooling; and the differences between the time readings give the intervals of time corresponding to the cooling; while also the readings themselves give the differences, at the beginning and end of the time intervals, between the temperatures of the globe and its surroundings, and also some mean between the pairs of readings gives the actual temperature corresponding to the emissivity to be calculated. The object of the subsequent calculation was to find the absolute emissivity at the temperatures and under the circumstances noted; but neither the absolute time nor the intervals of time appear in the final result.

In order to prepare the numbers thus recorded for calculation it was necessary to apply certain small corrections to the galvanometer readings.

1. As the reflecting galvanometer and a plane scale were used, and as the angle through which the ray of light turns is double that through which the mirror and magnet are turned by the current force, a number proportional to n^3/d^2 must be subtracted from the galvanometer readings in order to obtain numbers proportional to the magnet deflections (or their sines or tangents); n being the observed deflection and d the distance of the scale from the mirror. To do this a table of values of $n^3/(10152)^2$ was prepared; but very convenient tables for this purpose were published in 1890 by Dr. PAUL CZERMAK,[*] and ought now to form a part of the equipment of a physical laboratory.

2. The second correction consists in the subtraction of a small quantity proportional to the square of the deflection. It is for the term in the thermo-junction formula which is proportional to the square of the temperature. This correction was also applied with the help of a table or with the slide rule. (See footnote to p. 603 of the paper.)

These subtractions having been made, I obtain what I may call the "proportional deflections" and these stand opposite to the times observed. These numbers are proportional to the differences of the temperatures of the globe and the water-bath surrounding its enclosure at the corresponding times. The following copy from pages of my calculation book will show all details.

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[*] Footnote to p. 597 of paper.

CALCULATION of Results of April 2, 1890.

Sooted globe in vacuum ; thin coating. Vacuum $\frac{1}{1\cdot27}$ M. (Apparatus, fig. 2.)

(1) Time.	(2) Observed deflection.	(3) Corrections and sum of corrections.	(4) Proportional deflection.	(5) Log. of proportional deflection.	(6) Differences of logs. of proportional deflections.	(7) Emissivity (calculated by multiplying numbers of column 6 by $4\cdot323\times10^{-3}$)	(8) Log. of difference of temperatures of globe and surroundings.	(9) Difference of temperatures of globe and surroundings.
P.M. 3.40	1034·5	10·7 168·0 178·7	855·8	2·9324			2·3792	239°·4
					·0643	$2\cdot780\times10^{-4}$		
3.45	864·6	6·6 120·0 126·6	738·0	·8681			2·3149*	206·5
					·0468	2·024		
3.50	756·3	4·2 89·5 93·7	662·6	·8213			2·2681	185·4
					·0453	1·959		
3.55	669·9	2·9 70·0 72·9	597·0	·7760			2·2228	167·0
					·0438	1·894		
4.0	597·0	2·0 55·2 57·2	539·8	·7322			2·1790	151·0
					·0422	1·824		
4.5	535·9	1·5 44·6 46·1	489·8	·6900			2·1368	137·0
					·0408	1·764		
4.10	483·1	1·1 36·2 37·3	445·8	·6492			2·0960	124·7
					·0397	1·716		
4.15	436·7	0·8 29·0 29·8	406·9	·6095			2·0563	113·9
					·0372	1·608		
4.20	399·0	0·6 24·9 25·5	373·5	·5723			2·0191	104·5
					·0362	1·565		
4.25	365·2	0·5 21·1 21·6	343·6	·5361			1·9829	96·14
					·0347	1·500		
4 30	335·4	0·4 17·8 18·2	317·2	·5014			1·9482	88·76
					·0331	1·431		
4.35	309·4	0·3 15·1 15·4	294·0	·4683			1·9151	82·24
					·0317	1·370		
4.40	286·5	0·2 13·1 13·3	273·2	·4366			1·8834	76·45
					·0301	1·301		
4.45	266·5	0·2 11·3 11·5	255·0	·4065			1·8533	71·34
					·0286	1·237		
4.50	248·4	0·1 9·6 9·7	238·7	·3779			1·8247	66·79
4.55								
					·0526	1·137		
5.0	218·9	0·1 7·3 7·4	211·5	·3253			1·7721	59·17

* Vacuum not good enough at first reading or first and second.

Taking the common logarithms of the numbers in column 4 we have the fifth column of the table on p. 615.

The differences between the successive logarithms of deflections gives col. 6. To obtain the proper emissivities, these numbers must be multiplied by $(M \times c)/(300 \times S)$; where M is the modulus for reducing Neperian logarithms to common logarithms, c the capacity for heat of the globe, S the cooling surface, and 300 the interval of time in seconds. (See p. 604 of the paper.) The value of $Mc/300\,S$ is $4{\cdot}323 \times 10^{-3}$. The multiplication I perform by means of CRELLE's table.

Lastly, to find the difference of temperatures of globe and bath corresponding to the emissivities, I use the logarithms of the deflections in col. 5, and to each of these add the logarithm of the multiplier corresponding to the first term of the thermo-junction formula. In the formula of the note of p. 603 of the paper, this multiplier would be $\frac{1}{3{\cdot}38}$ and the *additive* logarithm $0{\cdot}4701 - 1$. In the present case April 2, 1890, the multiplier is 0·27976; and the additive logarithm $0{\cdot}4468 - 1$. Taking the anti-logarithms of the numbers so found, I obtain the difference of temperatures in Centigrade degrees between cooling body and surroundings.

Fig. 1.

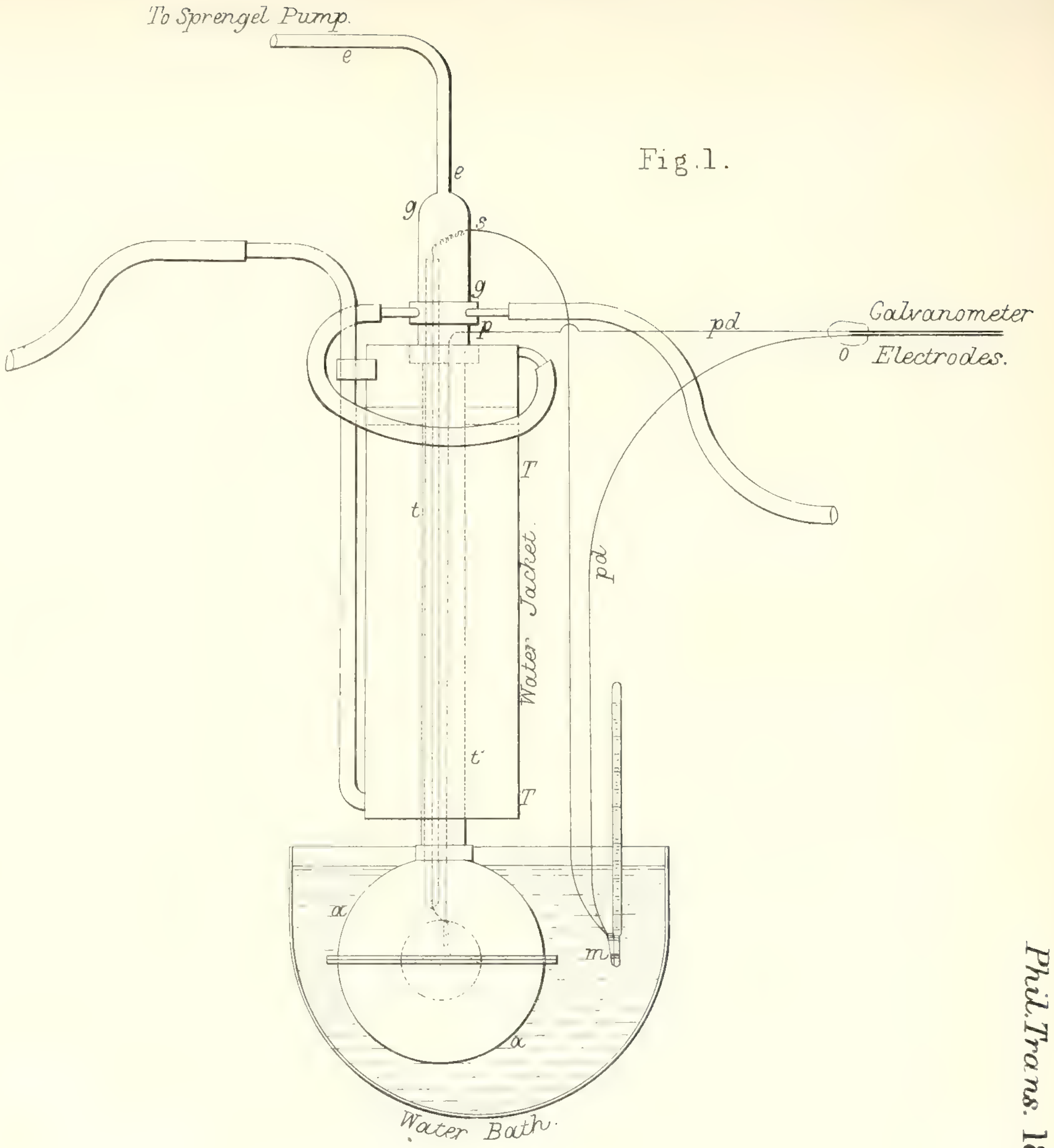
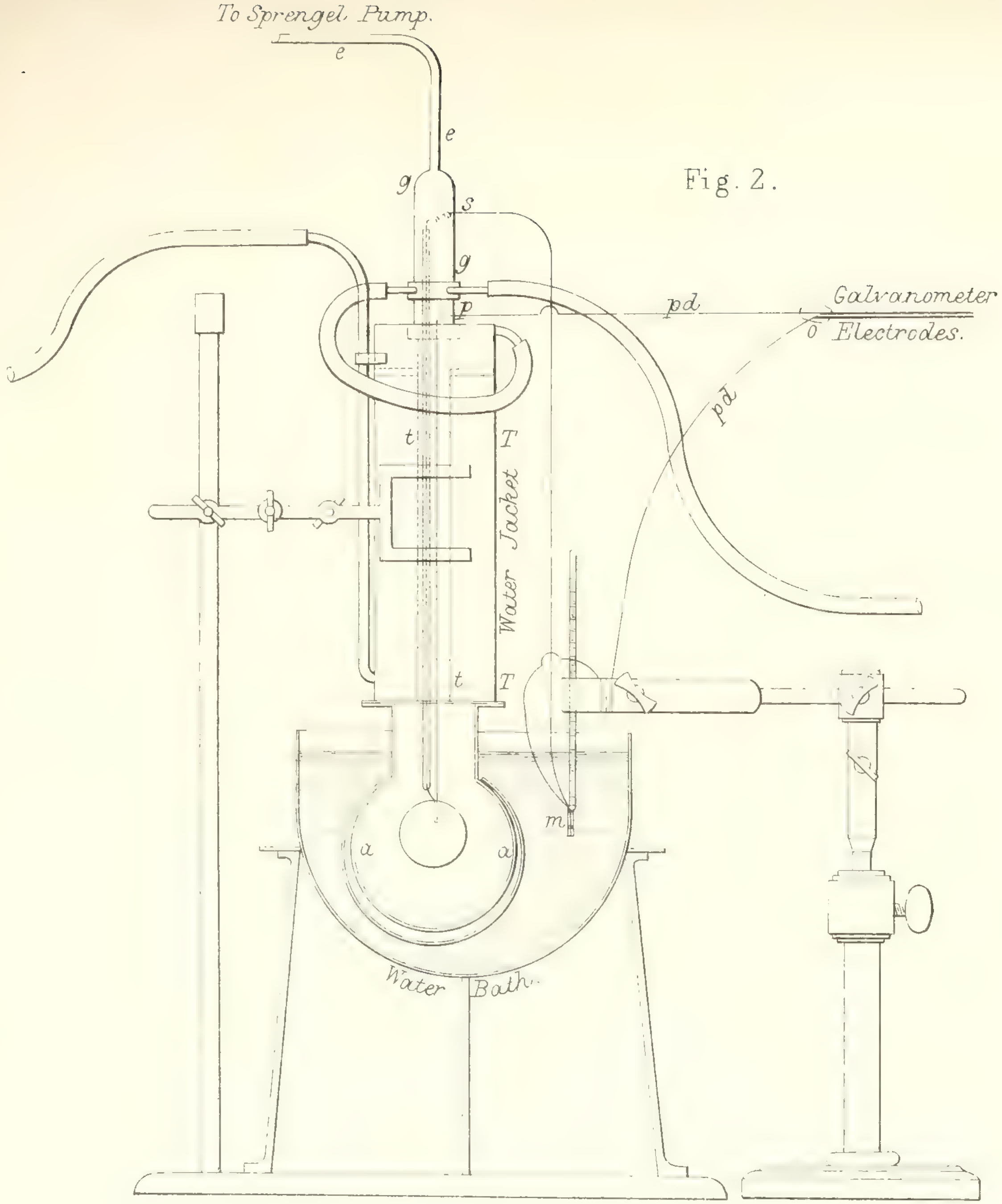

To Sprengel Pump.
e
e
g
s
g
p
pd
Galvanometer
o
Electrodes.
pd
T
t
Water Jacket.
t′
T
α
m
α
Water Bath.

Fig. 2.


To Sprengel Pump.
e
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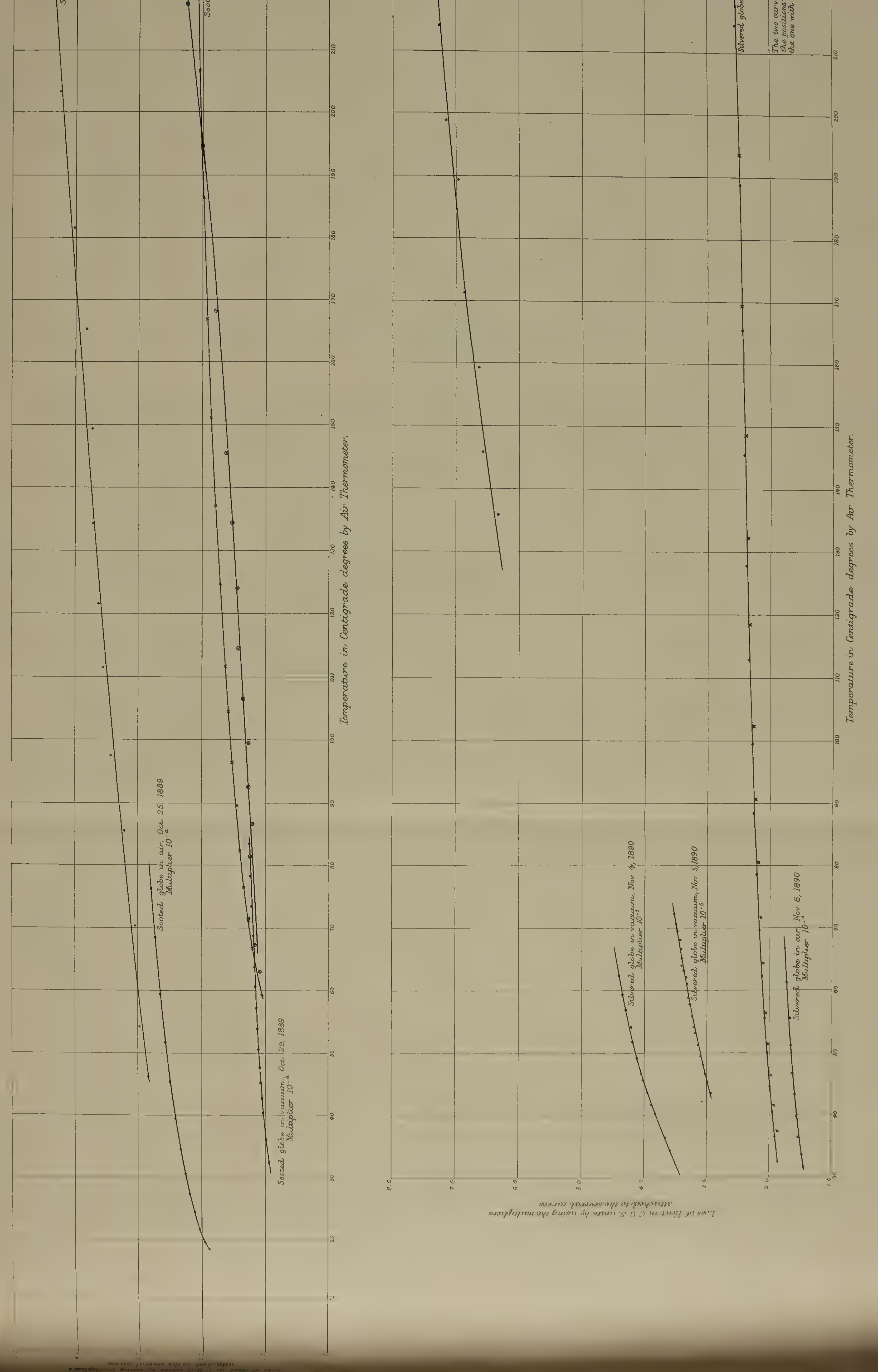
α
Water Bath.

Bottomley.


Sooted globe in air, Oct. 25, 1889
Multiplier 10⁻⁴
Sooted globe in vacuum, Oct. 29, 1889
Multiplier 10⁻⁴
Soot
S
Temperature in Centigrade degrees by Air Thermometer.
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
190
200
210
Silvered globe in vacuum, Nov 4, 1890
Multiplier 10⁻⁵
Silvered globe in vacuum, Nov 5, 1890
Multiplier 10⁻⁶
Silvered globe in air, Nov 6, 1890
Multiplier 10⁻⁴
Silvered globe
The two curv
the positions
the one with
Temperature in Centigrade degrees by Air Thermometer.
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
Loss of heat in C.G.S. units by using the multipliers
attached to the several curves

XII. *Harmonic Analysis of Hourly Observations of Air Temperature and Pressure at British Observatories.*—Part I. *Temperature.*

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[PLATES 19–23.]

THE Meteorological Council have lately published a volume entitled 'Harmonic Analysis of Hourly Observations of Air Temperature and Pressure at British Observatories.' It was thought preferable that this publication should be limited to the series of Tables giving the computed values of the harmonic constants, with a brief introduction explaining how the calculations had been carried out, and that the discussion of the results should be embodied in a separate memoir, which I hoped to communicate to the Royal Society, an intention which I now realize.

I have annexed to the present communication a selection of such of the Tables given in the volume referred to as appear necessary for my present purpose, and I have added a series of graphical representations of some of the results of the computations, which will facilitate the study of the subject.

I have also thought it convenient to reproduce, with some few modifications, the explanations, contained in the introduction of the same volume, of the method according to which the computations it contains were made.

The series of Tables to which I have referred give the results of computations commenced by myself, and completed in the Meteorological Office under the supervision of Mr. R. CURTIS.

The computations, as originally undertaken, were designed to supply the harmonic analysis of the hourly observations of temperature and pressure made at Greenwich Observatory, in each case for twenty years, which were published in 1878.

Subsequently it was determined to extend the investigation, as a part of the regular routine of the Meteorological Office, so as to obtain the harmonic constants from the photographic records of temperature and pressure of the self-recording instruments at the seven observatories maintained by the Meteorological Office, viz., Valencia, Armagh, Falmouth, Glasgow, Stonyhurst, Aberdeen, and Kew, for a series of twelve years.

With a view to reducing the necessary labour, the computations of the harmonic components, extending to the fourth order, have been made from the Greenwich data by the method proposed by myself (and explained in the 'Proceedings of the Royal Society,' vol. 42, pp. 61–79), with the help of the Tables to be found in the Appendix to the "Hourly Readings of the Self-recording Instruments at the Observatories of the Meteorological Council" for 1884.

The values of the constants for the seven observatories of the Meteorological Office, extending to the third order, were obtained from the photographic records by means of the mechanical analyser designed by Sir W. THOMSON, a description of which will be found in the 'Proceedings of the Royal Society,' vol. 27, p. 371, and as to the use of which and the degree of dependence to be placed on its indications reference may be made to the 'Annual Reports of the Meteorological Office,' for the years 1881, 1883, 1885, and 1890, and to the 'Proceedings of the Royal Society,' vol. 40, p. 382.

In these Tables, following the usual formula expressing the hourly value A_n in terms of its harmonic components,

$$A_n = p_0 + p_1 \cos n\,15^\circ + q_1 \sin n\,15^\circ + p_2 \cos 2n\,15^\circ, \text{ \&c.},$$

p_0 represents the mean value for the whole twenty-four hours; the coefficients of the cosines are designated by the letters p_1, p_2, p_3, &c., for the several orders of components; and those of the sines by the letter q_1, q_2, q_3, &c. The values are expressed in degrees Fahrenheit for temperature, and in inches for pressure.

The total amplitudes of the several components in the corresponding formula, involving sines only, are designated by the letter P_1, P_2, &c., where $P_r = \sqrt{(p_1^2 + q_1^2)}$, and

$$A_n = p_0 + P_1 \sin (n.15^\circ + T_1) + P_2 \sin (2n.15^\circ + T_2) + P_3 \sin (3n.15^\circ + T_3), \text{ \&c.}$$

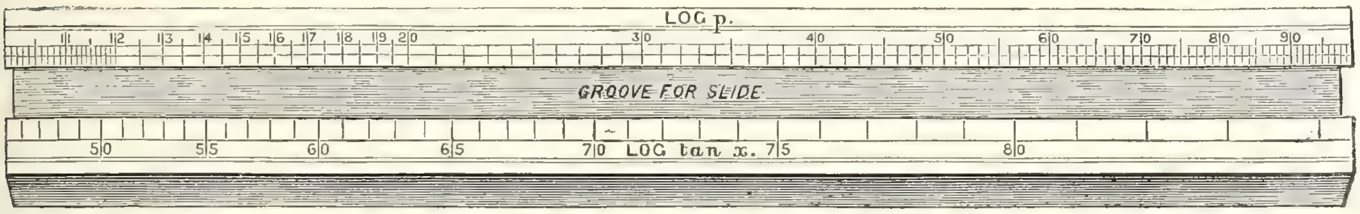
The Tables, however, instead of giving the values of the subsidiary angles T, show the epochs of the first maximum reckoned from midnight in angular measure, the letters μ_1, μ_2, μ_3, &c., indicating the epochs for the several components. This method of stating the phase or epoch of the components has the advantage of directly indicating their relation to the hour of the day, and, therefore, to the position of the Sun, to which the diurnal variations of temperature and pressure should obviously be referred.

If it be understood that $\tan x = p/q$, disregarding the signs of the coefficients, then the quadrants in which T will fall when regard is had to the signs of p and q, and the relation of μ to T and x, for the several components, are shown in the following Table.

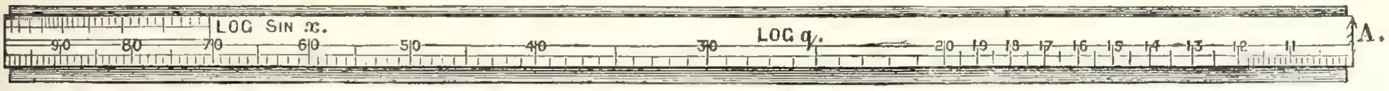
	$p+, q+$.	$p-, q+$.	$p-, q-$,	$p+, q-$.
Value of T	x 0 to 90°	$360^\circ - x$ 270° to 360°	$180^\circ + x$ 180° to 270°	$180^\circ - x$ 90° to 180°
Value of μ_1 . . .	$90^\circ - T_1$ $90^\circ - x$	$450^\circ - T_1$ $90^\circ + x$	$450^\circ - T_1$ $270^\circ - x$	$450^\circ - T_1$ $270^\circ + x$
Value of μ_2 . . .	$45^\circ - \frac{1}{2}T_2$ $45^\circ - \frac{1}{2}x$	$225^\circ - \frac{1}{2}T_2$ $45^\circ + \frac{1}{2}x$	$225^\circ - \frac{1}{2}T_2$ $135^\circ - \frac{1}{2}x$	$225^\circ - \frac{1}{2}T_2$ $135^\circ + \frac{1}{2}x$
Value of μ_3 . . .	$30^\circ - \frac{1}{3}T_3$ $30^\circ - \frac{1}{3}x$	$150^\circ - \frac{1}{3}T_3$ $30^\circ + \frac{1}{3}x$	$150^\circ - \frac{1}{3}T_3$ $90^\circ - \frac{1}{3}x$	$150^\circ - \frac{1}{3}T_3$ $90^\circ + \frac{1}{3}x$
Value of μ_4 . . .	$22\frac{1}{2}^\circ - \frac{1}{4}T_4$ $22\frac{1}{2}^\circ - \frac{1}{4}x$	$112\frac{1}{2}^\circ - \frac{1}{4}T_4$ $22\frac{1}{2}^\circ + \frac{1}{4}x$	$112\frac{1}{2}^\circ - \frac{1}{4}T_4$ $67\frac{1}{2}^\circ - \frac{1}{4}x$	$112\frac{1}{2}^\circ - \frac{1}{4}T_4$ $67\frac{1}{2}^\circ + \frac{1}{4}x$

The values of P and μ were throughout obtained from the calculated values of p, q, by means of a slide rule arranged for the solution of the equations, $\tan x = p/q$, and $P = p/\sin x$, as shown below :—

Fixed Rule.



Slide.



The upper scale of the *fixed* rule is graduated to show logarithms of numbers, and serves to indicate the value of $\log p$. The graduations of the lower scale of the same rule represent log tangents to radius equal 1 on the upper scale and indicate $\log \tan x$.

The graduations of the lower edge of the *slide* rule are logarithms of numbers in the reverse order to those of the *fixed* rule, and indicate the values of $\log q$. The upper edge of the *slide* rule is graduated to show log sines to the same radius as before, and indicates the value of $\log \sin x$.

The rule is used as follows :—The *slide* is moved in the groove till the zero point A, on its top edge, comes opposite to the graduation corresponding to $\log p$ on the *fixed* rule. Then opposite the graduation corresponding to $\log q$ on the lower edge of the *slide* will be found, on the scale of log tangents, the value of the angle x, or $\log \tan x = \log p - \log q$.

Then, without moving the slide, the value of P will be found on the scale, showing log p, opposite the graduation on the *upper* edge of the slide corresponding to log sin x; or log P = log p − log sin x.

The first series of Tables in the volume spoken of deal with Greenwich air temperature, the next refer to Greenwich pressure. Then follow the results obtained by the mechanical analyser for the seven observatories, first those dealing with temperature, and next those for pressure.

The Tables for each series (a few of which only are reproduced with this paper) are arranged as follows :—

No. I. gives the values of the p, q coefficients, in the case of Greenwich, for the first four orders for each year and for each month, and for the seven observatories of the Meteorological Office for the first three orders. The mean values, in the case of Greenwich, for each series of five years, as well as the means for the whole 20 years, are also given; and for the seven observatories for the period of 12 years.

No. II. is deduced from No. I., and gives in like form the values of P and μ for the four, or three components (as the case may be) for each year and month, and the corresponding means.

It will be observed that, in a few cases, μ has a negative sign, which implies, as will be further explained, that the epoch of maximum is thrown back *before* midnight.

No. III. is derived from No. I., and gives the mean annual values of the p, q coefficients for each year, with the five-year and 20-year means as before.

No. IV. is also derived from No. I., and gives the means of the several monthly values of the p and q coefficients for the 20 years. Table A, annexed to the present paper (p. 639), reproduces the results for the Greenwich observations, and Table C those for the seven observatories.

No. V. is derived from No. II., and gives the mean annual values of P and μ for each year, with the five-year and 20-year means.

No. VI. is also derived from No. II., and gives the means of the several monthly values of P and μ for the 20 years.

No. VII. brings together the monthly mean values of P and μ for the first four orders of components, for the four five-year periods over which the observations extend; the phase of maximum being in this Table stated in *apparent* time, instead of in *mean* time, as is done in the others. Table B, which is annexed to this paper, gives these results for the Greenwich observations, and Table D those for the seven observatories.

No. VIII. shows the mean monthly values of P and μ, obtained from the mean hourly values for each month and the entire year for the whole series of 20 years, as given in No. IX. These values will be seen to differ from those supplied by No. VI., and would be the values of P and μ, deduced from the mean values of p, q given in No. IV. The differences are due to No. VI. giving the means of the whole of the *separate* values obtained from the observations of the separate years and months, while the results in No. VIII. are obtained from the mean of the hourly values for the whole period in the aggregate, in which the effects of some of the fluctuations are necessarily lost. For the same reason, the mean value for the year, in No. VIII., is not the arithmetical mean of the 12-monthly values.

No. IX. gives the mean hourly values for each month of the year, obtained from the Greenwich hourly observations. This Table is extracted from the volume of Greenwich meteorological observations for 20 years.

No. X. gives the corrections that should be applied to the Greenwich observations on account of the non-periodical variation between the initial and final midnights of the daily period. The nature of this correction, and how the non-periodical variation affects the computations of the harmonic constants, are explained in the papers before referred to; (Proceedings of the Royal Society, vol. 42).

No. XI. gives the mean value for each hour and for each year, corrected as above, and the mean for the whole 20 years.

The Tables VIII. to XI. could not be prepared in the case of the seven observatories of the Meteorological Office, inasmuch as the numerical mean values of the hourly readings had not been obtained when the calculations now published were carried out.

The 20-years series for Greenwich temperature extends from 1849 to 1868, and that for pressure from 1854 to 1873, so that the results will only be strictly comparable for the 15 years from 1854 to 1868. The five-year means compared with those for the whole series of years serve to indicate how far the mean results, obtained from relatively short periods, are likely to deviate from those got for long periods. These computations also show the degree of consistency of the computed values of the quantities dealt with, from year to year, and from month to month, in a considerable series of years.

It is necessary to state that there has occasionally been some difficulty and uncertainty attending the calculation of the mean values.

In the first place, in the series of values of the p, q coefficients, if the signs change periodically, as will be seen to be the case in the third and fourth orders, the arithmetical mean can give no true indication of the mean extent of the variations that have actually taken place. For instance, if the positive and negative quantities are in the aggregate nearly equal, the mean result would be nearly zero, though, in fact, the variations may have been very marked, and the positive and negative sums considerable.

Again, as the several components of the diurnal curve of orders above the first are recurrent, there is in some cases an ambiguity arising from the difficulty of determining whether a change of phase in successive months takes place by occurring later or earlier. This, no doubt, is mainly due to the monthly periods, for which the means are computed, being too great to admit of following the several steps of a rapid change, such as takes place about the equinoxes in the components of the third and fourth orders. A somewhat similar difficulty arises at times in the case of what may be termed casual irregularities in the position of the phase of maximum, in dealing with some of which it becomes uncertain whether the maximum has been thrown forward or backward.

In dealing with the series of Greenwich temperature values, a doubt of this sort arises in relation to the abrupt changes in the value of μ_4 between January and February, and again between October and November, which might be attributed either to retrogression of the phase of maximum, or to progression. The uncertainty, however, in such a case hardly affects the mean monthly values, which are fairly consistent; but a doubt is introduced in determining an annual mean, the numerical value of which will depend on the conclusion adopted as to the way in which the changes take place. This hesitation is unavoidable in the absence of a series of values calculated for short periods while the change is going on.

Much greater uncertainty prevails in dealing with the barometric values than with the thermometric, the irregularities in the former being far more numerous and perplexing.

It may be noticed with reference to the quantities p, q, that their absolute amounts represent the extent of the *amplitudes* of the components, while their *signs* only indicate the phase, or epoch of maximum, in relation to the successive quadrants in each recurring series of components. A succession of signs in the order $+p$, $+q$; $-p$, $+q$; $-p$, $-q$; $+p$, $-q$, corresponds to an epoch of maximum gradually becoming later, and *vice versâ*. The mutual destruction of a series of positive and negative values of p, q will, therefore, merely signify that there is no true time of maximum, or that all positions are equally probable or uncertain.

When, as is not seldom the case, there are considerable variations in the values of μ, an assumption may become necessary, in computing the mean of these values, as to the direction in which the departure of the separate values from the mean should be reckoned, whether onward or backward.

For instance, in the Greenwich tables for temperature, the value of μ_2 in the month of May for the year 1885 would by the computation appear as 85° 1′. Inasmuch, however, as will be seen from the separate values, the mean value is evidently much nearer to the commencement of the first quadrant, to which μ_2 appertains, than to its termination, it has been assumed that the value instead of being 85° 1′ should be represented by $-$ 4° 59′; that is to say, that the departure of the maximum from the mean has taken place by occurring *earlier* and not *later*. All the entries of μ in the Tables marked with the minus sign indicate that an assumption of this description has been made.

PART I.—TEMPERATURE.

Greenwich Temperature.

The series of Tables dealing with Greenwich temperature shows that, with considerable variations in the absolute values of all the p, q coefficients for the several months, and from year to year, as well as of the total amplitudes P, and the epochs of maximum μ, there is still a strongly marked general consistency in the characteristics of the various elements. Reference to Table B annexed to this paper (which is Table VII. of the Greenwich series above described) shows very clearly that, with relatively few exceptions, chiefly to be found in the winter months, and among those components the absolute values of which are very small, the mean monthly epochs of maximum in the several five-year periods differ among one another by very small amounts, corresponding to intervals of time seldom as large as half-an-hour. As, however, the weight of the conclusions that may be drawn from these computations depends greatly on the internal evidence supplied by their greater or less consistency among themselves, it is important to dwell somewhat more on this point.

The requisite test will best be supplied by the values of μ which indicate the times of maximum phase, and are, therefore, immediately dependent on the Sun's action, and should follow his position; and this being subject to regular periodical change in the course of the year, it would follow that if the values of μ show corresponding regular, or nearly regular, periodical variations, they may be regarded as affording valid internal evidence of being trustworthy. The values of P, the total amplitude or variation of temperature, which are certainly subject to influences of a much more local and irregular character, will be liable to much greater and more irregular changes.

A convenient and succinct summary of the facts bearing on this point will be found in the following Table, which shows for the series of 20 years the number of times (reckoned as a percentage) in which the computed values of μ for single months fall in the several quadrants, the hours corresponding to which are as follows:—

TIME of Maximum according to Value of μ falling in successive Quadrants.

Components.	1st quadrant.	2nd quadrant.	3rd quadrant.	4th quadrant.
First . . .	Midnight to 6 A.M.	6 A.M. to noon	Noon to 6 P.M.	6 P.M. to midnight
Second . .	,, ,, 3 ,,	3 ,, 6 A.M.	6 A.M. to 9 A.M.	9 A.M. to noon
Third . . .	,, ,, 2 ,,	2 ,, 4 ,,	4 ,, 6 ,,	6 ,, 8 A.M.
Fourth . .	,, ,, 1.30 ,,	1.30 ,, 3 ,,	3 ,, 4.30 ,,	4.30 ,, 6 ,,

TABLE showing frequency of the Value of μ falling in the several Quadrants in different Months of the Year.

Month.	1st component.	2nd component.		3rd component.				4th component.				
	Quadrant.	Quadrant.		Quadrant.				Quadrant.				
	3	4	1	4	1	2	3	4	1	2	3	4
January	100	..	100	..	..	..	100	15	60	25		
February	100	..	100	..	..	..	100	25	5	..	70	
March	100	..	100	5	20	75	..	..	..	..	95	5
April	100	5	95	..	100	..	..	..	..	20	70	10
May	100	35	65	..	100	..	..	5	45	50		
June	100	45	55	..	100	..	..	..	100			
July	100	10	90	..	100	..	..	5	35	60		
August	100	..	100	..	100	..	..	..	..	55	45	
September	100	..	100	..	84	16	..	..	..	5	95	
October	100	..	100	..	..	40	60	..	..	10	85	5
November	100	..	100	..	..	..	100	25	40	25	10	
December	100	..	100	..	..	..	100	15	75	10		

NOTE.—Where the 4th quadrant is shown as preceding the 1st quadrant, it is assumed that the phase has fallen back, or occurred earlier.

These figures show that in their main features the variations in the value of μ for the second, third, and fourth components have a truly periodical character, and that as the year passes from winter to summer the maximum phase of the second and third travels backwards, that is, gradually occurs earlier in the day, while it returns in the opposite direction, becoming gradually later, in the change from summer to winter. In the fourth component corresponding changes take place between the equinoxes and the solstices.

A reference to Table B from the Greenwich series will show in more detail how well the consistency of the results is maintained in all the components.

The first component μ_1 is always in the first quadrant, *i.e.*, between noon and 6 P.M. The variation of the five-years mean from the 20-years mean is in no month more than $2\frac{1}{2}°$, or 10 minutes of time, and the average for all the months is less than half that amount.

In the second component the retrogression of the phase of maximum in the summer months leads to μ_2 (which from August onwards to March is always in the first quadrant, or between midnight and 3 A.M.) falling back into the fourth quadrant, that is occurring before midnight, and the greater frequency of this in May and June is therefore quite consistent with the regular periodical change which brings the mean value of μ_2 in June very nearly to zero, or midnight. The variation of the five-years mean from the 20-years mean is in no month greater than 6° or 24 minutes of time, and for the whole year the average variation is only $2°{\cdot}3$, or 9 minutes of time.

In the third component, from April to August, μ_3 is always in the first quadrant, or between midnight and 2 A.M. In September and October it changes rapidly to the third quadrant, in the last of these months 40 per cent. of the values being found in the second quadrant, and in November the value is established in the third quadrant, that is between 4 A.M. and 6 A.M., and there it remains till the end of February. In March μ_3 falls back to the first quadrant, 75 per cent. of the values being in the second quadrant and the rest in the first quadrant, with the exception of a single case which falls back to the fourth quadrant, or before midnight. The variation in any month of the five-years mean from the mean of 20 years in no case exceeds 5°, or 20 minutes of time, and the mean variation for all months is $2^\circ{\cdot}1$, or $8\frac{1}{2}$ minutes.

The fourth component shows double maxima and minima values of μ_4, the former occurring at the two equinoxes, the latter at the two solstices. In the winter months, November to January, the greatest frequency is in the first quadrant, between midnight and 1.30 A.M., with a considerable number before midnight or in the fourth quadrant, and a few in the second quadrant, or after 1.30 A.M. Again in May, June, and July the values of μ_4 are chiefly in the first and second quadrants, with a few cases in the fourth quadrant, while the values in June invariably fall in the first quadrant. From February to April, and again from August to October, the greatest frequency of μ_4 is in the third quadrant, or between 3 A.M. and 4.30 A.M., with deviations in the one direction to the second quadrant, and in the other to the fourth quadrant. In March and September 95 per cent. of the values fall in the third quadrant; February and April on the one side, and August and October on the other, being months of transition, between the equinoctial maxima and solstitial minima. The largest variation of the five-years mean from that for 20 years, for any month, is 10°, or 40 minutes of time, and the average for all months $4^\circ{\cdot}3$, or 17 minutes. The absolute values of the p_4, q_4 coefficients being very small, the maximum being less than $\frac{1}{4}^\circ$, and the average being less than $\frac{1}{10}$th degree Fahrenheit, the computed values of μ_4 are necessarily far more liable to error than those for the other components, and it is rather a matter for surprise that the variations should be so small, than that they should reach the actual quantities mentioned.

The values of μ in all cases depending on that of the angle whose tangent is p/q, the consistency of the value of μ implies consistency in the ratio of p to q. At the same time as $\mathrm{P} = \sqrt{(p^2 + q^2)}$ a considerable variation of the absolute value of P is quite compatible with consistent or invariable values of μ, and such variation in P will not tend to discredit the evidence in favour of such values of μ being trustworthy.

It may here also be noticed that as the value of P may obviously be expressed as $p/\sin x$, where $\tan x = p/q$, it follows that if the value of x, and consequently the epoch of maximum phase be regarded as fixed, for any time or place, then P will simply vary as p.

On the whole it may be regarded as sufficiently established that the computed components P and μ represent well marked truly periodical variations, even, as far as the fourth order, in which the maximum value of P hardly exceeds one-third of a degree Fahrenheit.

The component of the first order, which in the winter months is more than double the magnitude of any other, and in the summer more than ten times as large as any of them, gives the dominant character to the diurnal curve of temperature. There is an obvious tendency for high or low values of P_1 to be associated with high and low values, respectively, in the other components, though there are many deviations from such a rule.

In the series of 20 years instances may be found in almost every month, of variations in the value of P_1 for different years, of as much as 100 per cent., and the maximum values of the amplitude P in all components is frequently double the minimum. Nevertheless, these irregularities for the most part disappear even in the mean of a series of five years, and the monthly mean values for 20 years are as a whole remarkably consistent, the mean difference between the five-year and the 20-year values being less than 5 per cent.

The progression of the magnitude of P_1 in the course of the year follows approximately the Sun's meridional zenith distance, and the empirical formula $P_1 = 10 \cos z - \cdot 91$ gives a close approximation to the mean monthly values shown in the Tables, if a "lagging" of about eight or ten days is allowed in reckoning the zenith distance, as the following comparison will show:—

Month.	Meridional zenith distance.	Cos z.	Value of P_1.		Error of formula.
			From formula.	From tables.	
	° ′				
January . .	73 37	·282	1·91	1·90	+·01
February . .	66 47	·394	3·03	3·16	−·13
March . . .	55 42	·564	4·73	4·67	+·06
April . . .	44 31	·713	6·22	6·50	−·28
May	36 4	·808	7·17	7·30	−·13
June	29 23	·871	7·80	7·71	+·09
July	29 11	·873	7·82	7·75	+·07
August . . .	34 29	·814	7·23	7·17	+·06
September . .	45 39	·699	6·08	6·46	−·38
October . . .	57 20	·540	4·49	4·16	+·33
November . .	67 47	·378	2·87	2·80	+·07
December . .	73 38	·282	1·91	1·65	+·26

The amplitude of the component of the second order, P_2, has two clearly marked maxima, about the time of the equinoxes, with a principal maximum at midsummer. The component of the third order, P_3, varies in a converse manner, having two well marked minima at the time of the equinoxes, with a principal maximum at midsummer. In the fourth order, P_4 appears to combine the characters of the two former, having two maxima near the equinoxes, and a principal minimum in the winter.

The annual variation of the values of P_2 and P_3 will be represented, with a fair degree of approximation, by the expressions

$$P_2 = 1{\cdot}08 + {\cdot}20 \cos (\lambda + 126^\circ) + {\cdot}41 \cos (2\lambda - 2^\circ)$$
$$P_3 = {\cdot}42 + {\cdot}16 \cos (\lambda + 260^\circ) + {\cdot}10 \cos (2\lambda - 172^\circ),$$

in which λ is the Sun's longitude. The term involving cos 2λ gives rise, in the value of P_2, to the two maxima at the equinoxes, and in the value of P_3 to the two minima at the same seasons. The comparison of the observed and computed values of the two components for the several months is given below.

Month.	P_2.			P_3.		
	Observed.	Calculated.	Difference.	Observed.	Calculated.	Difference.
January . . .	·94	·91	− ·03	·36	·37	+ ·01
February . .	1·26	1·22	− ·04	·30	·30	·00
March . . .	1·34	1·38	+ ·04	·18	·29	+ ·11
April	1·21	1·17	− ·04	·48	·40	− ·08
May	·71	·76	+ ·05	·63	·54	− ·09
June	·56	·51	− ·05	·61	·62	+ ·01
July	·66	·70	+ ·04	·65	·58	− ·07
August . . .	1·11	1·18	+ ·07	·55	·46	− ·09
September . .	1·71	1·57	− ·14	·29	·37	+ ·08
October . . .	1·51	1·55	+ ·04	·29	·34	+ ·05
November . .	1·17	1·17	·00	·40	·39	− ·01
December . .	·76	·85	+ ·09	·33	42	+ ·09
		Sum .	·63	. .	. .	·69
		Mean .	± ·05	. .	. .	± ·06

It has already been remarked that the epoch of the maximum phase of the first three components appears to be earlier in the summer and later in the winter.

In the first order the mean value of μ is 214°, corresponding to 2 h. 26 m. P.M., the variation due to season being about 12°, or 48 minutes of time, by which amount the maximum is earlier in June and July than in December.

In the second order the first maximum in June is 20°, or 1 h. 20 m. earlier than in January. In the third order the difference in the same direction is 63°, or 4 h. 12 m. of time.

In the fourth order, as before noticed, there is some doubt as to the manner in which the change of positions of the summer and winter maximum is brought about. From March, when the first maximum occurs about 60° after midnight, or at 4 A.M., there is no doubt a retrogression, as in the other components, till June, when the maximum occurs 16° after midnight, or at 1 h. 4 m. A.M.; this is followed by a progression between June and October, when the maximum having become later, is again at about 60°, or 4 A.M. But a sudden change takes place in passing from October to November, which might be attributed either to the time of maximum rapidly advancing, that is, occurring later, or to its sudden recession, or becoming earlier; and, in whichever way it is brought about, one of the recurring epochs of maximum is established, from November to January, at about 10° after midnight. There is a like sudden change between January and February in the opposite direction, which again brings the maximum to about 60° from midnight, or 4 A.M. It will be seen that in February and November the absolute amplitude of this component is very small, and probably these sudden changes are coincident with the component becoming zero. (See Plate 23.)

Remembering that the fourth component of the diurnal curve includes a series of four undulations 90° apart, the explanation of what has just been stated, is probably to be found in a rapid change of conditions, under which the position of the first or earliest of these undulations recedes, until its place is taken by what was the second; so that as the maximum of the first undulation gradually becomes earlier, and at length occurs at 0°, or midnight, the maximum of the second undulation is approaching 90°, or 6 A.M., from which position it rapidly further recedes between January and February, in the last of which months it is found at 60°, or 4 A.M. The converse of such a process would explain the sudden change between October and November. On this hypothesis the numerical value of μ_4 in the months of November, December, and January, as given in the Tables, should be increased by 90° to render them properly comparable with the values in the other months. This would change the mean value for the year from 35° to 57°.

In proportion as the diurnal curve of temperature tends to greater simplicity, the magnitude of the component of the first order exceeds that of the others, and, in the months of May, June, and July, in which the amplitude of the first component is more than ten times that of any of the others, the temperature during the day, between the hours of 8 A.M. and 8 P.M., hardly deviates from a curve of sines. The mean temperature of the day coincides almost exactly with the temperature of the last-named hours, and the excess of the diurnal maximum over the mean hardly differs from the value of P_1, which, as before shown, is directly dependent on the Sun's meridional zenith distance. (Plate 21.)

The periodical variations in the amplitudes and epochs of the second, third, and fourth components, which are indications of the departures of the hourly temperatures from the simple curve of sines represented by the first component, are, without doubt, connected with the varying length of the day and other influences dependent on the time of year, among which the direction of the prevailing winds, the greater or less transparency of the atmosphere, and the amount of vapour will have places. All of these, however, will be connected in a more or less direct manner with the Sun's position and surrounding terrestrial conditions, though an expression for this connection may be difficult to discover or define.

The relation of the epoch of the first maximum in the third component to the time of sunrise—which, though directly dependent on the Sun's declination, may, of course, equally be regarded as a function of the Sun's longitude—seems to be decidedly marked; the former occurring, on the average of the whole year, at an interval of 48°, or 3 h. 12 m. earlier than the latter, or, what is the same thing, the first minimum occurring 12° or 48 minutes after sunrise, the mean deviation of the interval from that average being only 7°, or 28 minutes of time. A reference to Table B will illustrate this.

The periodical variation in the value of μ_3 leads to the third component having a *positive* maximum about 1 P.M. during the winter months, from October to February, which will be accompanied by maximum *negative* values four hours earlier and four hours later, corresponding with the *reduced* temperature in the mornings and afternoons of the *shorter* days. (Plate 23.)

In the summer months, from April to August, μ_3 has a maximum *negative* value about 1 P.M., instead of a *positive* maximum as in winter; and this, being accompanied by two *positive* maxima, one four hours earlier and the other four hours later, will in like manner correspond to the *higher* temperature in the mornings and afternoons of the *longer* day.

It will be seen that the positions of the midsummer and midwinter maximum phases correspond respectively to days of 16 hours with nights of 8 hours, or days of 8 hours with nights of 16 hours, and that at these seasons, when the variations of temperature due to these differences are greatest, the values of the amplitude of the component are also at a maximum. At the equinoxes, with a 12 hours day and night, the amplitude of the component has a minimum value, and the transition of the position of the maximum phase takes place as already described.

It might, perhaps, be expected that the change in the position of the maximum phase of the fourth component, μ_4, would take place in an analogous manner, in connection with days and nights of 6 and 18 hours duration respectively, corresponding to the epoch of the component. But such days and nights will only occur in higher latitudes than those of our observatories, and no data are available by which to test such a suggestion.

Although the several harmonic components of the temperature curve cannot be regarded as indication of specific physical efficient causes, acting at definite periodical harmonic intervals, the graphical representations of the series of monthly temperature curves and their components, which are given in the plates accompanying this paper, present some points to which attention may usefully be drawn.

As before noticed, the deviation of the *day* temperature from the first component is extremely small in the months of May, June, and July, during which between 8 A.M. and 8 P.M. the other components nearly cancel one another. From September to March however, the temperature during the day hours rises considerably above the first component, the second and third component either being both positive and operating in the same direction, or the third component if negative being overpowered by the other two, which from 10 A.M. to 4 P.M. are always positive.

The temperature curve between the hours of 4 A.M. or 5 A.M. and 7 A.M. or 8 A.M. always falls below the first component.

The first component having its maximum about 2 P.M. is always negative between 8 P.M. and 8 A.M.

The second component having a day maximum which varies between 11.30 A.M. to 1.30 P.M. is always negative between 4 A.M. and 8 A.M. In the summer months when the third component is largest, it is also negative between 4 A.M. and 7 A.M., and at other seasons, when it would operate in an opposite direction, its relatively small magnitude renders its effects unimportant.

The second component always having a maximum near noon will also have another near midnight.

In the summer months when the third component has a minimum shortly following noon, it will have a maximum shortly following midnight, that is nearly at the same time as the second component, and at this season the night temperature is found to be considerably above the first component. In the winter months the third component becomes negative during the hours when it was before positive, and the departure of the night temperature from the first component is much smaller.

In the summer months the time of morning mean temperature is nearly when the first component becomes zero, the second and third components then balancing one another.

In the winter the hour of morning mean temperature is later, and occurs when a positive value of the first component equals a negative value of the second.

The time of afternoon mean temperature throughout the year is somewhat either before or after 7 P.M., and almost exactly coincides with the time when the first and second components are equal but with opposite signs, the former being positive and the latter negative.

In the summer months the whole of the components are negative between the hours of 3 A.M. and 6 A.M., corresponding with the absolute minimum about this time.

In the winter the third and fourth components are not negative till after 6 A.M., while the negative branch of the first component is extended almost to 9 A.M., and the absolute minimum is found correspondingly later between 7 A.M. and 8 A.M.

The hour of absolute maximum is nearly coincident with that of the dominant first component, which is always between 2 P.M. and 3 P.M., the modifications due to the second component, which also has a maximum between noon and 2 P.M., being of secondary importance.

Sunrise in December is about an hour and-a-half before the time of mean temperature, while in June it is more than four hours before that time.

On the other hand, while sunset in December is rather more than three hours before the time of mean temperature, in June it is about half-an-hour after that time.

Also it will be found that in January whereas the temperature is above the mean of the whole 24 hours for not more than $8\frac{3}{4}$ hours, in June it is above the mean for nearly 12 hours.

It may here be pointed out that the *rationale* of some of the empirical rules for obtaining the mean daily temperature, from a limited number of observations made at certain stated hours, is supplied by reference to the harmonic expressions for the hourly deviations of the temperature from the mean of the whole day.

In the first place it is obvious, that by adding together the harmonic expressions for any two hours, twelve hours apart, the whole of the *odd* components disappear, and that the sum is twice the mean value added to twice the sum of the *even* components of the selected pair of hours, which are respectively equal to one another. Disregarding the components above the fourth order, as may be done in practice, if the hours selected be such that the component of the second order is zero, which will be the case when the hours correspond to $45° + \mu_2$, or $135° + \mu_2$, then half the sum of the observed temperatures at the selected pair of hours will be equal to the true daily mean added to the value of the component of the fourth order for the selected hours, which at English stations will hardly amount to so much as half a degree Fahrenheit. At Greenwich it will be found that in January the mean of observations at 4.30 A.M., and 4.30 P.M., or at 10.30 A.M., and 10.30 P.M., differ by less than $0°\cdot1$ from the true mean, and a similar result will be obtained in June by taking the mean of observations at 3 A.M. or 9 A.M., and at the corresponding afternoon hours.

For like reasons it will be seen that by taking the mean of observations at *any four* hours, at equal intervals of six hours, not only will the whole of the *odd* components disappear, but those of the *second* order also, so that the result will only differ from the true mean by the amount of the component of the fourth order for the selected hours. As the fourth component disappears at hours when $\mu_4 \pm 22\frac{1}{2}°$ equals $0°$ or $180°$, the hours at Greenwich which will give the best result will be found to be 2, 8, 14, and 20, or 5, 11, 17, and 23.

Again, if the mean of any *three* hours, at equal intervals of eight hours, be taken, it will be seen that the sums of the *first*, *second*, and *fourth* components disappear, and that the result will only differ from the true daily mean by the amount of the third component for the selected hours, which at English stations will hardly exceed three-quarters of a degree. The best result will be obtained when the selected hours correspond to $\mu_3 \pm 30° = 0°$, or $180°$, when the third component becomes zero, and at Greenwich for the greater part of the year this condition will be nearly complied with by the hours 3, 11, and 19, or 7, 15, and 23.

It will be readily understood that the true reason for such close approximations being so easily obtained, is the great preponderance of the first and second components of the temperature variation over all the rest.

Temperature at the Seven Observatories.

The mechanical analyser, by means of which the harmonic components of the temperature curves for these observatories have been obtained, is only adapted for computing the coefficients of the first three orders, so that the constants for the fourth order are not given in the published tables.

The values of P and μ were calculated from the values of the p, q coefficients, which are obtained by means of the instrument, in the same way as those for Greenwich.

An inspection of the table C and D, which are annexed, and contain a summary of the results, will at once show that in their main characteristics the results closely resemble those for Greenwich, and for this reason it will not be necessary to discuss them at great length.

The amplitude of the component of the first order P_1 is, however, in all cases below that of Greenwich, the lowest values being those of Valencia and Falmouth, no doubt due to their position on the coast, with means for the year of 2°·28 and 2°·35, compared with 5°·10 at Greenwich.

The Kew values most resemble those of Greenwich, but the mean maximum is more than one degree less, and the mean for the year about half a degree less. These differences are probably in a considerable degree due to differences of exposure, and local conditions of soil, &c.

The mean values of μ_1 for the year referred to local apparent time, lie between 205° and 220°, that for Greenwich being 214°. The summer values are somewhat below the mean, and the winter values somewhat above it, as is the case at Greenwich.

The amplitude of the first component conforms approximately, but not so closely as in the case of Greenwich, to the sine of the sun's meridian altitude, with a flattening of the curve in the summer months, and a tendency at some stations to a maximum value in May.

The component of the second order in all cases shows the double maximum at the equinoxes and the minimum value in summer. The numerical values of P_2 are, as a rule, below those at Greenwich, the yearly means varying between 0°·84 and 0°·61, against 1°·08 at Greenwich; Kew, as before, most closely resembling Greenwich.

The values of μ_2 generally follow the same law as at Greenwich, the epoch of maximum being earlier in the summer and later in the winter, this character being specially marked at Aberdeen, where the summer value of μ_2 is $-44°$, and the winter value 20°, a difference of 64°, or more than four hours of time, a result no doubt connected with the higher latitude of that station, the earlier sunrise and the longer day.

The component of the third order also in all cases closely resembles that at Greenwich, P_3 having the double minimum at the two equinoxes, and a maximum in summer. The mean yearly value is again somewhat below that for Greenwich, varying from 0°·23 to 0°·38, the Greenwich value being 0°·42.

The values of μ_3 accord very closely with those at Greenwich where the mean yearly value is 41°, the mean yearly values at the seven observatories ranging between 37° and 43°. The variations of the epochs of maximum from winter to summer are at all stations well marked, and at Aberdeen, as was the case with the component of the second order, μ_3 shows an exceptionally low value in the summer, differing about 70°, or nearly five hours of time, from the winter epoch.

The expressions for the annual variations of the values of P_2 and P_3, derived from the means of the observations at the seven observatories, differ little from those before given for the Greenwich values, allowance being made for the smaller amplitude of all the components. They are as follows:—

$$P_2 = {\cdot}68 + {\cdot}14 \cos(\lambda + 105^\circ) + {\cdot}21 \cos(2\lambda - 8^\circ),$$
$$P_3 = {\cdot}30 + {\cdot}13 \cos(\lambda + 260^\circ) + {\cdot}07 \cos(2\lambda - 166^\circ).$$

The comparison of the observed and computed monthly values of the components for several months is given below.

Months.	P_2.			P_3.		
	Observed.	Calculated.	Difference.	Observed.	Calculated.	Difference.
January .	·62	·62	·00	·22	·22	·00
February .	·78	·78	·00	·17	·18	+·01
March . .	·93	·85	−·08	·12	·19	+·07
April . .	·75	·63	−·12	·32	·30	−·02
May . . .	·42	·49	+·07	·53	·41	−·12
June . . .	·33	·33	·00	·43	·46	+·03
July . . .	·35	·40	+·05	·42	·41	−·01
August . .	·66	·65	−·01	40	·33	−·07
September .	·97	·90	−·07	·27	·25	−·02
October . .	1·02	·94	−·08	·15	·23	+·08
November .	·79	·78	−·01	·24	·28	+·04
December .	·59	·63	+·04	·22	·26	+·04
		Sum . .	·53		Sum . .	·51
		Mean . .	·04		Mean . .	·04

On the whole it may be regarded as well established by the substantial consistency of these results, that they truly indicate the chief characters of the diurnal inequality of temperature in the British Isles; and they leave no room to doubt that the variations which recur with such remarkable regularity from year to year, and are observed at so many different places, extending to quantities so small as those represented by the third and fourth order of harmonic components, are due to the operations of determinate physical causes, the effects of which this system of analysis enables us to distinguish and record with remarkable and unexpected precision.

I am not in a position to indicate, further than has already been done, any direct or precise relation between the observed thermometric results thus recorded, and the varying place of the Sun, on which in combination with the latitude and local influences of different descriptions they doubtless depend. The intermittent character of solar heat introduces a discontinuity of action which renders its connection with the resulting phenomena specially difficult of representation by algebraical expressions.

Acting on a suggestion of Professor G. DARWIN, I have calculated the harmonic constants that would reproduce a curve indicating an intermittent action such as that of the Sun, continuing only during a portion of the day, and commencing and ending abruptly at sunrise and sunset. The results are of some interest, and serve to throw light on the character and signification of the harmonic components of temperature that have been under discussion.

Such a calculation obviously disregards all cooling effects, and only deals with the Sun's heating action, which I have assumed to be proportional to the sine of his altitude ; and with a view to obtaining figures in some degree comparable with those obtained from actual observation, and following the analogy of the value of P_1, as previously pointed out, the total heating effect has been taken to be ten times the sine of the Sun's altitude, the power of a vertical Sun being that represented by ten.

On this assumption I have calculated the Sun's altitude for each hour of the day for midwinter, the equinox, and midsummer for certain selected latitudes, and corresponding heating effects. These computations supply a series of hourly values having their maximum at noon and becoming zero at sunrise and sunset, and disappearing so long as the Sun is below the horizon. Treating these in the usual manner, the resulting harmonic components as far as the fourth order are shown in the following Table F :—

TABLE F.

Latitude.	Midwinter.				Equinox.				Midsummer.			
	Components.				Components.				Components.			
	P_1.	P_2.	P_3.	P_4.	P_1.	P_2.	P_3.	P_4.	P_1.	P_2.	P_3.	P_4.
0°	−4·58	+1·92	0	− ·40	−5·06	+2·14	0	−·45	−4·58	+1·92	0·	−·40
20	−3·40	+1·70	− ·26	− ·30	−4·74	+2·02	0	−·43	−5·25	+1·74	+ ·29	−·30
30	−2·76	+1·42	− ·28	− ·11	−4·35	+1·68	0	−·37	−5·27	+1·55	+ ·38	−·21
40	−1·92	+1·23	− ·43	− ·06	−3·86	+1·60	0	−·35	−5·12	+1·22	+ ·43	−·05
45	−1·51	+1·03	− ·42	+ ·03	−3 52	+1·49	0	−·35	−5·19	+1·05	+ ·37	+·06
51½	−1·01	+ ·77	− ·39	+ ·12	−3·15	+1·33	0	−·30	−4·76	+ ·79	+ ·40	+·18
65	0	0	0	0	−2·21	+ ·95	0	−·20	−3·85	0	+ ·05	+·09

As from the nature of the hypothesis adopted the diurnal curve analysed is in all cases symmetrical on either side of noon, the values of q are all zero. Also the values of P and p become identical; the negative sign in the above table signifies that the component has a negative maximum value at midnight; and the positive sign that the component has a positive maximum value at the same hour. The values of μ for the several components will be seen to be as follows:—For the first component $\mu_1 = 180°$, or noon; for the second $\mu_2 = 0°$, or midnight; for the third $\mu_3 = 0°$ for the summer, and $\mu_3 = 60°$ for the winter, the change taking place at the equinox when the component becomes zero; for the fourth, in the lower latitudes, $\mu_3 = 45°$ at all seasons, in the higher latitudes at the equinox $\mu_4 = 45°$, passing to 0° both in the winter and summer.

For comparison with the above results, are given below the components of the diurnal temperature curve as actually observed at certain selected stations, varying in latitude, for the three seasons of the year.

TABLE G.

Stations.	Mid-winter.				Equinox.				Mid-summer.			
	Components.				Components.				Components.			
	P_1	P_2.	P_3.	P_4.	P_1.	P_2.	P_3.	P_4.	P_1.	P_2.	P_3.	P_4.
Singapore. Lat. 1° 15′ N.	−5·02	+1·78	−·28	−·18	−6·47	+2·04	+·68	−·48	−4·62	+1·54	+·32	−·51
Hong Kong. Lat. 22°18′ N.	−2·39	+1·01	−·67	−·81	−1·98	+·70	−·02	−·12	−1·86	+·63	+·09	−·13
Lyons. Lat. 45°46′ N.	−2·51	+1·01	−·41	+·07	−6·26	+1·49	+·11	−·31	−7·55	+·76	+·59	+·16
Greenwich. Lat. 51°30′ N.	−1·65	+·76	−·33	+·09	−4·67	+1·34	+·18	−·23	−7·71	+·56	+·61	+·21
Fort Rae. Lat. 62°40′ N.	−1·09	+·67	−·31	+·29	−7·71	+1·90	+·42	−·50	−6·01	+·59	+·19	+·06

The signs indicate the sign of the components at midnight.

It will hence be seen that while the intermittent curves representing the heating power of the Sun, under varying conditions of meridional altitude and length of day, are fairly well reproduced by combining the components of the first four orders having the values shown in Table F, these components bear a very close resemblance to those shown in Table G, obtained from the observed temperatures of places at the latitudes and seasons corresponding to those adopted in the computations from the intermittent curves.

The conclusion is unavoidable, that although both in the real and hypothetical cases the harmonic components, when combined, are truly representative of the peculiar forms of the curves from which they were derived, this affords no true indication of the existence of distinct physical influences operating in recurring cycles of 24, 12, 8, and 6, corresponding to the four orders of components; but that the results are, to a great extent, due to the particular form given to the analysis.

The component of the first order represents a variation from the mean value, which is symmetrical on either side of the epoch of its maximum, and necessarily involves a negative departure from the mean, equal to a corresponding positive departure at the hour of maximum, and twelve hours distant from the maximum.

As the diurnal curve of temperature is not symmetrical in relation to the mean value, the maximum day temperature being much more in excess of the mean than the minimum night temperature is below it, the first component, which is symmetrical in this respect, must be modified by the other components; and that of the second order, which has one of its maxima not far removed from the epoch of minimum of the first component, supplies the chief portion of the compensation necessary to correct the excess of that minimum over the true minimum.

Further, from the character of the analysis, the third component will be zero when the diurnal curve is symmetrical, but with contrary signs on either side of the hours half-way between noon and midnight, that is when the length of the day and night are each 12 hours. Any departure from this equality will introduce a component of the third order; with the result, that with a day shorter than 12 hours, one maximum phase will be between 6 A.M. and 6 P.M., and the other two in the hours before 6 A.M. and after 6 P.M.; while with a day longer than 12 hours, two maxima will be between 6 A.M. and 6 P.M., and the other in the hours after 6 P.M. and before 6 A.M. In the former case the two negative phases of the component will correspond with the reduced temperature in the morning and afternoon of the shorter day, and in the latter the two positive phases will correspond with the increased heat of the mornings and afternoons of the longer days.

It will also be apparent that with a day of 8 hours, and a night of 16 hours, or *vice versâ*, the epochs of the third component would tend to synchronize with the duration of the night and day, and its amplitude might be expected to be then greater, which is seen to be the case.

Analogous considerations will apply to the component of the fourth order. A positive maximum phase at or near midnight would give two maxima, near 6 A.M. and 6 P.M., respectively, corresponding to the long days in high latitudes in the summer, and also two negative minima at or near 9 A.M. and 3 A.M. respectively, corresponding to the short days in winter, in high latitudes. The Tables F and G both show P_4 with a positive sign under the circumstances just stated.

Again a negative phase at or near midnight would correspond to two negative phases at or near 6 A.M. and 6 P.M., respectively, which would correspond to the days and nights of nearly 12 hours in the lower latitudes and at the equinoxes, which conditions are also shown to exist by the two Tables before referred to, the sign of P_4 being in all cases negative at the equinox.

In connection with this view of the temperature components it may be noticed that if instead of reckoning the epochs of maxima as the phase nearest to midnight, that nearest noon were adopted, it would be found that the usual time of maximum phase of all the components is not far removed from noon, affording strong evidence that all these phases are closely dependent on the passage of the Sun over the meridian.

For Greenwich the following would be the results :—

	December.	March.	June.
	°	°	°
1st component	222	215	210
2nd ,, ...	200	198	181
3rd ,, ...	194	..	(193)
4th ,, ...	190	(193)	196

In case of the third and fourth components the figures enclosed in brackets ( ) are epochs of minimum.

In conclusion, I would observe that in the course of the preparation of the data on which this discussion has been based, I have computed the harmonic components of the temperature curves of many other places besides those here specially noticed, and that the results obtained from them entirely confirm those now set forth. I am not, however, in a position at the present time to present any satisfactory details of these investigations, involving as they do very tedious computations, which I have not yet been able to carry out in a systematic or complete shape.

The importance of the subject in connection with the study of the diurnal variations of atmospheric pressure cannot be overstated, and in the second part of this memoir, which will be devoted to that question, I shall have occasion to revert to it. Meanwhile I would invite the co-operation of persons interested in this branch of physics in the collection of data relating to temperature as complete as those now available in the case of atmospheric pressure.

NOTE.—In 1879 Sir G. STOKES communicated to the Meteorological Council, of which he was then a member, copies of letters addressed by him to Sir G. AIRY, containing the results of computations of the harmonic components of the Greenwich diurnal temperature curve, made in the ordinary manner, which differs somewhat from that adopted by me. These results supply the coefficients up to the fourth order for each of the 20 years of the Greenwich series, separately, and those of the fifth and sixth orders for the mean of the 20 years taken together. They are almost identical with the values given in the volume of tables published by the Meteorological Council, and the comparison of the means of the two sets of computations may be of some interest.

	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.	p_4.	q_4.	p_5.	q_5.	p_6.	q_6.
Sir G. STOKES	−4·280	−2·722	+·883	+·563	+·063	+·119	−·043	−·014	−·001	−·026	−·003	−·008
Meteorological Tables . .	−4·282	−2·738	+·885	+·557	+·066	+·118	−·046	−·015				

	P_1.	P_2.	P_3.	P_4.	P_5.	P_6.
Sir G. STOKES	5·072	1·047	·135	·045	·026	·009
Meteorological Tables . .	5·083	1·046	·134	·047		

It has not been thought worth while to repeat the detailed figures for the separate years, as they are virtually identical with those given in the Meteorological Office Tables.

Referring to a comparison that he had made between the mean observed hourly temperatures and the values obtained by computation from the daily means and the harmonic coefficients, Sir G. STOKES remarked that "the minute fourth order is swallowed up, even in a mean of 20 years' series, by the uncertainty in the amount of the diurnal fluctuation, not to speak of the much larger uncertainty in the mean temperature for a day. Nevertheless, it appears from the figures, that though this order is so minute, and though its effect on the mean temperature at any particular hour is trifling compared with the average variation from year to year of the same hour, it still emerges roughly with tolerable certainty from the mean of a good many years."

This conclusion is fully confirmed by the more complete examination of the results obtained from the mean *monthly* values, much of the strong evidence of the periodicity represented by the minute fourth order of components being lost by dealing with the mean *yearly* values.

TABLE A.—Mean Values (reckoned from Mean Midnight) of the p, q Coefficients for the First Four Orders of Harmonic Components of the Diurnal Inequality of Temperature at Greenwich, for each Month of the Year, obtained from 20 Years' Observations from 1849 to 1868.

Months.	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.	p_4.	q_4.	Mean temp. T.
January . . .	−1°·45	−1°·20	+ °·64	+ °·69	− °·24	− °·26	+ °·04	+ °·04	38°·9
February . .	−2·39	−2·06	+ ·88	+ ·89	− ·23	− ·17	·00	− ·07	39·7
March . . .	−3·71	−2·82	+1·02	+ ·87	− ·06	+ ·13	− ·12	− ·19	41·5
April	−5·46	−3·51	+1·04	+ ·56	+ ·27	+ ·36	− ·15	− ·08	47·2
May	−6·34	−3·61	+ ·63	+ ·17	+ ·50	+ ·38	− ·01	+ ·09	52·7
June	−6·69	−3·83	+ ·50	+ ·04	+ ·46	+ ·39	+ ·10	+ ·17	59·8
July	−6·58	−4·08	+ ·57	+ ·23	+ ·50	+ ·38	− ·01	+ ·12	62·5
August . . .	−6·06	−3·83	+ ·95	+ ·53	+ ·38	+ ·39	− ·16	+ ·01	61·9
September . .	−5·60	−3·21	+1·49	+ ·83	+ ·06	+ ·25	− ·23	− ·23	57·5
October . . .	−3·61	−2·04	+1·35	+ ·74	− ·25	− ·04	− ·10	− ·10	50·9
November . .	−2·29	−1·60	+ ·96	+ ·67	− ·35	− ·18	+ ·03	+ ·01	42·8
December . .	−1·22	−1·07	+ ·59	+ ·47	− ·25	− ·21	+ ·06	+ ·05	40·8
Mean 20 years	−4·282	−2·738	+ ·885	+ ·557	+ ·066	+ ·118	− ·046	− ·015	49·69

Table B.—Values of the Amplitude P, and the Apparent Time of the First Maximum μ, for the First Four Orders of Harmonic Components of the Diurnal Inequality of Temperature at Greenwich for each Month of the Year, for Five-Yearly Periods between 1849 and 1868.

	Five-yearly periods.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.	Year.
P_1	1st . .	1·90	3·25	4·94	5·80	6·97	7·76	7·39	6·85	6·20	3·94	2·74	1·44	4·93
	2nd . .	2·00	3·50	5·05	7·19	7·25	8·13	7·43	7·76	6·98	4·28	2·61	1·74	5·33
	3rd . .	1·99	3·17	4·42	6·05	7·24	6·55	7·62	6·75	5·82	3·93	2·89	1·79	4·86
	4th . .	1·71	2·71	4·27	6·94	7 74	8·40	8·57	7·30	6·74	4·47	2·97	1·62	5·29
	Mean . .	1·90	3·16	4·67	6·50	7·30	7·71	7·75	7·17	6·46	4·16	2·80	1·65	5·10
P_2	1st . .	·97	1·20	1·40	1·04	·66	·64	·72	1·05	1·57	1·50	1·09	·78	1·05
	2nd . .	1·00	1·33	1·43	1·45	·75	·63	·63	1·32	2·01	1·59	1·15	·81	1·18
	3rd . .	1·00	1·32	1·26	1·07	·82	·49	·57	·98	1·36	1·43	1·22	·77	1·03
	4th . .	·80	1·18	1·27	1 27	·59	·47	·72	1·07	1·83	1·66	1·22	·69	1·06
	Mean . .	·94	1·26	1·34	1·21	·71	·56	·66	1·11	1·71	1·54	1·17	·76	1·08
P_3	1st . .	·40	·28	·16	·49	·65	·73	·64	·56	·27	·23	·40	·28	·43
	2nd . .	·34	·31	·22	·49	·63	·58	·62	·52	·37	·35	·38	·35	·43
	3rd . .	·34	·28	·14	·47	·48	·43	·61	·46	·30	·23	·40	·36	·38
	4th . .	·35	·32	·19	·45	·77	·69	72	·67	·23	·35	·43	·33	·46
	Mean . .	·36	·30	·18	·48	·63	·61	·65	·55	·29	·29	·40	·33	·42
P_4	1st . .	·09	·12	·27	·18	·08	·15	·18	·20	·34	·14	·07	·10	·16
	2nd . .	·08	·11	·27	·24	·14	·24	·15	·15	·36	·15	·06	·08	·17
	3rd . .	·11	·08	·18	·19	·12	·17	·19	·15	·32	·16	·05	·10	·16
	4th . .	·09	·11	·19	·19	·13	·29	·17	·19	·33	·23	·11	·09	·18
	Mean . .	·09	·11	·23	·20	·12	·21	·17	·17	·34	·17	·07	·09	·16

TABLE B (continued).—Values of the Amplitude P, and the Apparent Time of the First Maximum μ, for the First Four Orders of Harmonic Components of the Diurnal Inequality of Temperature at Greenwich for each Month of the Year, for Five-Yearly Periods between 1849 and 1868.

	Five-yearly periods.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.	Year.
μ_1	1st . .	222°	217°	214°	212°	211°	208°	208°	213°	211°	212°	218°	223°	214°
	2nd . .	216	216	217	212	210	211	211	213	212	212	220	226	215
	3rd . .	213	216	214	214	212	209	211	212	210	213	218	217	213
	4th . .	217	220	215	113	210	211	212	213	211	216	218	222	215
	Mean . .	217	217	215	213	211	210	210	213	211	213	219	222	214
μ_2	1st . .	22	21	18	9	9	−8	0	16	19	19	22	18	14
	2nd . .	19	17	17	14	6	6	8	15	15	18	20	20	14
	3rd . .	23	19	22	12	17	9	11	17	13	18	20	23	17
	4th . .	20	20	16	17	−2	−1	13	14	16	17	21	20	14
	Mean . .	21	19	18	13	8	1	8	16	16	18	21	20	15
μ_3	1st . .	75	69	40	14	14	16	12	15	21	66	75	76	41
	2nd . .	74	70	30	19	14	12	12	18	36	68	73	73	41
	3rd . .	73	69	40	15	11	13	10	13	23	68	72	72	40
	4th . .	72	66	49	22	13	12	11	16	26	62	70	77	41
	Mean . .	74	69	40	18	13	13	11	16	27	66	73	74	41
μ_4	1st . .	12	56	58	47	25	16	23	46	58	62	25	12	58
	2nd . .	14	57	57	55	27	20	30	47	60	62	16	14	60
	3rd . .	10	65	56	48	16	17	24	38	57	51	−8	0	53
	4th . .	5	68	61	61	24	11	7	46	57	62	9	15	57
	Mean . .	10	62	58	53	23	16	21	44	58	59	11	10	57
	Time of sunrise	118°	106°	91°	77°	66°	58°	61°	72°	87°	101°	115°	122°	—

TABLE C.—Mean Values (reckoned from Greenwich Mean Midnight) of the p, q Coefficients for the First Three Orders of Harmonic Components, of the Diurnal Inequality of Temperature at the Seven Observatories of the Meteorological Office, for each Month of the Year, obtained from Twelve Years' Observations from 1871 to 1882.

	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.
Longitude correction local midnight later than Greenwich midnight	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′
Months.	p_1.							p_2.							p_3.						
January	− ·74	− ·74	− ·74	− ·69	− ·91	− ·79	−1·27	+ ·27	+ ·31	+ ·37	+ ·25	+ ·42	+ ·37	+ ·47	− ·03	− ·10	− ·11	− ·03	− ·12	− ·09	− ·15
February	−1·02	−1·36	−1·08	−1·18	−1·32	−1·40	−1·64	+ ·37	+ ·47	+ ·51	+ ·41	+ ·48	+ ·55	+ ·50	− ·02	− ·10	− ·09	− ·06	− ·09	− ·10	− ·04
March	−1·72	−2·35	−1·77	−2·17	−2·53	−2·06	−3·16	+ ·42	+ ·68	+ ·58	+ ·58	+ ·70	+ ·65	+ ·81	− ·03	− ·08	− ·05	− ·02	− ·03	− ·04	− ·01
April	−2·28	−3·38	−2·27	−3·37	−3·87	−2·56	−4·33	+ ·44	+ ·74	+ ·56	+ ·57	+ ·66	+ ·53	+ ·73	+ ·11	+ ·04	+ ·11	+ ·15	+ ·20	+ ·13	+ ·24
May	−3·02	−4·10	−3·08	−4·10	−4·69	−3·29	−5·09	+ ·34	+ ·45	+ ·44	+ ·15	+ ·40	+ ·10	+ ·26	+ ·21	+ ·21	+ ·30	+ ·41	+ ·35	+ ·33	+ ·45
June	−2·63	−3·83	−3·20	−4·30	−4·38	−3·31	−4·98	+ ·16	+ ·41	+ ·37	+ ·16	+ ·15	+ ·06	+ ·15	+ ·14	+ ·20	+ ·28	+ ·29	+ ·30	+ ·31	+ ·32
July	−2·46	−3·38	−3·39	−3·73	−3·95	−3·26	−4·77	+ ·22	+ ·34	+ ·39	+ ·19	+ ·26	+ ·10	+ ·11	+ ·14	+ ·12	+ ·28	+ ·29	+ ·25	+ ·28	+ ·33
August	−2·53	−3·26	−2·87	−3·57	−3·61	−3·05	−4·66	+ ·36	+ ·57	+ ·61	+ ·49	+ ·53	+ ·42	+ ·55	+ ·13	+ ·07	+ ·20	+ ·22	+ ·24	+ ·25	+ ·30
September . . .	−2·19	−2·82	−2·31	−2·91	−3·31	−2·99	−4·31	+ ·55	+ ·80	+ ·71	+ ·77	+ ·90	+ ·76	+ ·95	+ ·06	− ·03	+ ·05	+ ·06	+ ·13	+ ·09	+ ·18
October	−1·60	−2·12	−1·62	−1·99	−2·26	−2·07	−3·23	+ ·60	+ ·80	+ ·70	+ ·70	+ ·89	+ ·79	+1·13	− ·06	− ·18	− ·10	− ·11	− ·11	− ·13	− ·13
November . . .	−1·05	−1·27	−1·09	−1·15	−1·32	−1·12	−1·87	+ ·48	+ ·62	+ ·56	+ ·47	+ ·67	+ ·54	+ ·79	− ·12	− ·18	− ·18	− ·15	− ·21	− ·14	− ·26
December	− ·79	− ·84	− ·92	− ·54	− ·73	− ·67	−1·27	+ ·38	+ ·37	+ ·49	+ ·27	+ ·36	+ ·35	+ ·49	− ·05	− ·15	− ·15	− ·03	− ·13	− ·13	− ·15
Mean 12 years .	−1·84	−2·45	−2·03	−2·47	−2·	−2·21	−3·38	+ ·38	+ ·55	+ ·52	+ ·42	+ ·54	+ ·43	+ ·58	+ ·04	− ·02	+ ·05	+ ·09	+ ·07	+ ·06	+ ·09

TABLE C (continued).—Mean Values (reckoned from Greenwich Mean Midnight) of the p, q Coefficients for the First Three Orders of Harmonic Components, of the Diurnal Inequality of Temperature at the Seven Observatories of the Meteorological Office, for each Month of the Year, obtained from Twelve Years' Observations from 1871 to 1882.

	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.
Longitude correction local midnight later than Greenwich midnight	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′	10° 18′	6° 39′	5° 4′	4° 18′	2° 28′	2° 6′	0° 19′
Months.	q_1.							q_2.							q_3.						
January	−·48	−·90	−·59	−·81	−·92	−·66	−1·17	+·46	+·53	+·29	+·44	+·58	+·46	+·68	−·17	−·24	−·12	−·18	−·19	−·20	−·23
February	−·86	−1·31	−·84	−1·30	−1·21	−1·21	−1·63	+·55	+·71	+·38	+·63	+·60	+·61	+·80	−·10	−·20	−·05	−·15	−·15	−·16	−·17
March	−1·47	−2·18	−1·24	−2·07	−2·16	−1·43	−3·23	+·58	+·75	+·38	+·68	+·69	+·49	+1·03	+·07	+·06	+·08	+·05	+·10	+·09	+·09
April	−1·97	−2·88	−1·46	−3·06	−2·77	−1·50	−3·53	+·58	+·45	+·27	+·60	+·53	+·07	+·56	+·22	+·34	+·25	+·31	+·40	+·28	+·38
May	−2·30	−3·34	−1·73	−3·43	−3·06	−1·70	−4·13	+·29	+·14	−·03	+·31	+·04	−·37	+·10	+·40	+·43	+·37	+·45	+·49	+·18	+·51
June	−1·81	−2·97	−1·58	−3·28	−2·85	−1·56	−4·07	+·19	+·05	+·04	+·10	−·14	−·40	+·01	+·28	+·32	+·29	+·41	+·42	+·10	+·39
July	−1·74	−2·83	−1·67	−2·84	−2·62	−1·78	−4·44	+·27	+·17	+·07	+·23	+·08	−·31	+·07	+·28	+·32	+·32	+·38	+·37	+·20	+·50
August	−1·74	−2·72	−1·49	−2·82	−2·68	−1·85	−4·08	+·47	+·48	+·24	+·55	+·34	−·15	+·43	+·28	+·32	+·30	+·37	+·33	+·31	+·47
September . . .	−1·45	−2·38	−1·27	−2·30	−2·24	−1·90	−3·42	+·56	+·60	+·33	+·61	+·57	+·30	+·85	+·15	+·22	+·19	+·21	+·28	+·30	+·34
October	−·98	−1·50	−·92	−1·73	−1·45	−1·37	−2·46	+·52	+·71	+·34	+·72	+·64	+·54	+·80	−·04	−·08	+·01	−·10	−·03	−·01	+·03
November . . .	−·62	−1·17	−·65	−·99	−·93	−·76	−1·48	+·50	+·59	+·27	+·53	+·53	+·43	+·67	−·12	−·18	−·08	−·17	−·14	−·14	−·19
December	−·53	−·86	−·52	−·79	−·77	−·36	−1·00	+·49	+·47	+·34	+·42	+·42	+·30	+·56	−·23	−·22	−·13	−·13	−·15	−·15	−·20
Mean 12 years .	−1·33	−2·09	−1·16	−2·12	−1·97	−1·34	−2·89	+·46	+·47	+·24	+·49	+·41	+·16	+·55	+·09	+·09	+·12	+·12	+·14	+·07	+·16

TABLE D.—Mean Values of the Amplitude P, and of the First Maximum μ, reckoned from Local Apparent Midnight, for the First Three Orders of Harmonic Components of the Diurnal Inequality of Temperature at the Seven Observatories of the Meteorological Office, for each Month of the Year and for the entire Year, obtained from Twelve Years' Observations from 1871 to 1882.

Months.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.
	P_1.							P_2.							P_3.						
January . . .	0·91	1·20	0·96	1·09	1·31	1·05	1·74	·56	·62	·48	·52	·72	·60	·84	·18	·27	·17	·19	·23	·23	·28
February . .	1·34	1·94	1·37	1·76	1·81	1·87	2·32	·67	·86	·64	·75	·78	·82	·95	·11	·23	·11	·18	·17	·19	·19
March . . .	2·27	3·22	2·17	3·02	3·34	2·51	4·53	·73	1·02	·71	·92	1·00	·82	1·32	·10	·11	·11	·10	·13	·12	·14
April	3·02	4·45	2·72	4·57	4·76	2·97	5·60	·73	·79	·62	·84	·86	·55	·93	·25	·35	·28	·34	·45	·32	·46
May	3·80	5·29	3·54	4·85	5·60	3·71	6·55	·46	·49	·45	·38	·43	·43	·33	·45	·48	·47	·61	·61	·38	·68
June	3·19	4·85	3·57	5·42	5·24	3·67	6·44	·26	·42	·39	·27	·27	·42	·27	·33	·38	·40	·51	·51	·34	·51
July	3·01	4·41	3·78	4·70	4·74	3·73	6·52	·36	·41	·42	·34	·32	·37	·19	·32	·35	·42	·49	·45	·35	·60
August . . .	3·07	4·23	3·24	4·55	4·50	3·58	6·21	·60	·75	·66	·75	·64	·47	·73	·31	·33	·36	·44	·41	·40	·57
September . .	2·63	3·71	2·64	3·71	4·00	3·55	5·51	·79	1·00	·78	·99	1·08	·83	1·29	·18	·23	·22	·22	·32	·32	·39
October . . .	1·89	2·60	1·88	2·65	2·70	2·49	4·07	·81	1·08	·75	1·01	1·10	·96	1·40	·09	·21	·11	·18	·13	·14	·17
November . .	1·24	1·76	1·28	1·53	1·63	1·38	2·40	·69	·86	·64	·73	·86	·69	1·04	·19	·26	·21	·23	·26	·21	·33
December . .	0·97	1·21	1·07	0·97	1·06	0·79	1·62	·62	·60	·61	·51	·56	·48	·75	·24	·28	·20	·15	·20	·21	·26
Mean 12 years	2·28	3·24	2·35	3·23	3·39	2·61	4·46	·61	·74	·60	·67	·72	·62	·84	·23	·29	·26	·30	·32	·27	·38

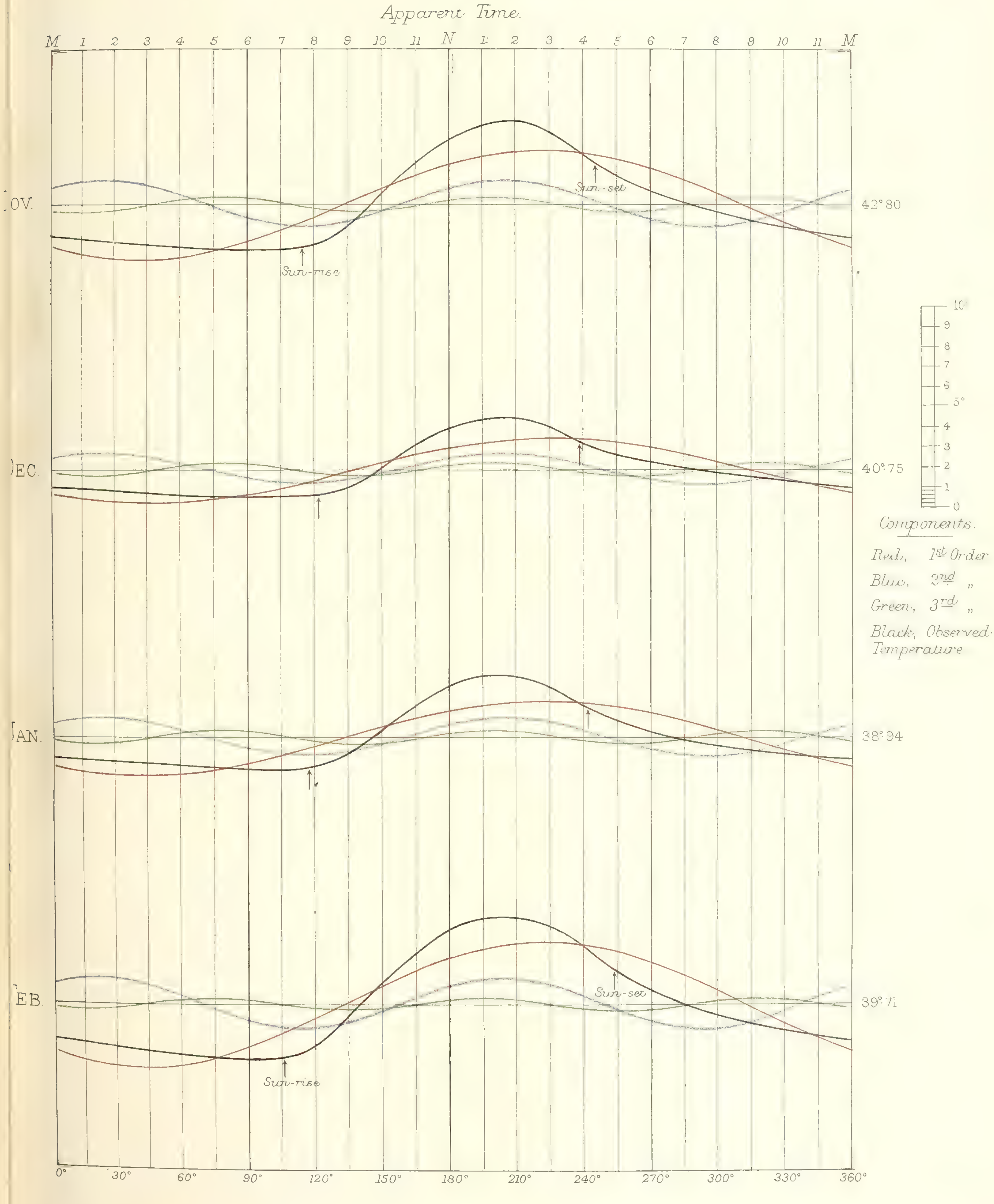
TABLE D (continued).—Mean Values of the Amplitude P, and of the First Maximum μ, reckoned from Local Apparent Midnight, for the First Three Orders of Harmonic Components of the Diurnal Inequality of Temperature at the Seven Observatories of the Meteorological Office, for each Month of the Year and for the entire Year, obtained from Twelve Years' Observations from 1871 to 1882.

Months.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.	Valencia.	Armagh.	Falmouth.	Glasgow.	Stonyhurst.	Aberdeen.	Kew.
	μ_1.							μ_2.							μ_3.						
January . . .	199°	220°	212°	222°	218°	215°	218°	17	21°	12°	22°	22°	21°	25	72°	73°	69°	78°	75°	76°	76°
February . .	206	215	209	220	216	215	221	13	18	10	21	20	18	25	69	70	61	77	74	72	83
March . . .	208	214	207	217	216	211	223	15	15	10	18	18	15	24	28	40	34	30	37	39	23
April	211	213	207	218	213	208	219	16	9	8	19	16	1	19	10	21	16	18	18	19	19
May	208	213	205	217	211	206	220	13	3	− 6	26	3	−35	8	12	15	13	13	16	9	17
June	204	211	201	213	210	203	219	17	− 3	− 1	26	−15	−44	5	11	13	10	14	15	5	17
July	203	212	200	211	210	205	220	14	5	− 2	20	3	−36	1	9	15	11	11	14	9	17
August . . .	203	213	202	213	213	208	220	15	13	5	19	14	−14	19	11	18	13	15	15	14	18
September . .	204	215	205	215	213	211	219	14	13	9	16	15	10	22	15	28	20	21	20	23	21
October . . .	204	212	208	220	213	214	220	13	18	11	22	19	18	21	67	65	57	74	67	61	58
November . .	203	220	209	220	216	216	221	16	19	13	23	20	21	23	67	72	66	76	72	76	75
December . .	204	220	205	231	225	202	219	17	20	14	25	22	19	25	77	73	70	84	75	75	79
Mean 12 years	205	215	206	218	215	210	220	15	13	7	21	13	0	18	37	42	37	43	42	40	42

TABLE E.—Mean Monthly Values of the Equation of Time, the Sun's Declination and Right Ascension, and of the Apparent Time of Sun Rise, at Greenwich and at the Seven Observatories for each Month of the Year. The time is in Angular Measure.

		Jan.	Feb.	March.	April.	May.	June.	July.	August.	Sept.	Oct.	Nov.	Dec.
		° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
Apparent time = mean time . . .		− 2 25	− 3 31	− 2 1	− 0 2	+ 0 52	− 0 4	− 1 20	− 0 40	+ 1 26	+ 3 30	+ 3 35	+ 0 57
Sun's declination		S.20 38	S.12 40	S. 1 4	N. 9 54	N.17 38	N.23 9	N.21 8	N.13 34	N. 2 45	S. 8 52	S.18 30	S.23 2
Sun's longitude.		296 0	326 0	356 0	26 0	55 0	85 0	114 0	143 0	173 0	203 0	234 0	265 0
Greenwich.													
Lat. 51° 29′ N. .	Sun rise A. T.	118 14	106 24	91 20	77 20	66 28	57 31	60 57	72 21	86 32	101 18	114 51	122 17
Long. 0° 0′ . .	,, merid. alt.	17 53	25 51	37 27	47 25	56 9	61 40	59 39	52 5	41 16	29 39	20 1	15 29
Kew.													
Lat. 51° 28′ N. .	,, rise A. T.	118 13	106 24	91 20	77 20	66 28	57 32	60 58	72 22	86 32	101 18	114 51	122 16
Long. 0° 19′ W. .	,, merid. alt.	17 54	25 52	37 28	47 26	56 10	61 41	59 40	52 6	41 17	29 40	20 2	15 30
Stonyhurst.													
Lat. 53° 51′ N. .	,, rise A. T.	121 1	104 9	91 28	76 11	64 13	54 11	58 3	70 43	86 14	102 20	117 15	125 35
Long. 2° 28′ W. .	,, merid. alt.	15 31	23 29	35 5	45 3	53 47	59 18	57 17	49 43	38 54	27 17	17 39	13 7
Falmouth.													
Lat. 50° 9′ N. .	,, rise A. T.	116 49	105 37	91 16	77 56	67 37	59 12	62 25	73 12	86 42	100 46	113 38	120 37
Long. 5° 4′ W. .	,, merid. alt.	19 13	27 11	38 47	48 45	57 29	63 0	60 59	52 25	42 36	30 59	21 21	16 49
Aberdeen.													
Lat. 57° 10′ N. .	,, rise A. T.	125 42	110 23	91 39	74 18	60 29	48 30	53 12	68 2	85 44	103 59	121 14	131 13
Long. 2° 6′ W. .	,, merid. alt.	12 12	20 10	31 46	41 44	50 28	55 59	53 58	46 24	35 35	23 58	14 20	9 48
Glasgow.													
Lat. 55° 53′ N. .	,, rise A. T.	123 46	109 23	91 35	75 4	62 1	50 52	55 13	68 8	85 26	103 19	119 36	128 52
Long. 4° 18′ W. .	,, merid. alt.	13 29	21 27	33 3	43 1	51 45	57 16	55 15	47 41	36 52	25 15	15 37	11 5
Armagh.													
Lat. 54° 21′ N. .	,, rise A. T.	121 40	108 16	91 29	75 55	63 42	53 25	57 23	70 20	86 10	102 34	117 49	126 21
Long. 6° 39′ W. .	,, merid. alt.	15 1	22 59	34 35	44 33	53 17	58 48	56 47	49 13	38 24	26 47	17 9	12 37
Valencia.													
Lat. 51° 55′ N. .	,, rise A. T.	119 43	106 40	91 22	77 8	66 4	56 56	60 27	72 4	86 29	101 29	115 17	122 52
Long. 10° 18′ W. .	,, merid. alt.	17 27	25 25	37 1	46 59	55 43	61 14	59 13	51 39	40 50	29 13	19 35	15 3

GREENWICH TEMPERATURE 1849-1868.



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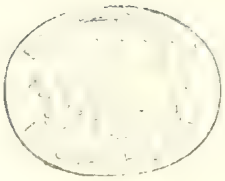

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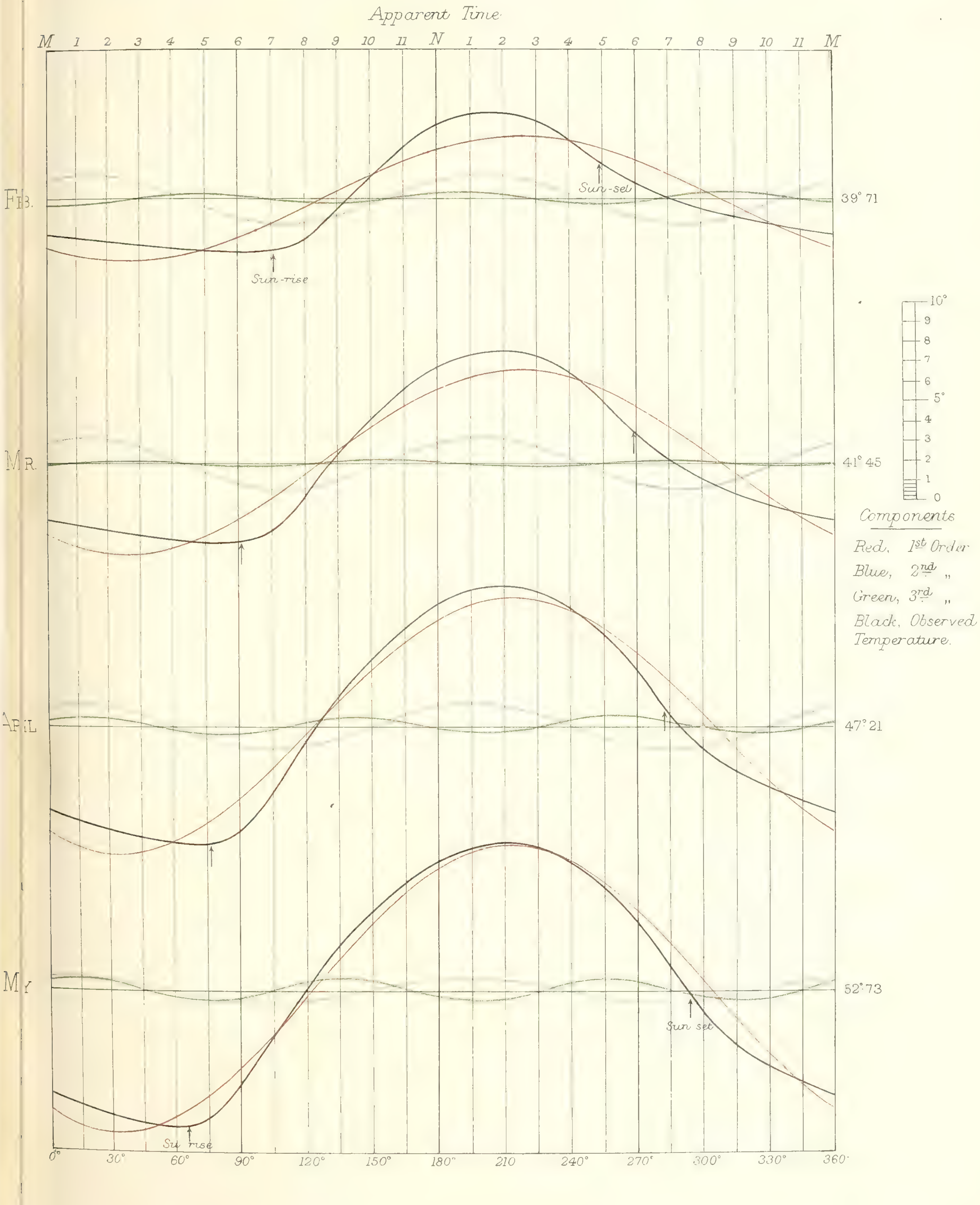

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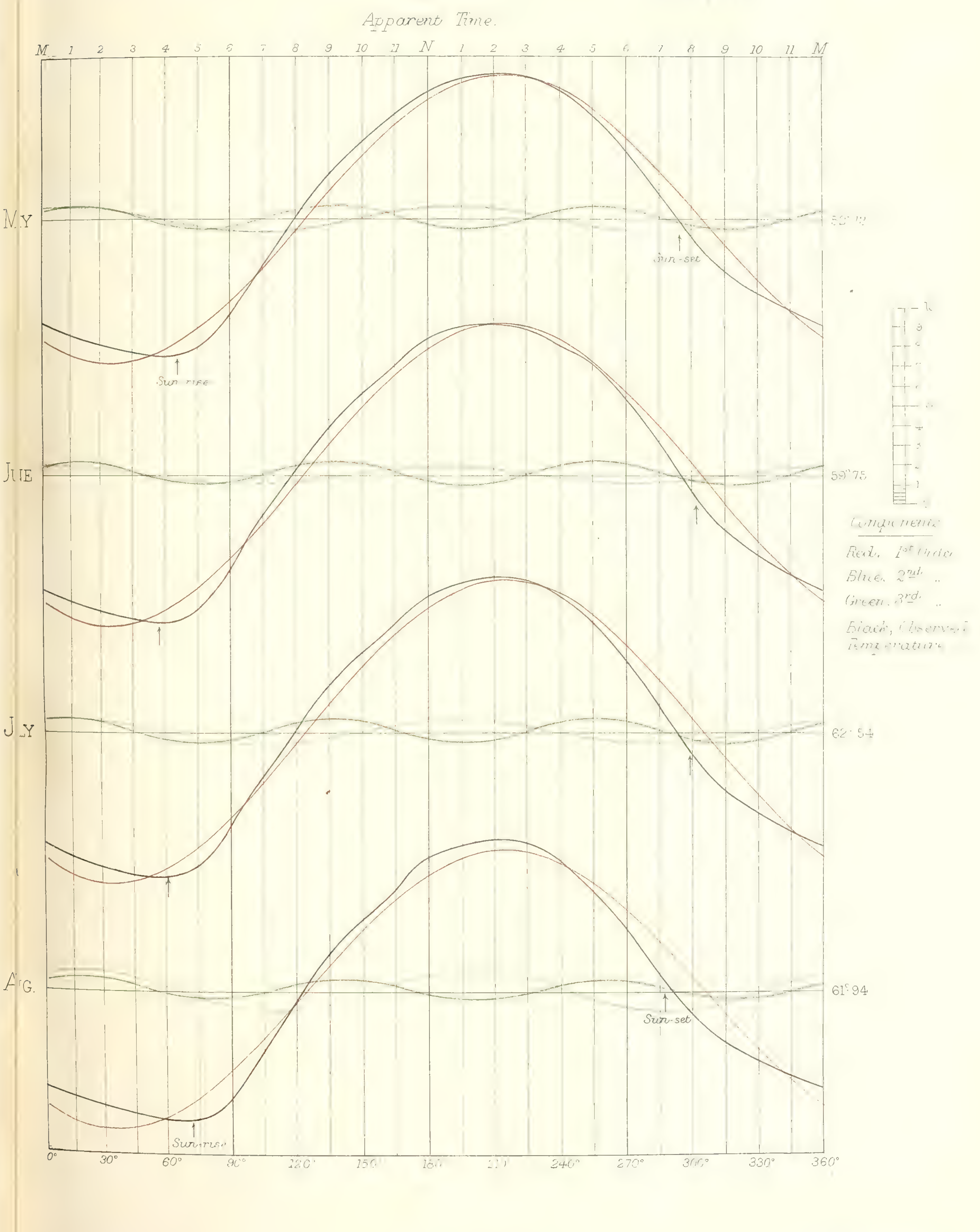

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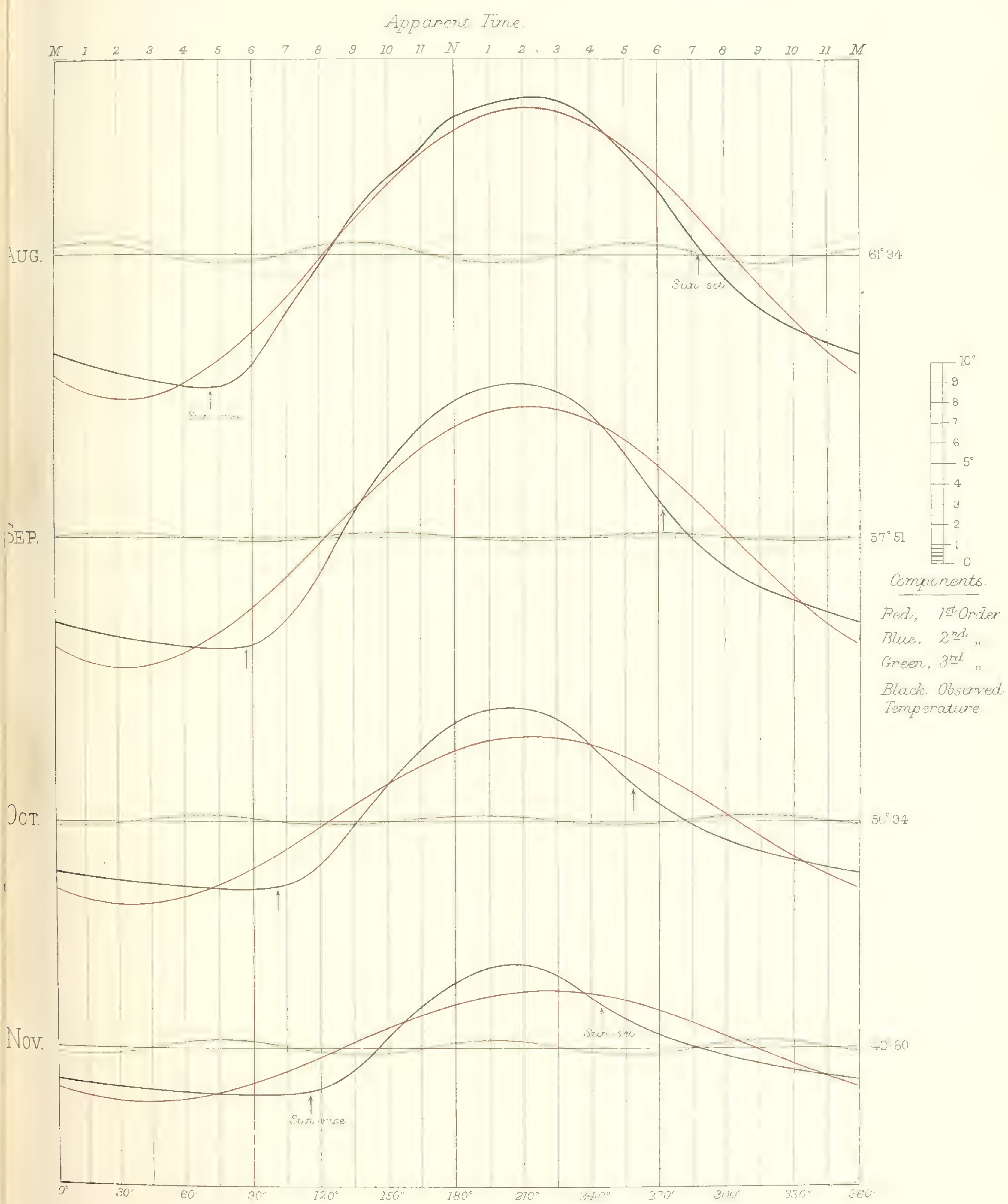

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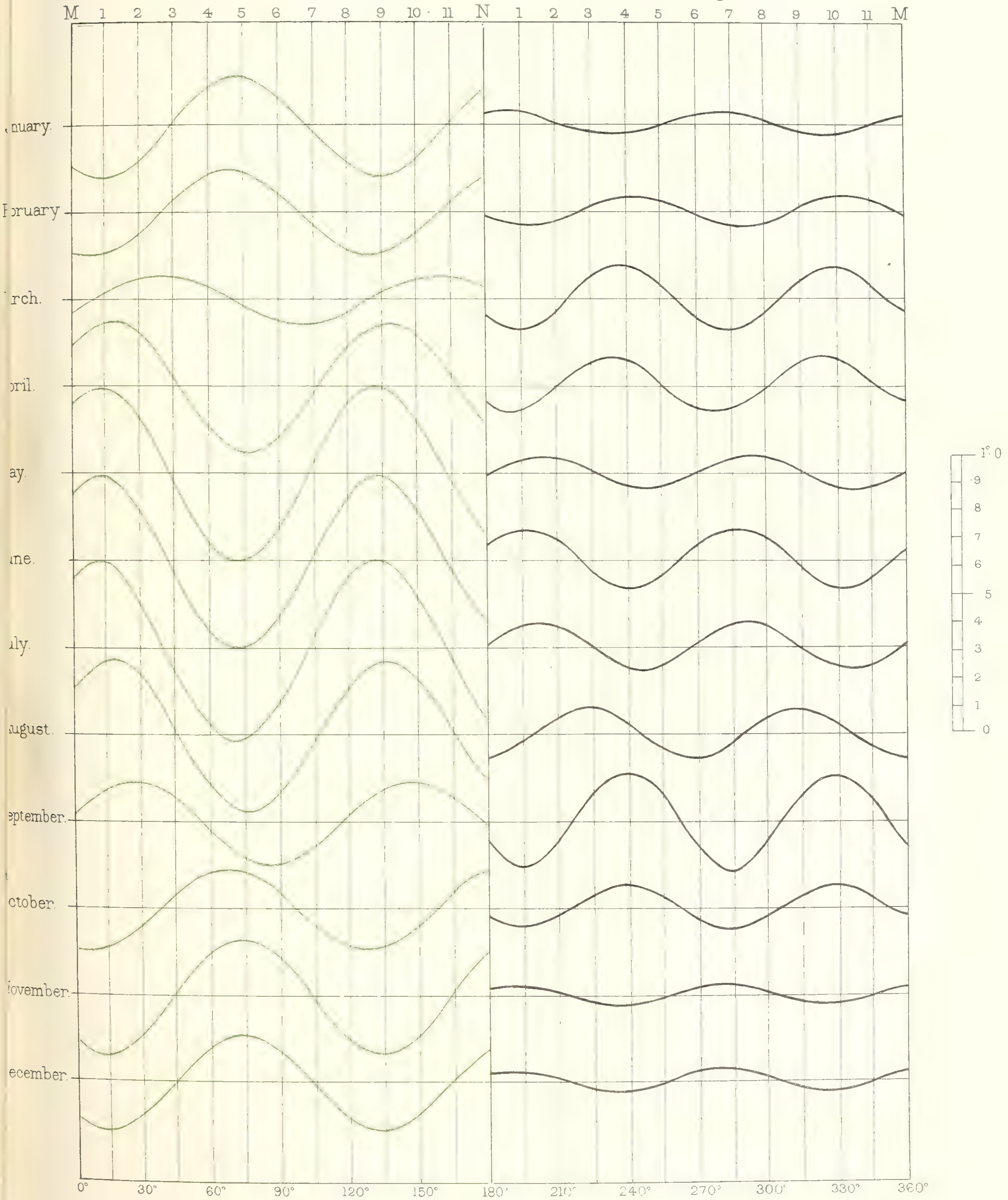
GREENWICH. TEMPERATURE. 1849-68. Apparent Time.

Third Component. Fourth Component.



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XIII. *The Variation of Molecular Surface-Energy with Temperature.*

By WILLIAM RAMSAY, *F.R.S., and* JOHN SHIELDS, *D.Sc., Ph.D.*

Received March 14,—Read March 23, 1893.

[PLATES 24, 25.]

INTRODUCTION.

1. The well-known relation of volume-energy to temperature, expressed by the equation

$$pv = \mathrm{R}\mathrm{T},$$

and the simple relations between p, v, and T when the substance examined is in the state of gas, together with the constancy in the value of R, if the weights of unit volumes of gases taken are proportional to their molecular weights, have led us to make experiments on the surface-tension of liquids in the hope of discovering relations as direct and simple. We were guided to this research by experiments by EÖTVÖS, which shall be discussed later.* We have not been disappointed; it appears that a similar formula expresses with moderate accuracy the variation of surface-energy with temperature; it may be stated in similar terms, using the symbols γ for surface-tension, *i.e.*, the force exerted along a linear centimetre of the liquid; s for a surface over which are distributed a number of molecules the same for all liquids; κ for a constant analogous to R in the gaseous equation; and τ for temperature measured in Centigrade degrees downwards, the critical temperature being taken at zero. An equation completely analogous to the gaseous equation, $pv = \mathrm{RT}$, should therefore be $\gamma s = \kappa\tau$.

That this is reasonable follows from the following consideration:—In the gas equation, $pv = \mathrm{RT}$, if v be kept constant, p decreases with fall of T, until where $p = 0$, $\mathrm{T} = -273°$, or absolute zero; the origin of the scale of absolute temperature is also the origin of the scale of pressure.

Correspondingly, in the liquid surface equation, if s be kept constant, γ decreases with τ, until where $\gamma = 0$, a condition of affairs which takes place at the critical

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* OSTWALD ('Lehrbuch der Allgemeinen Chemie,' p. 1130) has also pointed out the bearings of the experiments of EÖTVÖS in this direction, and has indicated the possibility of determining the molecular weights of liquids by investigation of this "colligative" property.

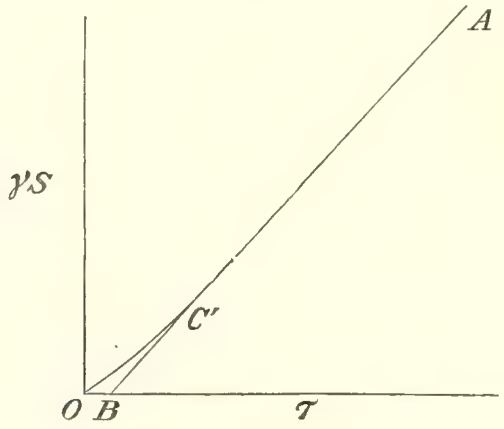
20.10.93.

temperature, τ is zero; hence the origin of the temperature-scale (τ) is also the origin of the scale of tension.

But the above equation, unmodified, represents facts with only approximate accuracy, for reasons which will now be adduced.

Let γs and τ be the axes of the curve AO, showing the relation between γs and τ. At O, the critical temperature, $\tau = 0$, and the value of γs is also 0. With rise of τ, *i.e.*, with fall of temperature, γs increases slowly at first, but soon attains a nearly steady rate of increase, pictured by the nearly straight line CA. The origin of the line CA, which may be regarded as a tangent to a curve at some point intermediate between A and C, is not O, but B, a temperature some degrees on the ordinary scale below O, the critical temperature. Hence, in representing the slope of the line CA by an equation, τ cannot be one of the factors, but τ diminished by the number of degrees between O and B. The equation then becomes, where d represents that number,

Fig. 1.



$$\gamma s = \kappa\,(\tau - d).$$

A similar correction might possibly be applicable to the gaseous equation, if we wish to exhibit the behaviour of gases at ordinary temperatures, which are far above the absolute zero of the thermo-dynamic scale. Were such a correction applicable, it would have the effect of merely altering the zero-point on the gas thermometer by some degrees.

That $d\,(\gamma s/dt)$ is a constant has been shown to be approximately true by Eötvös ('Wied. Ann.,' 27, 452); he also points out that s may be taken as equal to $(\mathrm{M}v)^{\frac{2}{3}}$, where v is the molecular volume of the liquid under investigation, *i.e.*, the volume in cub. centims. occupied by molecular weight of the compound taken in grammes. He deduces from the constancy of the differential the equation $\gamma\,(\mathrm{M}v)^{\frac{2}{3}} = \kappa\,(\mathrm{T} - \mathrm{T}')$, where T is the critical temperature of the liquid, and T′ the temperature at which γ and $(\mathrm{M}v)^{\frac{2}{3}}$ are observed, *i.e.* $(\mathrm{T} - \mathrm{T}') = \tau$. But, as we shall show, such an equation leads to erroneous results.

The modification to be introduced into the equation $\gamma (Mv)^{\frac{2}{3}} = \kappa (\tau - d)$ in order to secure concordance at temperatures (or values of τ) higher than those represented by the point C on the figure will be considered later.

In calculating the results of experiment, we have assumed with EÖTVÖS that $(Mv)^{\frac{2}{3}} = s$; for the surface of a liquid conceived to be contained in a cubical vessel is obviously a square on the line representing the cube-root of its volume. Such an assumption, if molecular volumes be employed, gives a comparison of surfaces on which are distributed equal numbers of molecules, and permits of comparison between different liquids.*

2. *Proof of the Validity of the Equation* $\gamma (Mv)^{\frac{2}{3}} = \kappa (\tau - d)$.

The data necessary in order to calculate the molecular surface-energy of a liquid are: (1) the height to which the liquid rises in a capillary tube of known diameter; (2) the density of the liquid at the temperature of observation; (3) the molecular weight of the liquid examined. These data will be found on p. 667, *et. seq.*

The liquids examined were: (1) ethyl oxide or ether; (2) methyl alcohol; (3) ethyl alcohol; the specific orthobaric volumes of these liquids at temperatures from that of the atmosphere to their critical temperatures were determined by RAMSAY and YOUNG ('Phil. Trans.,' 1887, A., 87; 313; 1886, A., 313); (4) methyl formate; and (5) ethyl acetate; the constants for these two liquids were kindly communicated to us by YOUNG, as they are not yet published; (6) carbon tetrachloride; (7) benzene; and (8) chlorobenzene, examined by YOUNG ('Trans. Chem. Soc.,' 1891, p. 932; 1889, p. 504; and 1891, p. 134); and acetic acid (*ibid.*, 1886, p. 790; and 1891, p. 909). These are at present the only liquids for which such data are available, and therefore they are the only ones of which the surface-energy and its variation with temperature can be determined.

They divide into two groups; to the first group belong ether, methyl formate, and ethyl acetate, carbon tetrachloride, benzene, and chlorobenzene. These liquids may be termed normal liquids, inasmuch as their behaviour resembles that of a normal gas. The second group comprises acetic acid and the alcohols; and these display properties which, as will afterwards be shown, make it more than likely that their molecules coalesce to form aggregates, as temperature falls, as indeed is known to be the case with acetic acid, even in the state of gas.

The first of these groups has the advantage of comprising liquids differing widely from each other chemically; they are therefore specially suitable for testing the truth of a physical law. They will be first considered, leaving till later the "associating" liquids, acetic acid and the alcohols.

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* It has been assumed that the distribution of molecules on the surface of a liquid is such that the average distance between any two molecules on the surface is equal to that in the interior. We can give no proof of the correctness of this assumption, save to refer to the results of this investigation which appear to justify it.

The following table contains numerical data regarding the variation of the molecular surface-energy of the "normal" liquids with temperature, and a comparison with numbers calculated by means of the equation $\gamma (\mathrm{M}v)^{\frac{2}{3}} = \kappa (\tau - d)$:—

Table I.

Ethyl oxide. (Crit. temp., 194°·5 C.) $\gamma (\mathrm{M}v)^{\frac{2}{3}} = 2·1716 (\tau - 8·5)$.				*Methyl formate.* (Crit. temp., 214° C.) $\gamma (\mathrm{M}v)^{\frac{2}{3}} = 2·0419 (\tau - 5·9)$.				*Ethyl acetate.* (Crit. temp., 251° C.) $\gamma (\mathrm{M}v)^{\frac{2}{3}} = 2·2256 (\tau - 6·7)$.			
t (cent.).	τ.	$\gamma (\mathrm{M}v)$.		t (cent.).	τ.	$\gamma (\mathrm{M}v)^{\frac{2}{3}}$.		t (cent.).	τ.	$\gamma (\mathrm{M}v)^{\frac{2}{3}}$.	
		Found.	Cal.			Found.	Cal.			Found.	Cal.
°	°	Ergs.	Ergs.	°	°	Ergs.	Ergs.	°	°	Ergs.	Ergs.
20	174·5	363·7	360·5	20	194	383·9	384·1	20	231	500·7	499·2
40	154·5	317·4	317·1	30	184	363·7	363·7	80	171	367·2	365·7
50	144·5	296·0	295·3	40	174	343·2	343·2	90	161	344·4	343·9
60	134·5	272·8	273·6	50	164	322·6	322·8	100	151	321·7	321·2
70	124·5	251·0	251·9	60	154	302·5	302·4	110	141	299·0	298·9
80	114·5	229·5	230·2	70	144	282·7	282·0	120	131	277·1	276·6
90	104·5	208·1	208·4	80	134	262·0	261·6	130	121	254·7	254·4
100	94·5	186·9	186·8	90	124	241·5	241·1	140	111	231·9	232·1
110	84·5	165·3	165·0	100	114	221·2	220·7	150	101	209·6	209·9
120	74·5	143·6	143·3	110	104	200·5	200·2	160	91	187·1	187·6
130	64·5	122·0	121·6	120	94	180·0	179·9	170	81	164·8	165·4
140	54·5	100·4	99·9	130	84	159·4	159·5	180	71	143·0	143·1
150*	44·5	78·7	78·2	140	74	138·3	139·1	190	61	120·4	120·8
160	34·5	58·6	56·5	150	64	118·0	118·6	200	51	98·9	98·6
170	24·5	39·0	34·8	160	54	97·1	98·2	210*	41	78·4	76·3
180	14·5	19·9	13·0	170	44	76·9	77·8	220	31	56·8	54·1
185	9·5	12·3	2·2	180	34	57·3	57·4	230	21	35·7	31·8
190	4·5	5·5	− 8·7	190	24	37·7	37·0	240	11	15·9	9·6
194·5	0·0	0·0		200*	14	19·2	16·5	245	6	7·2	− 1·6
				210	4	4·0	− 3·9	251	0	0·0	
				214	0	0·0					

TABLE I—(continued).

Carbon tetrachloride. (Crit. temp., 283° C.) $\gamma\,(Mv)^{\frac{2}{3}} = 2{\cdot}1052\,(\tau - 6{\cdot}0)$.				*Benzene.* (Crit. temp., 288°·5 C.) $\gamma\,(Mv)^{\frac{2}{3}} = 2{\cdot}1043\,(\tau - 6{\cdot}5)$.				*Chlorobenzene.* (Crit. temp., 360° C.) $\gamma\,(Mv)^{\frac{2}{3}} = 2{\cdot}0770\,(\tau - 6{\cdot}3)$.			
t (cent.).	τ.	$\gamma\,(Mv)^{\frac{2}{3}}$.		t (cent.).	τ.	$\gamma\,(Mv)^{\frac{2}{3}}$.		t (cent.).	τ.	$\gamma\,(Mv)^{\frac{2}{3}}$.	
		Found.	Cal.			Found.	Cal.			Found.	Cal.
°	°	Ergs.	Ergs.	°	°	Ergs.	Ergs.	°	°	Ergs.	Ergs.
80	203	414·6	414·7	80	208·5	425·1	425·1	150	210	423·2	423·1
90	193	393·7	393·7	90	198·5	404·5	404·0	160	200	401·3	402·3
100	183	372·3	372·6	110	188·5	384·0	383·0	170	190	381·8	381·5
110	173	351·7	351·6	110	178·5	362·9	361·9	180	180	360·5	360·8
120	163	330·3	330·5	120	168·5	341·6	340·9	190	170	340·0	340·0
130	153	309·4	309·5	130	158·5	320·3	319·8	200	160	319·1	319·2
140	143	288·1	288·4	140	148·5	299·0	298·8	210	150	298·2	298·5
150	133	267·4	267·4	150	138·5	278·1	277·8	220	140	277·0	277·7
160	123	246·7	246·3	160	128·5	256·9	256·7	230	130	256·2	256·9
170	113	226·0	225·3	170	118·5	235·2	235·7	240	120	234·9	236·2
180	103	204·8	204·2	180	108·5	213·8	214·6	250	110	213·9	215·4
190	93	183·3	183·2	190	98·5	193·4	193·6	260	100	192·8	194·6
200	83	162·3	162·1	200	88·5	172·5	172·5	270	90	171·9	173·9
210	73	140·8	141·1	210	78·5	151·9	151·5	280	80	150·6	153·1
220	63	118·9	120·0	220	68·5	131·0	130·5	290	70	128·9	132·3
230	53	96·8	99·0	230	58·5	110·1	109·4	300	60	109·7	111·5
240	43	76·5	77·9	240	48·5	89·0	88·4	310	50	90·3	90·8
250	33	55·5	56·8	250*	38·5	68·7	67·3	320*	40	71·5	70·0
260	23	35·9	35·8	260	28·5	48·6	46·3	333	27	46·7	43.0
270*	13	21·7	14·7	270	18·5	28·8	25·3	360	0	0·0	
283	0	0·0		275	13·5	19·9	14·8				
				280	8·5	9·0	4·2				
				288·5	0·0	0·0					

These results clearly show the validity of the equation $\gamma\,(Mv)^{\frac{2}{3}} = \kappa\,(\tau - d)$ over a very long range of temperature; from 20° C., in most cases, to within 20 or 30 degrees below the critical temperature. Deviation becomes apparent where an asterisk has been placed opposite the temperature. A complete formula, representing all the results with fair accuracy, will afterwards be adduced, but several important deductions may be drawn from the simpler one.

3. *Confirmation of these results at* — 89°·8.

Ethyl oxide, methyl formate, and ethyl acetate remain liquid, even when cooled to the boiling-point of nitrous oxide under atmospheric pressure. This temperature, which we have carefully determined by means of a thermometer filled with pure dry hydrogen ('Trans. Chem. Soc.,' 1893, p. 833), is — 89°·8. It was thought advisable to measure the surface-energy, and to find out whether the linear relation between molecular surface-energy and temperature persists even at that low temperature. The results are given in the Table which follows; the method of experiment is described on p. 671.

TABLE II.—Surface-energy at − 89°·8 C.

Substance.	$\gamma\,(Mv)^{\frac{2}{3}}$.		Percentage deviation.
	Found.	Calculated.	
Ethyl oxide . . .	611·2	599·4	+ 2·0
Methyl formate . .	557·0	608·7	− 8·5
Ethyl acetate . .	761·5	744·0	+ 2·4

The agreement is good except in the case of methyl formate; here, however, the negative value, judging from experiments with acetic acid and with the alcohols, may be due to molecular association. The positive value of the other two substances may be due to the fact that measurements were made in a different tube of wider capillary bore, and that the arrangements were necessarily less perfect than in determinations at higher temperatures.

We consider, therefore, that we are justified in stating that the linear relation between molecular surface-energy and temperature holds for ether within a range of 240°; approximately for methyl formate, of 270°; and for ethyl acetate, of 290°.

4. *Molecular Surface-energy of Acetic Acid and of Methyl and Ethyl Alcohols.*

The behaviour of acetic acid, and of methyl and ethyl alcohols, is exceptional; the surface-energy no longer shows a linear relation to temperature. Surface-energy, as has been stated, is the product of $v^{\frac{2}{3}}$ and γ, where $v^{\frac{2}{3}}$ represents one-sixth of the surface of a cube containing one gramme of liquid, and γ, the surface tension. It has been assumed, with EÖTVÖS, that the molecular volume raised to the two-thirds power $(Mv)^{\frac{2}{3}}$, gives a comparison of surfaces on which equal numbers of molecules lie. It is fair to assume that we have proved a linear relation to hold for "normal" liquids between $\gamma\,(Mv)^{\frac{2}{3}}$ and temperature. The question now arises—if certain liquids fail to show a linear relation, to what factor must the variation be ascribed?

Two of these factors, viz., γ and $(v)^{\frac{2}{3}}$, are deduced from actual measurements, viz., of the height of ascent in a capillary tube, and of the volume of one gramme of the substance. But M has been taken as the molecular weight of the simple molecules, $C_2H_4O_2$, CH_4O, and C_2H_6O. For acetic acid, at least, this assumption is acknowledgedly wrong, for it is well-known that its molecular weight, as determined from its gaseous density, is not a constant, but decreases with rise of temperature and fall of pressure. It has been shown by RAMSAY and YOUNG ('Phil. Mag.,' 1887, p. 206).

that the volume of one gramme of the vapour of acetic acid at 50°, the lowest temperature at which measurements were possible, and at a pressure nearly that of the saturated vapour (51·35 millims., instead of 56·56) is 3567 cub. centims., corresponding to the molecular weight, 109·6, instead of 60, the molecular weight of $C_2H_4O_2$, or 120, that of $C_4H_8O_4$.

The alcohols show no such deviation in their vapour densities at any observed temperatures and pressures; but that does not preclude the occurrence of association between simple molecules to form more complex molecular groups in the liquid substances. It is known that if the molecules in the gaseous state be caused to recede, dissociation takes place; for instance, at the same temperature, 50°, but at a pressure of 13·4 millims., one gramme of acetic acid gas occupied no less than 15,100 cub. centims., implying a molecular weight of 99·2, considerably greater than that of $C_2H_4O_2$ (60), but not so great as that of $C_4H_8O_4$ (120). And many other similar instances could be given.

Now, since association is promoted by the approach of molecules, it is clear that as molecules in the liquid state are much nearer each other than in the gaseous state, it is not unreasonable to suppose that some liquids may consist in whole or part of associated molecules, even although their gases show no signs of abnormal density.

These observations are illustrated by the following tables, which show the variation with temperature of the molecular surface-energy of acetic acid, and of methyl and ethyl alcohols, assuming that the molecular weight in each case is that of the simple molecule, $C_2H_4O_2$, CH_4O, or C_2H_6O. By choosing suitable molecular weights, implying a mixture of simple and complex molecules, the relation can, of course, be made a linear one.

TABLE III.—Surface-energy of acetic acid and of methyl and ethyl alcohols.

Acetic acid (Crit. temp., 321·5° C.).			*Methyl alcohol* (Crit. temp., 240·0° C.).			*Ethyl alcohol* (Crit. temp., 243·1° C.).		
t° C.	τ.	$\gamma (Mv)^{\frac{2}{3}}$.	t° C.	τ.	$\gamma (Mv)^{\frac{2}{3}}$.	t° C.	τ.	$\gamma (Mv)^{\frac{2}{3}}$.
°	°	Ergs.	°	°	Ergs.	°	°	Ergs.
20	301·5	371·2	20	220	271·4	20	223·1	331·0
130	191·5	261·5	70	170	216·2	40	203·1	307·3
140	181·5	250·2	80	160	206·6	60	183·1	284·8
150	171·5	238·4	90	150	196·3	80	163·1	261·2
160	161·5	226·3	100	140	186·7	90	153·1	247·1
170	151·5	213·9	110	130	176·7	100	143·1	235·0
180	141·5	200·2	120	120	166·3	110	133·1	221·7
190	131·5	187·9	130	110	154·8	120	123·1	208·0
200	121·5	174·9	140	100	142·9	130	113·1	193·3
210	111·5	160·5	150	90	131·3	140	103·1	178·8
220	101·5	146·9	160	80	118·1	150	93·1	163·0
230	91·5	132·1	170	70	104·8	160	83·1	147·2
240	81·5	117·5	180	60	91·0	170	73·1	130·1
250	71·5	101·5	190	50	76·1	180	63·1	112·6
260	61·5	86·0	200	40	60·6	190	53·1	94·9
270	51·5	71·6	210	30	45·4	200	43·1	75·7
280	41·5	54·8	220	20	29·2	210	33·1	57·1
290	31·5	39·9	230	10	13·4	220	23·1	39·2
300	21·5	25·1	234	6	7·6	230	13·1	19·8
310	11·5	11·3	240	0	0·0	234	9·1	13·3
320	1·5	0·82				236	7·1	9·9
321·5	0·0	0·0				240	3·1	3·7
						243·1	0·0	0·0

Observations of the rise in a capillary tube were made at − 89°·8 with methyl and ethyl alcohols in contact with their own vapours only.

The results were :—

	$\gamma (Mv)^{\frac{2}{3}}$.
Methyl alcohol	361·8 ergs.
Ethyl alcohol.	436·1 ,,

These numbers appear to lie concordantly on prolongations of the respective curves representing the surface-energy of the alcohols in relation to temperature. (See Plate 25.)

Now it is possible to calculate the amount of association at any temperature for acetic acid and the alcohols, on the assumption that κ is a constant, which is approximately true for the normal liquids examined, and that if no association occurred with the anomalous liquids, the value of κ would be similar to that which it possesses for the normal liquids.

The following values of κ were found :—

	κ.
Ethyl oxide	2·1716
Methyl formate	2·0419
Ethyl acetate	2·2256
Carbon tetrachloride	2·1052
Benzene.	2·1043
Chlorobenzene	2·0770
Mean value of κ	2·1209

Suppose acetic acid gas, consisting of molecules of $C_2H_4O_2$, to undergo molecular condensation, so that $2C_2H_4O_2 = C_4H_8O_4$, the pressure of the gas would fall to half of the original pressure, provided volume were kept constant; *i.e.*, unit volume would contain only half the original number of molecules. How would this affect the number of molecules on the surface of a cube representing unit volume? Clearly, if the number of molecules in unit volume is halved, the number of molecules on unit surface will be altered to $(\frac{1}{2})^{\frac{2}{3}}$; and a similar change will take place in the surface energy, one of the factors of which is molecular surface, or $(Mv)^{\frac{2}{3}}$. The value of the differential of surface-energy with temperature will be similarly altered; the amount of association can, therefore, be calculated by the equation

$$\frac{2{\cdot}12}{k} = x^{\frac{2}{3}},$$

where k is the differential found for the dissociating liquid at the temperature chosen, and x the unknown factor of association.

For example, calculating for acetic acid, and for methyl and ethyl alcohols the amount of association at $-$ 89°·8, 20°, 100°, and 150°, we obtain the following numbers:—

Acetic acid.				Methyl alcohol.				Ethyl alcohol.			
t.	k.	x.	$x \times 60 = $ M.W.	t.	k.	x.	$x \times 32 = $ M.W.	t.	k.	x.	$x \times 46 = $ M.W.
20°	0·8815	3·73	223·9	$-$ 89°·8	0·868	3·82	122·2	$-$ 89°·8	0·949	3·34	153·6
100	1·058	2·84	170·3	+ 20	0·932	3·43	109·8	+ 20	1·070	2·79	128·3
150	1·198	2·36	140·3	150	1·236	2·25	72·0	150	1·569	1·57	72·2

From these numbers it would appear that the limit to the amount of possible association is 4. With acetic acid at 20° the molecule is chiefly $(C_2H_4O_2)_4$; with methyl alcohol at $-$ 89°·8 $(CH_4O)_4$ is approached; ethyl alcohol has not reached so great a degree of association at such a temperature.

In seeking for other evidence pointing to similar conclusions, attention is naturally drawn to deductions of molecular weight from the depression in freezing-point of a solvent. But, in such cases, the concentration of the dissolved substance is comparatively small. Data by E. BECKMANN are available for ethyl alcohol dissolved in acetic acid and in benzene, and for acetic acid dissolved in benzene ('Zeitschr. Phys. Chem.,' vol. 2, p. 728). The molecular weight of C_2H_6O is 46, but BECKMANN found that a solution of alcohol in benzene, containing 32·45 per cent. of alcohol, froze at a temperature which implied a molecular weight of 318; *i.e.*, 6·9 simple molecules of C_2H_6O had coalesced to form complex molecules. Here, however, it was not proved that the crystals separating from the solution consisted of pure benzene, and not of a compound. The same solvent gave a molecular depression, when it contained 22·8 per cent. of acetic acid, corresponding to a molecular weight of 153 for acetic acid; by dividing this number by 60, the molecular weight of $C_2H_4O_2$, it appears that approximately three simple molecules have coalesced to form a complex molecule. The molecular weight of alcohol, determined by the depression in freezing-point of a solution of 14·2 per cent. of alcohol in acetic acid, gave 58 as the molecular weight of alcohol; and this implies a commencement of association.

Substances which have no, or only a slight, tendency to associate simple to complex molecules, however, such as naphthalene (molecular weight, 128), show no such large deviations, even in a concentrated solution. Thus BECKMANN found the molecular weight of naphthalene to be 132, even when 20·5 per cent. of naphthalene was dissolved in benzene.

That such abnormal results are not unknown, even in the gaseous state, is proved by the density of sulphur vapour at temperatures not far removed from its boiling-point under normal pressure; BILTZ's results ('Zeitschr. Phys. Chem.,' vol. 2, p. 920) show that whether the formula be accepted as S_8, or whether the association be regarded as proceeding to an indefinite limit, the fact remains that association takes place with fall of temperature. (See also THORPE and HAMBLY's results with hydrogen fluoride 'Chem. Soc. Trans.,' vol. 55, p. 163.)

As to the reason of such condensation, it is, perhaps, premature to speak. A recent paper on Valency, by FLAVITZKY ('J. Prakt. Chem.,' N.F., vol. 46, p. 57), however, appears to suggest a clue. It may be that the oxygen in such molecules as acetic acid and the alcohols has not exerted its maximum valency, on account of the comparatively high temperature, and the distance between the molecules when alcohol is in the gaseous state; and that valency increases either with fall of temperature, or with closer approach between the molecules.

A further proof that association is furthered by the liquid condition has been furnished by CUNDALL ('Chem. Soc. Trans.,' 1891, p. 1076). The association of nitric peroxide molecules (NO_2) is greatly promoted by the liquid condition, for while a 1·44 per cent. solution in chloroform contains only 0·274 per cent. of NO_2, the gas at similar temperature and pressure corresponding to such dilution contains 0·96 per cent. NO_2.

5. *Complete Equation expressing the relation between Molecular Surface-energy and Temperature.*

All the previous work has been on the assumption that the rectilinear equation of $\gamma v^{\frac{2}{3}} = \kappa(\tau - d)$ holds for the substances examined. But it was mentioned on p. 651 that near the critical point this simple relation no longer holds. There can be little doubt that as the liquid reaches a higher and higher temperature its surface is exposed to a more vigorous bombardment from the gaseous molecules. The molecular volume, and therefore the molecular surface, is rapidly increasing near the critical point; and the surface-tension, as well as the surface-energy, decreases less rapidly as the critical point is approached than at lower temperatures. The force required to extend a surface is less decreased by unit rise of temperature at high than at low temperatures. Here we have no analogy to guide us, for gases have never been investigated where p approximates to zero, the volume of one gramme being kept constant, or in other words, we know nothing of the behaviour of gases near the absolute zero of temperature.

It is, moreover, possible that the distribution of molecules on the surface of a liquid at high temperatures is different from that in its interior; and as we have no clear idea of the condition of matters at the surface of a liquid, we can form no theory which would conduct us to a reasonable, and not empirical form of equation.

It was necessary, therefore, to seek for some expression which would represent this deviation as a function either of molecular surface or of temperature, and Mr. ROSE-INNES has kindly assisted us in this. The expression

$$\gamma (\mathrm{M}v)^{\frac{2}{3}} = \kappa\tau - \kappa d(1 - 10^{-\lambda\tau})$$

reproduces the results with considerable accuracy. At high temperatures, *i.e.*, where τ is small, the correction becomes of importance; and at temperatures lower than the critical temperature by 25° or 30°, the correction becomes insignificant. The results near the critical temperature are reproduced for ethyl oxide, methyl formate, ethyl acetate, and benzene in the following table :—

Ethyl oxide.
$\kappa = 2·1716$.
$d = 8·5$.
$\lambda = 0·03$.

τ.	$\frac{\kappa d}{10^{\lambda\tau}}$.	$\gamma (Mv)^{\frac{2}{3}}$.	
		Cal.	Found.
0	18·45	0	0
4·5	13·6	4·9	5·5
9·5	9·6	11·8	12·3
14·5	6·8	19·8	19·9
24·5	3·4	38·2	39·0
34·5	1·7	58·2	58·6
44·5	0·9	79·1	78·7
54·5	0·4	100·3	100·4
64·5	0·2	121·8	122·0
74·5	0·1	143·4	143·6
84·5	0·05	165·1	165·3

Methyl formate.
$\kappa = 2·0419$.
$d = 5·9$.
$\lambda = 0·044$.

τ.	$\frac{\kappa d}{10^{\lambda\tau}}$.	$\gamma (Mv)^{\frac{2}{3}}$.	
		Cal.	Found.
0	12·1	0	0
4	8·1	4·2	4·0
14	2·9	19·4	19·2
24	1·1	38·1	37·7
34	0·4	57·8	57·3
44	0·1	77·9	76·9

Ethyl acetate.
$\kappa = 2·2256$.
$d = 6·7$.
$\lambda = 0·0312$.

κ.	$\frac{\kappa d}{10^{\lambda\tau}}$.	$\gamma (Mv)^{\frac{2}{3}}$.	
		Cal.	Found.
0	14·9	0	0
6	9·7	8·1	7·2
11	6·8	16·4	15·9
21	3·3	35·1	35·7
31	1·6	55·7	56·8
41	0·8	77·1	78·4
51	0·4	99·0	98·9
61	0·2	121·0	120·4
71	0·1	143·2	143·0

Benzene.
$\kappa = 2·1043$.
$d = 6·5$.
$\lambda = 0·029$.

τ.	$\frac{\kappa d}{10^{\lambda\tau}}$.	$\gamma (Mv)^{\frac{2}{3}}$.	
		Cal.	Found.
0	13·7	0	0
8·5	7·7	11·9	9·0
13·5	5·5	20·3	19·9
18·5	4·0	29·3	28·8
28·5	2·0	48·3	48·6
38·5	1·0	68·3	68·7
48·5	0·5	88·9	89·0
58·5	0·3	109·7	110·1
68·5	0·1	130·6	131·0
78·5	0·1	151·6	151·9

6. *Historical résumé.*

The first experiments on the influence of temperature on surface-tension were made by Frankenheim ('Jour. Prakt. Chem.,' vol. 23, p. 401). Laplace and Poisson, it is true, considered the question, but only in the sense that the diminution of density of the liquid would alter the length suspended. Frankenheim, however, found that the capillary ascent diminishes much more rapidly with rise of temperature than the density diminishes; and he discovered that the decrease in length of column supported, is approximately proportional to the rise of temperature. For water, he found that if the surface-tension, or coefficient of capillarity at 0° C., be placed equal to 1, the variation with temperature between 0° and 100° C. may be expressed with fair accuracy by the equation $\gamma = 1 - 0{\cdot}00191\, t$.

FRANKENHEIM was the first to point out that at a certain high temperature, differing for each liquid, the capillary ascent would cease; and he suggested that the state observed by CAGNIARD DE LA TOUR, now known as the critical point, should be that temperature.

BRUNNER, in 1847 ('Pogg. Ann.,' vol. 70, p. 514), and WOLFF, in 1857 ('Ann. Chim. Phys.' (3), vol. 49, p. 230), also found proportionality between capillary ascent and temperature; they observed, however, that such proportionality did not always hold. The next researches on this subject in order of time were by R. SCHIFF ('Annalen,' vol. 223, p. 47), in 1884; but as he measured the capillary heights at only two temperatures, no conclusion can be drawn from his work relative to the question. Moreover, the capillary rise was measured with the surface of the liquid in contact with air, and not with its own vapour.

The problem was again experimentally attacked by EÖTVÖS (*loc. cit.*), in 1886. Starting with VAN DER WAAL'S definition that corresponding states for different liquids are those at which

$$(1)\ \frac{V_1}{V_2} = \frac{v_1}{v_2} = \frac{p_2 T_1}{p_1 T_2},$$

(where V_1 and V_2 are the molecular volumes of two saturated vapours; v_1 and v_2; those of their liquids in contact with saturated vapours; p_1 and p_2, corresponding pressures; and T_1 and T_2, corresponding temperatures on the absolute scale), he reasons as follows:—

For a surface on which n molecules lie, the pressure of the vapour on that surface is $np_1v_1^{\frac{2}{3}}$, since $v^{\frac{1}{3}}$ is proportional to the mean linear distance between the molecules. On the other hand, the surface-tension across a line of m molecules is $m\gamma_1v_1^{\frac{1}{3}}$. The first force may be measured in dynesper square centimetre, the latter, in dynes per linear centimetre. The corresponding expressions for another liquid are $np_2v_2^{\frac{2}{3}}$ and $m\gamma_2v_2^{\frac{1}{3}}$.

EÖTVÖS next assumes that bodies which are in corresponding states should possess similar mechanical properties, especially those which relate to the forces which act between their parts, and to their energies. On this assumption, he imagines proportionality between $\gamma v^{\frac{2}{3}}$, the surface-energy, and the product pv the volume-energy; and combining the relations given above, he obtains

$$(2)\ \frac{\gamma_1}{p_1v_1^{\frac{1}{3}}} = \frac{\gamma_2}{p_2v_2^{\frac{1}{3}}}.$$

This equation seems to signify that the molecules are in a condition of equilibrium, inasmuch as it would imply that $\gamma_1 v_1^{\frac{1}{3}}$ and $\gamma_2 v_2^{\frac{1}{3}}$, the surface-tensions of the liquids, balance the pressures on the surfaces of the liquids; but as a vertical pressure cannot be balanced by a horizontal stress, perhaps it is right to assume that only proportionality is meant.

Equation (2), by appropriate multiplication, becomes

$$(3)\ \frac{\gamma_1 v_1^{\frac{2}{3}}}{p_1 v_1} = \frac{\gamma_2 v_2^{\frac{2}{3}}}{p_2 v_2}.$$

Now from (1)

$$\frac{p_1 v_1}{p_2 v_2} = \frac{T_1}{T_2},$$

hence

$$(4)\ \frac{\gamma_1 v_1^{\frac{2}{3}}}{T_1} = \frac{\gamma_2 v_2^{\frac{2}{3}}}{T_2}.$$

When two substances are in corresponding conditions they remain so, according to VAN DER WAALS, when the temperatures change proportionally to T_1 and T_2, or, in other words, when the change of condition with temperature at any corresponding temperatures is proportional to the temperatures. Hence EÖTVÖS argues that

$$T_1 \frac{d}{dT}\left(\frac{\gamma_1 v_1^{\frac{2}{3}}}{T_1}\right) = T_2 \frac{d}{dT}\left(\frac{\gamma_2 v_2^{\frac{2}{3}}}{T_2}\right).$$

This equation implies that the rate of change of temperature of the molecular surface-energy of any two liquids is the same.

We have deemed it necessary to enter at some length into the views of EÖTVÖS, because, although his conclusion is true for those liquids which we have investigated between wide ranges of temperature, his premises are scarcely justifiable.

This is best tested numerically; and although YOUNG has done so in numerous papers, it will be convenient to show the magnitude of the error for a specific case.

It is easy to test equation (1) stated in the form $p_1 v_1/T_1 = p_2 v_2/T_2$, from data given by RAMSAY and YOUNG ('Phil. Trans.,' 1887, A, pp. 81 and 85) for ether, and by YOUNG ('Trans. Chem. Soc.,' 1889, pp. 501 and 504) for benzene.

	Benzene.	Ether.
Corresponding temperatures	339·95° Absolute	282·95° Absolute
Pressures at these temperatures . . .	495·5 millims.	291·1 millims.
Molecular volumes at these temperatures	93·95 cub. centims.	101·86 cub. centims.
$\frac{pv}{T}$	136·94	104·79

These numbers, according to EÖTVÖS, should be equal, *i.e.*, 136·94 = 104·79.

The correspondence is better at higher temperatures.

	Benzene.	Ether.
Corresponding temperatures	521·5° Absolute	434·1° Absolute
„ pressures	21752 millims.	16088 millims.
„ volumes	137·81 cub. centims.	150·26 cub. centims.
$\frac{pv}{T}$	5748·2	5568·6

Here, again, Eötvös assumes that 5748·2 = 5568·6, an assertion unwarranted by fact.

We next proceed to test equation (3), derived from equation (1), on the supposition of numerical correspondence between pv and $\gamma v^{\frac{2}{3}}$.

	Benzene.	Ether.	Benzene.	Ether.
Corresponding temperatures	339·95°	282·95°	521·5°	434·1°
$\gamma v^{\frac{2}{3}}$ in dynes	453·6	382·6	70·7	54·2
pv.	46552	29651	2997600	2417400
$\frac{\gamma v^{\frac{2}{3}}}{pv}$	0·009744	0·01290	0·0000236	0·0000224

Again the want of correspondence is less striking at high than at low temperatures.

Lastly, it is possible to test equation (4), viz., $\gamma_1 v_1^{\frac{2}{3}}/T_1 = \gamma_2 v_2^{\frac{2}{3}}/T_2$. The same temperatures have been chosen, and the data are the same as those given above. The results are :—

	Benzene.	Ether.	Benzene.	Ether.
$\frac{\gamma v^{\frac{2}{3}}}{T}$	1·3343	1·3522	1·3557	1·2486

These results show that the basis of reasoning on which Eötvös founds his differential equation is a faulty one, and contrary to fact. The equation which he deduces from his differential equation, viz. :—

$$\gamma v^{\frac{2}{3}} = 0·227\ (T - T'),^*$$

where T is the temperature at which $\gamma v^{\frac{2}{3}} = 0$ (*i.e.*, the critical temperature), and T′ the temperature at which the observation is made, is identical with the first equation stated on p. 647, viz., $\gamma v^{\frac{2}{3}} = \kappa\tau$, and obviously does not hold.

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[*] The coefficient, 0·227, is expressed in milligrammes per square millimetre.

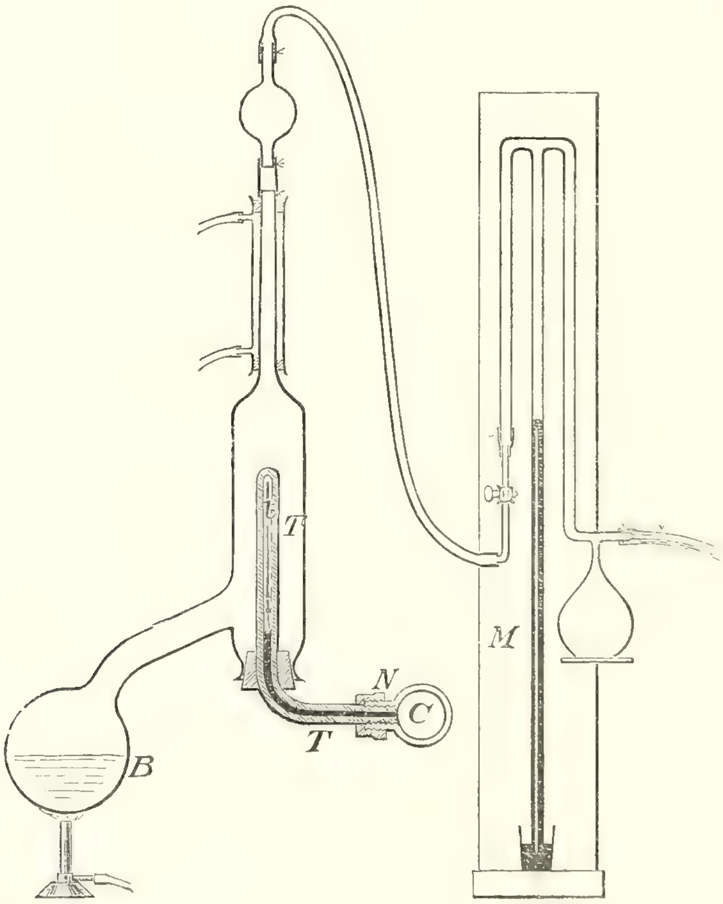
It should be remarked that we have confined our criticism to the rectilinear part of our work, and have not unfairly taxed the powers of Eötvös's equation in covering those parts of the observations where they approach the critical temperature and are represented by a curve. At best, Eötvös's reasoning leads only to a rough similarity, not to numerical identity.

Measurements of the surface-tension of ethyl oxide have recently been made by N. P. Kasterin, by a method which, from the figures and diagrams which he gives, evidently does not lead to trustworthy results. ('Jour. Russ. Phys. Chem. Soc.,' 24, Phys. Sect., p. 196.) De Vries, also, in an inaugural dissertation (Amsterdam, 1893), has obtained results with ether, which, although confirmatory of ours, are not sufficiently numerous to admit of any general conclusion.

7. *Experimental Details.*

It was necessary during the experiments on this subject to maintain the liquids investigated at constant temperatures, differing from each other by known amounts, and, when the temperatures were high, under considerable pressures. In order to satisfy these conditions, the apparatus consisted essentially of three parts, a tube for observing the phenomena, a heating apparatus, and a pressure apparatus.

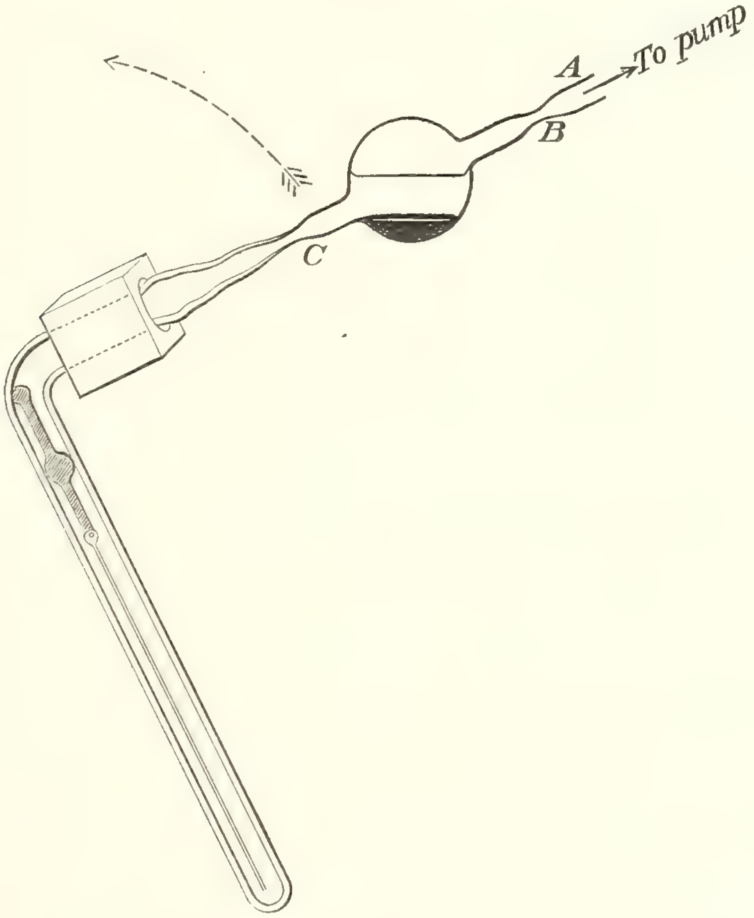
Fig. 2.



The liquid under experiment was enclosed air-free in a piece of thick walled barometer-tubing, *T*, secured in a neck *N* by means of an india-rubber joint compressed by a screw; it entered an iron cylinder *C*, carrying a screw plunger, not shown in the figure. The cylinder was filled with mercury, and by screwing the plunger in and out the pressure could be varied within wide limits, and the liquid in the tube could be exposed to any desired pressure, greater than that of the atmosphere. The temperature of the liquid could be controlled by means of the vapour-jacket, by the arrangement shown in the figure. The liquids in the bulb *B* boiled under known pressures, regulated by the gauge *M*, and the temperature could be varied at will. (For further details see 'Trans. Chem. Soc.,' vol. 47, p. 640, and 'Phil. Trans.,' 1887, A, p. 69.) As in the experiments of RAMSAY and YOUNG, the liquids employed to secure constant temperatures were:—(1) Alcohol (40° to 70°), (2) chlorobenzene (70° to 130°), (3) bromobenzene (120° to 150°), (4) aniline (150° to 180°), (5) quinoline (180° to 230°), (6) bromonaphthalene (220° to 280°), and (7) dibenzylketone (280° to 330°).

The method of filling the tube free from air, was as follows:—The tube was disconnected, the cap *N* slipped along *T* towards the bend, and the projecting end of the glass tube was sealed to a bulb, a constriction being left between the tube and the bulb, and also on the tube on which the bulb was blown, as shown in the figure. The liquid was then introduced into the bulb, some mercury was added, and the tube *A* was then connected with an air-pump, the liquid being heated, if necessary. After

Fig. 3.



it had boiled vigorously for some minutes, and all air had been expelled from tube and bulb, the tube was drawn off at the constriction *B*. The liquid in the barometer-tube was then repeatedly boiled so as to remove all traces of air from the walls of the tube ; the liquid forms pistons, which are pushed out by its vapour, and effectually expel all traces of air or dissolved gas. On cooling the barometer-tube, liquid enters ; when a sufficient quantity has entered, the tube is tilted in the direction shown by the arrow, to cause mercury to enter ; and when all is cold, the constriction *C* is cut, and after the cap has been slipped into position, the tube is inserted into the apparatus. It is easy to do this without admitting a trace of air, and the capillary ascent takes place, therefore, in presence only of the vapour of the substance.

The measurement of the capillary rise was effected by means of a very fine capillary tube *t*, enclosed in the wider tube *T*. This tube, about 6 centims. in length, had a minute hole blown in a small bulb at its lower end, and at this point it was attached to a shorter and less fragile rod, which served as a support, and on which the glass was thickened in several places, so that it maintained a constant position in the tube *T*. Matters were so arranged that the small bulb with the hole in it was completely covered by the liquid. The liquid thus gained access to the capillary tube, and the heights to which it rose were measured by screwing in the plunger, so as to raise the meniscus in the capillary tube *t* to within a millimetre or two of its upper end. By so doing, all the readings were taken at a portion of the capillary tube of which the diameter had been measured, and corrections for its probably varying bore were thus rendered unnecessary. The diameter of the capillary tube was originally measured by cutting off a small cylinder of about half a millimetre in length from its upper end, and measuring its internal diameter by a microscope provided with a micrometer scale, the length of whose divisions was determined at the same focus by reading against a standard scale. The diameter of the tube used was 0·2587 millim.

In this way a series of heights was obtained for the liquids investigated ; but these heights obviously do not represent the true capillary rise ; for the liquid in the barometer-tube *T* always stood at a higher level than if the barometer-tube had been of wider bore.

Corrections.—To correct the read heights so as to make them represent the heights in the capillary tube standing in a tube so wide that capillary rise may be neglected, the tube *T* was opened at its upper end, after the measurement had been completed, and its lower end was sealed to a tube bent twice at right angles, and connected with a tube of about 3 centims. diameter, the axis of which lay parallel to the barometer tube. Readings of the height to which the liquid in the tube *T* (with the capillary tube still enclosed) ascended above the level of liquid in the wide-bore tube, where capillary rise is too small to be appreciable, were then made with the cathetometer at the ordinary temperature, 15° to 20° C., the temperature being accurately observed at each measurement. Knowing that the capillary rise is zero at the critical point of the liquid, and knowing from our experiments that the relation between capillary ascent and temperature is a nearly linear one, the rise in the tube *T* could be calculated approximately enough for our purpose by constructing a diagram in which the rise at atmospheric temperature was made one end of a straight line connecting it with the critical temperature, where the rise is zero. At intermediate temperatures the rise in the tube *T* was read off from the diagram; it may be taken as practically accurate; for any divergence from truth is much less than the limits of error in reading the ascent in the narrow capillary tube. This rise was therefore added to the apparent rise in the capillary tube, and the sum was taken to represent the total rise.

No correction has been made for the increase in internal diameter of the capillary tube due to rise of temperature. The surface-tension is directly proportional to the diameter of the tube; but as the latter is increased by only 0·5 per cent. for a rise of temperature of nearly 400°, the correction was neglected.

Nor has any correction been introduced for the form of the meniscus. The total correction should have amounted to 0·05 millim. But readings were taken with the lower edge of the cross-line of the cathetometer on the lower surface of the meniscus, so that the minute error was thereby somewhat compensated.

In calculating the results of experiments so as to ascertain the surface-tension, it has been assumed that there is no angle of contact between the liquid and the glass. The reasons for this assumption are as follows:—[*]

A tube of capillary bore (0·65 millim.) contained ether free from air. It was connected with the pressure-apparatus, and its temperature was raised. Bubbles were made to appear in the tube, and it was found possible, at any given temperature, to keep them stationary; they did not ascend, or more correctly, the rate of ascent was exceedingly slow, so that a bubble might be kept for an hour, without rising a millimetre. A bubble in such a capillary tube is confined on its lower and upper surfaces by cups of liquid; the walls of the tube between these cups is wet, it is true, but by an exceedingly thin film of liquid; so thin that the transference of liquid from the upper to the lower part of the tube is very slow, and hence the almost stationary position of the bubble.

By decreasing the volume of the bubble on screwing in the plunger, the cups approached each other, the lower cup rising nearer to the dome confining the bubble of vapour on its upper side. The bubble began to ascend rapidly only when its vertical diameter had been made equal to its horizontal diameter; *i.e.*, when it was approximately spherical. Above 160° C. a bubble of ether refused to remain stationary in the tube, but at once ascended; and with ethyl alcohol, the ascent commenced above 220°. These temperatures are evidently conditioned by the stability of the upper dome of liquid; and with a narrower tube, the temperature above which the bubble could not be induced to remain stationary was much higher. In no case did it ascend until it had been made approximately spherical by compression.

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[*] Some of these experiments were made in conjunction with Mr. R. W. STEWART, B.Sc., with whom one of the authors made a preliminary survey of the field in 1891 and to whom he desires to express his acknowledgments.

Now, had there been an angle of contact, it would have been necessary to compress the bubble until its shape had become lenticular, *i.e.*, until its vertical diameter had become less than its horizontal diameter. This was never observed, and we conclude that there is, therefore, no angle of contact between a liquid which wets glass and the glass, if its surface is in presence only of its own vapour. But we have obtained evidence that the capillary rise in air differs to a small extent from that in the vapour of the liquid, and, under these circumstances, there may well be an angle of contact. For example, with ethyl oxide, the read height in air was 37·40 millims., while in contact with its own vapour the rise was 36·5 millims.; with methyl formate, the numbers are 40·9 and 39·9; with carbon tetrachloride, 27·5 and 25·4; with benzene, 51·9 and 49·6.

Such discrepancies might be due to the angle of contact existing in presence of air, but not in presence of vapour; or it might be due to a film of moisture on the surface of the liquids. But it is noteworthy that those liquids which are most easily volatile at ordinary temperatures give measurements in air which most nearly corroborate the rectilinear relation. The surface-energies, moreover, calculated from the rise of a liquid in contact with its own vapour, lie on the curve prolonged to − 89°·8; while those calculated from the capillary rise in air lie off the curves to a greater or less extent.

The reduction of the read heights in the capillary tube to dynes per linear centimetre was effected in the usual manner, by means of the equation

$$\gamma = \tfrac{1}{2} r.h.g.\,(\rho - \sigma),$$

where r is the radius of the tube, h the height to which the liquid is raised, g the gravitation constant, ρ the density of the liquid, and σ that of the vapour.

In the accompanying Tables the experimental results are stated. Column 1 shows the temperatures at which measurements were made; column 2, the observed capillary rise in the narrow tube (h_1), which is corrected in column 3 (h_2), so as to eliminate the effect of the barometer tube in which the measurements were made; column 4 (h_3) reproduces these numbers smoothed, so as to eliminate errors of experiment; ρ and σ are the weights of 1 cub centim. of the liquid and the gas respectively; and γ is the surface tension, calculated by the equation already mentioned.

Ethyl Oxide.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	37·4*	36·48	0·7143	0·00187	16·49
40	..	..	32·40	0·6894	0·00373	14·05
50	28·10	30·40	30·38	0·6764	0·00508	12·94
60	26·20	28·30	28·38	0·6658	0·00677	11·80
70	24·40	26·40	26·34	0·6532	0·00892	10·72
80	22·50	24·30	24·30	0·6402	0·01155	9·67
90	21·00	22·65	22·35	0·6250	0·01477	8·63
100	19·00	20·50	20·33	0·6105	0·01867	7·63
110	17·00	18·30	18·30	0·5942	0·02349	6·63
120	15·00	16·20	16·20	0·5764	0·02934	5·65
130	13·10	14·10	14·10	0·5580	0·03638	4·69
140	11·30	12·15	12·03	0·5385	0·04488	3·77
150	9·30	10 00	9·90	0·5179	0·05551	2·88
160	7·35	7·85	7·70	0·4947	0·06911	2 08
170	5·30	5·70	5·60	0·4658	0·08731	1·33
180	3·30	3·50	3·40	0·4268	0·11350	0·64
185	2·15	2·30	2·30	0·4018	0·13200	0·38
188	1·50	1·60	1·60	..	..	..
190	..	..	1·20	0·3663	0·16200	0·16
191	0·70	0·75	0·92	..	..	..
193	0·30	0·31	0·50	0·3300	0·20120	0·04
194·5	..	..	..	0·2636	0·26360	..

Methyl Formate.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	40·90	39·83	0·9745	0·0016	24·62
30	..	..	38·00	0·9598	0·0022	23·09
40	..	..	36·08	0·9447	0·0031	21·56
50	30·50	34·15	34·15	0·9294	0·0043	20·05
60	29·00	32·40	32·28	0·9133	0·0060	18·58
70	27·30	30·50	30·40	0·8968	0·0080	17·55
80	25·30	28·30	28·40	0·8803	0·0105	15·70
90	23·90	26·65	26·50	0·8635	0·0135	14·29
100	22·20	24·70	24·55	0·8452	0 0171	12·90
110	20·20	22·50	22·56	0·8264	0·0216	11·52
120	18·40	20·50	20·56	0·8070	0·0268	10·18
130	16·70	18·55	18·55	0·7860	0·0334	8·86
140	14·80	16·45	16·45	0·7368	0·0412	7·54
150	13·00	14·40	14·40	0·7403	0·0506	6·30
160	11·05	12·25	12·25	0·7136	0·0623	5·06
170	9·10	10·10	10·10	0·6844	0·0763	3·90
180	7·20	7·95	7·85	0·6521	0·0943	2·81
190	5·20	5·70	5·64	0·6148	0·1178	1·78
200	3·20	3·50	3·30	0·5658	0·1524	0·87
210	0·95	1·05	0·97	0·4857	0·2188	0·06
214	..	..	..	..	..	..

---

* Read in a wide tube, requiring no correction.

ETHYL Acetate.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	41·20	41·32	0·9005	0·0003	23·60
80	..	..	31·32	0·8245	0·0035	16·32
90	26·90	29·65	29·57	0·8112	0·0047	15·14
100	25·10	27·60	27·85	0·7972	0·0062	13·98
110	23·60	26·00	26·10	0·7831	0·0080	12·84
120	22·00	24·20	24·35	0·7683	0·0103	11·75
130	20·70	22·75	22·62	0·7533	0·0131	10·66
140	19·00	20·90	20·87	0·7378	0·0165	9·57
150	17·50	19·20	19·13	0·7211	0·0206	8·52
160	15·90	17·40	17·40	0·7033	0·0258	7·48
170	14·15	15·50	15·60	0·6848	0·0316	6·47
180	12·65	13·85	13·85	0·6653	0·0388	5·51
190	11·00	12·00	12·00	0·6441	0·0475	4·54
200	9·35	10·20	10·18	0·6210	0·0580	3·64
210	7·90	8·60	8·40	0.5944	0·0712	2·80
220	6·00	6·50	6·50	0·5648	0·0890	1·96
230	4·20	4·55	4·50	0·5281	0·1130	1·18
240	2·20	2·40	2·35	0·4778	0·1500	0·49
245	1·20	1·30	1·30	0·4401	0·1800	0·21
251	..	..	..	..	..	..

CARBON Tetrachloride.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	27·50	25·40	1·5940	0·0008	25·68
80	18·20	19·95	20·05	1·4765	0·0061	18·71
90	17·70	19·35	19·16	1·4554	0·0079	17·60
100	16·90	18·45	18·24	1·4343	0·0103	16·48
110	15·90	17·40	17·35	1·4125	0·0131	15·41
120	15·10	16·50	16·42	1·3903	0·0163	14·32
130	14·30	15·60	15·52	1·3680	0·0203	13·27
140	13·20	14·45	14·59	1·3450	0·0248	12·22
150	12·35	13·50	13·68	1·3216	0·0302	11·21
160	11·65	12·70	12·76	1·2982	0·0365	10·22
170	10·80	11·75	11·85	1·2734	0·0439	9·24
180	9·95	10·85	10·90	1·2470	0·0525	8·26
190	9·20	10·00	10·00	1·2192	0·0625	7·28
200	1·40	9·10	8·97	1·1888	0·0742	6·34
210	7·45	8·10	7·97	1·1566	0·0879	5·40
220	6·60	7·15	6·92	1·1227	0·1039	4·47
230	5·60	6·05	5·90	1·0857	0·1174	3·56
240	4·20	4·55	4·80	1·0444	0·1464	2·74
250	3·25	3·55	3·92	0·9979	0·1754	1·93
260	2·25	2·45	2·60	0·9409	0·2146	1·20
270	1·20	1·30	1·55	0·8665	0·2710	0·59
283	..	..	..	0·5578	0·5578	..

Benzene.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
80	..	..	39·45	0·8127	0·0027	20·28
85	35·30	38·70	38·60	..	..	..
90	34·60	37·90	37·72	0·8042	0·0036	19·16
100	33·00	36·15	36·03	0·7928	0·0047	18·02
110	31·40	34·40	34·28	0·7810	0·0060	16·86
120	29·70	32·50	32·50	0·7692	0·0076	15·71
130	28·00	30·65	30·72	0·7568	0·0095	14·57
140	26·60	29·10	28·95	0·7440	0·0118	13·45
150	25·05	27·35	27·18	0·7310	0·0144	12·36
160	23·00	25·15	25·38	0·7185	0·0173	11·29
170	21·60	23·60	23·60	0·7043	0·0209	10·20
180	19·85	21·65	21·80	0·6906	0·0249	9·15
190	18·20	19·85	19·97	0·6758	0·0298	8·16
200	16·70	18·20	18·13	0·6610	0·0355	7·17
210	14·90	16·20	16·25	0·6432	0·0421	6·20
220	13·20	14·35	14·37	0·6256	0·0503	5·25
230	11·60	12·60	12·45	0·6066	0·0599	4·32
240	9·45	10·25	10·45	0·5852	0·0715	3·41
250	7·80	8·45	8·50	0·5610	0·0857	2·56
260	6·00	6·50	6·42	0·5327	0·1040	1·75
270	4·05	4·35	4·22	0·4985	0·1289	0·99
275	3·10	3·30	3·18	0·4708	0·1450	0·66
280	2·30	2·45	2·05	0·4514	0·2209	0·29
288·5	0·00	0·00	0·00	0·3043	0·3043	0·00

Chlorobenzene.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
150	26·8	29·30	29·17	0·9599	0·0054	17·67
160	25·5	27·85	27·90	0·9482	0·0068	16·62
170	24·5	26·75	26·63	0·9355	0·0083	15·67
180	23·3	25·45	25·33	0·9224	0·0102	14·66
190	22·0	24·00	24·05	0·9091	0·0124	13·69
200	21·0	22·90	22·77	0·8955	0·0150	12·72
210	19·5	21·25	21·48	0·8802	0·0180	11·75
220	18·4	20·05	20·15	0·8673	0·0215	10·81
230	17·4	18·90	18·84	0·8518	0·0255	9·88
240	16·0	17·40	17·50	0·8355	0·0301	8·94
250	14·8	16·10	16·16	0·8196	0·0355	8·04
260	13·3	14·50	14·80	0·8017	0·0418	7·14
270	12·2	13·25	13·46	0·7835	0·0494	6·27
280	11·1	12·05	12·05	0·7639	0·0580	5·40
290	9·9	10·70	10·76	0·7440	0·0667	4·54
300	8·8	9·50	9·43	0·7220	0·0778	3·79
310	7·4	8·00	7·95	0·6988	0·0909	3·05
320	6·0	6·45	6·47	0·6703	0·1075	2·35
333	4·7	5·00	4·80	0·6274	0·1360	1·47

Acetic Acid.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	38·80	38·80	0·9532	0·0001	23·46
130	25·60	27·90	27·75	0·9233	0·0043	16·18
140	24·50	26·70	26·64	0·9091	0·0055	15·32
150	23·40	25·45	25·52	0·8961	0·0067	14·46
160	22·50	24·45	24·40	0·8818	0·0089	13·58
170	21·30	23·10	23·23	0·8689	0·0111	12·71
180	20·25	21·95	22·03	0·8555	0·0137	11·77
190	19·20	20·80	20·80	0·8413	0·0168	10·93
200	18·00	19·45	19·55	0·8264	0·0205	10·05
210	16·95	18·30	18·25	0·8117	0·0249	9·11
220	15·80	17·00	16·95	0·7943	0·0300	8·22
230	14·50	15·60	15·52	0·7758	0·0361	7·28
240	13·05	14·05	14·03	0·7576	0·0433	6·36
250	11·60	12·45	12·43	0·7359	0·0518	5·40
260	10·00	10·75	10·84	0·7138	0·0621	4·48
270	8·50	9·10	9·20	0·6897	0·0742	3·59
280	7·00	7·50	7·50	0·6597	0·0912	2·71
290	..	..	5·76	0·6337	0·1076	1·92
300	..	..	3·95	0·5956	0·1332	1·16
310	..	..	2·13	0·5423	0·1720	0·49
320	..	..	0·25	0·4615	0·2956	0·32
321·5	..	..	..	0·3523	0·3523	..

Methyl Alcohol.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	45·90	..	0·7905	0·0002	23·02
70	34·20	37·20	37·35	0·7460	0·0015	17·64
80	33·20	36·05	35·88	0·7355	0·0021	16·70
90	31·75	34·40	34·30	0·7250	0·0029	15·72
100	30·50	33·00	32·93	0·7140	0·0040	14·80
110	29·15	31·45	31·40	0·7020	0·0054	13·85
120	27·55	29·70	29·72	0·6900	0·0071	12·88
130	26·15	28·10	27·95	0·6770	0·0094	11·84
140	24·50	26·30	26·10	0·6640	0·0122	10·79
150	22·70	24·30	24·30	0·6495	0·0156	9·77
160	21·00	22·40	22·20	0·6340	0·0199	8·65
170	19·00	20·25	20·10	0·6160	0·0253	7·53
180	16·90	17·95	17·85	0·5980	0·0319	6·41
190	14·30	15·20	15·36	0·5770	0·0401	5·23
200	12·00	12·70	12·70	0·5530	0·0508	4·05
210	9·30	9·85	10·03	0·5255	0·0652	2·93.
220	6·80	7·15	7·05	0·4900	0·0864	1·80
230	3·45	3·65	3·75	0·4410	0·1187	0·77
234	..	..	2·40	0·4145	0·1381	0·42
235	1·85	1·95	2·05	..	..	..
236	..	..	1·72	0·3955	0·1505	0·27
240	..	..	..	0·2712	0·2712	..

ETHYL Alcohol.

t.	h_1.	h_2.	h_3.	ρ.	σ.	γ.
°	millims.	millims.	millims.			dynes.
20	..	44·00	43·80	0·7926	0·0001	22·03
40	..	..	41·14	0·7754	0·0003	20·20
60	..	..	38·44	0·7572	0·0008	18·43
80	33·60	36·40	35·56	0·7377	0·0017	16·61
90	31·40	34·05	34·05	0·7283	0·0025	15·63
100	30·00	32·50	32·50	0·7174	0·0035	14·67
110	28·50	30·80	31·00	0·7057	0·0049	13·69
120	27·20	29·30	29·30	0·6925	0·0066	12·68
130	25·50	27·45	27·56	0·6789	0·0088	11·63
140	24·00	25·80	25·70	0·6631	0·0115	10·59
150	22·20	23·80	23·76	0·6488	0·0149	9·52
160	20·40	21·80	21·75	0·6329	0·0192	8·45
170	18·35	19·60	19·65	0·6165	0·0245	7·34
180	16·10	17·20	17·45	0·5984	0·0311	6·23
190	14·10	15·00	15·08	0·5782	0·0397	5·13
200	11·75	12·50	12·40	0·5558	0·0508	3·99
210	9·30	9·90	10·00	0·5291	0·0655	2·91
220	6·80	7·20	7·20	0·4958	0·0854	1·87
230	4·00	4·20	4·20	0·4550	0·1135	0·91
234	..	..	3·06	0·4292	0·1296	0·59
235	2·30	2·45	2·80	..	..	..
236	..	..	2·42	0·4184	0·1401	0·43
240	1·60	1·65	1·10	0·3825	0·1715	0·15
243·1	..	..	..	0·2756	0·2756	..

Experiments at Low Temperatures.

To measure the capillary rise at − 89°·8, the temperature of nitrous oxide boiling under atmospheric pressure, an arrangement different from that described was resorted to. A tube, about 2 centims. wide and 20 centims. long, was sealed at one end; into it was inserted a capillary tube of known diameter, provided with three glass struts or knobs at each end, so as to make it stand vertically in the wider tube. After the capillary tube had been introduced, the wider tube was drawn off at its upper end, to form a narrow neck, through which it could be exhausted. The pure liquid was then introduced, and, after the tube had been rinsed with it several times, 3 or 4 cub. centims. were left in; the tube was then exhausted, and the liquid was made to boil under reduced pressure for some time, so that its vapour might expel all traces of air.

The capillary neck was then sealed. Observations were made of the capillary ascent at the ordinary temperature, the whole tube having been immersed in water for this purpose. It was then placed in a double-walled tube containing liquid nitrous oxide, with a few shreds of platinum at the bottom, so as to keep the liquid in constant ebullition. Such double-walled tubes, from the space between the walls of which the air has been exhausted as recommended by DEWAR, make it possible to keep a considerable quantity—say, 100 centims.—of nitrous oxide liquid for several hours; and if the liquid be filtered into the tube, to remove particles of ice, it is absolutely limpid and clear; and the radiation of heat through the walls is so slow that it is only occasionally necessary to wipe off the dew which deposits. Readings may be made with a cathetometer with perfect ease. It is, of course, necessary to have the nitrous oxide at a level considerably higher than that of the liquid in the capillary tube, so as to ensure that the *surface* of the liquid in the capillary tube shall be at the temperature − 89°·8, and some time must be given to allow the liquid to reach the low temperature. The rise is a slow one; but in half-an-hour the position of the liquid in the capillary tube is permanent.

The defect of this method is that readings must be taken at different parts of the capillary tube, for there is no means of adjusting the level of the liquid so that its surface is always at a definite point in the capillary. The differences which we have observed between the actual values of surface energy and the calculated values, which amount to about 2 per cent., are probably due to this cause, and to a slight error in determining the diameter of the tube; it was found by weighing the mercury which filled a known length, while the diameter of the other tube was found by the much surer method of direct reading.

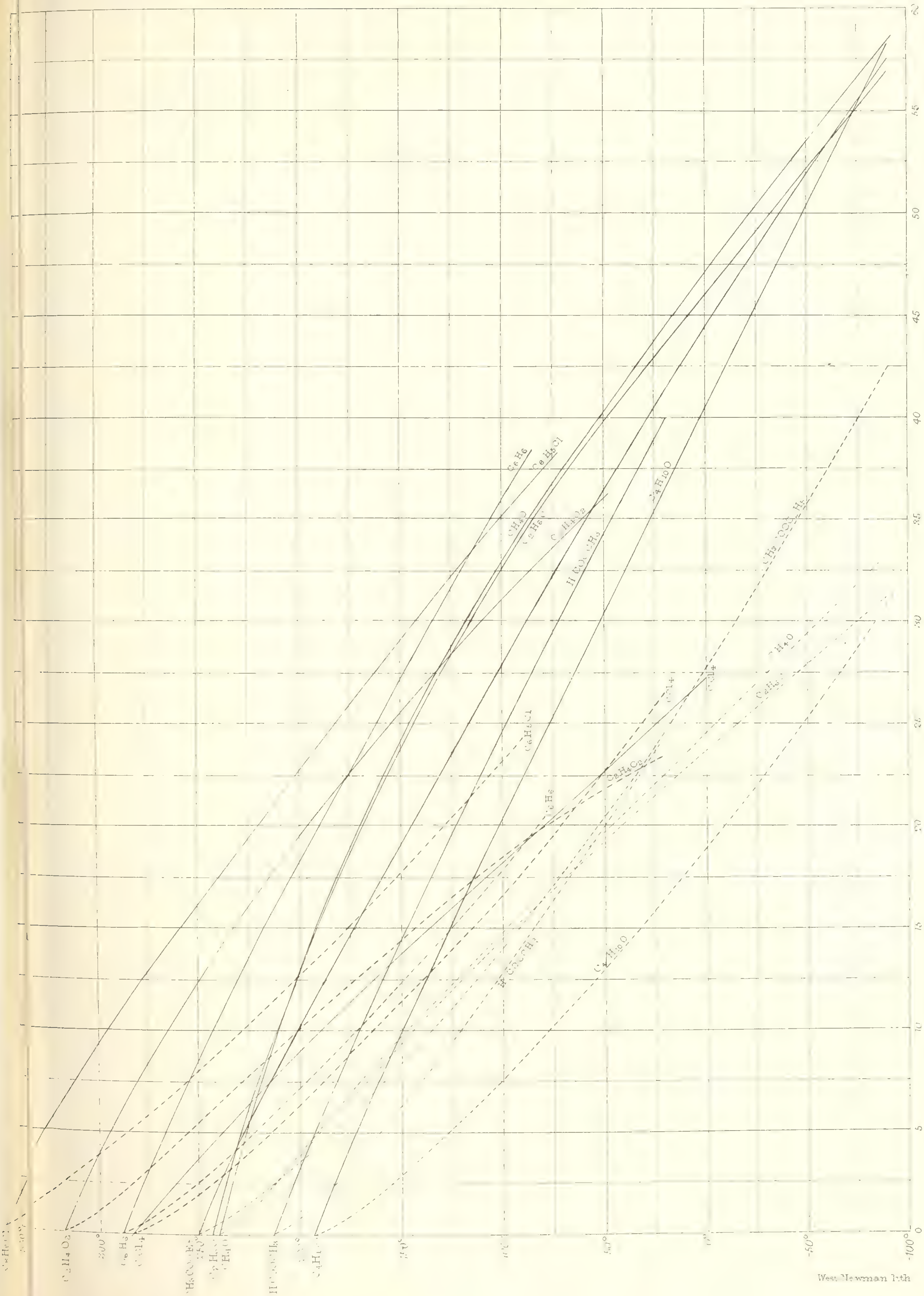
Description of Plates 24 and 25.

Plate 24 shows the height of ascent of liquid in the capillary tube, in millimetres, mapped against temperature as ordinates. Two points call for special notice. First, the continuous lines, indicating the rise, end abruptly at the respective critical points. In no case is there any change of curvature as the critical point is approached. Hence, although observations near the critical temperature are difficult to make, the correctness of readings from the curve is very probable, inasmuch as the curve could not follow any other course than that drawn. This enabled us to calculate the surface energy of chlorobenzene, a liquid which was not examined above 333°, and, moreover, it confirms the observations at lower temperatures of that and the other liquids. Second, it is to be remarked that the lines showing the difference of heights with temperature of acetic acid and of the two alcohols are much more curved than those representing the other substances.

The broken lines represent the surface tension in dynes. Here it is to be noticed that acetic acid and the alcohols give lines of double curvature, while the other substances give continuous curves.

It is difficult to explain the behaviour of methyl formate below 20°. Readings taken on two separate days at − 89°·8 gave practically the same height, viz., 39·8 and 40·1 millims. rise, in the wider capillary; this corresponds to 54·5 and 54·9 millims. rise in the narrow capillary with which the other observations were made. But such observations demand abrupt change of curvature between + 20° and − 90°. As we have no intermediate observations to guide us, the lines are not drawn.

Plate 25 exhibits the variation of surface-energy with temperature. The close approach to parallelism of the straight lines is noteworthy; κ, in the equation $\gamma (\mathrm{M}v)^{\frac{2}{3}} = \kappa (\tau - d)$ has nearly the same numerical value for all substances, except, of course, for those which associate. The steeper slope of the curves of acetic acid and of the alcohols is well marked, and also the reflex curvature; for at temperatures near the critical temperatures, the deviation, $\kappa d (1 - 10^{-\lambda\tau})$ begins to tell; while at low temperatures the lines are curved owing to association of simple molecules to complex molecular groups.



C_6H_6
C_6H_5Cl
CH_4O
C_2H_6O
$C_2H_4O_2$
$H\,CO_2CH_3$
$C_4H_{10}O$
$CH_3COOC_2H_5$
CCl_4
CH_4O
C_2H_6O
$C_2H_4O_2$
C_6H_5Cl
C_6H_6
$H\,CO_2CH_3$
$C_4H_{10}O$
$C_2H_4O_2$
300°
C_6H_6
CCl_4
250°
C_2H_6O
CH_4O
$C_4H_{10}O$
200°
150°
100°
50°
0°
-50°
-100°
0
5
10
15
20
25
30
35
40
45
50
55
60

XIV. *On the Photographic Spectra of some of the Brighter Stars.*

By J. Norman Lockyer, *F.R.S.*

Received November 3,—Read December 8, 1892.

[Plates 26–30.]

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30.—Spectra of Arcturus and the Sun.

I. INTRODUCTION.

In the Bakerian Lecture for 1888* I brought together the various observations of the spectra of stars, comets, and nebulæ which had been made up to that time, and showed that the discussion suggested the hypothesis that all celestial bodies are, or have been, swarms of meteorites, and that the difference between them is one of condensation only. The new classification of the heavenly bodies according to their spectra, rendered necessary by this hypothesis, differed from previous ones inasmuch as the line of evolution followed, instead of locating the highest temperature at its commencement as demanded by Laplace's hypothesis, placed it much later. Hence bodies of increasing temperature were demanded as well as bodies of decreasing temperature.

These conclusions were necessarily based on observations made by others, for the reason that my own work up to that time had been chiefly directed to the Sun.

So soon, however, as my solar work rendered it necessary to determine the Sun's true place among the stars in regard to its temperature and physical conditions, arrangements were made both at Kensington and Westgate-on-Sea to photograph the spectra of stars and nebulæ.

The present communication gives an account of certain parts of this inquiry so far as they have been carried; it is based upon 443 photographs of 171 of the brighter stars.

Having this new and accurate basis of induction, the objects have been to determine :—

(1.) Whether the hypothesis founded on eye observations is also demanded by the photographs.

(2.) In the affirmative case to discover and apply new tests of its validity, or otherwise.

The results as yet obtained are not sufficient to permit a discussion of all points bearing upon the hypothesis, but since most of the crucial ones are covered by the photographs already obtained, it appears desirable that the publication of the work done during the last two years should be no longer delayed.

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* 'Roy. Soc. Proc.,' vol. 44, p. 1.

II. METHODS OF WORK.

(1.) The Objective Prisms.

At Kensington arrangements have been made for photographing the spectra by means of objective prisms. This method was originally suggested and employed by **Fraunhofer**, in 1814, for observing stellar spectra. For photographic purposes the method, which is obviously the best, has been revived with conspicuous success by Professor **Pickering**, of Harvard College.

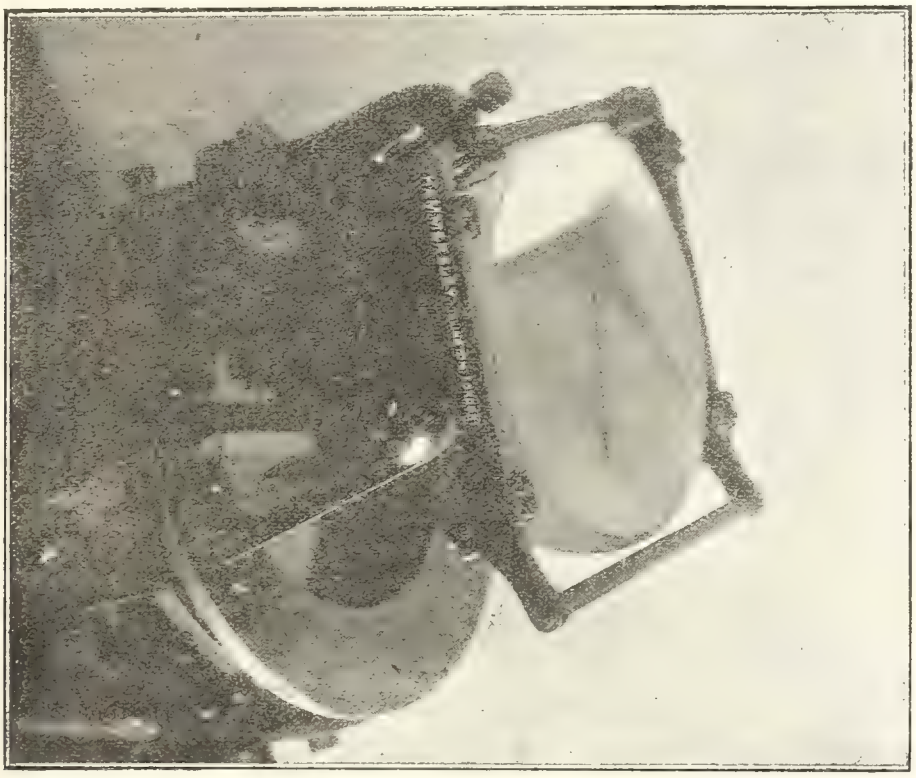
An ordinary telescope with a prism in front of the object-glass becomes a complete star spectroscope, the rays coming from a star being already parallel, so that both slit and collimating lens can be dispensed with. As the image of a star is a point, the spectrum thus obtained will have no breadth, and for eye observations it is necessary to use a cylindrical lens in conjunction with the eye-piece of the telescope. For photographic purposes, however, a different method is adopted. The prism is so placed that its refracting edge lies in a parallel of declination, and then it is only necessary to allow the driving clock to be slightly in error in order to obtain the necessary width.

Instrument A.—The main instrument employed in the work at Kensington has been a 6-inch refracting telescope, with an object-glass made and corrected for G by the Brothers **Henry**. This was at first used in conjunction with a prism of $7\frac{1}{2}°$ by **Hilger**, and, for convenience, this combination will be afterwards referred to as *Instrument A.* The object-glass and prism are fixed at the end of a wooden tube, which is attached to the side of the 10-inch equatorial, at such an angle that the spectrum of a star falls on the middle of the photographic plate when its image is at the centre of the field of the larger instrument. The camera is arranged to take plates of the ordinary commercial size, $4\frac{1}{4} \times 3\frac{1}{4}$ inches. The spectra obtained with this instrument are 0·6 inch long from F to K. An excellent photograph of the spectrum of a first magnitude star can be obtained with an exposure of 5 minutes.

Instrument B.—A 6-inch prism, with a refracting angle of 45°, was obtained from the Brothers **Henry** in May last, and, since then, this has been in constant use with the Henry 6-inch object-glass. The combination will be referred to as *Instrument B.* The spectra obtained with this instrument are 2 inches long from F to K, and the definition is exquisite. In some photographs the calcium line at H is very clearly separated from the line of hydrogen, which occupies very nearly the same position. It is unnecessary to swing the back of the camera in order to get a perfect focus from F to K. The deviation of the prism is so great that it would be very inconvenient to incline the tube which supports it at the proper angle to the larger telescope. When photographing the spectrum of a star, the star is therefore first brought to the centre of the field of the large telescope, and the proper deviation is then given by reading off on the declination circle. This method has been found to work quite satisfactorily.

With this combination the exposure required for a first magnitude star is about 20 minutes. The method of mounting the prism is shown in fig. 1.

Fig. 1.



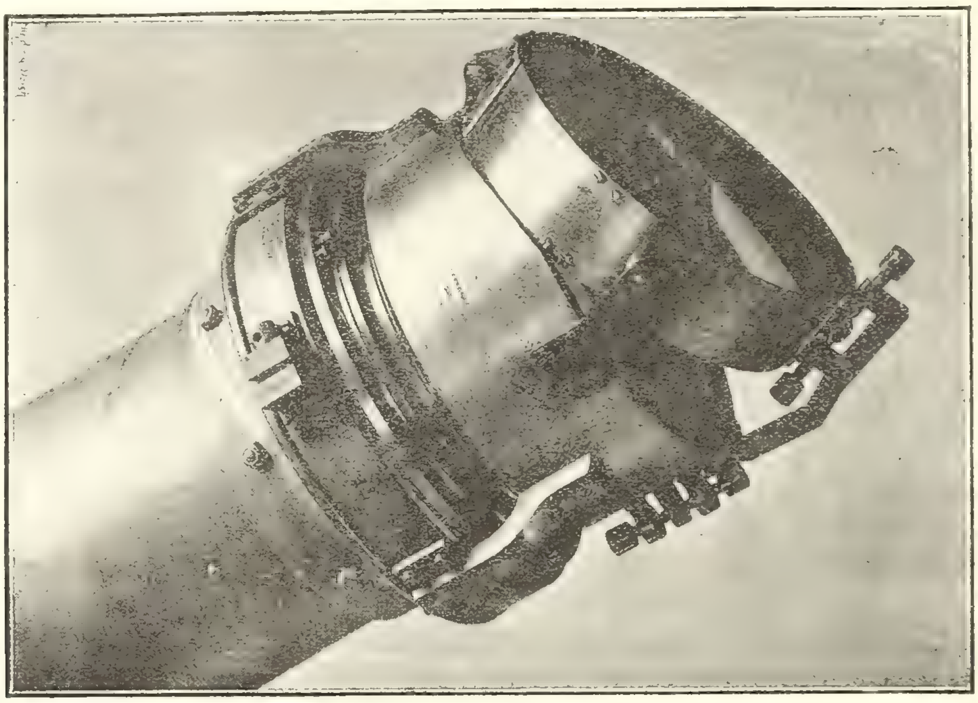
Objective prism of 45° fitted to 6-inch object-glass.

Instrument C.—Two prisms of $7\frac{1}{2}°$ each and 10 inches clear aperture have been provided for use with the 10-inch Cooke refractor, but, as this telescope is required for the instruction of students during the greater part of the year, they have not been largely employed. The prisms are circular in form, and are fitted in cells provided with adjusting screws. A camera taking plates $4\frac{1}{4} \times 3\frac{1}{4}$ inches replaces the eyepiece of the telescope.

To facilitate the adjustment of the spectrum, the 3-inch finder can be inclined to the body of the telescope, so that when a star is brought on the cross-wires its spectrum falls in the proper place on the photographic plate. The focal length of the 10-inch objective is about 11 feet, and the spectra obtained are about $1\frac{1}{2}$ inches long from F to K. With this dispersion and aperture, a normal first magnitude star, like Altair, requires an exposure of about 10 minutes on a fairly good night to give a fully exposed spectrum $\frac{1}{10}$th of an inch wide. The results so far obtained with this instrument do not equal those obtained with Instrument B. Fig. 2 shows the way in which the prisms are attached to the cell containing the object-glass.

Instrument D.—For the fainter stars, the 6-inch prism of $7\frac{1}{2}°$, which was dismounted from Instrument A, has been adapted to a Dallmeyer rectilinear lens of 6 inches aperture and 48 inches focal length, the tube with its appendages having been mounted in place of the tube of the Dallmeyer photoheliograph. This, also, is fitted with a finder inclined at an angle to correct for the deviation of the prism. The spectra taken with this instrument are only 0·3 inch long from F to K, and the time of exposure is reduced in proportion. Even a fifth magnitude star may be photographed in 30 minutes.

Fig 2.



Objective prisms of $7\frac{1}{2}$° each attached to 10-inch object-glass.

The Clock Rate.

The proper regulation of the clock error and consequent "trail" of the spectrum across the plate parallel to itself are essential to the success of photographs taken by the objective prisms. The spectrum of a bright star must obviously be made to trail more quickly than that of a fainter one, and a shorter exposure is sufficient. Since for the same clock error, and in the same time, a star near the pole will give a shorter trail than one nearer the equator, declination must also be taken into account. Keeping a constant clock error, equal widths of spectrum for stars of different declinations may be obtained by lengthening the time of exposure for stars away from the equator, but in that case, the stars near the pole would be over-exposed in relation to those nearer the equator.

The exposure given to stars of equal magnitudes should evidently be the same, no matter in what part of the sky they may be situated, and the clock error should, therefore, be increased in proportion to the secant of the angle of declination.

The light-ratio of stars being $2{\cdot}512^n$, where n expresses the difference in magnitude, the time of exposure must vary in the same proportion, and the clock error in inverse proportion. Thus, where 5 minutes' exposure is sufficient for a first magnitude star, 31 minutes is required to obtain a fully-exposed spectrum of a star of the third magnitude. This law, however, only applies to photographic magnitudes, and must be modified according to the type of spectrum or the colour of the star.

For conveniently adjusting the exposures, tables have been constructed which show at a glance the position of the regulator for a star of given magnitude and declination.

It is obvious that with an instrument of high dispersion, the number of stars it is possible to photograph is very limited, as the long exposures required for the fainter stars are impracticable, and, even if possible, the definition of the lines would be destroyed by atmospheric tremors.

Hence, it is at present only possible to photograph the spectra of the faint stars on a very small scale. With an objective of 8 inches aperture and 44 inches focal length, and a prism of 13° refracting angle, Professor PICKERING has photographed the spectra of stars down to the eighth magnitude. These spectra are about one centimetre long and a millimetre broad, and though they do not show a very great amount of detail, they are sufficient to reveal the type of spectrum.

With an instrument capable of photographing faint stars, a large number of spectra may be taken at one exposure; but, with the instruments of larger dispersion, this is not generally the case, as there are few bright stars of nearly equal magnitude sufficiently close together.

The red stars, being much weaker in blue and violet rays than the yellow or white stars, require much longer exposures than white stars of equal magnitude. To obtain a spectrum of β Pegasi extending to the K line, for example, at least three times the exposure required by a white star of similar magnitude must be given.

Electrical Control.

In consequence of the great accuracy required in the driving of the telescope when long exposures are necessary, the 10-inch equatorial has been fitted with a simple and inexpensive form of electrical control. This is a modification of that designed by Mr. RUSSELL, of the Sydney Observatory.* The existing driving gear has been altered so that the driving rod performs its revolution in a second, and the motion is then communicated to the driving screw through a small worm wheel. The driving rod is vertical and in two parts, the lower portion ending in a faced ratchet wheel, 3 inches in diameter, and with 200 teeth. The upper part of the rod ends in an arm at right angles to itself, and this arm carries a ratchet of suitable shape held down by an adjustable spring. An electro-magnet, connected with the controlling pendulum, is arranged so as to only permit the ratchet to pass it once a second (see fig. 3). If the clock be driving too quickly, the ratchet is held until the stop is raised by the pendulum. When held in this way the ratchet is lifted out of the teeth, and the driving clock itself is not affected.

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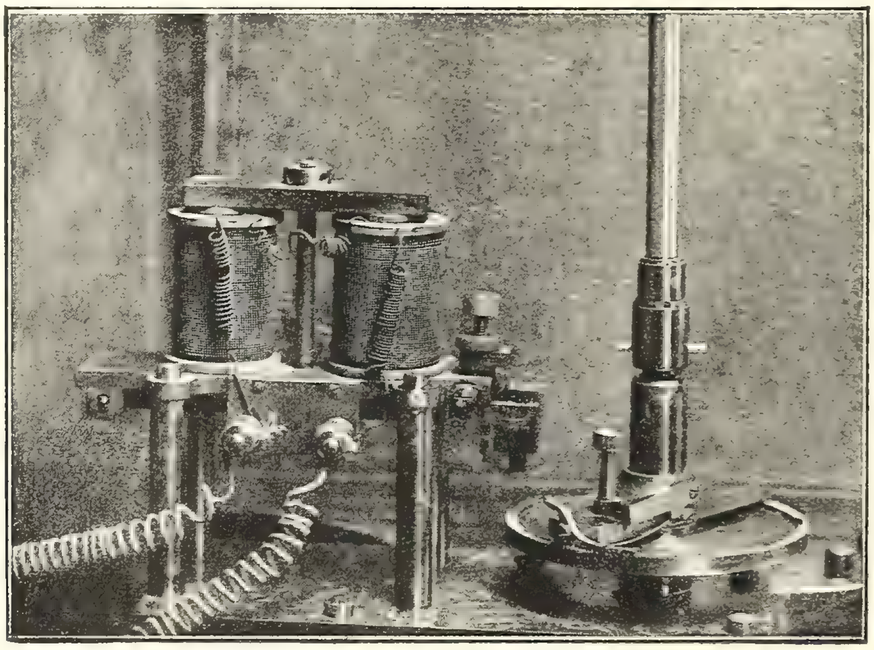
* 'Monthly Notices,' vol. 51, p. 43, 1890–91.

In order that this form of control may be effective, it is essential that the clock should be going too quickly, as it is only capable of retarding the driving rod.

The controlling pendulum is, of course, regulated to the rate required for the particular star which is being photographed.

Fig. 3.



Electrical control for 10-inch equatorial.

In Mr. RUSSELL's form of control the two parts of the driving rod are connected by friction plates. It was found, however, on testing this arrangement, that when the upper portion was held by the electro-magnet the rate of the governors was seriously retarded. I therefore asked Messrs. HAMMERSLEY (who carried out the necessary alterations of the driving clock), to provide some more efficient substitute, and the ratchet wheel was the result.

(2.) THE 30-INCH REFLECTOR.

Instrument E.—Before the erection of the 3-foot reflector, the arrangements at Kensington did not provide for the use of a very large aperture, and I was led to erect a 30-inch reflector at Westgate-on-Sea, having received grants in aid from the Government Grant Fund. The mirror, which is a very fine one of 11 feet 3 inches focus, was figured and presented to me by my friend Dr. COMMON; for the flat (7 inches in diameter) I am indebted to other friends, the Brothers HENRY, and I am anxious to take this opportunity of expressing my great obligations to them for this magnificent help in my work.

The Slipping-plate.

The telescope is of the Newtonian form, the eye end being provided with a slipping-plate, which is a modification of that designed by Dr. Common for use with his 5-foot reflector.* This is adapted to take a plate holder for photographing direct images, and also for use with eye-pieces and spectroscopes. The advantage gained by its use is that a small amount of motion can be given to the photographic plate or spectroscope without in any way disturbing the large mass of the whole telescope. The slipping-plate is so arranged that one screw corrects any small error in right ascension, and the other in declination. A small eye-piece is used as a follower when photographing directly. It is capable of adjustment so that it may be used to observe any small star near the edge of the field, and being provided with cross wires illuminated by a small incandescent lamp it enables the observer to correct any small errors in right ascension or declination.

The Slit Spectroscopes.

In addition to a Maclean star spectroscope and a small direct vision spectroscope, two others have been employed. One is by Hilger, and is fitted with one prism of 60° and two half prisms. This is provided with the usual arrangements for observing and photographing the spectra of celestial bodies. The other is a four-prism spectroscope by Hammersley, and is arranged so that one or two prisms may be removed at pleasure. It is provided with an automatic adjustment for minimum deviation, and is supplied with a bright line micrometer, which is illuminated electrically. A small telescope for observing the spectrum by reflection from the surface of the second prism, a camera to replace the eye-piece, and arrangements for comparison spectra, complete the instrument. The whole spectroscope can be rotated round the axis of the collimator. In order to reduce the weight of the spectroscope the more massive parts are made of aluminium. The weight of the whole instrument, with the four prisms, is only eight pounds.

A grating spectroscope, which has been obtained for the solar work, can also be adapted to the reflector for special investigations. The grating is one of Rowland's, with 14,438 lines to the inch.

(3.) ENLARGEMENTS OF THE NEGATIVES.

Many of the negatives taken have been enlarged about nine times on glass, and further copies have been taken on bromide paper, bringing the enlargement up to about 25 times the size of the original.

Owing to various causes the photographic spectra obtained by the method of trails show irregularities resembling the lines along the spectrum observed when the slit of a spectroscope is partly clogged with dust. It has been noticed that the period of the irregularities is equal to the time of revolution of the main driving screw of the telescope, and hence they may be accounted for by supposing the driving gear to be mechanically imperfect. In that case some of the parallel lines which, by their juxtaposition form the broadened spectrum, are superposed, while others are drawn apart, thus giving rise to dark and bright lines parallel to the length of the spectrum. These lines are more apparent in the case of bright stars than fainter ones. If the telescope were driven with perfect regularity and the atmosphere were quite steady, we should obtain a spectrum of uniform intensity along its width. This condition has very nearly been obtained in some cases.

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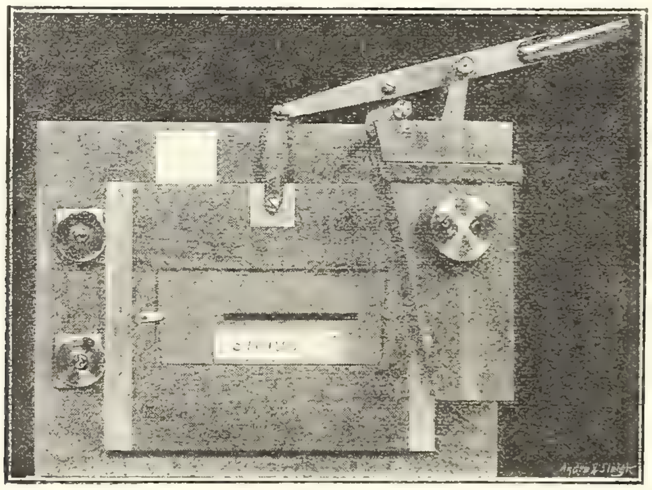
* 'Monthly Notices,' vol. 49, p. 297, 1888–89.

The irregularities above described are eliminated in the enlarged negatives by giving them an up-and-down motion during exposure in a direction parallel to the lines of the spectrum. This was originally done by hand, but a negative holder has been constructed in which the necessary motion is given to the negative by a small driving clock.

Fig. 4.



Negative holder used in enlarging.

A diagram of the arrangement is given above. The only drawback to this method is that defects of the film are apt to produce, by a succession of their images on the enlarging plate, lines (generally very faint) which have a semblance of the true spectrum lines.

To distinguish the real lines from the artificial ones, a direct enlargement of the spectrum is made on the same plate alongside the other, the to-and-fro motion being dispensed with. By a comparison of the two enlarged strips, one can see at a glance which are the true lines of the spectrum, and which are those produced by small irregularities on the film. It may be stated that Dr. Scheiner has also used a somewhat similar method to the one described, the only difference being that he caused the plate on which the enlargement was to be taken to have the oscillating motion, instead of the original negative. The method employed at Kensington, though no account of it had been published, had been in use for some time before Dr. SCHEINER's method was announced.*

III.—LISTS OF SPECTRA PHOTOGRAPHED.

My object has not been so much to obtain photographs of the spectra of a large number of stars, as to study in detail the spectra of comparatively few. Hence many of the stars have been photographed several times with special exposures and foci for different regions of the spectrum.

The following lists of 443 photographs of the spectra of 171 stars will show what material has been available for discussion. The stars are arranged in each table in the order of their right ascension.

(1.) STARS photographed with Instrument A (6-inch prism of $7\frac{1}{2}°$).

Name of star.	Number of photographs.	Name of star.	Number of photographs.
α Andromedæ	1	α Canis Minoris	2
β Cassiopeiæ	1	α Hydræ	1
γ Pegasi	1	α Leonis	4
α Cassiopeiæ	2	γ Leonis	1
γ Cassiopeiæ	19	θ Leonis	1
β Andromedæ	2	β Ursæ Majoris	1
δ Cassiopeiæ	1	α Ursæ Majoris	1
β Arietis	1	δ Leonis	3
γ Andromedæ	1	γ Ursæ Majoris	1
α Arietis	2	γ Virginis	1
α Ceti	5	ε Ursæ Majoris	1
β Persei	1	α Virginis	2
α Persei	1	ζ Ursæ Majoris	1
The Pleiades	1	η Ursæ Majoris	1
α Tauri	3	α Bootis	16
α Aurigæ	16	ε Bootis	1
β Orionis	5	β Ursæ Minoris	1
β Tauri	1	α Coronæ	1
Nova Aurigæ	6	α Herculis	1
γ Orionis	2	α Ophiuchi	1
δ Orionis	3	γ Draconis	2
λ Orionis	1	α Lyræ	2
ι Orionis	1	β Lyræ	12
ε Orionis	2	γ Lyræ	2
ζ Tauri	1	α Aquilæ	4
ζ Orionis	1	δ Cygni	1
κ Orionis	1	γ Cygni	1
θ Aurigæ	1	α Cygni	2
α Orionis	9	ε Cygni	1
μ Geminorum	6	α Cephei	1
γ Geminorum	2	ε Pegasi	2
α Canis Majoris	4	η Pegasi	1
β Canis Minoris	1	β Pegasi	5
β Canis Majoris	1	α Pegasi	1

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* 'Nature,' vol. 42, p. 303, 1890.

(2.) STARS photographed with Instrument B (6-inch prism of 45°, and 6-inch Henry object-glass).

Name of star.	Number of photographs.	Name of star.	Number of photographs.
α Andromedæ	1	α Orionis	3
γ Cassiopeiæ	6	α Canis Minoris	1
β Andromedæ	1	α Canis Majoris	2
γ Andromedæ	1	α Virginis	1
α Arietis	1	α Bootis	11
β Persei	1	α Ophiuchi	1
α Persei	1	α Lyræ	2
α Tauri	1	β Lyræ	7
α Aurigæ	5	γ Lyræ	1
β Orionis	3	α Aquilæ	5
γ Orionis	1	γ Cygni	3
ε Orionis	1	α Cygni	3
ζ Orionis	1	β Pegasi	2

(3.) STARS photographed with Instrument C (two 10-inch prisms of 7½°).

Name of star.	Number of photographs.	Name of star.	Number of photographs.
γ Cassiopeiæ	6	α Geminorum	1
α Arietis	1	α Canis Minoris	3
β Persei	1	β Geminorum	2
α Tauri	2	α Leonis	2
α Aurigæ	2	α Ophiuchi	2
β Orionis	7	χ Virginis	1
β Tauri	1	α Bootis	4
γ Orionis	4	α Coronæ	2
δ Orionis	2	α Lyræ	15
ε Orionis	1	α Aquilæ	5
ζ Orionis	1	α Cygni	2

(4.) Stars photographed with Instrument D (6-inch prism of $7\frac{1}{2}^\circ$ on 6-inch Dallmeyer lens).

Name of star.	Number of photographs.	Name of star.	Number of photographs.
ζ Cassiopeiæ	1	α Serpentis	1
β Ceti	1	π Scorpii	1
η Ceti	1	δ Scorpii	1
α Ursæ Minoris	1	β Scorpii	1
τ Ceti	1	δ Ophiuchi	1
γ Ceti	1	ε Ophiuchi	1
α Ceti	1	σ Scorpii	1
ρ Persei	1	γ Herculis	1
κ Persei	1	α Scorpii	2
ν Persei	1	β Herculis	2
The Pleiades	1	λ Ophiuchi	1
ζ Persei	1	τ Scorpii	1
ε Persei	1	ζ Ophiuchi	1
48 Persei	1	ζ Herculis	1
α Aurigæ	1	η Herculis	1
γ Orionis	1	ι Ophiuchi	1
β Aurigæ	1	κ Ophiuchi	1
α Canis Minoris	1	ε Herculis	1
α Lyncis	1	η Ophiuchi	1
ε Hydræ	1	δ Herculis	1
ρ Hydræ	1	π Herculis	1
ζ Hydræ	1	θ Ophiuchi	1
ε Leonis	1	ρ Herculis	1
α Leonis	2	β Ophiuchi	2
ψ Ursæ Majoris	1	μ Herculis	2
12 Comæ Berenicis	1	γ Ophiuchi	1
13 Comæ Berenicis	1	ν Ophiuchi	1
14 Comæ Berenicis	1	67 Ophiuchi	1
16 Comæ Berenicis	1	70 Ophiuchi	1
δ Corvi	1	72 Ophiuchi	1
17 Comæ Berenicis	1	ο Herculis	1
γ Virginis	1	η Serpentis	1
α Canum Venaticorum	2	σ Sagittarii	1
δ Virginis	1	ζ Aquilæ	1
ε Virginis	1	ν Aquilæ	1
α Virginis	2	σ Cygni	1
ζ Virginis	1	29 Cygni	1
η Bootis	1	P Cygni	1
α Bootis	3	ζ Delphini	1
λ Bootis	1	β Delphini	1
λ Virginis	2	α Delphini	1
γ Bootis	2	δ Delphini	1
ε Bootis	1	γ Delphini	1
α Libræ	2	α Cygni	4
β Bootis	1	ζ Cygni	1
β Libræ	1	δ Capricorni	1
β Coronæ	1	ζ Aquarii	1
υ Bootis	1	ζ Pegasi	1
θ Coronæ	1	μ Pegasi	1
γ Coronæ	2		

(5.) SPECTRA taken with Instrument E.

Name of star.	Number of photographs.	Name of star.	Number of photographs.
Orion Nebula	5	α Aquilæ	1
α Orionis	2	α Boötis	1
α Coronæ	1	α Aurigæ	1

IV.—TABULATION OF SIMILAR SPECTRA.

As in the case of stellar spectra observed in the usual way, the photographic spectra vary very considerably in passing from star to star, and well-marked groups may be recognised.

In the classification of stars which has been adopted from a consideration of the visual observations, only the broader differences in the spectra have been taken into account. Professor PICKERING, however, has suggested a provisional classification in connection with the Henry Draper Memorial photographs of stellar spectra, but this chiefly relates to photographs taken with small dispersion. Now that it has become possible to obtain large dispersion photographs of the spectra, much more detail is revealed, and it becomes necessary to deal with the presence or absence of individual lines to a greater extent than by Professor PICKERING.

Hence, in the first instance, I have arranged the various stars of which the spectra have been photographed at Kensington in tables, without reference to any of the existing classifications, and taking into account the finer details.

The basis upon which the first grouping is founded is the extent of the continuous spectrum. Such a distinction was not possible in the case of eye observations, and it is only by a consideration of the photographs that the classification from this point of view can be made.

Some spectra show a remarkable continuous absorption either in the ultra-violet or violet, in others this absorption extends to about K, whilst in a third class it reaches as far as G. This practically amounts to the same thing as classifying the spectra according to the thickness of the hydrogen lines, for it is found that where the continuous absorption is least the hydrogen lines are broadest. But it must not be forgotten that the broadness of these lines depends, to an as yet undetermined extent, upon the time of exposure, and it may further be that other causes besides temperature may be effective in producing the broadening.

(1.) TABLE A.—Stars in which there is no remarkable continuous absorption either in the ultra-violet or violet. In these stars the hydrogen lines are broad.

(α.)	(β.)	(γ.)	(δ.)
The spectra differ widely from the solar spectrum. There is only a small number of lines, and the hydrogen lines are not of extreme thickness.	The spectra do not differ very widely from the solar spectrum. There is a large number of lines, but the hydrogen lines are not of extreme thickness, though much thicker than in the solar spectrum.	The spectra show very broad hydrogen lines, and, in addition, faint representatives of some of the lines seen in stars of sub-division α.	The spectra show very broad hydrogen lines, and, in addition, faint representatives of the lines seen in stars of sub-division β.
γ Pegasi	β Cassiopeiæ	α Andromedæ	θ Aurigæ
ζ Persei	α Canis Minoris (Procyon)	ζ Cassiopeiæ	γ Geminorum
ε Persei	δ Cassiopeiæ	β Persei (Algol)	α Geminorum
β Orionis (Rigel)	ν Persei	48 Persei	α Canis Majoris (Sirius)
β Tauri	β Arietis	β Aurigæ	γ Ursæ Majoris
γ Orionis (Bellatrix)	α Persei	β Canis Minoris	α Canum Venaticorum
δ Orionis	δ Leonis	η Leonis	α Coronæ
λ Orionis	12 Comæ Berenicis	α Leonis	α Lyræ (Vega)
ι Orionis	13 Comæ Berenicis	β Ursæ Majoris	
ε Orionis	14 Comæ Berenicis	ι Ophiuchi	
ζ Tauri	γ Virginis	δ Capricorni	
ζ Orionis	γ Herculis	γ Lyræ	
κ Orionis	α Ophiuchi	α Pegasi	
β Canis Majoris	α Aquilæ (Altair)	The Pleiades	
η Ursæ Majoris	α Cephei		
α Virginis	ζ Aquarii		
π Scorpii			
δ Scorpii			
β Scorpii			
σ Scorpii			
τ Scorpii			
ζ Ophiuchi			
67 Ophiuchi			
σ Sagittarii			
δ Cygni			
α Cygni			

Many of the photographs are on too small a scale or are too ill-defined to show more than that there is no remarkable absorption in the violet or ultra-violet, and that the hydrogen lines are broad. These are as follows :—

π Ceti.
γ Ceti.
16 Comæ Berenicis.
δ Corvi.
17 Comæ Berenicis.
θ Coronæ.
λ Ophiuchi.
ε Herculis.
η Ophiuchi.
ρ Herculis.

ρ Hydræ.
β Leonis.
ε Ursæ Majoris.
ζ Ursæ Majoris.
ζ Virginis.
λ Bootis.
λ Virginis.
γ Bootis.
α Libræ.
β Libræ.
γ Coronæ.
β Coronæ.
δ Herculis.
γ Ophiuchi.
27 Ophiuchi.
o Herculis.
ζ Aquilæ.
29 Cygni.
ζ Delphini.
β Delphini.
α Delphini.
γ Delphini.
ζ Pegasi.

(2.) TABLE B.—Stars in which there is a continuous absorption in the ultra-violet, extending to about K. In these stars the hydrogen lines are relatively thin.

(α.) The spectra differ considerably from the solar spectrum, although showing a large number of fine dark lines.	(β.) The spectra have a very close resemblance to the solar spectrum.
γ Cygni	α Cassiopeiæ β Ceti α Ursæ Minoris α Arietis κ Persei α Aurigæ (Capella) β Geminorum ε Hydræ ζ Hydræ ε Leonis γ Leonis α Ursæ Majoris ε Virginis α Bootis η Bootis ε Bootis β Bootis ε Ophiuchi β Herculis ζ Herculis η Herculis π Herculis β Ophiuchi μ Ophiuchi ν Ophiuchi 70 Ophiuchi γ Aquilæ ζ Cygni η Pegasi μ Pegasi

(3.) TABLE C.—Stars in which there is a continuous absorption in the ultra-violet or violet, extending to about G. In these stars the hydrogen lines are narrow.

(α.) In the spectra there are indications of dark flutings fading away towards the less refrangible end of the spectrum.	(β.) There are no indications of flutings in the spectra.
β Andromedæ	η Ceti
α Ceti	α Lyncis
ρ Persei	γ Andromedæ
α Orionis (Betelgeux)	α Tauri (Aldebaran)
μ Geminorum	α Hydræ
δ Virginis	ψ Ursæ Majoris
ν Bootis	β Ursæ Minoris
δ Ophiuchi	α Serpentis
κ Ophiuchi	γ Draconis
α Scorpii (Antares)	η Serpentis
α Herculis	o^2 Cygni
β Pegasi	ε Cygni
	ε Pegasi

(4.) TABLE D.—Stars which are not included in any of the preceding Tables.

γ Cassiopeiæ.
β Lyræ.
P Cygni.
Nova Aurigæ.

It will be seen, on reference to the Tables, that the bright-line stars of the Wolf-Rayet type, and the red stars of Group VI., have not yet been photographed at Kensington. All the stars of these types are very faint, and the dispersion hitherto employed has been too great to successfully cope with them. A larger aperture, with smaller dispersion, appears to be essential; and this has not yet been available. It is intended, however, to utilize the three-foot reflector for this work as soon as possible.

All the stars of Table D show special features in their spectra, and will not be further considered in the present communication.

V.—DISCUSSION OF STARS IN TABLE A.

The stars included in this Table are characterised by the absence of any remarkable continuous absorption either in the violet or ultra-violet, and by the presence in their spectra of broad lines of hydrogen. As already shown, they may be conveniently classified into four sub-divisions, which we may now consider separately. The photographs, which only show the lines of hydrogen, owing to small dispersion or defective definition, will be omitted from the discussion.

(1.) Sub-division *α*.

Characteristics of the Spectra.

In all the stars in question the hydrogen lines are well marked, though not of such extreme breadth as in stars like Sirius. The total number of lines in the spectra is remarkably small, as compared with the number seen in stars which resemble the Sun. The lines seen in addition to those of hydrogen vary very considerably in passing from star to star, though some lines are common to them all. It will, therefore, be convenient to further sub-divide the stars, according to the presence or absence of individual lines. Such a classification is given in the following Table.

Stars in which there occur strong lines at 4025, 4120·5, 4143, 4388, 4471, in addition to hydrogen lines of moderate breadth.

(1)	(2)	(3)	(4)	(5)	(6)
There are strong lines at 4481, 4233, 4127, and 4130. The lines at 4025 and 4471 are very feeble. A few lines of iron are added.	The lines at 4025 and 4471 are much intensified, the last as strong as 4481. There are no lines of iron.	The lines at 4481, 4127, 4130, are now faint, while 4025 and 4471 are very strong.	The lines at 4025 and 4471 are still very strong, and lines at 4414·5 and 4088 are seen.	The line at 4414·5 is intensified, while that at 4088 is feebler.	Stars which show no lines of importance except the five typical lines.
α Cygni	β Orionis β Tauri 67 Ophiuchi δ Cygni	γ Orionis β Canis Majoris	δ Orionis ι Orionis ε Orionis ζ Orionis κ Orionis	ε Persei α Virginis γ Pegasi	ζ Persei λ Orionis ζ Tauri η Ursæ Majoris π Scorpii δ Scorpii β Scorpii σ Scorpii τ Scorpii ζ Ophiuchi σ Sagittarii

The preceding Table only refers to the more important lines in the spectra, and in order to facilitate comparisons, the following more complete one has been compiled. The wave-lengths were determined in the first instance from photographs taken with the two 10-inch prisms (Instrument C), and have since been confirmed by photographs taken with the larger dispersion of the 6-inch Henry prism of 45° (Instrument B). The values are believed to be accurate to four figures, and in cases where decimals are added, greater accuracy was attainable. The numbers following the wave-lengths indicate as nearly as possible the intensities of the lines, 6 being the strongest, and 1 the feeblest.

[Note, Aug. 15, 1893.—The wave-lengths have been determined by reference to the lines of hydrogen (marked *H*, *h*, and *G* in the plates), which appear in the spectra of all the stars which have been photographed.]

Wave-lengths of principal lines in spectra of Stars in Table A, sub-division α.

1.	2.		3.	4.	5.
	Kensington.	Potsdam.			
			3919 (2)		
			3926 (3)		
3933 (6)	3933 (6)		3933 (3)	3933 (1)	3933 (1)
3961 (6)					
	3963 (2)		3963 (3)	3963 (2)	
3968 (6)	3968 (6)		3968 (6)	3968 (6)	3968 (6)
	3994 (1)		3994 (3)	3994 (2)	
	4008 (2)		4008 (5)	4008 (2)	4008 (1)
4024 (2)					
4025 (1)	4025 (3)	4026·6 (6)	4025 (6)	4025 (4)	4025 (4)
			4040 (2)		
			4069 (2)	4069 (2)	4069 (2)
			4071 (2)		
			4075 (2)	4075 (2)	4075 (2)
				4088 (5)	
				4095 (2)	
4101 (6)	4101 (6)	4102·0 (6)	4101 (6)	4101 (6)	4101 (6)
			4104 (2)		
				4114 (4)	
			4119 (2)		
	4120·5 (2)	4121·2 (2)	4120·5 (4)	4120·5 (2)	4120·5 (2)
4121·5 (2)					
4127 (3)	4127 (3)	4128·3 (6)			
4130 (3)	4130 (3)	4131·2 (4)			
4143 (1)	4143 (2)	4143·8 (3)	4143 (5)	4143 (2)	4143 (2)
			4168 (3)		
4172 (4)	4172 (1)		4172 (1)		
4177 (4)	4177 (1)	4179·0 (2)	4177 (1)		
4233 (5)	4233 (2)	4233·3 (3)			
4241·5 (2)			4241·5 (2)		
4246 (2)					
			4253 (2)		
	4267 (2)	4267·7 (2)	4267 (4)	4267 (1)	4267 (1)
4298 (3)					
4302 (3)					
4307 (3)					
4314 (3)				4314 (1)	4314 (1)
4337 (2)					
4340 (6)	4340 (6)	4340·7 (6)	4340 (6)	4340 (6)	4340 (6)
			4345 (2)		
4351 (4)	4351 (1)		4351 (2)	4351 (1)	
4383 (3)					
	4388 (3)	4388·4 (4)	4388 (5)	4388 (2)	4388 (4)
4394·3 (3)			4394·3 (2)		
			4414·5 (2)	4414·5 (1)	4414·5 (3)
4417 (3)			4417 (2)		
		4439·6	4437 (3)		
4471 (1)	4471 (4)	4472·4 (6)	4471 (6)	4471 (5)	4471 (4)
4481 (5)	4481 (5)	4482·2 (6)	4481 (3)		
4541 (2)				4541 (1)	4541 (1)
4549 (4)					
4555·5 (3)			4553 (3)		
4558·5 (3)					

Wave-lengths of principal lines in spectra of Stars in Table A, sub-division α—(continued).

1.	2.		3.	4.	5.
	Kensington.	Potsdam.			
			4567 (2)		
4571 (2)					
			4574 (2)		
4583 (4)					
			4613 (2)		
4629 (3)			4629 (2)		
			4643 (2)		
			4650 (2)	4650 (6)	
			4714 (3)		

A list of the lines seen in three photographs of the spectrum of Rigel, taken at Potsdam, has already been published by Dr. Scheiner,* and, allowing for the difference between the Potsdam scale and that of Ångström, which I have employed, there is a fair agreement between the two series of measures.

The spectra of four typical stars of this sub-division, namely: α Cygni, β Orionis, γ Orionis and ζ Orionis, are shown in Plate 27. Many of the finer lines recorded in the table have been obliterated in the process of reproduction, but all the more important lines are present.

As far as the work has yet gone, sub-division (1) is only represented by α Cygni. The spectrum of this star is a very remarkable one. It shows many more lines than such stars as Rigel, but not nearly so many as stars which more or less resemble the Sun. The hydrogen lines are only of moderate thickness, and there are a few other strong lines, in addition to many fainter ones. On reference to Plate 27, it will be seen that it has many lines in common with Rigel and Bellatrix.

The K line of calcium is common to all the stars in question, but the strong blue line of calcium, at λ 4226, is absent.

With the exception of the K line, the lines of hydrogen, and the high temperature line of magnesium, at λ 4481, all the lines may be said to be at present of unknown origin. Some of the lines fall near lines of iron, but the absence of the strongest lines indicates that the close coincidences are probably accidental.

The Spectrum of γ Orionis.

A very fine photograph of the spectrum of this star was taken on September 17, 1892, with Instrument B. This shows many very fine lines in addition to those given in the preceding Tables. A complete list is given below, the intensities being indicated as in the previous tables.

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* 'Ast. Nach.,' No. 2923, p. 328.

LINES in the spectrum of γ Orionis (Bellatrix).

Wave-length.	Intensity.	Wave-length.	Intensity.
3919	2	4267	4
3926	3	4271	1
3933	3	4276	1
3963	3	4340	6
3968	6	4345	2
3982	1	4351	2
3994	3	4373	1
4008	5	4388	5
4025	6	4394·3	2
4040	2	4414·5	2
4069	2	4417	2
4071	2	4437	3
4075	2	4442·5	1
4088	1	4447	1
4101	6	4471	6
4104	2	4475·5	1
4119	2	4481·2	3
4120·5	4	4517	1
4127	1	4528·0	2 a broad faint band.
4130	1	4530·5	
4143	5	4553	3
4149	1	4566	2
4153	1	4574	2
4168	3	4591	1
4172	1	4602	1
4177	1	4613	2
4227	1	4620	1
4235	1	4629	2
4241·5	2	4643	2
4253	2	4650	2
		4714	3

Sequence of Spectra in these Stars.

The stars of Table A, Sub-division α, may be arranged in a continuous series. The sequence is shown in Plate 27.

The four typical stars shown are—

ζ Orionis.
γ Orionis.
β Orionis.
α Cygni.

Early observations of these stars seemed to suggest that the lines change their intensities—even to disappearance—from time to time. No such change, however, has appeared at Kensington. In the case of Rigel photographs have been taken October, 1890, and October, 1892, and the spectrum has remained constant. Duplicates of γ, δ, ϵ, and ζ Orionis and β Tauri have also been obtained, without any changes being detected.

It has been a common practice to call the stars of this type "Orion stars," it being supposed that they were confined to the constellation Orion. We now know, however, that they are distributed in nearly all parts of the northern celestial sphere.

(2.) SUB-DIVISION β.

Characteristics of the Spectra.

In the stars of this sub-division the hydrogen lines are not quite so broad as in stars like Sirius, though they are much broader than in those like the Sun. Besides the lines of hydrogen, all the principal lines of the solar spectrum can be made out, but they are much finer than in the stars like the Sun. The metallic lines appear to be stronger when the hydrogen lines are finer and *vice versâ.*

Examples are shown in Plate 28.

α Aquilæ and α Ophiuchi have been included in this list, although they have particularly hazy lines in their spectra. These lines, however, agree in position with those of stars in Sub-division γ. In the case of α Aquilæ Professor PICKERING has also noticed that the lines are hazy, and has suggested that this may be due to the very rapid rotation of the star, the axis of rotation being at right angles, or nearly so, to the plane passing through the star and the Sun.* It is evident that such a rotation would broaden all the lines, and that the fainter lines would most probably disappear.

A special series of photographs of the spectrum of α Aquilæ is being taken at Kensington in order to determine whether the haziness of its spectrum lines is invariable.

Arrangement in Order of Differences.

The stars of this sub-division do not differ essentially from each other, the only difference being in the relative thicknesses of the hydrogen and the metallic lines.

Arranging the stars in the order of differences they may be further sub-divided as follows :—

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* 'Annals of Harvard College Observatory,' vol. 26, p. 21.

(1.) The hydrogen lines are very broad, but a great number of fine metallic lines are visible.	(2.) The hydrogen lines are not so broad as in (1), and all the chief solar lines are clearly visible.	(3.) The hydrogen lines are still broader than in the solar spectrum, but not so broad as in (2). All the solar lines are very distinct.
δ Cassiopeiæ β Arietis δ Leonis 13 Comæ Berenicis 16 Comæ Berenicis α Ophiuchi α Aquilæ	β Cassiopeiæ 14 Comæ Berenicis γ Herculis α Cephei ζ Aquarii	α Canis Minoris 12 Comæ Berenicis γ Virginis

A complete list of the lines seen in the spectra of these stars would be superfluous, as it would be practically a list of the lines which appear in the solar spectrum.

(3.) SUB-DIVISION γ.

Characteristics of the Spectra.

In the stars of this sub-division, the hydrogen lines are extremely broad, and the additional lines are correspondingly faint. The characteristic lines occupy the same positions as lines in the spectra of stars in Sub-division α, the principal ones being those at λλ 4025, 4471, 4481. The stars given in the list bear a close resemblance to each other, but two further sub-divisions may be recognised. In the first of these, of which β Persei may be taken as a type, the lines at 4471 and 4481 are almost of equal intensity; in the second, *e.g.*, α Andromedæ, 4471 is very faint, and other additional faint lines appear.

Further Sub-division.

(1.) The strongest additional lines are at λλ 4024, 4471, and 4481, the latter pair being of equal intensity.	(2.) The intensity of the line at λ 4481 diminishes, whilst the line at λ 4471 is unaffected and other faint lines appear.
ζ Cassiopeiæ β Persei η Leonis α Leonis ι Ophiuchi γ Lyræ Pleiades { Atlas, Alcyone, Merope δ Capricorni	α Andromedæ 48 Persei β Aurigæ β Canis Minoris β Ursæ Majoris α Pegasi Pleiades { Maia, Taygeta, Electra

(4.) SUB-DIVISION δ.

Characteristics of the Spectra.

In these stars the hydrogen lines are very broad, and the additional lines, as before remarked, are chiefly faint representatives of those which appear in the solar spectrum. There is a complete inversion of the intensities of the hydrogen and metallic lines as compared with the solar spectrum. The most strongly-marked line, next to those of hydrogen, is generally the K line of calcium. A line of magnesium at 4481 is also usually easily distinguished. In the brighter stars, such as α Lyræ, a trace of the line at 4024 appears. In this case, however, it is accompanied by the lines of iron, whereas in Sub-division γ no iron lines have been detected so far.

It will be evident that no sharp line can be drawn between these stars and those of Sub-division β. The difference lies only in the relative intensities of the hydrogen and metallic lines. When all the principal solar lines are easily seen in the photographed spectrum the star has been classed in Sub-division β, but when only a comparatively small number of lines is seen, it has been placed in Sub-division δ.

It has not been found desirable or, indeed, possible to further sub-divide the stars included in Sub-division δ.

VI.—DISCUSSION OF STARS IN TABLE B.

In these stars there is a considerable amount of continuous absorption in the ultra-violet, and the spectra beyond K are very difficult to photograph with the instruments employed, as compared with the stars of Table A. The thickness of the hydrogen lines does not greatly differ from that observed in the solar spectrum.

(1.) SUB-DIVISION α.

Characteristics of the Spectra.

The only star of this sub-division which has yet been photographed at Kensington is γ Cygni. The spectrum shows a large number of dark lines, but it presents a very different appearance from that of the Sun. The characteristic grouping of lines about G is entirely absent, and many lines have very different intensities. The lines of calcium, including H and K, are only of moderate thickness.

Among the more prominent lines are the following:—

4383
4372
4351
4340 (G)

4325
4314
4233
4227
4176
4172
4126
4101 (*h*)
4118

The photographed spectrum is reproduced in Plate 26, and it will be seen that in addition to the lines given above there is a great number of less prominent lines.

(2.) SUB-DIVISION β.

Characteristics of the Spectra.

All the stars in this sub-division have spectra which very closely resemble the solar spectrum. In the spectrum of Arcturus, for example, over one hundred lines have been photographed in the region between G and *h*, and each line has its counterpart in the solar spectrum, as will be seen on reference to Plate 30. It appears, therefore, that all the substances which are present in the absorbing atmosphere of the Sun, are also present in the absorbing atmospheres of the stars of this sub-division.

The spectra show only small differences when compared with each other. Perhaps the greatest is between Capella and Arcturus. In the former the hydrogen lines are slightly thicker than in Arcturus, while in the latter some of the lines, especially those of calcium, are slightly intensified. The spectrum of Capella more nearly resembles the solar spectrum than does that of Arcturus. In Arcturus, also, the continuous absorption extends a little further towards G than it does in Capella.

We may therefore further sub-divide the stars of this sub-division as follows :—

Further Division into Two Sub-Classes.

(1.) Stars with spectra which cannot be distinguished from that of the Sun.	(2.) Stars with spectra which differ from the Sun in having some of the lines, especially those of calcium, slightly intensified. The lines of hydrogen are somewhat thinner than in (1).
α Cassiopeiæ	β Ceti
α Ursæ Minoris	κ Persei
α Arietis	ζ Hydræ
α Aurigæ (Capella)	α Bootis (Arcturus)
β Geminorum (Pollux)	β Bootis
ε Hydræ	π Herculis
ε Leonis	γ Aquilæ
γ Leonis	
α Ursæ Majoris	
ε Virginis	
η Bootis	
ε Bootis	
ε Ophiuchi	
β Herculis	
ζ Herculis	
η Herculis	
β Ophiuchi	
μ Herculis	
ν Ophiuchi	
70 Ophiuchi	
ζ Cygni	
η Pegasi	
μ Pegasi	

It will be evident that these two sub-divisions must be placed in juxtaposition whatever classification may be finally adopted. The spectra of Capella and Arcturus, which may be taken as types, are reproduced in Plate 29.

Carbon Absorption.

A question of considerable interest, as will appear later, is, whether in these stars there is any evidence of carbon absorption. In a communication to the Royal Society in 1878,* I showed that in the solar spectrum there is a dark fluting of carbon commencing at wave-length 3883. This has since been confirmed by Messrs. TROWBRIDGE and HITCHINS,† who also found that this fluting was the only one which remained visible when a complicated metallic spectrum was added to that of carbon.

It is therefore to be expected that indications of carbon absorption will be found in the spectra of the stars under discussion.

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* 'Roy. Soc. Proc.,' vol. 27, p. 308.

† 'Phil. Mag.,' series 5, vol. 24, p. 148.

The photographs already taken do not generally extend beyond K, and hence the fluting in question is not open to investigation. In the spectrum of Arcturus, however, this region has been photographed, and the fluting is apparently present, but its intensity does not differ greatly from its intensity in the solar spectrum.

In a photograph taken by the Brothers Henry two years ago, there is a decided darkening in the region of another group of carbon flutings commencing at λ 4215, but this does not appear in the Kensington photographs.

VII.—DISCUSSION OF STARS IN TABLE C.

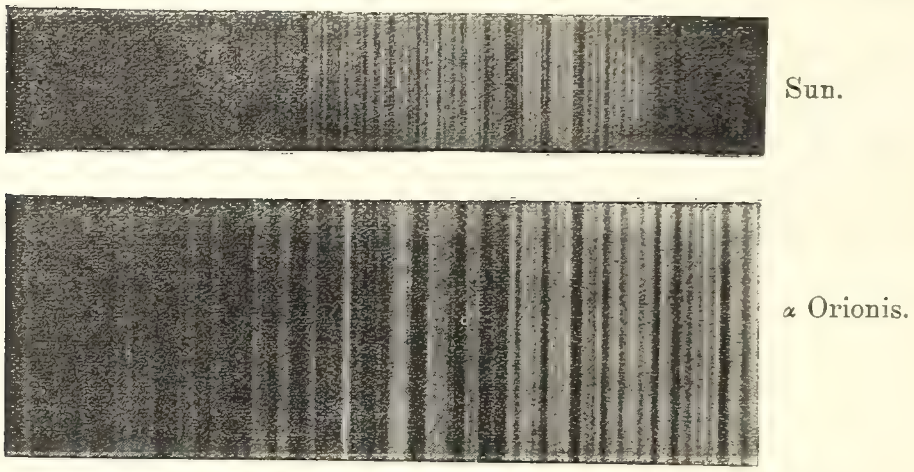
In all the stars included in this Table there is a very considerable amount of continuous absorption in the violet, extending to about G. Long exposures are therefore required, in order that their spectra may be photographed. The brightest star in the Table is α Orionis, and even this requires an exposure of 35 minutes with the instrument A to give a spectrum one-tenth of an inch wide, extending to the K line. With the 30-inch reflector and slit spectroscope, an exposure of one hour is barely sufficient. It is accordingly a matter of considerable difficulty to photograph the spectra of these stars, as most of them are below the third magnitude.

(1.) Sub-division α.

Characteristics of the Spectra.

Notwithstanding the fact that the visual spectra of these stars are characterised by conspicuous flutings, the region more refrangible than F consists almost entirely of dark lines. Indeed the only indication of flutings in this region in any of the stars named are near wave-lengths 4763 and 4585, and these vary in intensity in the different stars. The flutings are stronger in α Herculis than in any of the other stars in the Table, as will be seen on reference to Plate 26, and they are still stronger in the spectrum of Mira Ceti, as seen in a photograph kindly forwarded to me by Professor Pickering. So far as the line spectrum is concerned, α Orionis may be taken as a type of the remainder. Lines of hydrogen, iron, manganese, calcium, chromium, cobalt, titanium, and strontium are common to them all. The calcium lines, it will be seen, are considerably intensified as compared with the corresponding lines seen in the solar spectrum. This applies also to the lines of chromium and manganese, so that the spectrum presents a very different appearance from that of the Sun. Many of the lines are at present unidentified. There is a remarkable difference in the lines at G, the grouping of which is so characteristic of the solar spectrum. This will appear from fig. 5.

Fig. 5.



Comparison of the G region of the Spectrum of α Orionis and the Sun.

The Dark Flutings in these Stars.

By the use of EDWARDS' isochromatic plates, photographs of the spectra of some of the stars under discussion have been obtained which extend as far in the green as λ 550. These consequently show some of the flutings which have been mapped by DUNÉR, VOGEL, and others.

The wave-lengths of the flutings have been determined by comparison with the spectrum of Arcturus photographed under the same instrumental conditions. These, compared with the mean values determined from eye observations by DUNÉR* and VOGEL,† are as follows:—

DUNÉR'S number of band.	DUNÉR (eye observations).	VOGEL (eye observations).	LOCKYER (photographs).
10	..	4607	4585
9	4770	4766	4763
8	4958	4959	4958
7	5168	5168	5165
5	5451	5452	5455

The positions given for the bands are for their more refrangible edges, and they are the same in all the stars of this group of which the spectra have been photographed at Kensington. The positions determined by VOGEL and DUNÉR, however, for the same band in different stars vary considerably. For band 5, VOGEL found the wave-length 5444 in α Orionis, and 5458 in α Herculis. These differences are, it is possible, due to the difficulties of observation, and in a small degree to the differing velocities in the line of sight.

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[*] 'Sur les Étoiles,' p. 120.

[†] 'Bothkamp Beob.,' Heft 1, 1872.

The complete discussion of this region is reserved until better and more numerous photographs have been obtained.

Detailed Discussion of the Spectrum of α Orionis.

As α Orionis is the brightest star given in the Table under discussion, a special investigation has been made of its spectrum. Several photographs have been taken at Kensington, and one at Westgate, with the 3-prism slit spectroscope. The latter was exposed for one hour on November 30, 1891, and, as the slit was very narrow, the spectrum is well-defined. All the lines shown in the Westgate photograph have been reproduced in a later photograph taken at Kensington with the instrument B. The intensity of the violet is somewhat feeble, and the reduction has been limited to the region F to λ 4029. On reference to the photograph (Plate 28), it will be seen that the part of the spectrum extending from the violet to λ 4860 (F) consists almost entirely of lines, over 100 being distinctly visible between F and G. There are distinct groupings of lines, however, which correspond with the flutings seen in α Herculis and other stars of the same type.

For the reduction of the photograph a spectrum of daylight photographed with the same instrumental conditions has been employed.

It is easy to recognise well-marked lines which serve for the construction of the wave-length scale by means of the usual projection curve.

The hydrogen lines at F, G, and *h*, are almost as strong as in the spectrum of the Sun.

The investigation of the origins of the various lines includes also an inquiry into the probable temperature of the absorbing regions in the star. Thus, when we come to investigate the lines due to iron, we find that they do not correspond in relative intensities with those which appear in the arc spectrum. This will be clear on reference to the map, which shows the strongest lines in the arc spectrum, and all the lines of the flame spectrum of iron. Particular attention may be drawn to the four lines in the arc spectrum at wave-lengths 4201·6, 4203·5, 4206·3, and 4209·8. It will be seen that there is a complete inversion in intensities if we take the arc spectrum of iron, and many other cases are equally evident.

When compared with the flame spectrum of iron, which has been specially photographed for this investigation, a smaller number of lines is explained, but there are no inversions of importance. The feeble line at 4132 in the flame spectrum is comparatively strong in the spectrum of the star, but this is in all probability due to the superposition of the line of lithium which has the same wave-length. The only important divergence is in the case of the line 4143, which is much stronger in the star than in the flame spectrum. It is, however, a strong line in the arc spectrum of iron.

These comparisons lead to the conclusion that the temperature of the most important iron-absorbing region in α Orionis is nearer that of the oxy-coal gas flame than that of the electric arc. Still, as many of the arc lines appear to coincide with lines in the spectrum of the star, though they have different relative intensities, it is probable that the average temperature is intermediate between that of the arc and that of the flame, but nearer to the latter.

It is evident then that the flame spectra of other substances should also be compared with the spectrum of the star. Those which show any coincidences are indicated on the map. They include manganese, chromium, calcium, magnesium, strontium, cobalt, and lithium. Here, again, the arc spectra, especially in the case of manganese, show many inversions, and hence the conclusion that the temperature of the absorbing regions is lower than that of the electric arc is strengthened.

The results of the investigations into the origins of the lines are shown in Plate 28.

Elements identified in α Orionis.	
H	Mg
Fe	Co
Mn	Sr
Cr	Li
Ca	C

Many of the lines remain for the present unidentified, although careful comparisons have been made with the flame spectra of all the principal elements.

Amongst the strongest unknown lines are those at wave-lengths 4393, 4406, and 4763. There are also three very strong lines at λ 4038, 4053, and 4164. These are either entirely absent from the solar spectrum or only appear as very faint lines. That at λ 4406 forms a close double with the slightly more refrangible iron line λ 4405. It is strongly marked in the spectrum of α Tauri, as photographed at Kensington, and also appears in the spectra of β Andromedæ, α Herculis, β Pegasi, and other stars of Table C.

Bright Flutings.

In the spectra of α Orionis and α Herculis which have been photographed at Kensington on isochromatic plates there is the bright edge of a fluting at λ 5165 as near as can be determined, and this very closely agrees with the wave-length of the brightest edge of the carbon fluting λ 5164·8, according to measures made from my large scale photographic solar comparisons. If this were the only criterion for bright carbon, the evidence might be considered conclusive. There are, however, two other bright maxima in this compound carbon fluting, at wave-lengths 5128·5 and 5098·3, and we should also expect to find these in the spectrum. In α Herculis, two secondary bright flutings are clearly visible, and the following table will show their positions relatively to the carbon flutings.

Carbon.	α Herculis.
5164·8	5165
5128·5	5136
5098·3	5107

It will be seen that the coincidences are not very exact for the secondary maxima, but the question is complicated by the fact that several dark lines appear in the same region, and the secondary maxima may therefore be masked.

Further photographs will be taken with special reference to this region, special exposures being required, as the photographic plates in use are very slow for this part of the spectrum, and the instrument has to be specially focussed.

Other regions of the spectrum have also been examined for bright flutings, more particularly with reference to the flutings of carbon. From a photograph of the spectrum of Mira Ceti taken by Professor PICKERING, there is strong evidence of the existence of the carbon group, commencing at wave-length 4215·6, but there are no decided indications of it in α Orionis. In the latter spectrum the dark lines are much more numerous than in Mira, and hence the bright fluting, if present, may be masked. It seems more likely, however, that the increased temperature, indicated by the greater number of lines in α Orionis, is the true explanation of the disappearance of the fluting, only the most persistent one—that at λ 5165—remaining. There are no certain indications of the bright fluting of carbon commencing at λ 4736 (see Plate 28).

Sequence of Spectra of these Stars.

Plate 26 shows the spectra of α Herculis, β Pegasi, α Orionis, and β Andromedæ, arranged in the order of differences. This order has been chiefly determined by the flutings near λ 4763 and λ 4585, which gradually thin out in passing from α Herculis to β Andromedæ. It will be seen also that the lines of calcium and some other lines thin out in the same order.

Including the stars of which the spectra have not been reproduced, they have been arranged as follows, those bracketed together being indistinguishable from each other.

$$\left\{\begin{array}{l}\beta \text{ Andromedæ.}\\ \alpha \text{ Ceti.}\\ \nu \text{ Bootis.}\end{array}\right.$$

α Orionis.

$$\left\{\begin{array}{l}\beta \text{ Pegasi.}\\ \alpha \text{ Scorpii.}\\ \rho \text{ Persei.}\end{array}\right.$$

$\left\{\begin{array}{l}\alpha \text{ Herculis.} \\ \delta \text{ Virginis.}\end{array}\right.$

$\left\{\begin{array}{l}\delta \text{ Ophiuchi.} \\ \mu \text{ Geminorum.}\end{array}\right.$

(2.) SUB-DIVISION β.

Characteristics of the Spectra.

All the spectra of the stars of Sub-division β of Table C are practically identical, and only that of α Tauri is therefore reproduced.

This is shown in Plate 26, and on comparison with the spectrum of α Orionis the close similarity in the photographic region will be evident.

In all the stars named the intensity of the spectrum beyond G is feeble, and long exposures are required in order to obtain photographs in that region. The falling off of intensity, however, is not so strongly marked as in the stars of Sub-division α.

As in α Orionis, the origins of the chief lines have been traced to hydrogen, iron, calcium, manganese, chromium, magnesium, titanium, cobalt, and strontium. The lines of calcium, especially the low temperature line at λ 4226, are particularly broad. The manganese lines near λ 4029, and the iron line near λ 4045, are remarkably conspicuous. The latter is relatively much stronger than the other iron lines in the same region, as compared with stars like the Sun, and this suggests that it may be strengthened by a line of some unknown substance which has very nearly the same wave-length. On reference to the photograph it will be seen also that the two iron lines λ 4383 and λ 4405 are also considerably intensified in this group of stars. These two pairs of lines, with the strong line of calcium at λ 4226 and the weakness of the spectrum beyond G, serve for the ready identification of all stars resembling α Tauri.

DUNÉR* describes the spectrum of α Hydræ as more resembling that of the Sun than that of α Tauri, but the photographic spectrum is practically identical with α Tauri. The calcium line λ 4226 is very strong, as are also the other characteristic lines to which reference has been made.

Possible Variation in the Spectrum of α Tauri.

An interesting point has followed from the comparison of the Kensington photographs of α Tauri with one taken and kindly forwarded to me by Professor PICKERING. The strong line, about λ 4406, in the Kensington photographs is hardly visible in the photograph taken by Professor PICKERING, and it would, therefore, appear that the spectrum must be variable. This particular line, it may be remarked, probably forms a close double with the iron line λ 4405, being slightly less refrangible. It is a very strong line in a photograph of the spectrum of α Tauri taken at Kensington, on November 20, 1888, and is seen in the spectra of all the stars resembling α Tauri or α Orionis. A photograph taken on September 15, 1892, with the instrument B shows the line strongly marked.

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* 'Sur les Étoiles,' &c., p. 88.

It will be evident from a comparison of the photographs that these stars bear a close relationship to those of Sub-division α. The increased brightness of the violet end of the continuous spectrum, in conjunction with the fading out of the flutings, may be taken as an indication of increased temperature in passing from Sub-division α to Sub-division β.

VIII.—THE GENERAL SEQUENCE OF THE SPECTRA OF THE STARS NOW UNDER DISCUSSION.

(1.) SEQUENCE OF SUB-DIVISIONS.

We have now before us the facts relating to the various spectra which have been photographed at Kensington, and the next thing to do is obviously to attempt to trace the relationships of the various sub-divisions to which reference has been made.

Two Series of Spectra.

One important fact comes out very clearly, namely, that, whether we take the varying thicknesses of the hydrogen or of the lines of other substances as the basis for the arrangement of the spectra, it is not possible to place all the stars in one line of temperature. For example, in the stars of Table A, Sub-division α, the hydrogen lines are of the same average thickness as in the stars of Table A, Sub-division β, but the remaining lines are almost entirely different; and the two sub-divisions cannot be placed in juxtaposition. It is, therefore, necessary to arrange the stars in two series. This has been done, and the general results of the investigation are shown in Plates 26, 27, and 29, where the spectra of eighteen typical stars are arranged in order of differences. The corresponding sub-divisions are as follows:—

Reference to Plates 26 *and* 27.

Typical stars.	Sub-divisions.	
α Andromedæ	Table A, sub-division γ	2
β Persei	,, ,, ,, ,,	1
ζ Orionis	,, ,, ,, α	4
γ Orionis	,, ,, ,, ,,	3
β Orionis	,, ,, ,, ,,	2

Typical stars.	Sub-divisions.				
α Cygni	Table	A,	sub-division	α	1
γ Cygni	,,	B	,,	α	
α Tauri	,,	C	,,	β	
β Andromedæ α Orionis β Pegasi α Herculis	,,	C	,,	α	

Reference to Plate 29.

Typical stars.	Sub-divisions.				
α Andromedæ	Table	A,	sub-division	γ	2
α Canis Majoris α Geminorum	,,	,,	,,	δ	
α Persei	,,	,,	,,	β	2
α Canis Minoris	,,	,,	,,	,,	3
α Aurigæ	,,	B	,,	β	1
α Bootis	,,	,,	,,	,,	2

(2.) Variations Observed.

It will be seen, on reference to Plates 26 and 27, that, starting with α Herculis, we have a mixed spectrum of lines and flutings, the latter gradually disappearing in β Pegasi, α Orionis, and β Andromedæ, until, in α Tauri, nothing but lines remain. At the same time the continuity of the lines is unbroken, so that if we were to arrange the stars according to the thicknesses of, say, the lines of calcium, the same order would be deduced.

Next in the series is γ Cygni, the spectrum of which has much in common with that of α Tauri, but there is less continuous absorption, and many of the lines of α Tauri thin out.

The next step, to α Cygni, is rather a long one, but it seems very probable that, if more photographs were available, intermediate spectra would be found. It will be seen, however, that in α Cygni the hydrogen lines are intensified as compared with γ Cygni, and that all the important lines of α Cygni agree in position with prominent lines in γ Cygni.

At this stage there is a great diminution in the number of lines, as will be seen on referring to the spectrum of Rigel (β Orionis), which is the next star in the series. In passing to Rigel, the more important lines of α Cygni are retained, and a few new lines make their appearance.

The next star in the series is Bellatrix, the line at λ 4471 increasing in intensity, while the line at λ 4481 has almost disappeared.

In the next stage, represented by ζ Orionis, an important new line at λ 4088 makes its appearance, but all the chief lines of Bellatrix are retained.

α Virginis is the next star in the series, and here again it is clear that the continuity is unbroken, the hydrogen lines also being broader than in the stars already dealt with.

β Persei and α Andromedæ are the representatives of the sub-divisions which come next in the sequence, and in these also it will be seen that there is perfect continuity.

From α Herculis to α Andromedæ, we thus have a continuous series of spectra, the dark flutings first disappearing, and afterwards most of the lines of the more common metals such as iron and manganese, lines of unknown origin gradually replacing them. At the same time, the amount of continuous absorption is gradually diminishing, and the lines of hydrogen are increasing in intensity.

We now come to the second series (Plate 29), and here it is quite an easy matter to begin where the first series left off, namely, with such a star as α Andromedæ. In these stars, as already pointed out, there are no certain indications of iron. There are other stars, however, such as Sirius and α Geminorum in which there are all the important lines of α Andromedæ, and, in addition, some of the lines of iron.

In the stars which come next in the sequence, represented by β Arietis, there is an intensification of the metallic lines, and a slight diminution in the breadth of the hydrogen lines. This goes on through the various sub-divisions represented by α Persei, α Canis Minoris, α Aurigæ, and α Bootis, the hydrogen lines thinning out as the metallic lines thicken. Continuous absorption in the ultra-violet commences in the later stars of the series.

Thus, from α Andromedæ to Arcturus there is a perfectly continuous series of spectra.

IX.—DISCUSSION OF RESULTS IN RELATION TO THE METEORITIC HYPOTHESIS.

(1.) Phenomena to be Expected on the Hypothesis.

Reference to Classification based on Eye Observations.

We are now in a position to consider the various divisions of the photographic spectra already arrived at, in relation to the groups which were previously suggested from a discussion of eye observations. This classification was as follows :—[*]

Group I.—Radiation lines and flutings predominant. Absorption beginning in the last species.

Group II.—Mixed radiation and absorption predominant.

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[*] 'Roy. Soc. Proc.,' vol. 44, p. 26.

GROUP III.—Line absorption predominant with increasing temperature. The various species will be marked by increasing simplicity of spectrum.

GROUP IV.—Simplest line absorption predominant.

GROUP V.—Line absorption predominant, with decreasing temperature. The various species will be marked by decreasing complexity of spectrum.

GROUP VI.—Carbon absorption predominant.

GROUP VII.—Extinction of luminosity.

The fundamental difference between this and other classifications is that it demands the existence of bodies of increasing as well as bodies of decreasing temperatures.

We have, therefore, to inquire how far this condition is satisfied by the mass of new facts at our disposal. This involves the consideration of some points in connection with the meteoritic hypothesis to which brief reference alone has been made in my previous communications.

The Complex Origin of the Spectra of Nebulæ.

On the hypothesis, the bright lines seen in the nebulæ should have three origins.

(1.) The lines of those substances which occupy the greatest volume (or largest area in a section); in other words, the lines of those substances which are driven furthest out from the meteorites and occupy the interspaces, when possibly they may be rendered luminous by electricity. Chief among these, from laboratory experiments, we should expect hydrogen, and next, from the same experiments, we should expect gaseous compounds of carbon.

(2.) We are justified in assuming that the most numerous collisions will be partial ones—grazes—sufficient only to produce comparatively slight rises in temperature. The nebula spectrum, so far as it is produced by this cause, will therefore depend upon the phenomena produced in greatest number, and we may therefore expect to find the low temperature lines of various metallic substances.

(3.) In addition to the large number of partial collisions there will be a relatively small number of end-on collisions, producing very high temperature,* and, so far as this cause is concerned, there will be some lines produced which are associated with very high temperatures.

Combining these conclusions, in the spectra of nebulæ we should expect to find evidence of

Hydrogen.
Compounds of carbon.
Low temperature metallic lines and flutings.
Lines which are only produced at very high temperatures.

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* 'Roy. Soc. Proc.,' vol. 43, p. 150.

The Passage to Bright-line Stars.

On the hypothesis, the lines seen in the spectra of bright-line stars should, in the main, resemble those which appear in nebulæ. They will differ, however, for two reasons :—

(1.) Owing to partial condensation of the swarm the hydrogen area will be restricted, and the bright lines of hydrogen will lose their prominence; the volume occupied by the carbon compounds will be relatively increased, and the brightness of the carbon bands will be enhanced.

(2.) On account of the increased number of collisions, more meteorites will be rendered incandescent, and the continuous spectrum will be brighter than in nebulæ.

Stars of Increasing Temperature.

Initially, each pair of meteorites in collision may be regarded as a condensation. Ultimately, when all the meteorites are volatilized, there will only be one condensation, in the shape of a spherical mass of vapour. Between these points there must be other conditions.

(Stage 1.)—At the stage of condensation immediately following that of the bright-line stars, the bright lines from the interspaces will be masked by corresponding dark ones produced by the absorption of the same vapours surrounding the incandescent meteorites. One part of the swarm will give bright lines, another dark lines at the same wave-lengths, and these lines will therefore vanish from the spectrum. The interspaces will be restricted so that absorption phenomena will be in excess, and the first absorption will be that due to low-temperature vapours, that is, fluting absorptions of various metals. The radiation spectrum of the interspace will now be chiefly that of the compounds of carbon. Under these conditions we know from laboratory experiments* that the amount of continuous absorption at the blue end will be at a maximum.

(Stage 2.)—With further condensation the radiation spectrum of the interspaces will gradually disappear, and the fluting absorptions will be replaced by dark lines, for the reason that the incandescent meteorites will be surrounded by vapours produced at a higher temperature, the number of violent collisions per unit time and volume being now greatly increased. This dark line spectrum need not necessarily resemble that of the Sun.

(Stage 3.)—The line absorption and the continuous absorption at the blue end of the spectrum will diminish as the condensations are reduced in number, for the reason that only those vapours high up in the atmospheres surrounding the condensations will be competent to show absorption phenomena, in consequence of the bright continuous spectrum of the still disturbed lower levels of those atmospheres.

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* LOCKYER and ROBERTS-AUSTEN, 'Roy. Soc. Proc.,' 1875, p. 344.

Among the more important lines which will disappear at this stage will be those of iron, for the reason that there will be bright lines from the interspaces occupying the same positions as the dark lines produced by the absorption of the vapour surrounding the stones.

The number of violent collisions per unit time and volume being further increased, we should expect the absorption of very high temperature vapours.

The Hottest Stars.

Ultimately, then, we should expect that the order of the absorbing layers will follow the original order of the extension of the vapours round the meteorites in the first condition of the swarm, and the lines seen bright in nebulæ, whatever their origins may be, should therefore appear almost alone as dark lines in the hotter stars, and the hydrogen especially should have its lines broadened with each increase of depth in the atmosphere. The continuous absorption at the violet end of the spectrum will be at a minimum. If, when the hydrogen lines are thickest the swarm is not yet completely condensed, that is, if there be nebulous matter surrounding the central mass of vapour, a fine bright line will be seen down the centre of each dark one.

Stars of Decreasing Temperature.

When we consider the cooling condition, that is, what happens when the temperature of the mass of vapour is no longer increased by the fall towards the centre of meteorites composing the initial swarm, we should expect to find the phenomena indicated below.

(Stage 1.)—The hydrogen lines will begin to thin out, on account of the diminishing depth of the absorbing atmosphere, and new lines will appear.

As I have already pointed out,* the new lines will not necessarily be the same as those observed in connection with stars of increasing temperature. In the latter there will be the perpetual explosions of the meteorites affecting the atmosphere, whereas in a cooling mass of vapour we have to deal with the absorption of the highest layers of vapours. Those lines which will first make their appearance, however, will be the longest low temperature lines of the various chemical elements.

(Stage 2).—The hydrogen lines will continue to thin out, and when the absorption of the hotter lower layers makes itself felt the spectra will show the high temperature spectra of the various chemical elements, showing many more lines. The difference between these and the lines seen in stars of increasing temperature should be one due to the different percentage composition of the absorbing layers, so far as the known lines are concerned.

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* 'Roy. Soc. Proc.,' vol. 45, p. 382.

With this increasing line absorption there will be a recurrence of the continuous absorption in the ultra-violet.

(Stage 3).—With the further thinning of the hydrogen lines and reduction of temperature of the atmosphere, the absorption flutings of the compounds of carbon should come in.

So much, then, for what we should expect, assuming the hypothesis to be true.

I now proceed to show how far these requirements are satisfied by the mass of new facts now at our disposal.

(2.) The Actual Phenomena Recorded on the Photographs.

Nebulæ.

A preliminary account of the photographs of the spectrum of the Orion Nebula taken with Instrument E has already been published,[*] and though the discussion is not yet completed, it may be stated that the lines recorded in the spectrum are at wave-lengths which approximate very closely to the lines of hydrogen, to flutings which appear in the spectra of compounds of carbon, to a fluting of magnesium at 5006, and to the longest flame lines of iron, calcium, and magnesium.

The chromospheric line designated D_3 has been recorded in the visual spectrum of the Orion Nebula by Dr. Copeland,[†] and the observation has since been confirmed by Mr. Taylor.[‡]

The line which is always associated with D_3 in the spectrum of the chromosphere, viz., that at λ 4471 (Lorenzoni's *f*), is also shown in the photograph of the spectrum of the Orion Nebula.

The requirements of the hypothesis with regard to nebulæ are therefore met in every point so far considered by the new facts.

Dividing up the lines into the three groups of origins suggested, we have in the case of the Orion Nebula :—

(*a*.) Spectrum of large interspace (= that of non-condensable gases driven out of the meteorites)	=	Lines of hydrogen ; flutings of carbon.

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[*] 'Roy. Soc. Proc.,' vol. 48, p. 199.

[†] 'Monthly Notices,' vol. 48, p. 360.

[‡] *Ibid.*, vol. 49, p. 124.

(*b.*) Spectrum of vapours produced by the large number of partial collisions	=	Fluting of magnesium at λ 5006;[*] low temperature, lines of iron, calcium, and magnesium.[†]
(*c.*) Spectrum of the vapours produced at a very high temperature by the relatively small number of end-on collisions. The solar chromosphere may be taken as indicating the spectrum associated with this very high temperature	=	Chromospheric lines, $D_3 + \lambda$ 4471.

A detailed reference to those nebulæ showing the spectra of carbon bands alone is reserved for a future communication.

Bright-Line Stars.

Professor PICKERING has shown that the Draper Memorial Photographs (copies of which he has very kindly forwarded me) prove that bright-line stars are intimately connected with the planetary nebulæ, the lines in the spectra being almost identical, as the following table will show :—[‡]

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[*] My suggestion of the coincidence of the brightest line of the nebular spectrum with the fluting of magnesium at λ 5006·5, has given rise to discussion, and some observations have been made by Professor KEELER, with the three-foot Lick refractor and the third order spectrum of a diffraction grating, on this subject. When I first suggested magnesium as the origin of the chief line in the nebular spectrum, the measurements then available were quite sufficient to justify it as regards position, and there were other and stronger grounds for the suggestion, depending upon laboratory experiments and upon the frequently observed fluted character of the line. There are so many pitfalls attending the delicate measurements involved in such an inquiry, that I am not yet convinced that the absolute wave-length of the nebular line has been determined. The Astronomer-Royal has recently pointed out ('Monthly Notices,' vol. 52, p. 245) that the varying flexure of the telescope, when presented to different parts of the sky, has an effect on the measurements, and in the spectroscopic determinations of the motions of the stars in the line of sight at Greenwich, the displacement of the lines in the spectrum of a star was found to give clear indications of the existence of the systematic error referred to. Further, I have convinced myself of the fluted nature of the line by new observations made with instruments best fitted to show it, while the Lick telescope is perhaps the ideal telescope *not* to employ in such an inquiry. Hence, although the visibility of magnesium in nebulæ is not fundamental for my argument, I still hold that it is more probably the origin of the nebular line than an unknown form of nitrogen.

[†] I have previously given evidence deduced from eye observations, indicating the presence of other low temperature flutings of manganese and magnesium.

[‡] 'Ast. Nach.,' No. 3025, p. 1.

PROFESSOR PICKERING's Table of Bright Lines.

Planetary nebulæ.	Bright-line stars. Type I.	Bright-line stars. Type II.	Bright-line stars. Type III.
501			
486	486	486	
470	469	469	
..	462	464	464
..	454	455	455
..	..	451	451
447	..	447	
..	..	..	443
434	434	434	434
..	420	420	421
410	410	410	412
..	406	406	407
..	402	402	
397	398	397	
..	395	..	395
388	389	389	

The main point of difference is that the chief nebular line near λ 5006 is not seen in the spectrum of bright-line stars, and this no doubt is due to the relative absence of feeble collisions as condensation goes on. The brightening of this line in the spectra of Nova Cygni and Nova Aurigæ, as the stars faded away, is sufficient evidence that it is associated with low temperature, and hence it is not surprising to find that it is absent from the spectra of the bright-line stars, which on this hypothesis are hotter than the nebulæ, since they are more condensed.

I have stated that we should expect—

(*a.*) The carbon flutings, and
(*b.*) The continuous spectrum to be brighter than in nebulæ, while
(*c.*) The hydrogen lines are fainter.

(*a.*) I may add, by way of reminder, that in my previous discussion of these bodies [*] I showed that there was evidence of a very considerable amount of carbon radiation in the visible region of the spectrum. Subsequent work at Kensington and Westgate, and an examination of Professor PICKERING's photographs have strengthened this view. The photographs show a band which agrees very closely in position with the band of carbon commencing at λ 4736, but which under certain conditions has its brightest part at λ 468.[†] A full discussion of this question will form the subject of a future communication.

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[*] 'Roy. Soc. Proc.,' vol. 44, pp. 33–43.

[†] *Ibid.*, p. 38.

(*b.*) There can be no question as to the continuous spectrum being brighter in bright-line stars than in nebulæ.

(*c.*) The hydrogen lines also are decidedly less prominent. Indeed they were not recorded at all in the eye observations of γ Argûs (Arg.-Oeltz., 17681), or WOLF and RAYET'S 2nd and 3rd stars in Cygnus,[*] but they are shown in Professor PICKERING'S photographs.

Stars of Increasing Temperature.

Stage 1.—We should expect the spectra to show :—

(*a.*) Absence of bright lines.
(*b.*) The presence of dark metallic flutings.
(*c.*) The presence of bright carbon flutings.
(*d.*) Continuous absorption in the violet.

We know that there are spectra—those of stars in Table C, Sub-division α—which answer these requirements.

(*a.*) They show no bright lines under normal conditions, but if the stars are variable, the disturbances which bring about the change of luminosity at maximum, produce bright lines in the spectrum as in the case of the spectrum of Mira Ceti photographed by Professor PICKERING.

(*b.*) In my discussion of DUNÉR'S observations[†] I showed that most of the dark flutings observed in the part of the spectrum visible to the eye approximated very closely to the flutings seen in the flame spectra of manganese, lead, and iron. As already stated (p. 702) some of these dark flutings have been photographed at Kensington, but they have not yet been completely investigated.

(*c.*) In my discussion of DUNÉR'S[‡] eye observations, I also showed that in addition to dark metallic flutings we had to deal with radiation flutings of carbon.[§] The evidence afforded by the photographs has already been referred to (p. 704) and although there is now no reason to doubt the actual presence of carbon radiation, further photographs are being obtained to carry on the inquiry.

The stars of this class which have already been photographed at Kensington are well advanced in condensation, as indicated by the numerous dark lines, and all the flutings, both bright and dark, are confined to the region less refrangible than G. We should therefore not expect to get the more refrangible carbon flutings, as I have pointed out in a previous communication.* It is among the least condensed stars that we should expect the bright carbon to be more manifest, and, indeed, in the spectrum of Mira Ceti photographed by Professor PICKERING, there is strong evidence of the presence of one of the more refrangible carbon bands commencing at λ 4215.

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[*] 'Roy. Soc. Proc.' vol. 44, pp. 33–43.

[†] *Ibid.*, p. 49.

[‡] *Ibid.*, p. 52.

[§] Subsequent observations by myself and Mr. FOWLER seemed to leave no doubt as to the

(*d.*) That there is a very considerable amount of continuous absorption in the ultra-violet or violet has already been stated, and the photographs reproduced in Plate 26 fully demonstrate this.

It is evident that the sequence of the spectra photographed should resemble that deduced from eye observations. In the case of the stars already photographed (see Table C, Sub-division α) this order was as follows, the group of stars being divided into fifteen species :—

α Orionis ν Bootis	Species 15.
α Ceti δ Virginis α Herculis	,, 10.
α Scorpii β Pegasi δ Ophiuchi	,, 9.
μ Geminorum ρ Persei	,, 8.

There is evidence that β Andromedæ, which was not included by DUNÉR in his Tables, should also be included in Species 15. GOTHARD observed the spectrum of this star in 1882,† and stated its spectrum as VOGEL's Class III.—*a.* (Group II.); DUNÉR, however, re-examined the spectrum‡ and recorded that it was like that of α Tauri, in the red there was "almost a band," but the remainder of the spectrum consisted entirely of lines. My own observations at Westgate, and Mr. FOWLER's observations at Kensington, in the latter part of 1888, showed that the spectrum

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presence of these bright carbon flutings.[1] Dr. COPELAND had previously made important observations of "Nova" Orionis with reference to this point,[2] and he identified one of the bright bands as "the great hydrocarbon band seen in the spectrum of every comet that has been examined under favourable circumstances." Referring to his observations of α Orionis, Mr. MAUNDER[3] states that "the carbon band at 5164 was coincident (within the limits of observation with this dispersion) with the bright space towards the blue of DUNÉR's band 7."

* 'Roy. Soc. Proc.,' vol. 44, p. 59.

† 'Public. des Astr. Obs. zu Herény,' Heft 1, p. 50.

‡ 'Sur les Etoiles,' p. 86.

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[1] 'Roy. Soc. Proc.,' vol. 47, p. 40.

[2] 'Monthly Notices,' vol. 46, p. 112.

[3] 'Greenwich Spect. Observations,' 1889, p. 22.

was then intermediate between α Orionis and α Tauri, and that it should, therefore, be placed in Species 15. It is, perhaps, possible that the spectrum is slightly variable.

Other eye observations at Kensington,* showed that α Ceti should be transferred to Species 15. Including β Andromedæ and this star we should, therefore, have the following relation :—

Order from eye observations.	Order from photographs.
β Andromedæ. α Ceti.	β Andromedæ. α Ceti. ν Bootis.
α Orionis. ν Bootis.	α Orionis.
α Herculis. δ Virginis.	α Scorpii. β Pegasi. ρ Persei.
α Scorpii. β Pegasi. δ Ophiuchi.	α Herculis.
μ Geminorum. ρ Persei.	δ Virginis.
	δ Ophiuchi. μ Geminorum.

It will be seen that very nearly the same order as that derived from eye observations is arrived at from the photographs, and the wonderful thing is that the observations of DUNÉR will bear the severe test which has thus been applied to them.

(Stage 2.)—At this stage we should expect--

(*a*.) Diminution in the amount of continuous absorption.

(*b*.) Spectrum consisting of dark metallic lines, but possibly differing from the solar spectrum.

These conditions are fulfilled by the stars of Table C, Sub-division β, and Table B, Sub-division α, of which α Tauri and γ Cygni may be taken as types. The continuous absorption is least in latter. These spectra show numerous metallic lines, but they do not exactly resemble the solar spectrum. The hydrogen lines are comparatively thin, while other lines have very different intensities as compared with lines in the solar spectrum.

In these stars we have to deal with the varying volatilities of the meteoritic constituents of the swarm, while in the case of stars which are cooling we have to deal with successive combinations rendered possible by the fall of temperature in a gaseous mass. Hence differences in the spectra are to be expected.

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* 'Roy. Soc. Proc.,' vol. 45, p. 385.

(Stage 3.)—The phenomena which would be expected on the hypothesis, at this stage, are fully satisfied by the stars of Table A, Sub-division α. In these stars there is no continuous absorption in the violet or ultra-violet, and the spectrum is one with simple line absorptions, the iron lines quite disappearing after such a star as α Cygni is passed. The new lines which now make their appearance include the chromospheric line at λ 4471, and possibly a few others. It is important to note that the photographic region of the spectrum of the chromosphere has not yet been fully investigated, and hence a fair comparison with the spectra of these stars in the region F to K is not yet possible. M. Deslandres and Professor Hale have photographed the chromospheric spectrum in the region more refrangible than H, but have not as yet published any account of the spectrum in the region now under discussion.

The Hottest Stars.

The conditions required by the hypothesis with regard to the stars at this stage are satisfied by the stars of Table A, Sub-division γ.

In these stars we have—

(*a.*) Broad lines of hydrogen, and

(*b.*) Other absorption lines agreeing in position with some of the bright lines which appear in nebulæ. This will be seen from the following Table, showing the coincidences of the lines which appear in α Andromedæ with the lines of the Orion Nebula:—

α Andromedæ.	Orion Nebula.
3933 (K)	3933
3968 (H)	3968
4025	4025
4066	4068
4101 (*h*)	4101
4130	4130
4143	4142
4152	4155
4168	4167
4187	4188
4202	4200
4268	4268
4340 (G)	4340
4471	4471

Professor Pickering's photographs have shown that in the spectrum of Pleione there are bright lines down the middle of the broad dark lines of hydrogen, and we know from the photographs taken by Mr. Roberts and others that this star is involved in uncondensed nebulous matter. Hence, another possible condition in the hottest stars is satisfied by the photographic facts at our disposal.

It will be seen then, that these considerations of the conditions of increasing temperature demanded by the hypothesis, have enabled us to determine that the series of spectra represented in Plates 26 and 27 is in all probability a series in *ascending* order of temperature. All the phenomena we should expect, on the hypothesis, are met with among the spectra of celestial bodies.

We have next to consider the phenomena connected with stars of decreasing temperature.

Stars of Decreasing Temperature.

(Stage 1.)—We have seen that with the failure of the supply of meteorites falling into the centre of the now vaporized mass, cooling will commence, and the longest lines in the spectra of the various chemical elements should make their appearance. This condition is met with in the stars of Table A, Sub-division δ. The following Table will show that this is true in the case of the iron lines, if we take the spectrum of Sirius as a type.

Some of the Iron Lines in the Spectrum of Sirius.

4045
4063
4071
4215
4298
4307
4383
4404
4414

(Stage 2.)—The conditions at this stage of cooling are satisfied by the stars of Table A, Sub-division *β*. In these stars, as already pointed out (p. 696), we get, in addition to fairly broad lines of hydrogen, nearly all the lines which appear in the solar spectrum, and these it is well known agree in the main with the arc spectra of the various chemical elements.

(Stage 3.)—The stars of Table B, Sub-division *β*, represent the conditions which are required by the hypothesis at this stage of cooling. The metallic line absorption is again at a maximum, and we find the lines of the various chemical elements similar to those seen at Stage 2 of the ascending series, but with different intensities and with different amounts of continuous absorption at the violet end of the spectrum. This difference, so far as the known lines are concerned, I have already pointed out will be due to a different percentage composition of the absorbing mass of vapour (p. 713).

Continuous absorption in the violet begins at this stage, as will be seen on reference to the spectrum of Arcturus. The question of carbon absorption at this stage has already been referred to (p. 700). There is undoubted evidence of its presence in the solar spectrum, and in the spectrum of Arcturus—the only star which has yet been investigated with special reference to this point.

Hence, it seems probable, as I stated in a former paper,* that "the indications of carbon will go on increasing in intensity slowly, until a stage is reached, when, owing to the reduction of temperature of the most effective absorbing layer, the chief absorption will be that of carbon."

It is evident that all such stars will be dim, and hence their spectra have not been met with in this preliminary survey of the photographic spectra of the brighter stars.

The phenomena we should expect on the hypothesis, in stars of decreasing temperature, therefore actually appear in the series of spectra represented in Plate 29.

The general result of the above discussion then, as far as it goes, is as follows:—Among the 171 stars already considered there are really two series of spectra, one representing the changes accompanying the increase of temperature, while the other represents the effects of decreasing temperature. The fundamental requirement of the meteoritic hypothesis is, therefore, fully justified by the discussion of the photographs.

A very important point in connection with the two series of successive spectra is that one spectrum, such as that of α Andromedæ, possesses characteristics common to both, and we might, therefore, connect the two series together by this spectrum. In that case we should find, if we commence with the first spectrum in Series 1, say that of α Herculis, that the continuous absorption diminishes and that the breadth of the hydrogen lines regularly increases, until such a spectrum as that of α Andromedæ is reached (Plates 26 and 27). Then the condition would be reversed, the breadth of the hydrogen lines diminishing and the continuous absorption in the ultra-violet increasing in extent until such a star as Arcturus is reached (Plate 29).

I have previously discussed the question of what spectrum is associated with the hottest stars,† and, generally speaking, it appears to be that in which the continuous radiation at the violet end of the spectrum is greatest, and the hydrogen absorption lines are broadest.‡

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* 'Roy. Soc. Proc.,' vol. 43, p. 155.

† Bakerian Lecture, 'Roy. Soc. Proc.,' vol. 44, p. 26.

‡ An erroneous idea with regard to the indications of the temperature of the stars has been held by those who have not considered the matter specially. It has been imagined that the presence of the series of hydrogen lines in the ultra-violet was of itself sufficient evidence of a very high temperature. The experiments of CORNU[1] however, have shown that the complete series of lines can be seen with an ordinary spark without jar. Hence, the high temperature of such a star as Sirius is not indicated by the fact that its spectrum shows the whole series of hydrogen lines, but by the fact that there is bright continuous radiation far in the ultra-violet.

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[1] 'Journal de Physique,' vol. 10, 1886.

The stars with flutings in their spectra are probably among the coolest, as we know from laboratory experiments that flutings are always associated with low temperatures. Assuming these distinctions to be true, it is seen that when the two series of ascending and descending spectra are united in this way, the hottest stars, as we should expect, fall in the middle of the combined series.

(3.) Relation of the Groups to the Tabular Divisions.

We may next proceed to inquire into the relationships of the groups in the classification first suggested by the eye observations to the various sub-divisions in the tables of photographs which have already been given.

The Ascending Series.

It has already been shown (p. 707) that the most probable sequence of spectra in what we have now demonstrated to be the ascending series is as follows, the stars of highest temperatures being placed at the head of the list.

Table A, Sub-division γ, 1.
,, ,, α, 5.
,, ,, α, 4.
,, ,, α, 3.
,, ,, α, 2.
,, ,, α, 1.
,, B, ,, α.
,, C, ,, β.
,, ,, α.

If, as in the original classification, we take Group II. to include all the stars with dark flutings, Group III. to include all the stars of increasing temperature which have line spectra, and Group IV. to include the hottest stars, we shall evidently have the following relation :—

Table A,	Sub-division	γ, 2	Group IV.
,,	,,	γ, 1	
,,	,,	α, 5	Group III.
,,	,,	α, 4	
,,	,,	α, 3	
,,	,,	α, 2	
,,	,,	α, 1	
,, B,	,,	α	
,, C,	,,	β	
,,	,,	α	Group II.

It will be seen that the sequence now determined from the photographs follows exactly the same order as the groups originally suggested by the hypothesis, from a discussion of the eye observations. That is, it is not necessary to interchange any of the groups in order to obtain agreement with the photographic results.

The Hottest Stars. (Group IV.)

At this point it becomes necessary to consider what spectra shall be included in Group IV. In the Bakerian Lecture I wrote as follows with regard to it :—

"The next group, the fourth, brings us to the stage of highest temperature, to stars like α Lyræ ; and the division between this group and the prior one must be more or less arbitrary, and cannot at present be defined. One thing, however, is quite clear, that no celestial body without all the ultra-violet lines discovered by Dr. HUGGINS can claim to belong to it."

We are met by a similar difficulty when we attempt to draw the dividing line between this group and the next (Group V.).

The photographic spectra, however, enable us to more clearly define the group, and it will be convenient to include the stars of Table A, Sub-division γ, 1 ; Table A, Sub-division γ, 2 ; and Table A., Sub-division δ. The stars in the first of these sub-divisions are still increasing their temperatures, though having almost reached their maximum, and they are distinguished from stars of Group III. by the reappearance of the line at λ 448, and in having broader hydrogen lines than those stars.

The stars of Table A, Sub-division γ, 2, represent, so far as we can judge at present, the stars of highest temperature. They are distinguished from those of the previous sub-division by the disappearance of the line at λ.

Stars of Table A, Sub-division δ, which include α Lyræ and Sirius, are marked by the appearance of iron lines in the spectrum. They probably represent the earliest stage of cooling.

The dividing line between Group IV. and Group V. may very conveniently be taken by the absence or presence of the blue line of calcium at λ 4226·6, this line making its first appearance in the stars of Table A, Sub-division *β*.

We have then the following relation to the tables :—

$$\left.\begin{array}{llr}\text{Table A, Sub-division} & & \gamma, 1\\ \text{,,} & \text{,,} & \gamma, 2\\ \text{,,} & \text{,,} & \delta\end{array}\right\}\text{Group IV.}$$

The Descending Series.

If we treat the descending series in the same way, accepting the original definition of Group V., we have the following relation :—

$$\left.\begin{array}{llr}\text{Table A, Sub-division} & & \gamma, 2\\ \text{,,} & \text{,,} & \delta\end{array}\right\}\text{Group IV.}$$

$$\left.\begin{array}{llr}\text{Table A, Sub-division} & & \beta, 1\\ \text{,,} & \text{,,} & \beta, 2\\ \text{,,} & \text{,,} & \beta, 3\\ \text{,, B,} & \text{,,} & \beta, 1\\ \text{,,} & \text{,,} & \beta, 2\end{array}\right\}\text{Group V.}$$

Here again, the photographs, so far as they go, fully justify the grouping suggested by the meteoritic hypothesis.

Combining the two series, we thus get exactly the same order from the photographic spectra, as that originally deduced, on the hypothesis, from eye observations.

(4.) Extension of the Original Classification.

Formation of Sub-groups.

These considerations enable us to provisionally extend the classification originally suggested by the eye observations, as there are now many more details at our command. Further sub-division into species was attempted in the case of Group II., and Group I. was divided into the two well-defined sub-groups of nebulæ and bright-line stars. In the case of the other groups, a finer division was not possible from the available facts.

Group II. was originally divided into fifteen species from a consideration of the eye observations. Since the photographs open up the blue for our investigation, the sub-division may be based on extent of the fluting-absorptions in the way which will be indicated.

Taking the groups in order, perhaps the most useful sub-division will be as indicated below.

GROUP I.—Radiation lines and flutings predominant.

Sub-group α.*—Nebulæ.

Sub-group β.—Bright-line stars.

GROUP II.—Mixed fluting radiation and absorption predominant. Much continuous absorption in the violet.

Sub-group α.—Dark flutings probably extending from the red end to the region more refrangible than G† (λ 434).

Sub-group β.—Dark flutings extending as far as G. *E.g.*, Mira Ceti.

Sub-group γ.—Stars in which the most refrangible dark fluting is at λ 4585. *E.g.*, α Herculis.

Sub-group δ.—Stars in which the most refrangible dark fluting is at λ 4763. In addition to the flutings there is a large number of dark lines. *E.g.*, α Orionis.

GROUP III.—Line absorption predominant, with increasing temperature. Less continuous absorption at the violet end.

Sub-group α.—Stars with line spectra resembling those of Group II., Sub-group δ, but with only a single fluting in the red remaining. *E.g.*, α Tauri.

Sub-group β.—Continuous absorption in violet less than in Sub-group α. The calcium lines are less intense, while lines at λλ 4172, 4233, and 4177, have their intensity increased. *E.g.*, γ Cygni.

Sub-group γ.—Stars with spectra consisting of a relatively small number of dark lines. The hydrogen lines are of only moderate breadth, and among the additional lines are some which are seen bright in the solar chromosphere. In this sub-group the continuous absorption in the violet is almost a minimum. *E.g.*, β Orionis.

GROUP IV.—Simplest line absorption predominant, the hydrogen lines being very broad.

Sub-group α.—The spectra are marked by the presence of fine lines at wave-lengths 4024, 4471, 4481, the two latter being of almost equal intensity. *E.g.*, β Persei.

Sub-group β.—Highest temperature. The spectra show additional faint lines, and 4471 almost disappears. *E.g.*, α Andromedæ.

Sub-group γ.—The lines of iron make their appearance, but the line of calcium at λ 4226 is not yet distinct. *E.g.*, Sirius.

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* The Greek letters have been adopted to avoid confusion with the symbols employed in VOGEL'S classification.

† None of the stars of this sub-group have yet been photographed at Kensington, but their existence is indicated by the discussion of DUNÉR'S observations, and on the hypothesis, there must be an intermediate stage between the bright-line stars and stars like Mira Ceti.

GROUP V.—Line absorption predominant, with decreasing temperature.

Sub-group α.—The lines of hydrogen are still broad, and the line of calcium at λ 4226 is clearly visible. The grouping of lines about G, which is so characteristic of the solar spectrum, is not visible. *E.g.*, β Arietis.

Sub-group β.—All the solar lines are now clearly visible, but the hydrogen lines are broader than in the solar spectrum. The grouping at G is only partially developed. *E.g.*, Procyon.

Sub-group γ.—The spectra very closely resemble the solar spectrum, the characteristic grouping of lines about G being fully developed. Carbon absorption commencing in the ultra-violet. *E.g.*, Capella.

GROUP VI.*—Carbon absorption predominant.

Sub-division into Species.

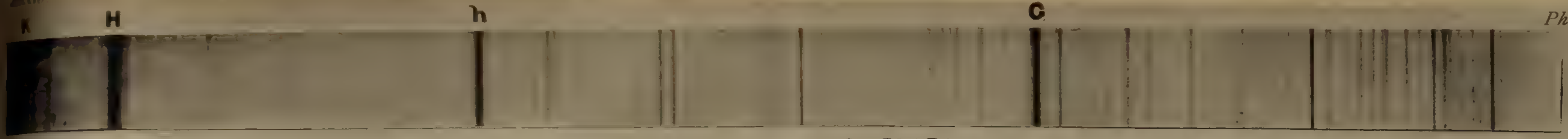
As more photographs become available for detailed examination, it will be possible to sub-divide the various sub-groups into species. Thus, in the case of the bright-line stars, Professor PICKERING, agreeing with me in classifying the bright-line stars with the nebulæ, has already divided the former into three types. In consequence, however, of there being less than 200 stars included in the present discussion, I have not thought it desirable to attempt the further sub-division in the present communication. It is clear already that the species of Group II. given in my former communication will hold good.

In concluding this communication, I am anxious to express my obligations to those who have assisted me in the different branches of this inquiry.

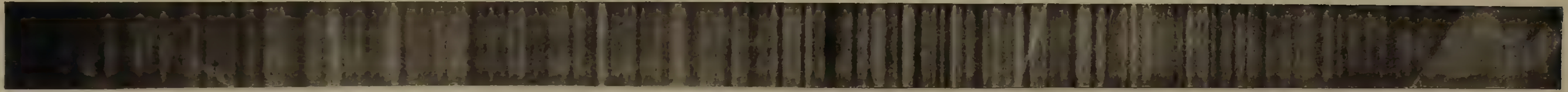
The chief labour entailed in taking the photographs has been borne by Messrs. FOWLER, BAXANDALL, and SHACKLETON. It is impossible to overrate the zeal and the patience they have shown in the almost hopeless observing conditions of the last two years. Repeated visits to the observatory on the same night have often been made. Some of the photographs obtained at Westgate-on-Sea were taken by Mr. W. J. S. LOCKYER. Mr. FOWLER and Mr. BAXANDALL are chiefly responsible for the determination of wave-lengths, Mr. FOWLER having also largely aided in the final discussion of the photographs. Messrs. GREGORY and FOURNIER have assisted from time to time, and the photographic enlargements have been made by Sergeant KEARNEY and Corporal HASLAM.

---

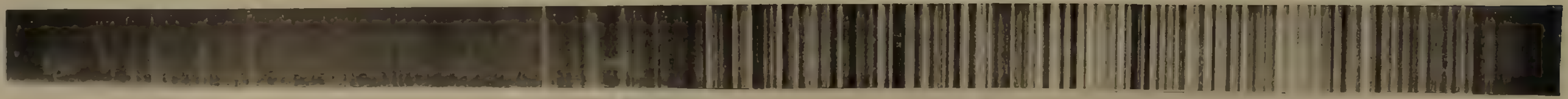
* No photographs of the spectra of stars of this group have yet been photographed at Kensington. So far as they are yet known, they are all below the fifth magnitude, and the investigation of their spectra is therefore almost impossible with the present means. With the instrument D the spectrum of 152 Schj. could not be photographed with an hour's exposure.



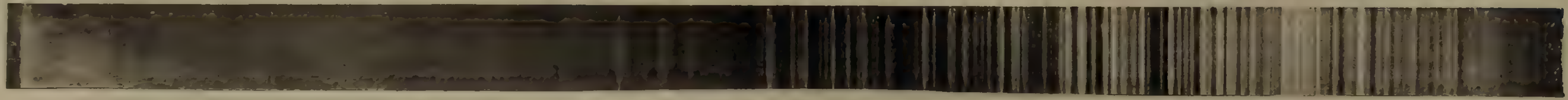
α CYGNI



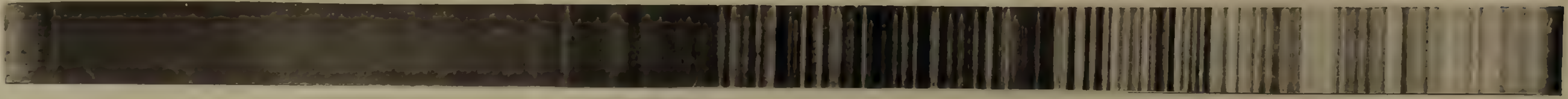
γ CYGNI



α TAURI



β ANDROMEDÆ



α ORIONIS

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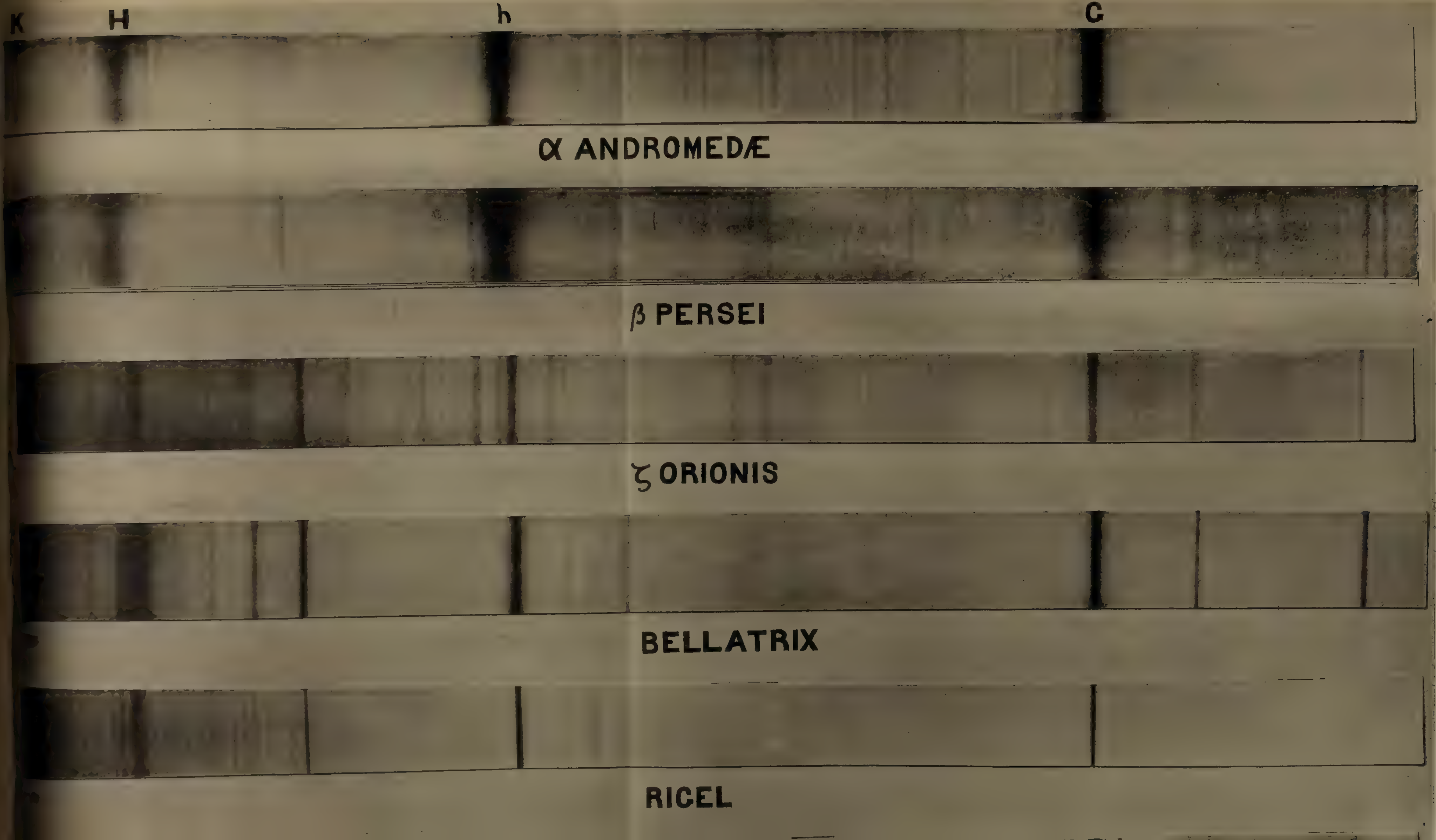
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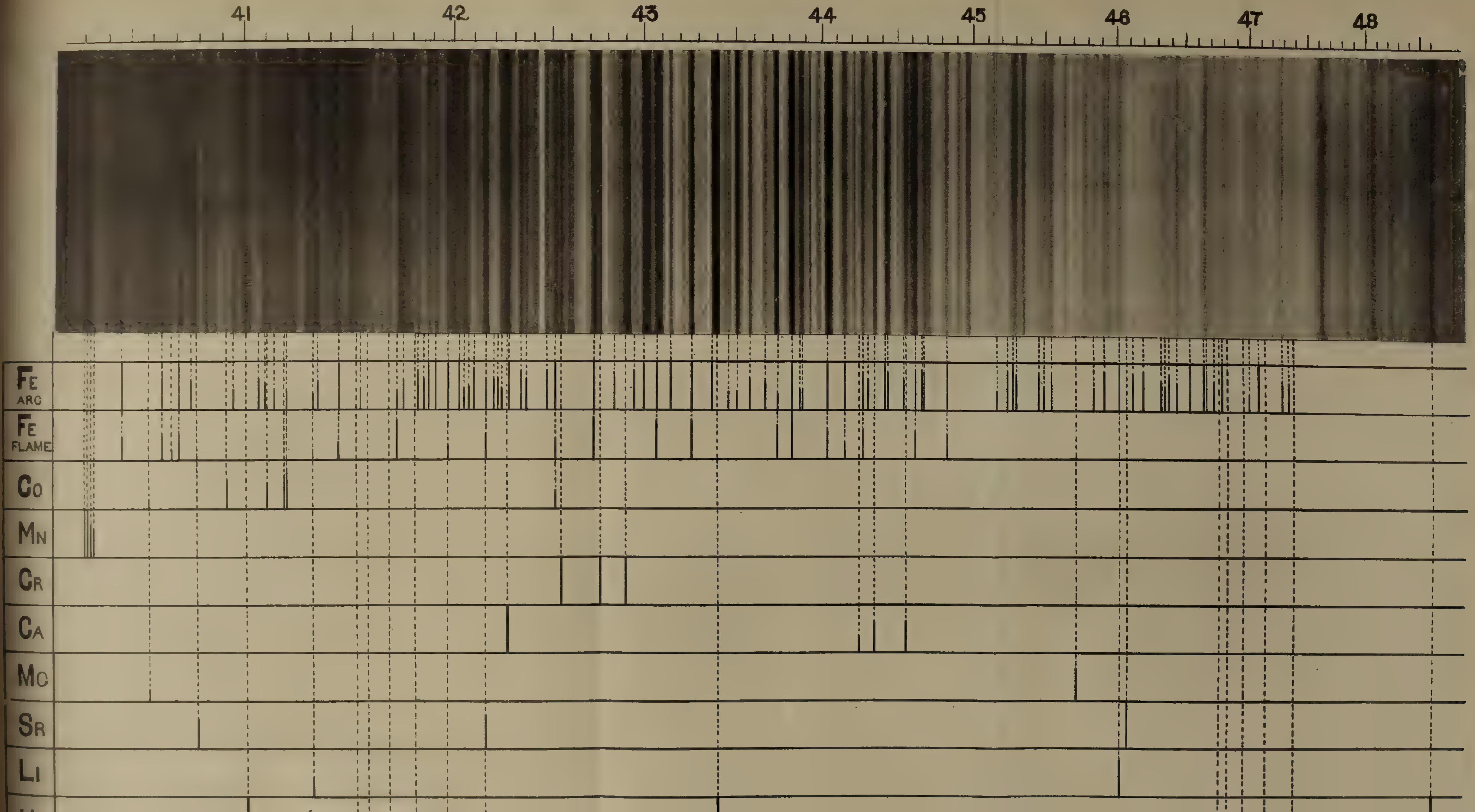
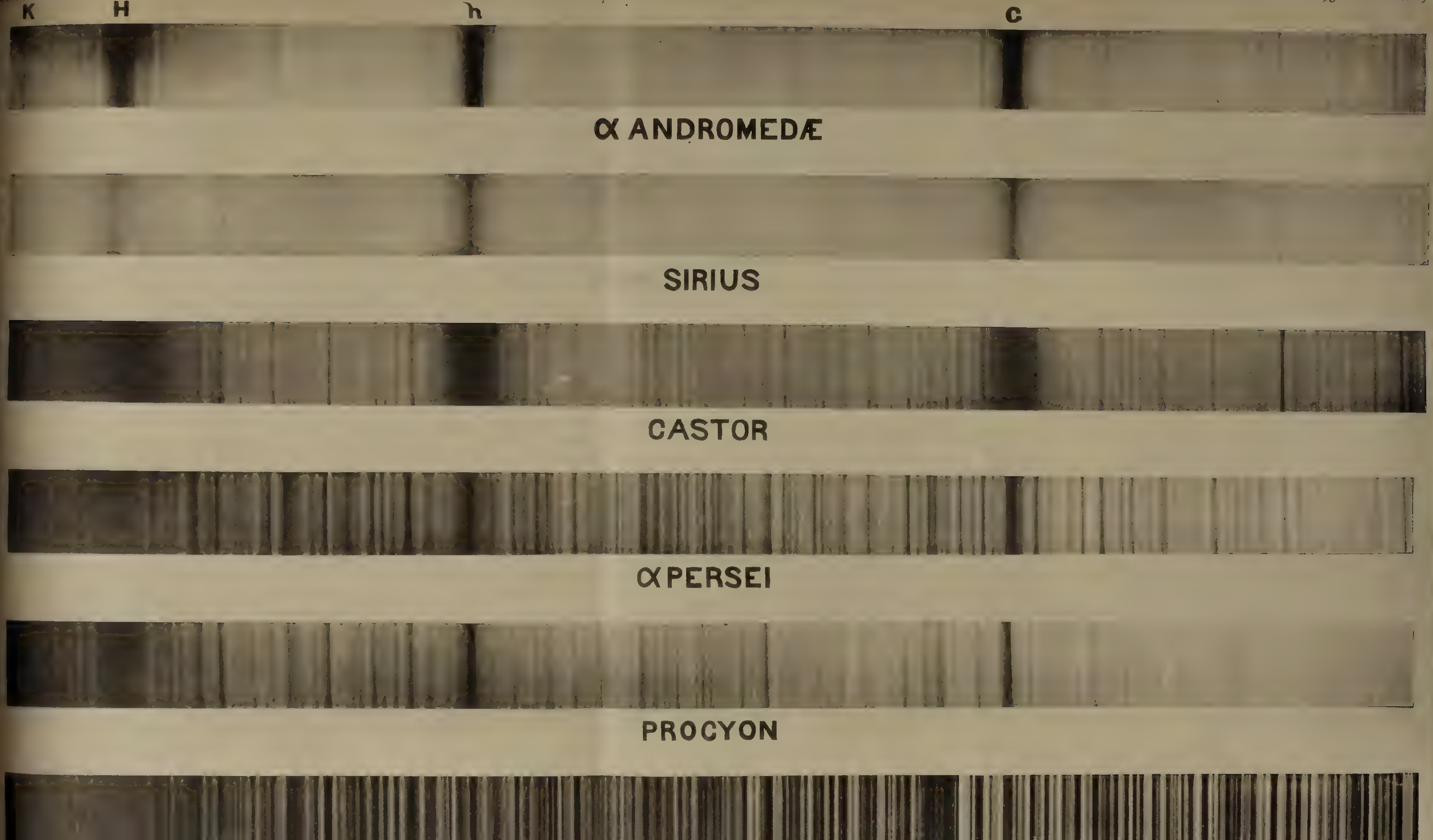
Lockyer.

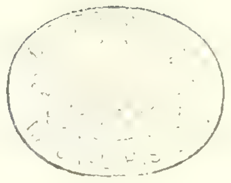
α ORIONIS

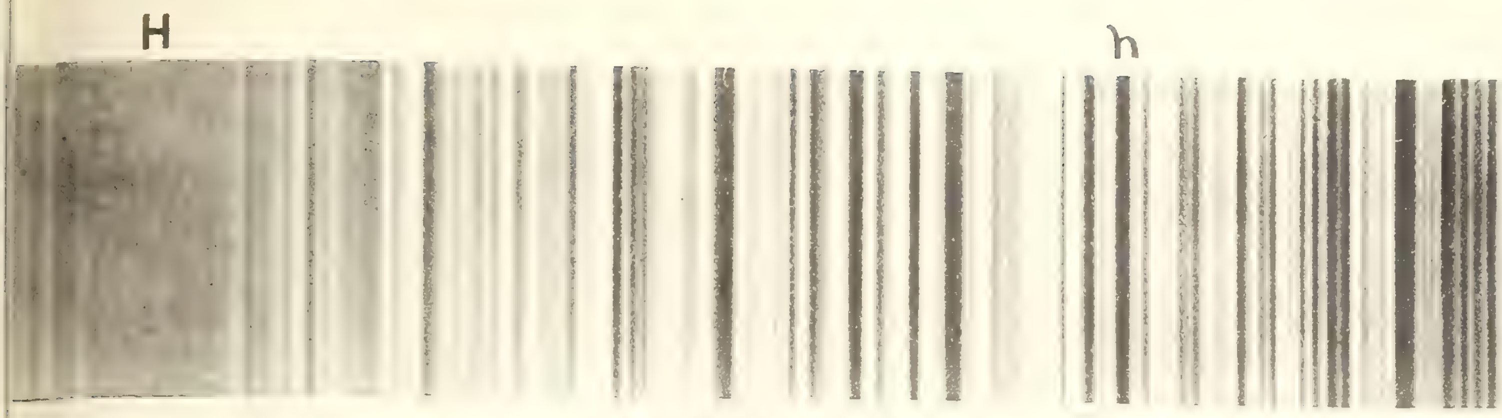
Phil. Trans. 1893. A. Plate 28.



Lockyer.







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XV. *Aberration Problems.—A Discussion concerning the Motion of the Ether near the Earth, and concerning the Connexion between Ether and Gross Matter; with some new Experiments.*

By OLIVER J. LODGE, *F.R.S.*

Received and Read March 31, 1892.—Revised July 17, 1893.

[PLATES 31, 32.]

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1. THE nature of the connexion between ether and gross matter is one of the most striking physical problems which now appear ripe for solution, and as a preliminary to the undertaking of fresh experiments I propose to review the subject in order to realize our present position in connexion with it.

The subject may be attacked either optically or electrically. It first prominently presented itself historically in connexion with the earth's motion through space and the finite velocity of light; and it will be convenient to consider the matter first on this side, and to realize precisely what ought to be expected to happen on any simple hypothesis concerning the ether; working it out, however, in most cases with accuracy,* and by no means ignoring as negligible small quantities of the second order (hundred-millionths), which MICHELSON has practically shown to be nowadays within the limits of highly-refined observation.

Necessary Hypotheses or Postulates.

2. There are one or two hypotheses regarding the ether so elementally obvious that they may be regarded as almost axiomatic, such as the following:—

(i.) *In interstellar space the ether is free;* that is, its properties in no way depend on the existence or motion of gross matter. It may, therefore, be called at rest. Whether it is in absolute rest or not appears to be a question which can hardly be put into an intelligible shape. If it be moving relatively to itself, we have in those regions no obvious means of ascertaining the fact. But just as it is natural to assume that its properties in free space are uniform, so it is natural to assume that its motion there, whatever it is, is perfectly uniform, and it may be *defined* as absolute rest. When I speak of the ether anywhere as "stationary," I mean stationary with respect to interstellar or free ether. When I speak of the ether anywhere as "free," I mean that its properties are identical with the interstellar ether enormously distant from all gross matter. And this is the condition of ordinary space, except for the presence of meteoric particles, whose influence, if any, we at present legitimately ignore.

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* Whenever equations are approximate only, the symbol $\approx$ is used instead of the symbol $=$.

The only hypothesis which at first sight *appears* to assume infinitely distant ether to be affected by the motion of, say, the earth, is that of Sir GEORGE STOKES, in 1845, where an irrotational motion, zero with respect to the earth, was postulated for it. But he must have seen some way in which so impossible an assumption could be avoided; and the question how far any kind of irrotational motion can be conceived of as allowing rest at infinity, and yet no slip at the earth's surface, will be discussed later, § 31.

(ii.) *Inside material bodies the ether is modified.*—We learn this by direct experiment and observation.

For transparent bodies we learn it by optical experiments, which proves that light travels more slowly through their modified ether than it does in free ether; while at the same time there is no doubt but that the ether inter-penetrates them, because material substance itself is wholly incompetent to transmit anything possessing the properties and the speed of radiation.

In metallic bodies we find great opacity combined with anomalous dispersion and other complex effects. In them evidently the ether is intensely modified, if it exists at all.

I shall call the ether inside gross matter of any kind "modified ether," but as to the particular way it is modified I make no assumption. [Electrostatic experiments *suggest* that inside transparent bodies, something which may be called its "virtual elasticity" is diminished. Magnetic experiments suggest that inside several opaque substances it is loaded, so as to increase what may be called its "virtual density;" and there is a temptation to identify $4\pi/K$ with the one, and $4\pi\mu$ with the other, of these two ethereal constants. Further, electrokinetic experiments suggest that inside metallic conductors the ether has a virtual viscosity, whereby its motion through matter is resisted precisely as the first power of the velocity. But none of these doubtful hypotheses shall here be obtruded.]

Rate of Travel of "Modified" Ether.

3. Defined in this way it is quite obvious that "modified ether" travels at the same steady pace as its material encasement. For lift a lump of glass or of copper from one side of a table to the other, the modified ether which was in one place is now in another, and has necessarily accompanied the material body. If the modification of ether by matter requires time, there would be some lag during epochs of acceleration; but during steady velocity there would even so be no difference in speed between modified ether and matter, only a slight lag in space.

Ignoring this possible finite speed of affection of ether by matter, unless circumstances make us revert to it, the question faces us, what is meant by the travelling of the modified ether?

It is not a question easy to state without some looseness of language, but we may ask:

(*a*) Does it mean that the identical stuff inside the matter travels from one place to the other? If so, the free ether which it has displaced must stream back round the body in the same way as a material fluid would have to do.

(*b*) Or does it mean that no ether travels at all, that the mere presence of the matter causes the modification wherever it is, so that it is only the modification or affection which travels? If so, the ether abandoned by the matter becomes free *in situ*, while the ether encroached on by the matter becomes modified *in situ*, and there is no question as to its motion.

On hypothesis (*b*) the whole ether is fixed and imperturbable by the motion of matter. The portion enshrouded by matter at any instant has properties differing from those of free ether, but the modification is only connected with the matter causing it in the same sort of way as a shadow is connected with the object casting it.

Of the two hypotheses, there can be no question but that the second is the simpler and considered as a hypothesis is preferable, but we must enquire whether it is competent to sustain the weight of all known facts.

FRESNEL'S *Hypothesis*.

4. It is notorious that the hypothesis at present holding the field is not exactly either of these, but is some form of the bold and picturesque idea of FRESNEL; viz., that in addition to the free and undisturbed ether of space existing equally everywhere and flowing through the pores of gross matter, there is an extra quantity of bound ether fixed to the matter and travelling with it; this additional quantity being $(1 - 1/n^2)^{\text{th}}$ of the whole.

This idea of FRESNEL'S seems, at first sight, essentially to involve the condensation of ether by matter, so that its density inside bodies is n^2; for the fixed ether is superposed upon the normal ether of space. (Certainly the converse is true; viz., that extra ethereal density involves FRESNEL'S law, as will be shortly shown.)

Now the facts of gravitation, and many electrostatic experiments, suggest that the ether is practically incompressible; hence the notion of any actual increase of density inside gross matter is repugnant.

FRESNEL, however, himself pointed out, in a subsequently written postscript to his original letter to ARAGO[*] promulgating his famous hypothesis, that the extra density need not be taken too literally. (As this postscript seems rather to have been overlooked it may be worth while to quote it).

"*Note additionnelle à la lettre de* M. FRESNEL *à* M. ARAGO, *insérée dans le dernier Cahier des Annales.*

En calculant la réfraction de la lumière dans un prisme entraîné par le mouvement terrestre, j'ai supposé, pour simplifier les raisonnements, que la différence entre les vitesses de la lumière dans le prisme et dans l'éther environnant provenait uniquement d'une différence de densité, l'élasticité étant la même de part et d'autre ; mais il est très possible que les deux milieux diffèrent en élasticité comme en densité. On conçoit même que l'élasticité d'un corps solide peut varier avec le sens suivant lequel on le considère ; et c'est très probablement ce qui occasionne la double réfraction, comme l'a observé le Dr. YOUNG. Mais quelle que soit l'hypothèse que l'on fasse sur les causes du ralentissement de la marche de lumière dans les corps transparents, on peut toujours, pour résoudre le problème qui m'était proposé, substituer par la pensée, au milieu réel du prisme, un fluide élastique en équilibre de tension avec l'éther environnant, et d'une densité telle que la vitesse de la lumière soit précisément la même dans ce fluide et dans le prisme supposés en repos ; cette égalité devra subsister encore dans les deux milieux entraînés par le mouvement terrestre : or, telles sont les bases sur lesquelles repose mon calcul."—('Ann. de Chim. et de Phys.,' 1818, t. 9, p. 128 or 286.)

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[*] 'Ann de Chim. et de Phys.' (2), vol. 9, p. 56.

And Mr. GLAZEBROOK ('Phil. Mag.,' December, 1888) shows that in the interaction of ether and matter, a term depending on relative acceleration is sufficient to sustain the results achieved by FRESNEL'S hypothesis. In other words, that a *virtual* density, or loading of the ether by matter, is quite enough without true condensation.

It is, however, still appropriate to speak of the extra ethereal density inside matter; meaning the coefficient of this acceleration term.

5. A plausible mode of exhibiting the naturalness of FRESNEL'S law is as follows :—

The constant which determines the speed of electromagnetic waves through any medium is μK ; by the differential equation to wave motion.

In a dense body the value of this constant is $\mu' K'$.

Shift a lump of this body from one place to another. Its constant $\mu' K'$ has been shifted in position too, but the ordinary space-value μK remains behind ; so the resultant shift of the property determining the velocity of light (the effective medium) is a fraction $(\mu' K' - \mu K)/\mu' K'$, of the shift of the body.

So, if the lump moves with velocity u, the property of it concerned with the velocity of light shifts with velocity $(\mu' K' - \mu K)/(\mu' K').u$; that is with speed $(1 - 1/n^2)u$.

And, as in all probability the velocity of wave motion *relative to its medium* is unaltered, this may be taken as the extra speed of the light caused by the motion of the matter.

6. It is here assumed that the medium simply carries the wave motion with it as air carries sound. It is not customary to doubt that wave motion must be affected by any motion of its medium in that simple manner. But a singular investigation by Professor J. J. THOMSON ('Phil. Mag.,' April, 1880) seems to show that on electromagnetic principles the speed of ether waves is affected with only *half* the velocity of the medium conveying them.

This extraordinary result is not at present positively contradicted by the FIZEAU experiment, even as repeated by MICHELSON, because the value of $1 - 1/n^2$ for water is not sufficiently different from $\frac{1}{2}$ to afford a certain criterion ; and water is the only substance for which a positive result has as yet been obtained. Certainly the negative result obtained for *air* by both FIZEAU and MICHELSON is in accord with FRESNEL'S theory and not in accord with J. J. THOMSON'S. But a definition of what is meant by "moving medium" seems necessary before we can adequately test the question whether electromagnetic waves in it move with it or lag behind.

I suppose that it must be desirable to examine substances other than water, especially those with a much higher refractive index. I hope to do this, though it may be noted that the value of n which would make FRESNEL'S and THOMSON'S theories *exactly* agree, is 1·4142, and that the available range of refractive indices of liquids and solids affords but a narrow margin for discrimination between the two hypotheses.

The balance of evidence is at present strongly in favour of FRESNEL'S hypothesis, and I propose ordinarily to assume its truth. I cannot, indeed, understand the possibility of THOMSON'S theory, though I detect no flaw in his work, for it seems to require a distinction between the case of source or receiver moving through a medium, and the case of medium flowing past source or receiver; that is, it seems to demand a knowledge of *absolute* velocity.

FRESNEL'S *Law.*

7. The statement of FRESNEL'S law can be thrown almost into the form of hypothesis (*b*), § 3, and at the same time its apparent licence of language about "free" and "bound" ether can be lessened, by supposing that the "modification" induced by the encroachment of matter on the ether is really a condensation, in the ratio $1 : n^2$; no motion in the ether other than what is necessarily involved in that act being postulated. On this method of statement the ether outside a moving body is absolutely stationary, but, as the body advances, ether is continually condensing in front, and, as it were, evaporating behind, while inside it is streaming through the body in its condensed condition at a pace such that what is equivalent to the normal quantity of ether in space may remain absolutely stationary. To this end its speed relatively to the body must be v/n^2, and accordingly its speed in space must be $v(1 - 1/n^2)$.

Thus, instead of saying that a portion of the ether is moving with the full velocity of the body while the rest is stationary, it is probably preferable to say that the whole internal ether is moving with a fraction of the velocity of the body.

One or other form of statement is absolutely involved in the Fresnellian idea of increased ethereal density, as may be rigorously shown (*vide* Lord RAYLEIGH, 'Nature,' March, 1892; *vide* also EISENLOHR), thus:—

Consider a slab moving forward flatways with velocity v, let its internal ethereal density be n^2, and let the external ether, of density 1, be stationary. Let the speed of the internal ether *through space* be xv, and consider that the amount of ether enclosed between two planes moving with the slab, one outside and one inside, must be constant; it follows at once that

$$v = n^2 (v - xv)$$

whence

$$x = 1 - \frac{1}{n^2}.$$

Now whatever may be the inner meaning of this statement concerning the velocity of the internal ether, it certainly agrees with, and is at once suggested by, the fact, thoroughly established by both negative and positive experiments, that light travels down a running stream of matter at a pace

$$\frac{V}{n} + v\left(1 - \frac{1}{n^2}\right).$$

The negative experiments supporting this are such as the achromatic prism experiment suggested and tried by ARAGO, repeated more elaborately by MAXWELL and by MASCART; the water-telescope observations suggested by BOSCOVICH, tried by AIRY and by HOEK; interference experiments of BABINET and of HOEK; and several other experiments by MASCART. The positive experiment establishing it is the very beautiful and well-known one of FIZEAU, now repeated and confirmed beyond the reach of any but quantitative cavil by MICHELSON.*

Whether any ether is moved by moving matter may still be an open question, but that the speed of light is affected in a fairly ascertained way by the motion of transparent matter through which it is passing, is certain.

8. But the specific motion of the internal ether is not the whole of FRESNEL's hypothesis; there is the fixity of the external ether to be verified too. And that has not yet been done. In fact, one important experiment, to be discussed later on, throws grave doubt upon it, at least for large moving bodies like the earth.

But unless the fixity of external ether be granted, our argument from density concerning the value of the internal velocity breaks down. Consider again two planes moving with a slab of matter, one inside and one outside the mass, and let the space motion of the outside ether at the position of outside plane be affected by the motion of the slab to the extent yv, then all we can say is that

$$v - yv = n^2 (v - xv),$$

or,

$$x = 1 - \frac{1-y}{n^2},$$

wherefore it is possible for x and y to be unity together.

We may take it, however, that the quantitative accuracy of the FIZEAU experiment renders anything of this sort very unlikely, and that we are bound to suppose the ether immediately outside moving matter to be stationary, *i.e.*, to be completely unaffected by its motion, unless we are directly forced by facts to admit the contrary.

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* 'Amer. J. Sci.,' vol. 31, p. 377.

The two parts of FRESNEL's law, the motion of internal ether, and the fixity of external ether, can and ought to be verified separately. The FIZEAU experiment has verified the one. I propose to attempt the other. To this end I am passing a beam of light, split into two equal halves, very near a rapidly rotating disk (in fact between a pair of rotating disks clamped together), so that one half the light travels with the mechanical motion and the other half travels against it. The two half beams, after several journeys round and round, are united, with interference effects, and the observation consists in watching the system of bands for any shift caused by the motion. For description of this experiment see §§ 33–47 below.

Phenomena Resulting from Motion of Source, Receiver, or Medium.

9. The phenomena which can be appealed to as evidence of a state of motion, and which necessarily result from that motion if of a suitable kind, are four, viz.:—

(1) Changes or apparent changes in direction of ray, as observed by telescope with cross-wires; the change commonly called "aberration" proper.

(2) Changes or apparent changes in frequency of vibration, as observed by the pitch or colour appreciated by an observer, or by the shifted position of lines in a spectroscope; a change which may be referred to as the Doppler effect.

(3) Changes or apparent changes in the time taken over a fixed journey, as observed by the relative lag in phase between two portions of a split beam and the consequent shift of interference fringes when they are re-united.

(4) Changes or apparent changes in the intensity of radiation in different directions, as observed by the amount of energy received by a given area exposed normal to the rays at a given distance from a source, but having different aspects with respect to the line of motion.

Or, briefly summarizing them, the possible phenomena caused by motion are changes in direction, in period, in phase, and in amplitude.

Apparent Direction as Affected by Motion in General.

10. Consider the subject first from a corpuscular or projectile point of view, first ignoring the medium. A gun travelling broadside on must be aimed behind the object, and its shot will travel in a skew direction (keeping always straight in front of the muzzle, but not travelling along the axis of the gun) with a velocity compounded of the speed of projection and the speed of the gun. The apparent position of the source, as recognized through a hole in the target, will therefore be its true position at time of firing, but not its position at time of hit.

Whether we choose to call this an aberration or not is a matter of nomenclature merely.

If the gun is fixed, with the target moving across the line of fire, the gun must be aimed in front of the object. The shot will go straight along the barrel produced, but the hole in the target will indicate a gun in front of its true position; this error being aberration proper.

Fig. 1.

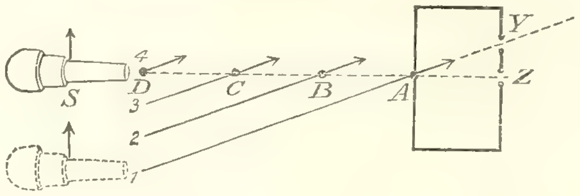

Diagram of shot fired from a moving cannon; piercing a target, at Y if stationary, at Z if moving at same pace as gun. ABCD is the locus of successive shots, but is not the line of fire.

If both gun and target are travelling at the same speed everything occurs as if they were at rest, unless a stagnant medium has to be taken into account. Relative motion of the medium causes windage, as is well known.

Since motion of the medium causes a shift of the line of fire, it may be expected to produce a miss, but this is not a true aberration, it only appears to be such because of the fire being limited to one line; suppose instead of a single gun a broadside of guns or a number of guns firing from a turret, then the effect of a cross-wind is, indeed, to displace all the shots, but not to prevent the target being hit by one which would otherwise have missed it, and the hole in the target will indicate the position of the gun really firing the shot.* Hence, even on a corpuscular theory, a wind across the line joining source and receiver, will not cause any effective aberration. Neither can a steady tail wind deliver a stream of bullets from a machine-gun more frequently than they are emitted.

If guns are fired from a revolving turret, the paths of the shot will not be radial, but will be skewed by an amount depending upon the peripheral velocity.

Watching the beams of a revolving lighthouse, tracking their way to a distance and brandished rapidly round, it is not at once quite evident whether the shape of those beams is not a spiral of enormous pitch (see below). We see, however, that on the corpuscular view the paths will be straight, though not radiating from the precise centre; for instance, the rays from the Sun, whose peripheral velocity is nearly 5000 miles an hour, would if regarded as projectiles, be inclined to their radius at an angle of $\frac{5}{186\times3600}$ radian, or about $1\frac{1}{2}$ seconds of arc; and the Sun's centre would be, apparently displaced through a fraction of this angle, equal to Sun's radius/Sun's distance; *i.e.*, through about the $\frac{1}{150}$th part of a second.

11. But now, proceeding to look at the matter from the point of view of waves, there are many differences; principally depending on the fact that there is no question of initial velocity of projection about a wave: it crawls through the medium,

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* As these projectile examples are only used for illustration, I simplify matters artificially by omitting all curvatures of path. The subject of aberration in general is illustrated more fully in a Royal Institution Lecture, 'Proc. R. I.,' April 1, 1892; also reported in 'Nature,' vol. 46, p. 497

self-propelled, at its own definite velocity. No aberrational effect can be produced by any cause which does not act on a wave-front for a finite time.

Fig. 2.

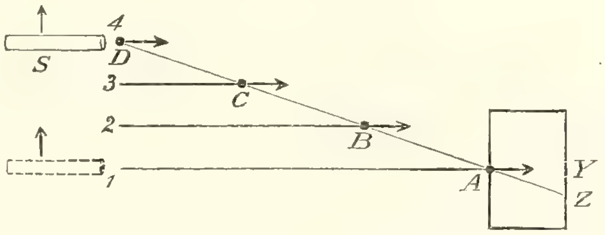

Diagram of disturbances or waves emitted without momentum from a moving source; leaving target or telescope, at Y if stationary, at Z if moving. The line ABCD is the locus of successive disturbances, but is not the ray or real path. The diagram may also be taken to represent the effect of a cross stream of medium, with source stationary.

Hence waves emitted by a revolving source advance just as they would if it were stationary; any peculiarity on the surface, say a Sun spot, is depicted in a precisely radial direction, and there will be no displacement of the Sun's centre. So also with light from a flying star: the star will be seen in its position at time of emission, just as it is seen in the physical state corresponding to that instant, not to the instant of vision.

Fig. 3.

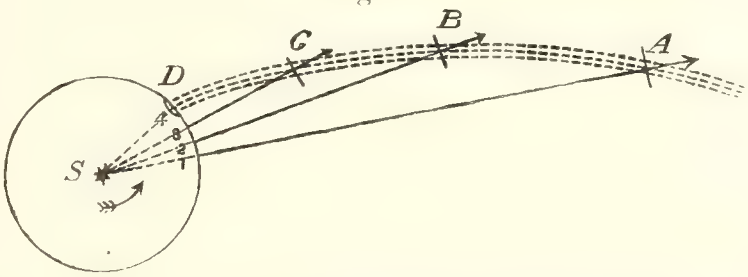

Diagram of parallel beams emitted from a revolving lighthouse. ABCD is the apparent beam, but AS, BS, &c., are the lines of vision or true rays.

As to a beam from a revolving lighthouse, it is not the path of a labelled disturbance, or true ray, which displays itself by illuminating dust particles, but it is the locus of successive disturbances sent out from a given moving point; so if the source has revolved through an angle θ while the light travels a distance r, $\theta = (\omega/\mathrm{V})\,r$, and their shape *is* a spiral of Archimedes as suspected above; though the direction of vision is not tangential to them, but is truly radial as already stated.

The analogy between rays of light and lines of force is fairly close, and just as it is convenient to say that a rotating source revolves its rays, so it is convenient to say that a rotating magnet revolves its lines of force. The induction phenomena obtained from a magnet spinning on its own axis are a sufficient justification of this statement.*

In an old note-book of date 1876, I find a suggestion for measuring the speed of magnetic propagation, by rotating a long bar magnet on its axis and observing its action on a distant magnetic needle;

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* See also Mr. Tolver Preston, 'Phil. Mag.,' February and March, 1885.

the idea being that with a finite speed of propagation the lines of force would lag, and thereby acquire a curvature out of the magnet's meridian; so that a distant needle instead of pointing straight at the magnet would be tangential to these lines, and would therefore be slightly deflected during the spin.

We now see, however, that no such aberrational effect is to be expected, except on a corpuscular view of magnetic propagation.

Concerning the effect of motion of other kinds, certain things are experimentally known; *e.g.*, motion of the receiver is known to cause aberration, however the fact be precisely accounted for; and motion of the medium alone is known not to cause aberration of any perceptible magnitude, else would terrestrial surveying operations be inaccurate. But no experimental data as yet obtained are evidence concerning small quantities of the second order, and it will be well to examine critically and geometrically the whole subject of wave motion from a moving point to a moving telescope through a uniformly moving medium, all the velocities being possibly different in magnitude and direction. So far as steady and uniform motion is concerned this may be considered the most general case.

Convenience of attributing Relative Motion to Medium.

12. Before considering separately the phenomena mentioned in § 9, it may be convenient to consider what it is which must be in motion in order to produce one or other of them. And, first, which of them a motion of the medium alone causes.

Nothing can be more certain than that relative motion is all we are concerned with, so that whether a source travels through a medium, or the medium drifts past the source, comes to precisely the same thing. Sometimes one mode of expression is convenient, sometimes the other. It may be most natural to contemplate the medium as stationary, and to throw all motion on source and receiver, but I find that it is often very simple and helpful to invert this order, and to think of the ether of space as drifting past the earth, or other body, supposed stationary.

We shall not invariably use this device, but whenever a number of things—source, mirrors, telescope, and observer—have to be thought of as moving all precisely alike through the ether, it is simpler to think of the ether as streaming past them.

Case of Fixed Source in Moving Medium.

13. Consider now a fixed point-source in a uniformly moving medium.

Spherical wave-fronts are thrown off and immediately begin to drift, so that their centres get displaced a distance, vt, while their radii enlarge by an amount, Vt; and the distance through space which a disturbance has by that time travelled in the direction θ will be compounded of these two distances, and will be inclined to the radius, or direction of travel if all were stationary, by an angle ϵ, which may be called the aberration angle. The velocity with which light journeys over the radius vector, r, is

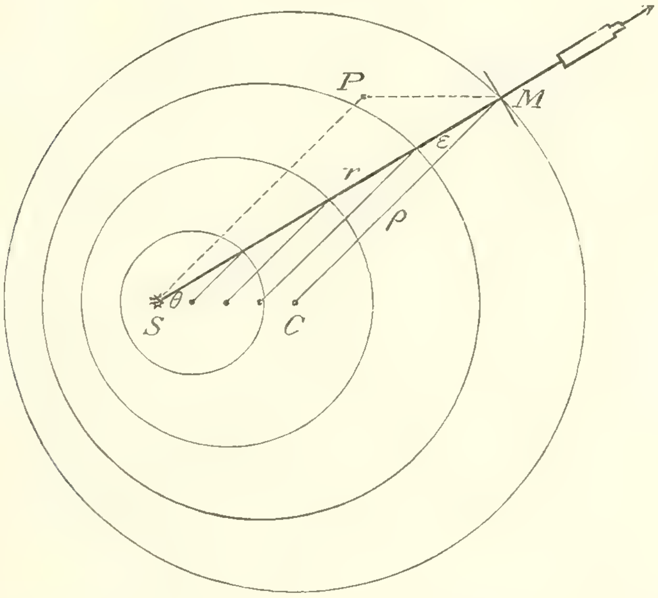
$$\mathrm{V} \cos \epsilon + v \cos \theta, = \mathrm{V}_1 \text{ say}:$$

the time of the journey being simply t, as before.

The angle ϵ is defined by the equally obvious geometrical relation

$$\mathrm{V} \sin \epsilon - v \sin \theta = 0.$$

Fig. 1.



Successive waves emitted by a fixed source S into a drifting medium. The row of dots SC represent the respective wave-centres. The figure also represents waves in a stationary medium, emitted by a source moving from C to S.

Here is a picture of the source and successively-emitted and abandoned drifting wave-fronts. SM is the path of a labelled disturbance, and is to be considered as a ray; it is inclined at angle ϵ to the corresponding wave-normals.

SP is what would have been the light journey in the same time if the medium had been stationary; PM or SC represents the drift.

The result of the state of things exhibited in the diagram may or may not be appreciated by a spectator—that depends on what his own motion is,—but if he is moving simply with the medium, he perceives the following:—

(1) An aberration, ϵ, in any direction inclined at angle θ to the motion, such that

$$\sin \epsilon = \frac{v}{\mathrm{V}} \sin \theta = \alpha \sin \theta,$$

it being convenient to denote the ratio of velocities, v/V, by a single symbol α, and to call it the aberration constant. A telescope moving with the medium and placed with its object glass tangential to the advancing wave-fronts, will focus the image on its cross wires, and will be pointing not to the object, but to the centre of the wave it is receiving; its collimation axis coincides with a radius or wave-normal, not with a ray.

(2) A Doppler alteration of wave-length in every direction; as is obvious in the figure, from the distribution of drifted wave-fronts. It is positive on one side, and negative on the other side, of a certain direction, θ_0, such that the radius vector is equal to the radius, or

$$\cos\theta_0 = \frac{v}{2\mathrm{V}} = \tfrac{1}{2}\alpha\,;$$

the aberration angle for this particular case of no Doppler effect being twice the complement of θ_0.

A spectator moving with the medium will perceive this change of wave-length as a change of pitch (or colour) of value

$$\log\frac{n}{n'} = \log\frac{\lambda'}{\lambda} = \log\,(\cos\epsilon + \alpha\cos\theta) \approx \alpha\cos\theta.$$

An observer travelling with the medium will not observe any modification in interference or diffraction effects, nor will he experience any change of intensity due to motion; for the waves will be brought him at the customary time periods, and be subject to the ordinary flux of energy, as if everything were stationary.

Case of moving Source in fixed Medium.

14. The same figure (fig. 4) serves to illustrate the common case of medium and observer stationary, and source alone moving.

But we must be careful to note that ϵ is only the aberration *angle*, and that whether it is to be called "aberration" or not depends on the meaning attached to that term. The source emits spherical waves in its successive positions, and leaves them to expand at their normal rate. The fixed telescope, pointing to centre of advancing wave, is therefore pointing to the source at the instant when it emitted that light; and, since it is thus seen in its true place at instant of emission, it is most natural to say that the aberration caused by moving source alone is *nil;* for that it may have moved by the time of vision, is obvious.

There is not much more to be said on this head, for the source after throwing off a wave may do what it likes, the light will convey information as to where and how it was at the time of emission. Phenomena depending on a *succession* of waves, *e.g.*, changes of pitch, are of course produced, see fig. 4.

The question arises whether the waves thrown off from a moving source are really spherical shells: whether the motion of the source does not affect its vibration? It is not easy to answer this thoroughly and accurately, but practically there can be no doubt that the emission of light cannot be affected by any feasible terrestrial motion;

for, in the time of one vibration, the earth, which is the quickest available vehicle, has only moved a distance of $\frac{1}{10000}$ of a wave-length; which is equivalent to a middle C fork sounding and creeping along at the rate of 15 inches an hour. No practical question as to imperfection in spherical form of wave from moving source is therefore likely to arise. See however § 19, for discussion of a question not of *shape* but of *intensity*.

There happen to be one or two interesting things connected with the reflexion of light from a moving source when there is some connexion established between the reflected ray and the subsequent position of the source, *e.g.*, as when a ray is reflected back upon itself, with the object of causing interference; these are specially dealt with in §§ 59, 60.

Case of Source and Receiver moving together through Stationary Medium; or, correlative case of Medium drifting past fixed Source and Receiver.

15. Consider a telescope fixed relatively to source, and medium drifting freely past both. The object-glass must be set skew to the wave front, but normal to the advancing ray or radius vector.

In fig. 4, SM is the axis of the telescope, and it points straight at the source. There is no resultant aberration, the object is seen in its true position.

It is also seen of its right colour, for the waves are carried to the receiver at their accustomed frequency: there is no Doppler effect. A steady wind alone is powerless to influence either direction or pitch.

But what about interference phenomena, depending on the *time* of a given journey? Manifestly a motion of the medium will be able to affect this, and may accordingly bring about the displacement of fringes representing hurry or lag of phase.

Consider a telescope fixed relatively to the source and placed so as to receive light along the radius vector r.

If the medium is stationary, the light journey is accomplished in the time

$$T = \frac{r}{V},$$

but if moving, the time of the journey is

$$T' = \frac{r}{V \cos \epsilon + v \cos \theta},$$

and so there is a hurrying up of phase

$$\frac{T}{T'} = \cos \epsilon + \alpha \cos \theta,$$

or

$$T - T' \approx \alpha T \cos \theta.$$

The wind therefore causes a positive or negative change of phase in every direction except that whose cosine is $\frac{1}{2}v/V$, the same direction as that already (§ 13) indicated as possessing a zero Doppler effect.

But the *observation* of the lag of phase thus caused by motion of the entire ethereal medium is not so easy as might appear, and, in fact, it has not yet been detected; for the simple reason that it is liable to affect both the interfering rays equally: as we now show.

Devices for Observing the Lag of Phase.

16. The possible ways in which change of phase, produced by a moving medium, may be looked for, are:—to split a beam of light into two halves, and then—

(1) Make the medium flow with one half beam and against the other.

This is successful, and is the FIZEAU experiment; but it entails control over the medium, and artificial motion of it; the terrestrial orbital motion cannot be utilized in this way.

(2) Send the two beams, not parallel, but round contours in two different planes; or, say one across the line of ether motion, and the other along.

This is MICHELSON'S experiment; but it only attempts an effect whose magnitude is the second order of aberration magnitudes; because, before the beams can be brought together again to interfere, a reversal or complete circuit is necessary.

(3) Make the medium flow at different rates along the two beams: as for instance, by interposing a dense substance in one of them.

But, on FRESNEL'S hypothesis, this ought to fail; because the free ether, which is the only ether in motion, is unaffected by the dense substance. The only way to move either more or less than the normal quantity of ether in any given space, is to move bodily a dense substance occupying that space. So long as that is stationary, with respect to source and receiver, motion of the whole produces no effect.

To prove that on FRESNEL'S law, no dense substance can cause different interference effects when moving than it causes when stationary, we can proceed to calculate the virtual thickness of a slab immersed in an ether stream, or the time retardation it causes in a beam.

Interference Effects as modified by Ether Motion through Dense Stationary Bodies.

17. The calculation of the lag in phase caused by FRESNEL'S ethereal motion is a very simple matter. A dense slab of thickness z, which would naturally be traversed with the velocity V/μ, is traversed with the velocity $(V/\mu)\cos\epsilon + (v/\mu^2)\cos\theta$; where v is the relative velocity of the ether in its neighbourhood; whence the time of journey through it is

$$\frac{\mu z}{V\left(\cos\epsilon + \frac{\alpha}{\mu}\cos\theta\right)}, \qquad \text{instead of} \quad \frac{\mu z}{V},$$

or the equivalent air thickness, instead of being $(\mu - 1)z$, is

$$\frac{\mu z}{\cos\epsilon + \frac{\alpha}{\mu}\cos\theta} - z = \left(\frac{\mu\cos\epsilon - \alpha\cos\theta}{1 - \left(\frac{\alpha}{\mu}\right)^2} - 1\right)z,$$

or, to the first order of minutiæ, $(\mu - 1)z - \alpha z\cos\theta$; θ being the angle between ray and ether drift inside the medium.

So the extra equivalent air layer *due to the motion* is approximately $\pm\,\alpha z\cos\theta$, a quantity independent of μ.

Hence, no plan for detecting this first-order effect of motion is in any way assisted by the use of dense stationary substances; their extra ether, being stationary, does not affect the lag caused by motion, except indeed in the second order of small quantities, as shown above.

Direct experiments made by HOEK,* and by MASCART, on the effect of introducing tubes of water into the path of half beams of light, are in entire accord with this negative conclusion.

Thus, then, we find that no general motion of the entire medium can be detected by changes in direction, or in frequency, or in phase; for on none of them has it any appreciable (*i.e.*, first-order) effect even when assisted by dense matter.

The remaining possible effect that may be looked for is a change of energy.

Effect of Motion on Intensity of Radiation in Different Directions.

18. At first sight it looks as if there ought to be an unequal distribution of energy round a source past which the medium is streaming. For when the waves are drifting along, their energy moves too, and it can thus be distributed unsymmetrically round the source.

The energy emitted per second, or the *power* of the radiation, is

$$P = 4\pi\rho^2 Vq,$$

where q is the energy per unit volume at distance ρ from the wave centre; supposing that radiating power is unaffected by the motion. So at a place r, θ, reckoning from source as origin, and line of drift as initial line (as in fig. 4), since $r = \rho\ (\cos\epsilon + \alpha\cos\theta)$,

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* 'Archives Néerlandaises' (1869), vol. 4, p. 443, or 'Nature,' vol. 26, p. 500.

$$q = \frac{P(\cos\epsilon + \alpha\cos\theta)^2}{4\pi V r^2}$$

$$\approx q_0 (1 + 2\alpha\cos\theta + \alpha^2\cos 2\theta - \tfrac{1}{2}\alpha^3 \sin\theta \sin 2\theta),$$

q_0 being the energy at the same place when there was no drift.

So the energy received per second by a given small area A at that place, facing the source, *i.e.*, normal to the rays, is

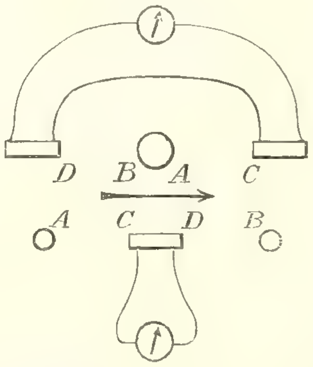
$$qV\,A\cos\epsilon = \frac{A\cos\epsilon}{4\pi r^2}\,P\,(\cos\epsilon + \alpha\cos\theta)^2.$$

The radiation at distance r from the source is, in fact, the same as what the radiation would be at distance ρ in a stationary medium; except for the small inclination ϵ.

So a pair of similar thermopiles, fore and aft, at equal distances from a source, will, on this hypothesis, receive unequal radiation; the difference being equal to $4\alpha\,(PA/4\pi r^2)$, or proportional to 4α.

FIZEAU suggested this method, but I am not aware of its having been tried yet.[*]

Fig. 5.



Thermopile experiment suggested by FIZEAU; in two alternative forms.

19. But it is a serious question whether the reasoning establishing the effect is quite sound. It is not unlikely that motion may affect the radiating power of a source. In fact, the theory of exchanges almost necessitates something of the kind, else the two faces of an enclosure would become unequal in temperature by reason of mere motion through the ethereal medium.[†]

Hence, if, as in fig. 5, we consider a pair of thermopiles with a hot body half-way between them, or a pair of equally hot bodies with a thermopile half-way between them, all subject to an ethereal drift in the direction of the arrow, we may assert that although the radiation from A is carried down stream in undue proportion towards C, the amount actually emitted in this direction is diminished in a compensatory manner, so that the resultant flux of energy remains unaffected by the motion.

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[*] The suggestion is quoted in a comprehensive, but to me not very intelligible, treatise on the whole subject of aberration: 'Astronomische Undulationstheorie,' by Professor Dr. KETTELER, of Bonn.

[†] BALFOUR STEWART ('Brit. Assoc. Report,' 1871, Sects. p. 45), argued that this inequality of temperature actually occurred; and, since motion thus afforded an available heat engine, he deduced an ethereal friction, dissipating energy. But, as Lord RAYLEIGH points out (in his Article on "Aberration," 'Nature,' March 1892), it is far more likely that motion should alter radiating and absorbing powers than that it should disturb equality of temperature.

It is not necessary to suppose that motion disturbs the equality which otherwise exists between radiating and absorbing powers. It is true that if a surface like A radiates less than when the medium is stationary, a surface like C facing the stream must radiate more; but then it may absorb more also. So that in all respects the balance may be undisturbed by the motion of the medium.

It is probable, therefore, that even by this intensity method, nothing more than the second order of aberration magnitude is effective for displaying a general drift of the medium as a whole.

At the same time it seems desirable that an experiment with thermopiles, like that suggested by FIZEAU, should be tried, in order to verify the above deductions from the theory of exchanges, combined with the supposed persistent uniformity of temperature of an enclosure whether at rest or in motion; for thereby the absence of friction or dissipation of energy by motion of solids through ether would be verified.

Case of only Receiver Moving.

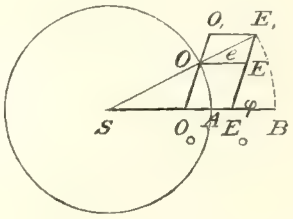
20. If the receiver be not fixed relatively to the medium, nor relatively to the source, but be moving on its own account, the effects due to this motion must be added to the preceding effects. First suppose both source and medium stationary.

The source S emits waves in spherical shells, whose radii are also rays. Any motion of the receiving telescope can be resolved tangentially and radially. Radial motion gives Doppler effect only; tangential motion gives aberration only—both of the commonplace type.

If the telescope were stationary, its object-glass must be tangential to the wave front, but directly it moves it must encounter the wave front obliquely, with the same obliquity ϵ as if it were stationary and the medium drifting (fig. 4), and the eye-piece will then be brought to the light at the right instant. Revolution of a radial telescope about the source would effect this in the simplest way, without introducing any Doppler effect or change in focal length.

Consider a telescope O_0E_0 pointing straight at a source S (fig. 6), and at the instant a given luminous disturbance starts from S, let the telescope begin moving in a direction ϕ with a velocity u. Let it thus reach the position OE by the time the light has got as far as O, *i.e.*, to the spherical wave front indicated in the diagram. Then it follows that by the time the telescope has reached the position O_1E_1 the light will have reached E_1, too, and will accordingly have passed along the collimating axis by reason of the combined motions.

Fig. 6.



A telescope receiving light from S and moving from OE to O_1E_1 while light traverses OE_1.

A stationary telescope receiving the same ray at the same instant would have had to occupy the position OE_1, and would have looked straight at the object (with a slightly greater focal length). Hence the angle O_1E_1O or OSO_0 is the angle of aberration, the amount by which the object appears to be displaced in the direction of motion. A telescope which had been revolving round the source, instead of being translated, would have gone from AB to OE_1 in the time, and have rotated through this same angle. Call it *e;* it is such that

$$\frac{\sin e}{\sin \phi} = \frac{EE_1}{OE_1} = \frac{u}{V} = \beta, \text{ say,}$$

the medium, remember, being stationary.

The focal length of the moving telescope differs from that necessary for a fixed one; being OE instead of OE_1, or

$$f' = f(\cos e - \beta \cos \phi)\,;$$

but this is best regarded as part of the Doppler effect, since its principal term represents radial motion. With a non-achromatic lens the change of refrangibility due to motion tends to compensate* this effect. But whereas the change of refrangibility is produced equally by motion of source or motion of receiver, this change of focal length seems to be caused only by motion of receiver. It is a shortening of focus as a telescope recedes from the light. I suppose it is too small to observe, else it would seem able to discriminate motion of earth from motion of star, and give absolute motion of telescope through the ether.

A terrestrial source (*e.g.*, a sodium flame) might be used, and a perfectly achromatic lens; but surely no focussing could be delicate enough to discriminate such sort of difference as exists between the two sodium emissions ?

The way in which motion of receiver to or from source causes an apparent change of frequency, *i.e.*, a real change in the frequency with which waves are *received*, is too well known and simple to be more than mentioned. Its amount in any direction is

$$\log \frac{n}{n'} = \log\,(\cos e + \beta \cos \phi) \approx \beta \cos \phi,$$

where $\beta = u/V$, $\sin e = \beta \sin \phi$, and u is the velocity of the telescope at angle ϕ with the ray.

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* This was originally written "exaggerate."

Summary.

21. Collecting these statements together, we may summarize them thus:—

Source alone moving produces—
- A real and apparent change of colour;
- A real but not apparent error in direction;
- No lag of phase, except that appropriate to altered wave-length;
- A change of intensity corresponding to different wave-length.

Medium alone moving, or
Source and receiver moving together } gives—
- No change of colour;
- No change of direction;
- A real lag of phase, but undetectable without control over the medium;
- A change of intensity corresponding to different virtual distance, but probably compensated by change of radiating power.

Receiver alone moving gives—
- An apparent change of colour;
- An apparent change of direction;
- No change of phase, except that appropriate to extra virtual speed of light;
- A change of intensity corresponding to different virtual velocity of light.

Thus the interference effect and the Doppler effect do not occur together. Motion of the medium produces one; motion of source or of receiver produces the other.

Aberration of direction and of pitch occur simultaneously, but are complementary to each other, since one depends on motion across the line of sight, the other on motion along it. One varies as the sine, the other as the cosine, of the inclination. Further discussion of the Doppler effect is deferred to §§ 53–58.

22. It is noteworthy that not one of the methods is able to establish the existence or non-existence of a general ethereal drift near the earth; for, as shown above, *uniform motion of the entire medium produces no observable first-order effect of any kind.* It plainly becomes the more necessary to attend minutely to possible second-order effects.

In a paper in the 'Archives Néerlandaises,' vol. 21, Professor H. A. LORENTZ discusses, with much power, the whole subject of ether movement; the idea of the following method of treatment is derived from that paper.

Definition of Ray.

23. In § 13 we defined a ray as the path of a labelled disturbance,* for it is that which enables an eye to fix direction, it is that which determines the line of collimation of a telescope. Now in order that a disturbance from A may reach B, it is necessary that adjacent elements of a wave front at A shall arrive at B in the same phase; hence the path by which a disturbance travels must satisfy this condition from point to point, viz., that disturbances arriving at any point from a preceding point of a ray agree in phase. This condition will be satisfied if the time of journey down a ray and down all infinitesimally differing paths is the same.

The equation to a ray is therefore contained in the statement that the time taken by light to traverse it is a minimum; or

$$\int_{\mathrm{A}}^{\mathrm{B}} \frac{ds}{\mathrm{V}} = \text{minimum.}$$

If the medium, instead of being stationary, is drifting with the velocity v, at angle θ to the ray, we must substitute for V the modified velocity $\mathrm{V}\cos\epsilon + v\cos\theta$, and so the function that has to be a minimum in order to give the path of a ray in a moving medium is

$$\int_{\mathrm{A}}^{\mathrm{B}} \frac{ds}{\mathrm{V}(\cos\epsilon + \alpha\cos\theta)} = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{V}\cos\epsilon - v\cos\theta}{\mathrm{V}^2(1-\alpha^2)}\, ds = \text{minimum.}$$

Path of Ray, and Time of Journey, through an Irrotationally Moving Medium.

24. Writing a velocity-potential ϕ in the above equation to a ray, that is putting

$$v\cos\theta = \frac{\partial\phi}{\partial s},$$

and ignoring possible variations in the minute correction factor $1 - \alpha^2$, between the points A and B, it becomes

$$\text{Time of journey} = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\cos\epsilon}{1-\alpha^2} \cdot \frac{ds}{\mathrm{V}} - \frac{\phi_{\mathrm{B}} - \phi_{\mathrm{A}}}{\mathrm{V}^2(1-\alpha^2)} = \text{minimum.}$$

Now the second term depends only on end points, and therefore has no effect on path. The first term contains only the second power of aberration magnitude; and hence it has much the same value as if everything were stationary. A ray that was straight, will remain straight in spite of motion; whatever shape it had, that it will retain. Only cos ϵ, and variations in α^2, can produce any effect on path, and effects so produced must be very small, since the value of cos ϵ is $\surd(1-\alpha^2\sin^2\theta)$. A second-order effect on direction may therefore be produced by irrotational motion, but not a first-order effect. A similar statement applies to the time of journey round any closed periphery.

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* [It has been objected that a bit of wave-front cannot be labelled, because of diffraction effects. This seems to me only a practical difficulty, and a more practical definition based upon preserved phase-connexion follows a few lines later in the text; but the meaning conveyed by the convenient phrase "labelled disturbance" can equally well and I think unobjectionably be expressed by calling a ray the path of a definite, or identical, portion of energy—the direction of energy-flux.—July, 1893.]

MICHELSON'S *Experiment.*

25. We conclude, therefore, that general ethereal drift does not affect either the path of a ray or the time of its journey round a complete contour, to any important extent. But that taking second-order quantities into account, the time of going to and fro in any direction inclined at angle θ to a constant drift is, from the above expression, § 24,

$$\frac{2\,\mathrm{T}\cos\epsilon}{1-\alpha^2}=\frac{\surd(1-\alpha^2\sin^2\theta)}{1-\alpha^2}\times 2\mathrm{T},$$

where 2T is the ordinary time of the double journey.

Hence, by this means, interference effects due to drift would seem to be possible, since the time depends subordinately on the inclination of ray to drift (*cf.* §§ 59–62)

The above expression applies to MICHELSON'S[*] remarkable experiment of sending a split beam to and fro, half along and half across the line of earth motion; and is, in fact, the theory of it. There ought to be an effect due to the difference between $\theta=0$ and $\theta=90°$, but he does not observe any. Hence, either something else happens, or the ether near the earth is dragged with it, so as not to stream through our instruments. When α is constant I see no way out of this conclusion, except hypothetical disturbance at reflexion of some minute kind, one of the mirrors being normal and the other tangential to the drift; but I perceive no adequate reason for this suggestion (see § 60). It is true that if the earth is carrying the ether with it, α will *not* be constant, at different distances from its surface; but, then, the plane of MICHELSON'S experiment was horizontal.

If the ether is dragged along near moving matter it behaves like a viscous fluid, and a velocity-potential must (save by some exceptional theory, § 31) be abandoned; but, as this would involve the curvature of rays striking the earth and much complication, it seems a pity to abandon it until compelled by direct experimental evidence to recognize ethereal viscosity.

The experiment of MICHELSON'S raises a strong presumption in favour of such viscosity, nevertheless his negative result is conceivably explicable in other ways; one of which has been ingeniously suggested by Professor FITZGERALD, viz., that the cohesion force between molecules, and, therefore, the size of bodies, may be a function of their direction of motion through the ether; and accordingly that the length and breadth of MICHELSON's stone supporting block were differently affected, in what happened to be, either accidentally or for some unknown reason, a compensatory manner.

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[*] 'Phil. Mag.,' Dec., 1887.

26. There is already one experiment, which I have never seen criticised either way, tending in a sense precisely contrary to MICHELSON's. FIZEAU* observed the polarization produced by a pile of plates, and considered that he had proved that the azimuth of the plane of polarization varied with the direction of orbital motion of the Earth, and hence that the ether was streaming past them. If so, polarization by reflexion is the only phenomenon known which is capable of showing a first-order effect of the general ethereal drift. The experiment seems to me extremely difficult, but to be well worthy of repetition by other observers. [I believe that Lord RAYLEIGH's objection to the experiment as performed by FIZEAU is that the effect was unseen until an illegitimate or unsafe magnifying device was employed.]

Meanwhile I shall hope to examine the question of ether motion near moving matter in a simpler fashion (§ 33).

Assuming for the present that the ether is not disturbed in a viscous manner by the motion of gross matter through it, we can make the following assertions:—

General Statements Concerning Aberration.

27. A ray is straight whatever the motion of the medium, unless there are eddies, and accordingly no irrotational currents of ether can divert a ray. But, if the observer is moving, the apparent ray will not be the true ray, and accordingly the line of vision will not be the true direction of object.

In a stationary ether, wave-normal and ray coincide, but the line of vision of a moving observer slants across both (§ 20).

In a moving ether, wave-normal and ray enclose an angle, and line of vision depends upon motion of observer. If the observer is stationary his line of vision is the ray; if he moves at the same rate as the ether his line of vision is the wave-normal (§ 13).

The line of vision, in fact, always depends on the motion of the observer, not at all on the motion of the ether so long as it has a velocity-potential. Hence nothing can be simpler than the theory of aberration if this condition is satisfied.

A similar but more general condition (to be obtained in the next section) suffices to secure the straightness of a ray whatever happens, or more generally that whatever the path of a ray may be by reason of reflexion or refraction in a stationary ether, the same it shall be in a moving one; and readily accounts for the absence of all effect on direction due to the general relative drift of the medium, whether in the presence of dense matter (water-filled telescopes) or otherwise (*cf.* 'Nature,' vol. 46, p. 498).

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* 'Ann. de Chim. et de Phys.,' 1859, vol. 57, p. 129.

However matter affects or loads the ether inside it, it cannot on this theory be said to hold it still, or carry it with it. The general ether stream must remain unaffected, not only near, but inside matter, if rays are to retain precisely the same course as if it were relatively stationary.

But it must be understood that the ethereal motion here contemplated is the *general drift of the entire medium,* or its correlative the uniform motion of all the matter concerned. There is nothing to be said against aberration effect being producible or modifiable by motion of *parts* of the medium, as, for instance, by sliding one portion of the ether past another portion, as by the artificial motion of slabs and other partitioned-off regions. These matters are to some extent mixed up with the law of refraction, which we consider later, but the general ideas concerning them have been already given. Artificial motion of matter may readily alter both the time of journey and the path of a ray (*cf.* §§ 7 and 52).

Effect of placing Ordinary Matter in the path of a ray in a Drifting Medium. FRESNEL'S *Law a special case of a universal Potential-function.*

28. Inside a transparent body light travels at a speed V/μ; and the ether, which outside drifts at velocity v making an angle θ with the ray, inside may be drifting with velocity v' and angle θ'.

Hence the equation to a ray inside such matter is

$$T' = \int \frac{ds}{(V/\mu)\cos\epsilon' + v'\cos\theta'} = \text{min.}, \qquad \text{where } \frac{\sin\epsilon'}{\sin\theta'} = \frac{v'}{V/\mu} = \alpha'.$$

This may be written

$$T' = \int \frac{\cos\epsilon'\,ds}{V/\mu\,(1-\alpha'^2)} - \int \frac{v'\cos\theta'\,ds}{V^2/\mu^2\,(1-\alpha'^2)};$$

the second term alone involves the first power of the motion, and assuming that $\mu^2 v'\cos\theta' = d\phi'/ds$, and treating α'^2 as a quantity too small for its possible variations to need attention, the expression becomes

$$T' = \mu T\,\frac{\cos\epsilon'}{1-\alpha'^2} - \frac{\phi'_B - \phi'_A}{V^2\,(1-\alpha'^2)},$$

T being the time of travel through the same space when empty. Now, if the time of journey and course of ray, however they be affected by the dense body, are not to be more affected by reason of ethereal drift through it than if it were so much empty space, it is necessary* that the difference of potential between two points A and B should be the same whether the space between is filled with dense matter or not (or, say, whether the ray-path is taken through or outside a portion of dense medium); in other words (calling ϕ the outside and ϕ' the inside potential-function), in order to secure that T′ shall not differ from μT by anything depending on the first power of motion, it is necessary that $\phi'_B - \phi'_A$ shall equal $\phi_B - \phi_A$, *i.e.*, that the potential inside and outside matter shall be the same up to a constant, or that $\mu^2 v' \cos\theta' = v \cos\theta$; which for the case of drift along a ray is precisely FRESNEL's hypothesis.

Another way of putting the matter is to say that to the first power of drift velocity

$$\mathrm{T}' = \mu\mathrm{T} - \int(\mu^2 v' \cos\theta - v\cos\theta)\ ds/\mathrm{V}^2,$$

and that the second or disturbing term must vanish.

29. Hence FRESNEL's hypothesis as to the behaviour of ether inside matter is equivalent to the assumption that a potential-function, $\int\mu^2 v \cos\theta\, ds$, exists throughout all transparent space, so far as motion of ether alone is concerned.

Given that condition, no first-order interference effect due to drift can be obtained from stationary matter by sending rays round any kind of closed contour, nor can the path of a ray be altered by ethereal drift through any stationary matter.

As soon as matter is locally moved, however, its motion may readily produce an effect, for it has no potential conditions to satisfy; it may easily be moved in a closed contour. Suppose it moves with velocity u, always with the light, the relative drift of ether thereby caused in it must, as above, be u/μ^2, and so it may be said to virtually carry the ether inside it forward with velocity $u - u/\mu^2$; for that is the amount by which it affects the time of journey of a ray. This does not mean that it carries with it any ether of space; in fact, it definitely means that it does *not* appreciably disturb the ether of space (*cf.* § 3, *b*).

The equation to a ray in *moving* matter, subject to an independent ether drift, is

$$\int\frac{ds}{\mathrm{V}/\mu\cos\epsilon + v/\mu^2\cos\theta + u[1-(1/\mu^2)]\cos\phi} = \text{const.}$$

30. It is noteworthy that almost all the observations which have been made with negative results as to the effect of the Earth's orbital motion on the ether are equally consistent with complete connexion and complete independence between ether and

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* [The argument has here been slightly expanded since the MS. was sent in to meet a suggestion of inadequacy made by Dr. SCHUSTER, to whom I am also indebted for an objection to the term "velocity-potential" at first applied to this function ϕ'. As Professor FITZGERALD has observed, it is more general than a velocity-potential, though it reduces to that when the medium is homogeneous, or when $\mu = 1$. The text has been altered accordingly.—July, 1893.]

matter. If there is complete connexion, the ether near the earth is relatively stagnant, and negative results are natural. If there is complete independence, the ether is either absolutely stationary or has a velocity-potential, and the negative results are thereby explained.

Ordinary astronomical aberration, and all other phenomena concerned with vision through strata high above the earth, so far as they have been accurately observed, are consistent with complete independence, but not with a viscous drag.

On the other hand, the negative result of Mr. MICHELSON'S attempt to detect a second-order effect appears only to be consistent with relative stagnation.

A doubtful positive result, supposed to be obtained by FIZEAU (§ 26), on a change in the azimuth of the plane of polarization effected by transmission through oblique plates, would, if established, support relative motion between earth and ether.

31. Is it possible for a sphere to move through a fluid without disturbing it rotationally and propagating rotary motion into space?

It is not possible for an ordinary solid moving through an ordinary fluid. Diffusion of motion, or viscosity, is bound to occur.

It is possible for a vortex ring or assemblage of vortex rings, because at their surface there is no slip. It is possible also if the sphere be a solidified portion of the fluid, which condenses in front and evaporates behind (as already mentioned).

Professor STOKES seems to say, that though not possible to retain a velocity-potential with any *viscosity*, yet with some kind of *rigidity* it may be possible, because deviations from irrotational motion go off into space with the speed of light. If so, the earth might possibly carry some ether with it, and yet a ray be straight.

I do not see any way in which it can abstain from rotationally disturbing the fluid if at the same time it has to carry some with it. Neither, I think, do Mr. HICKS or Mr. LARMOR, to whom I wrote.

Lord KELVIN, however ('Papers,' vol. iii., p. 436), has invented an "ether," or kinematically rigid incompressible ideal substance, which satisfies electromagnetic equations and magnetic boundary conditions, whose equations of motion are like those of an elastic solid, and which yet permits locomotion of smooth solids filling vesicular hollows in it, and which in general "takes precisely the same motion for any given motion of the boundary as does a frictionless incompressible liquid in the same space showing the same boundary."

The experiment now to be described proves, I think, that by the motion of ordinary masses of matter the ether is appreciably undisturbed, and raises a presumption in favour of the earth's motion being equally impotent.

The one thing in the way of the simple doctrine of an ether undisturbed by motion is MICHELSON'S experiment, viz., the absence of a second-order effect due to terrestrial movement through free ether. This experiment may have to be explained away; perhaps as suggested above (end of § 25).

Consequences of Rotational Motion.

32. If the condition of a velocity potential is not satisfied, it follows from equations in §§ 23, 24, that a ray through an ether possessing rotational motion will not be straight.

An ether in the slightest degree carried along in the neighbourhood of moving matter must by all analogy be called viscous. Now a ray approaching a moving body in a viscous ether will in general be entering strata moving with increasing speed, and will accordingly be curved in the direction of the motion (§ 52). A negative or lagging real aberration would therefore occur, in addition to the simple positive apparent aberration caused by motion of the observer; and the direction actually perceived would be the resultant of the two. The motion of an observer is practically constant all over the earth, but the drift of ether strata would be different at different aspects to the earth's orbital motion. Hence the observed value of stellar aberration ought to vary with the time of day, and with the latitude of the observer.*

It becomes important to ascertain definitely whether the ether is viscous or not—whether moving matter can, in the smallest degree, drag or shear the ether in its neighbourhood. If it does, either the theory or the observations of astronomical aberration must be overhauled. But experiment is necessary to answer the question.

Details of Experiment to determine how much, if any, effect on the Velocity of Light is exerted by the Motion of Gross Matter near the Ray.

33. After considering the motion of belts, of fly-wheels, and of double plates rotating oppositely, as in the Holtz machine, I decided to try a pair of plates clamped together with a disk-shaped space between them, and to reflect a split beam of light several times round in this space, half the light in one direction, and half in the other, while the pair of plates were revolving at a high speed. MICHELSON'S device for obtaining two equal beams of light travelling in opposite directions round a contour, by means of a semi-transparent mirror, is plainly the most suitable for any case where the effect of motion is to be observed, and where great length of path is desirable.

Accordingly I sent to Mr. ADAM HILGER a strongly braced stout wooden frame, or hollow square, 38 inches in the side, internal measurement, and asked him to fit it—(1) with three plane mirrors, each 6 × 2 inches, supported in a specified fashion, and silvered on the front; (2) with a fourth mirror, 4 × 2 inches, supported rather differently, bevelled to 45° at two of its edges, and likewise fully silvered on the

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[* It has been pointed out also by Professor FITZGERALD that, if such stratified motion existed, the top of a tower or mountain should exhibit aberrational effects when viewed from below. This might be tested with greater accuracy than is possible in celestial observations.—July, 1893.]

front; (3) with telescope and collimator holders at 45° to the frame; and (4) with a holder for a thinly silvered piece of optically plane glass, 4 × 2 inches, at intersection of axes of telescope and collimator.

The drawings in Plate 31 sufficiently illustrate this part of the apparatus.

The telescope and collimator were a pair ($1\frac{1}{2}$-inch aperture 1 foot focal length) given to the Univ. Coll., Liverpool, by Mr. I. ROBERTS, F.R.S. They happened to have quartz lenses, which was unnecessary, but otherwise were well adapted for the purpose; the slit or aperture of the collimator having especially convenient motions. To the eye end of the telescope, in addition to its own cross-wire eyepiece of low power (which was useful for setting), I adapted an excellent micrometer by COOKE, belonging to a $4\frac{1}{2}$-inch telescope, presented to the College by Mr. GEORGE ROGERSON. It has a pair of independent micrometer heads, each divided into 100 parts, moving respectively a vertical spider line and an X. It also has eyepieces of various powers: the one commonly used for the measurements here recorded being marked "200."

Fig. 7.

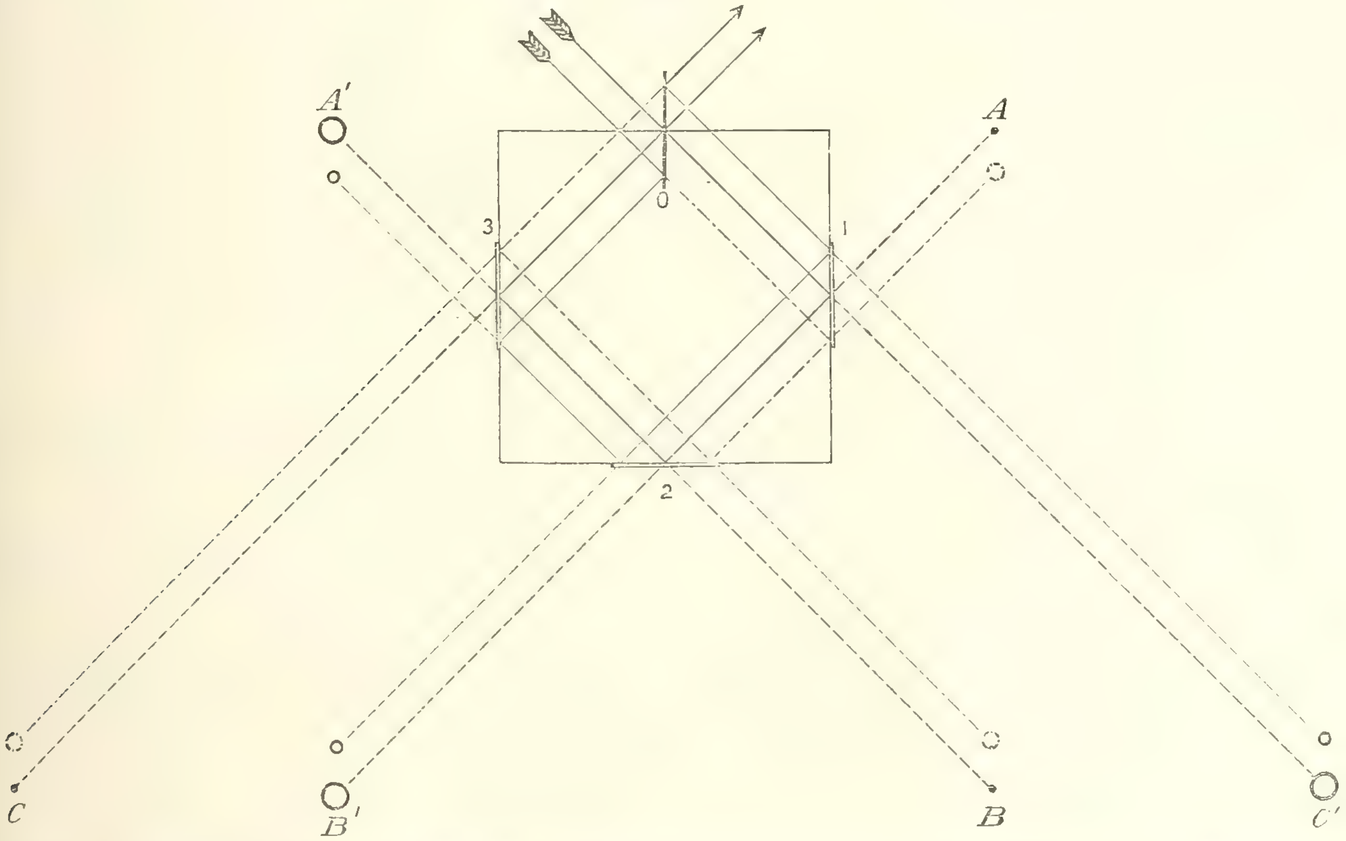

Diagram showing images in simplest case, with three mirrors, and beam going only once round optical square. The points to be imaged are the splitting points on semi-transparent mirror O. Two such points are shown, one imaged by a dot, the other by a ring. A represents the pair of images in first mirror for the transmitted ray; B the image of these in second mirror; C the image in mirror 3. Dashed letters mark the corresponding images for the reflected ray; and the final function of the semi-transparent mirror is to make C′ coincide with C so as to give interference.

Examining the path of a beam of light from collimator to telescope round the frame, as shown most simply in fig. 7, it appears that, by reason of there being an odd number of reflexions, the interfering rays do not travel identical paths in opposite directions, but only parallel paths.* They enter the square at one point of the semi-transparent mirror, and they leave it at another point, having meanwhile travelled side by side. I therefore designed a compensator, consisting of a plane piece of optical glass, with its faces not parallel but inclined at a very small angle ; this was mounted in a round cell, and made capable of measured rotation in its own plane. By introducing this normally into the beam, and rotating it into the right position, it was supposed that accidental inequalities of path could be compensated ; and also that the bands could be shifted by a measured amount. Hitherto, however, no use of the compensator has become necessary, and I have some doubts as to whether it would act in the way supposed, or whether it would not merely double the number of bands in some positions.

The object of the fourth or front mirror, shown in fig. 8, is to enable the light to go more than once round the frame. This mirror has to stand a little forward, in advance of the square defined by the planes of the other three, and the amount by which it stands forward regulates, at the same time, the width of the beam and the number of journeys it makes round the frame. Everything else can be permanently set.

If each beam is of breadth b, and travels n times round the frame ; if the length of this fourth mirror is l, and the amount by which it stands forward out of the square is d ; then it is easy to see, by fig. 8, that

$$l = (n-1)\sqrt{2}\,.\,b, \qquad \text{and} \qquad b = \sqrt{2}\,.\,d;$$

also that the centre of the effective part of the semi-transparent mirror, *i.e.* the intersection of axes of telescope and collimator, is a fixed distance, viz. $\frac{1}{2}l$, behind the square.

Hence the only thing that requires re-adjustment in order to vary the number of times light goes round, is d, the setting forward of the front mirror.

The 45° bevel at the ends of this mirror is to enable the whole of its (silvered) face to be utilised, and to allow a beam which just misses it to graze past it unimpeded into the telescope. (See Plate 31).

With the front mirror 4 inches long, and the centre of semi-transparent mirror 2 inches back from the 38-inch square formed by the other three mirrors, the most frequent adjustment has been to set the front mirror 1 inch forward. A parallel beam incident on the centre of semi-transparent mirror, at 45°, must now go three times round the frame, rebounding three times from each of the three mirrors, and twice from the front mirror; and the width of the beam may be as much as 1·4 inch. The pair of plates between which the light has to go are, however, only 1 inch apart, so this limits the effective aperture in a direction perpendicular to the plane of reflexion. A narrower beam may be sent more times round, by setting back the fourth mirror the proper amount.

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* This fact makes the bands more susceptible to some kinds of disturbance, *e.g.*, irregularities of temperature or density ; more stable fringes can be obtained by using an even number of reflexions, *e.g.*, a triangle instead of a square (*cf.* 'Nature,' vol. 46, p. 500), but then the light will not go more than once round.

Fig. 8.

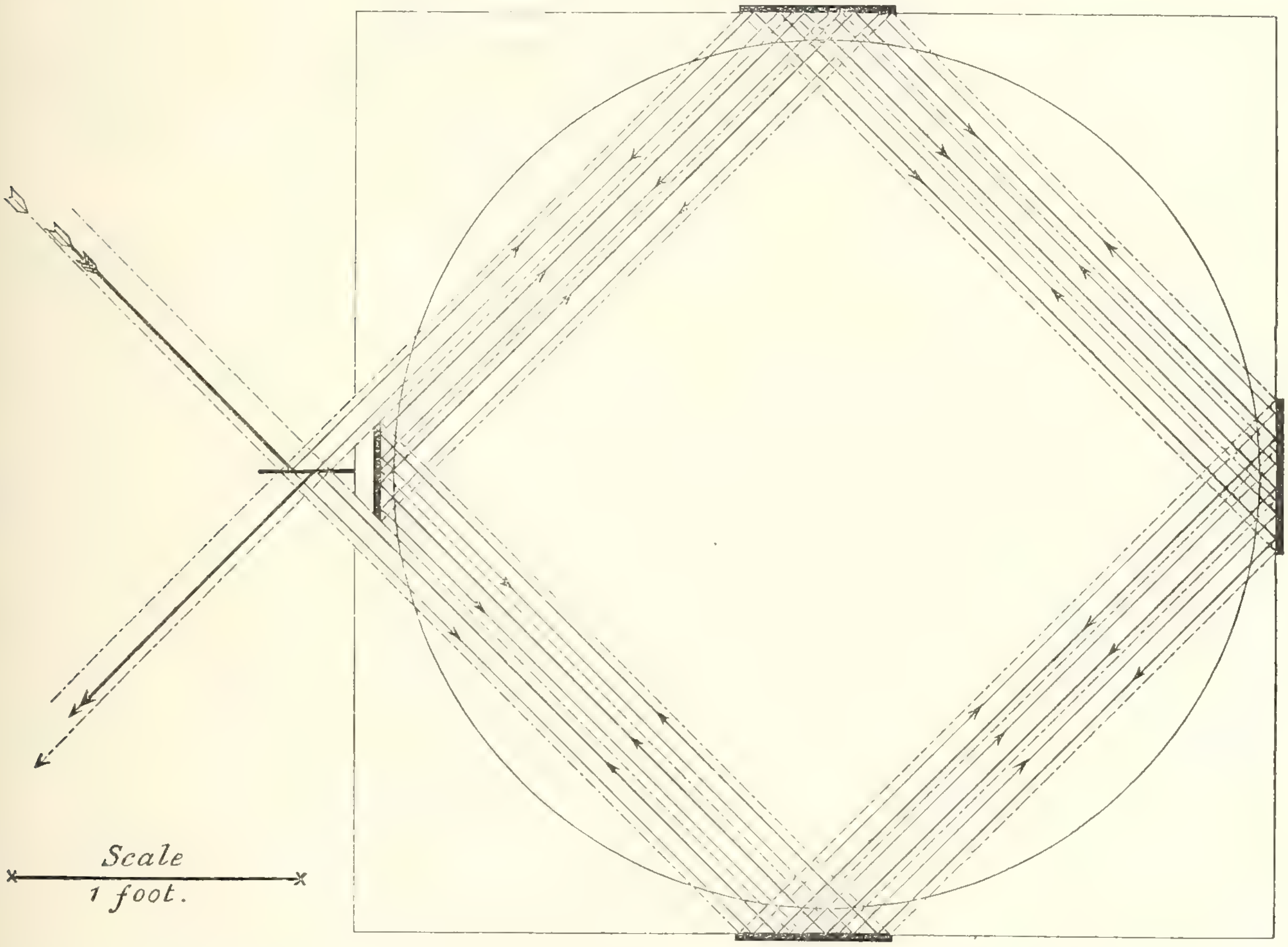

Diagram showing arrangement of mirrors for the usual case of beam going three times round. Interfering rays travel parallel but not identical paths, and re-unite at a point different from that at which they split. The greatest possible size of beam, with a 4-inch fourth mirror, is represented. The centre of the semi-transparent glass, *i.e.*, the intersection of telescope and collimator axes, has to be 2 inches behind square formed by the three mirror faces. The fourth or extra mirror has to be 1 inch in front of the same square.

Whirling Machine.

34. For the pair of plates I use a couple of steel circular-saw disks, one yard in diameter, of best hammered steel, and bevelled down somewhat from centre to edges, braced up at centre with wrought-iron cheeks and bolts. The plates were specially made by SEEBOHM and DIECKSTAHL, of Sheffield, and are stated to be able to stand sixty-seven tons to the square inch.

At this strength they would fly to pieces at 8800 revolutions per minute, supposing them simply cylindrical and unbraced at the centre hole.* The bevelling and the clamping ought to afford margin enough to run them up to 6000 (though the makers recommend no speed over 4000); but hitherto I have not spun them at more than 3000 revolutions to the minute, and even at this speed, I now have a screen or sentry-box of double boiler plate (consisting of a small iron boiler cut in halves longitudinally and one half fitted inside the other) for the observer, whose eye is in the plane of the disks, to look through.

The use of steel disks is sufficiently justified by the high speed they will stand, but it may be also held that iron is the most probable great constituent of the earth; and further, that as there are so many other ways of experimenting on transparent matter, opaque matter is appropriate in this experiment.

Since steadiness of rotation was very essential, I arranged to rotate the disks horizontally on a vertical shaft balanced on a steel point in an oil vessel, and with a slightly flexible or elastic bearing near the top, so as to get the whole to sleep like a tee-totum; and in order to avoid any lateral strain, as of driving belt, to drive electrically by a dynamo armature on the axle itself (fig. 9).

Messrs. MATHER and PLATT were good enough to undertake this part of the work, using their smallest size Manchester dynamo as motor, with its axle set up on end, the armature being wound with less wire than usual, and being extra strengthened against centrifugal force by steel wire. The ordinary bearings of the dynamo remain, with oiling wicks inserted, and hence there is a little nicety required to get all three bearings in a precise line. It is also needful for the shaft to be vertical, to avoid any attempt at gyroscopic precession.

Fearing lest some damage to the disks might occur from sudden application of too great moment to the armature, especially by reason of some accidental jamb or other sudden stoppage, I imitated a device employed in some milk-whirling machines, and introduced a wooden cup or friction coupling between the top of the dynamo shaft and the bottom of the disk spindle. The cup being made of hard wood grips the rounded end of the disk spindle, and thus applies to it sufficient rotating moment, but permits slip in preference to violent acceleration. This plan is, I think, on the whole advisable, and is certainly a safeguard. It may seem to spoil the tee-totum idea, but the dynamo axle, which is supported at each end, and constrained to rotate about rather a long axis, is to be regarded as a driving machine only; the "top" begins from the wood cup upwards. The brass collar of the upper bearing is let into an india-rubber cylindrical socket, so as to afford some very slight play; and just above the wooden cup is a safety collar or loose guide not touching the shaft.

My assistant, Mr. B. DAVIES, had some trouble in getting and keeping the two shafts accurately aligned, especially since any wear of the wooden cup tends to throw them out again. Also the bearings of the armature are, at present, hardly tight enough. Difficulties such as these have hitherto prevented the whirling machine from being quite satisfactory. At about 800 revolutions a minute a tremor begins. At high speeds it steadies itself, but at specific speeds the tremor re-occurs, and at some of the highest speeds is rather alarming. At others, however, the tee-totum action steadies the spin, and at these only could observations be made.

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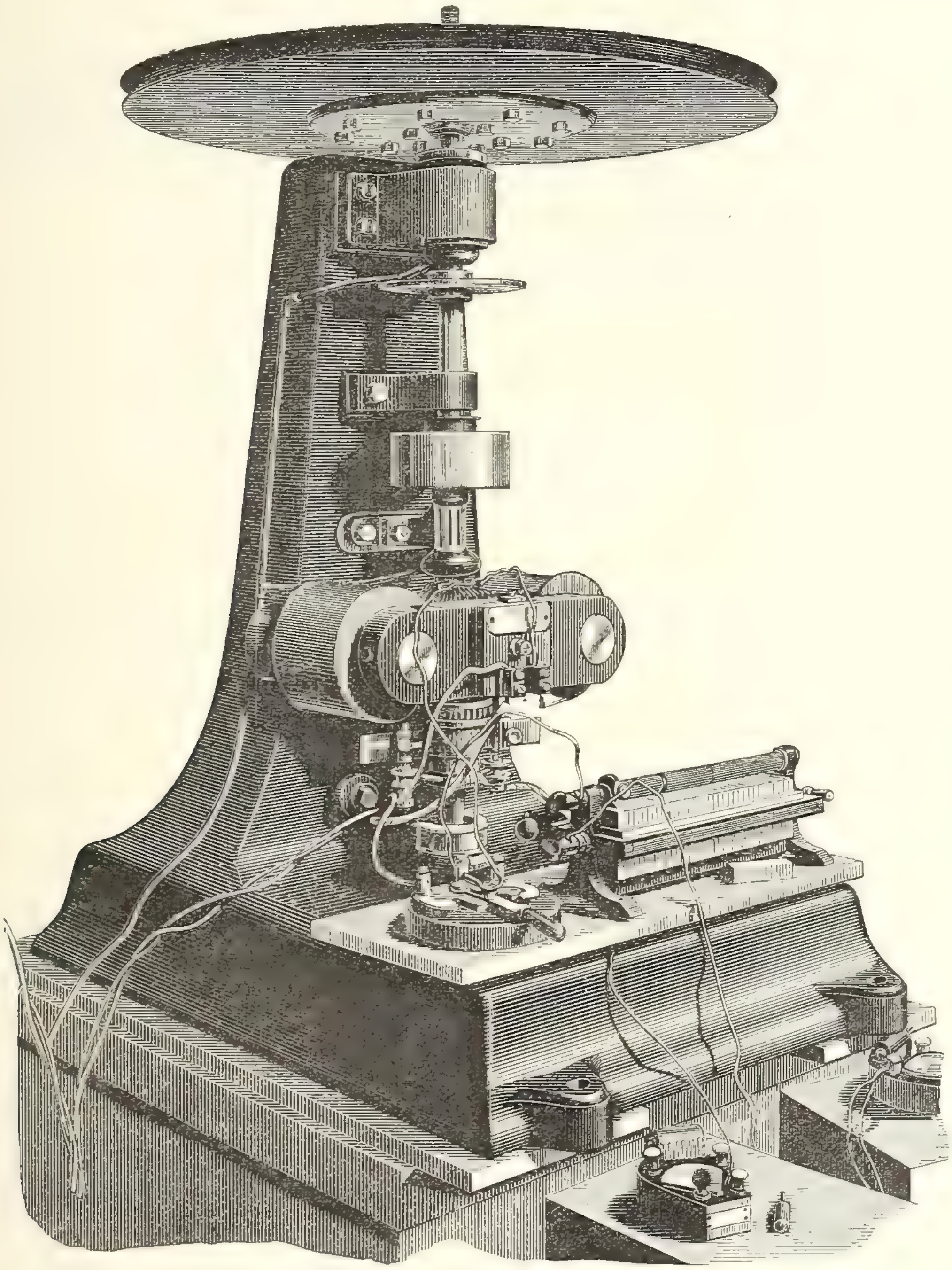
* The connexion between tenacity and maximum peripheral speed for a ring is $T = \rho v^2$; while, for a uniform disk with a small hole in it, EWING adapts GROSSMANN to show that it is $T = \frac{1}{4}(3 + \mu)\rho v^2$ where ρ is density of material, and μ is POISSON's ratio. See 'Nature,' vol. 43, pp. 462, 514, 534.

Fig. 9.



Whirling machine for ether experiment, with pair of steel disks, 1 yard in diameter. From a photograph taken during preliminary tests, before it was bolted down to stone pier. Full voltage is always supplied to the field magnets, variable resistance is in armature circuit only. The brass tube conducts away surplus oil. Drawings of the machine are given in Plate 32.

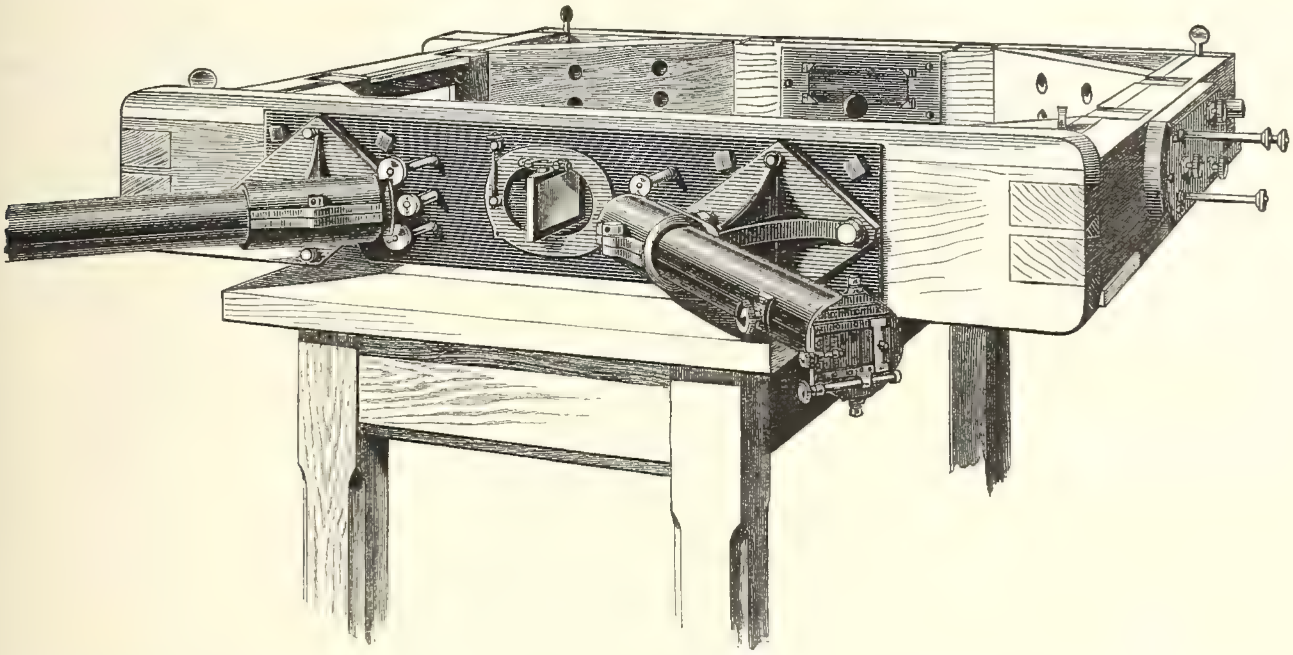
A perforated steel disk, pierced with circles of 3, 5, 9, 15, and 25 holes, and fixed on the spindle, gives the speed quite accurately by the musical note from an air jet through it, or even a card held against it. But the particular octave was occasionally checked by a wheel speed-counter; and recently one of the mercury speed-indicators of Messrs. NAPIER BROS. has been fitted on, and works very well for approximate readings.

The current is supplied usually from about twenty small storage cells, which I happened to have. The regulation is effected by one of ROUSSEAU'S very convenient carbon rheostats introduced into the armature circuit, the whole voltage of the battery being always supplied to the field magnet.

When the whirling machine arrived it was mounted on a stone pier in the middle of my laboratory, a pier built up from the sandstone rock beneath, and it was bolted down to timbers embracing the pier. The disks were then put on, a pulley gearing from the ceiling being convenient for raising and lowering them; and after some preliminary work the battery was applied. The field magnet took only 4 or 5 ampères from the 20 cells with gas-engine going; 10 ampères began to drive the machine slowly, and 30 ampères gave a speed of 800 revolutions up to 1400 revolutions a minute, though the last figure was not reached before the disks were encased. The volts actually on the terminals were from 20 to 30. The bearings, however, were not at this time quite easy, and less power now suffices. Thus the numerous spins at 1250 revolutions per minute taken during March, 1892, required, to maintain full speed, 385 watts, viz., 27·5 volts and 14 ampères.

35. The optical parts were the first to arrive (about June, 1891), and were the subject of much preliminary experiment. After a few alterations, such as planing down the base of the telescope-holders to a better angle, there was no particular difficulty in getting the light of an oxyhydrogen lantern to go three times round the frame and then enter the telescope. A dark room and strong light were useful for making this adjustment, for the course of the ray could then be tracked without difficulty; but after the adjustments were made, a paraffin lamp could be substituted for the lantern without too much enfeebling the image. On now inserting in its place the semi-transparent mirror, a second image made its appearance. Removing the lens of the collimator, the two images of the "slit," or square aperture usually employed, were seen very small and separate. By adjusting the semi-transparent mirror, which only moved the reflected image, the two were made to coincide. The collimating lens was then re-inserted, and the telescope focussed for infinity. Bands at once appeared. They were usually slant, but when best defined became horizontal. In these early trials vertical bands were only got with difficulty; they seemed to prefer being horizontal. Shaking the whole table on which the frame rested did not hurt them, but pressing gently on the wooden frame distorted them. It was easier to get them with light which had gone only once or twice round the frame, but there was no real difficulty with three times round. Introducing the wedge compensator and turning it

round caused them to shift. Introduction of ordinary sheet glass into the beam distorted them till they were like the water-mark on cloth.



View of the optical frame, supporting the mirrors, telescope, and collimator: detached from its position round the steel disks, where it is shown in fig. 11.

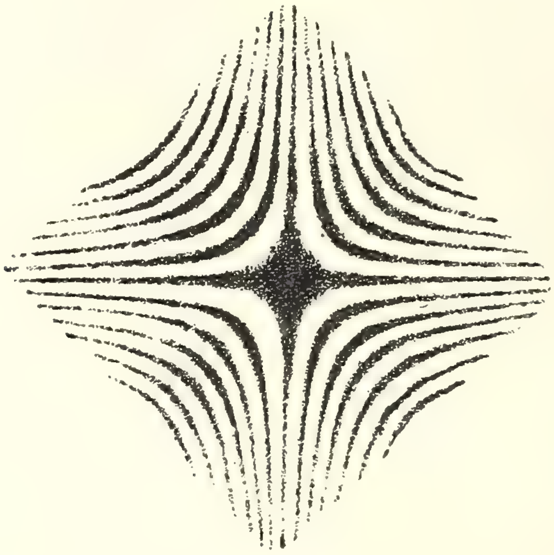
Later experience makes it absurdly easy to get the bands and to arrange that they shall be vertical, well defined, of any convenient width, and with the centre white band symmetrical among the coloured ones. Tilting the mirrors, pressing on the frame, or touching the semi-transparent plate, makes the coloured fringes move with a concertina-like motion towards or from the middle band, but the middle band is not easily shifted by anything. Altering the angle between the mirrors widens or narrows the bands, and when they get very wide a double system of hyperbolæ usually makes its appearance.

I conjecture that the horizontal bands first seen were the right or left branches of fig. 10, and that the widening or narrowing of the vertical bands may be expressed as an up or down motion of the figure.

Sometimes by pushing in the eye-piece of the telescope another set of bands could be seen horizontal, in exchange for the vertical bands which had, as it were, gone out of focus. These horizontal bands were more tremulous than the others and tilted readily. They occurred in the proper focal plane for infinity (assuming the mirrors to be accurately plane), and are probably what I saw first. The vertical bands do not become visible till the eye-piece is pulled several inches out. The probable meaning of this double set is that the mirrors have a very slight cylindrical curvature, the generating line of the cylinder being vertical. A pair of opposite mirrors, though very nearly parallel, will intersect in a line, and the bands will be parallel to this line. If this line is horizontal the bands will be horizontal, and for this direction the mirrors seem accurately plane. But for the vertical line of intersection the *length* of the mirrors is effective, and this seems to be slightly curved, so as to throw the focal plane of the vertical bands further out.

This does not account for the hyperbolæ. But with sodium light, systems of hyperbolæ, and also ellipses, can be seen in Brewster or Jamin plates, with different angles of setting (see LUMMER, 'Wiedemann's Annalen,' vol. 24, p. 417), and I expect the theory is somewhat similar.

Fig. 10.



One frequent appearance of the bands.* (But light and dark should be interchanged in the figure.)

36. The following observations were made recently as to the effect of various movements on each set of fringes :—

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* Cassini ovals are just as easy to get as these. The bands are always curves of the fourth degree and are, of course, sections of surfaces of constant retardation. The virtual sources may be taken as in fig. 7, viz., the nearly coincident images, C and C'.

Operation.	Effect on vertical bands.	Effect on horizontal bands.
Pressing lightly and intermittently downwards on corner of large wooden frame.	Concertina action; no shift of middle band.	Either rotation of whole set about a point, or shift up and down of whole set, or mere blurring.
Turning slightly one of the supporting screws of front or "fourth" mirror.	Concertina action and tilting of all bands, except the middle one; no shift of middle one	Shifting of entire set, and disturbance of bands.
Vertical screw supporting semi-transparent plate turned so as to tilt its plane about a horizontal axis, the axle being two inches to one side of the plate.	No concertina action, but a rapid blurring of the bands.	Concertina action, and tilting of all bands but the middle one; no shift of middle one.
Horizontal screws supporting semi-transparent plate turned so as to tilt it about a vertical axis, the axle being near one end of the plate.	Concertina action; no shift of middle band.	No concertina action, but a rapid blurring of the bands.

The effects of vertical and horizontal screws were thus complementary on the two sets, as was natural.

37. In order that the dark bands shall be really dark, a nice adjustment of the thickness of silver film on the semi-transparent plate is necessary. It can only be hit on by a sort of chance, for when once taken out of the silvering bath it is useless to put it back again if not sufficiently done.

The equality of transmission and reflexion at 45° is readily *tested* by two gas flames a yard or two apart, with the plate at the corner of a right-angled isosceles triangle at whose acute angles are the flames. On looking into the plate, one flame and the image of the other are seen side by side, and ought to be of equal brightness.

The film, however, by gradual tarnishing gradually becomes more transparent, so it is best to slightly overdo the plate, and let it age till right. Or an overdone film may be thinned down with potassic cyanide if wanted quickly. The unpermanence of these plates is a little troublesome; I should prefer to deposit a thin platinum film by "electrical evaporation," after the manner of Mr. CROOKES.

Certainly the bands can be *seen* when the images are very unequal, but they are on a background of spurious or non-interfering illumination, and for *measurement* it is desirable to get the bands exceedingly sharp.

The unsilvered side of the semi-transparent plate, of course, reflects some light, and gives another image. With a short course for the beam this useless image comes into the field, but it need not be superposed on the other unless the plate happens to be of very uniform thickness; and, if superposed, it can be got rid of by using a Nicol. With a very long course for the beam a Nicol is useless, for all the light seems polarized after so many reflexions; but fortunately, although the plate should be of good quality and uniformly thick, it is extremely unlikely for anything like superposition to occur after so long a journey, and often the useless image is not even in the field—never with a high power.

I must say that the satisfactory behaviour of the optical arrangements is due to the skill of Mr. ADAM HILGER in working glass to true planes. It is a difficult matter, for he says they are apt to change after being taken off the tool. He has several times tried to improve on the first set of four mirrors he sent me, but without success.

Those now sent usually have radii of curvature from three to eight hundred metres, and are not at all satisfactory, though their curvature is too small to detect with a spherometer.* Judging by their behaviour the original set must be very good. I expect they are of superior, or older glass. They are, of course, mounted so as not to strain them in the least.

38. To support the optical frame over the whirling machine, with the plane of the light between its two disks, a substantial wooden structure was erected, from brick piers coming up through the floor, entirely independent of any support from the whirling table or its stone pier. To this the frame was fixed, and it was supplied with a lid and floor, to box in the disks and make them easier to drive. The lid had a domed cavity for the top of the spindle ; the floor had a hole edged with thin india-rubber sheeting to permit the spindle bearing to pass through air-tight without transmitting vibrations.

In order that the semi-transparent plate might not be affected by the blast from the whirling plates, a couple of optical glass windows were inserted to screen it completely. I feared lest the blast would have some effect upon the mirrors themselves, but they were substantially backed by thick brass plates bearing steadily against three accurate screws in a strong frame (see Plate 31), and I hoped it might not.

39. On the 21st of July, 1891, a first complete spin was taken. The bands being vertical, the cross wires were set on one of the dark ones, and the speed increased until a shift of three bands might have occurred. The shift actually observed was $1\frac{1}{2}$ band, and they recovered their old position very fairly when the motion ceased. Strongly suspecting this shift to be spurious, I had the brushes of the dynamo reversed, and later in the same day was able to take a reversed spin. The shift was approximately the same in amount and in the same direction. The centrifugal force of the blast evidently did affect the mirrors. Pressing their supporting plates by hand, a similar shift could be got : the screws did not hold them with absolute firmness, and it seemed as if the end held by only one screw yielded more than the end held by two, so as to produce a minute tilt.

To see if the pressure of the blast distorted the frame as a whole, or only tilted the mirrors, the box was made air-tight, like an organ chest, and air was pumped into it by foot bellows: but this made very little difference to the bands, though the pressure was 6 centims. of water. A water gauge was also used to measure the centrifugal force of the blast: it was about 3 centims. of water, but was not easy to measure satisfactorily. Evidently the blast acted mainly on the mirrors. I contemplated a vacuous chamber, but shrank from some obvious difficulties, besides realizing that the residual air must give trouble anyhow. I decided to risk interposing transparent matter in the beam.

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* I measure it by focussing a telescope on the image they give of an object at a considerable known distance.

So some plate-glass protectors were cut and framed, one in front of each of the three mirrors, the fourth at present being left still exposed, partly because it seemed less likely to be affected by the wind, and partly because of the very narrow space available between it and the disks.

Seeing that each half-beam of light, in going three times round, has to go through each piece of plate-glass twice at every reflexion, or eighteen transmissions altogether, the intensity of the emergent light was feeble, and the bands were distorted by imperfections in the glass. Still they could be got clear, though curled, and the cross wires could be set on them. On 24th and 25th of July, spins at 1380 revolutions a minute were taken, and no shift so great as $\frac{1}{10}$th of a band could be detected; whereas, with the light only going twice round, there might have been a shift of $2\frac{1}{4}$ bands if the ether had been carried full tilt. Hence it would appear that the ether was not carried round with the disks by so much as $\frac{1}{20}$th of their speed.

The alignment of the shafts and other mechanical details were now attended to, so as to make possible higher speeds of rotation. Fresh arrangements for holding and adjusting the semi-transparent plate (the setting of which is a delicate matter) were made, and the micrometer was more firmly fitted by a much longer tube into the telescope. All sorts of steadyings were attended to. A pair of wires were laid across to the Walker Engineering Laboratory, where Professor HELE SHAW had a large Crompton dynamo; and so in December, 1891, my assistant, Mr. DAVIES, got the speed up to 2800 revolutions per minute.

43 amperes and 75 volts on the terminals (4·3 H.P.) gave 2500 revolutions a minute with lid of box off. With lid on, it went up to 2800 revolutions, but now the air in the box got quite hot, and a new difficulty arose from oil. It crawled up in an imperceptible film, notwithstanding the traps arranged for it, and, flying from the under surface of the lower disk, bespattered the mirrors and spoilt them.

More elaborate oil-catchers, to spray the oil off before it got to the disks, were therefore arranged, and the disks were kept whirling many days to get rid of all traces of the oil that had already soaked between the steel and the wrought-iron cheeks. Having cured the oil difficulty, fresh and improved cover glasses for the mirrors were got from Mr. HILGER, one extra large and extra thin one for the front mirror included, and these were carefully framed and placed in position. The light was then with difficulty, and considerable skill on the part of DAVIES, got three times round through all the glasses, and, during Christmas, 1891, spins were taken, but the highest speeds shook too much to give good observations. At 1800 revolutions, a shift of half a band occurred. Reversing the rotation, the shift was appreciably the same. It was probably due to pressure of the blast on the frame itself.

Hence, ordered a light wooden circular drum to be put inside the frame to catch the air pressure, and also to keep hot air from the mirrors if possible.

DAVIES took out the old wooden socket from the axle, and made a new boxwood one to connect the two shafts. We also now got the large boiler-plate protector for the eye end of the observing telescope to protrude through. It just shows at the back of fig. 11.

Resilvered the mirrors, and got the optical arrangements more perfect. Also ordered a much heavier mass of metal to whirl in subsequent experiments. For the present, however, we go on with the steel disks.

40. During February, 1892, the various preparations were made. The drum was a very satisfactory mahogany structure, octagonal outside and circular inside, with long slit windows glazed with optical glass to catch the blast. The drum was fitted inside the frame, with $\frac{1}{8}$th-inch clear space all round between the two. It was secured to the floor of the frame and joined with the lid so as to be fairly air-tight.

Still observed a shift, often of about $\frac{3}{4}$ths band. At speeds above 2000 the bands usually disappeared, from shaking. Removed the lid of the frame for the air to escape, and the shift was smaller. It was specially noticeable that it lagged. It took a little time to reach its maximum, and when the disks stopped the bands continued to recover for some time afterwards. It was just as if a solid had experienced a strain and sub-permanent set. All the effects were irreversible. Gradually made the drum more air-tight and supported it by wooden bars from above. A shift of $\frac{1}{4}$th band was still got at a speed of 1000, with the light three times round. Strains in the drum still seemed to be transmitted somehow to the frame.

Made a saw-cut all round the floor of the frame, so as to leave the floor on the drum instead of on the frame; also supplied a lid to the drum, and supported it by this lid from long wooden girders, as shown in fig. 11, keeping it and its supports everywhere out of contact with the frame or its supports, except the upright posts from the tables.* Studied the bands more particularly and observed the double hyperbolæ system. Found also the great advantage of the middle white

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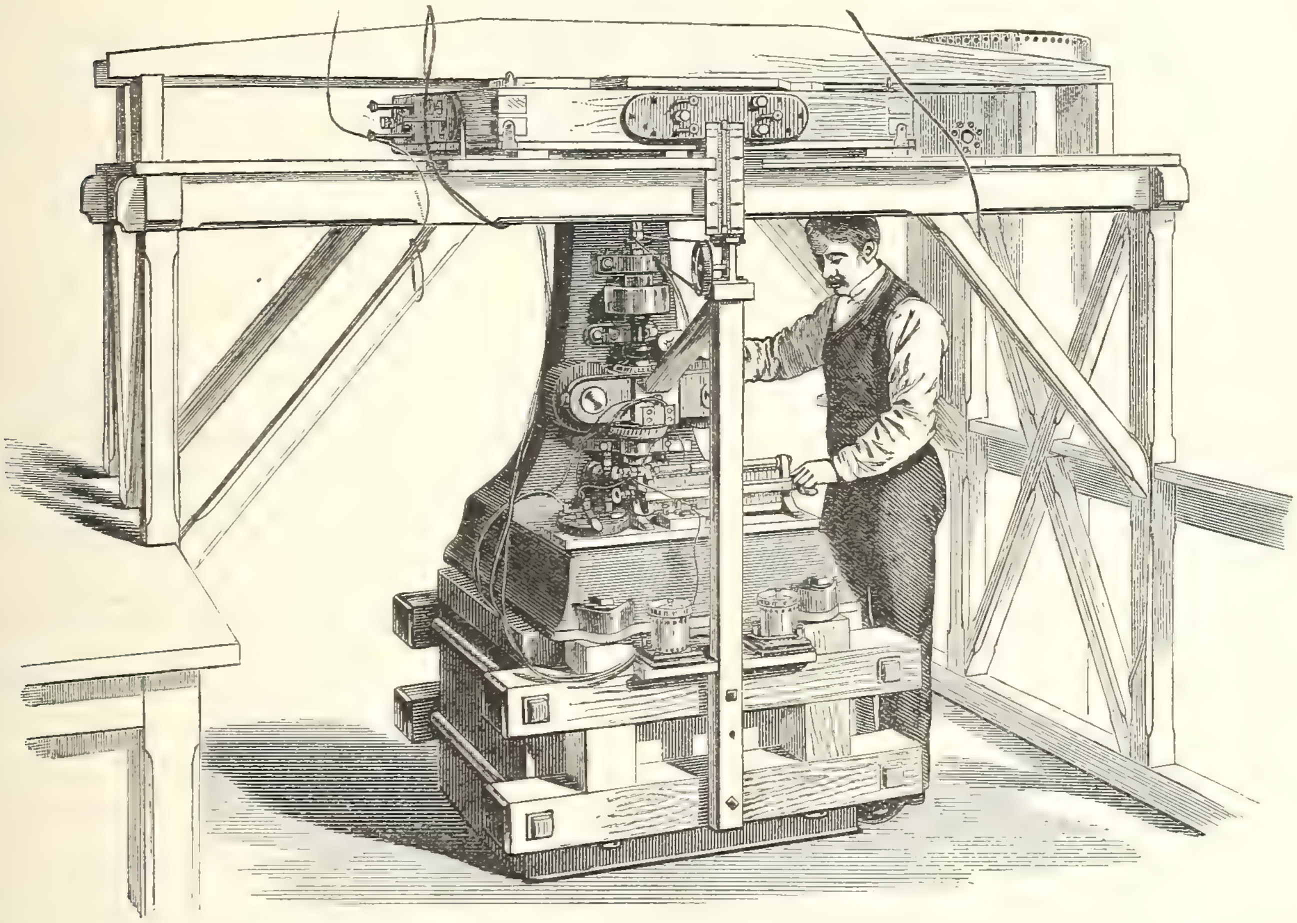
* Quite recently (March, 1892) I have taken a series of spins without the drum, and with no cover glasses on except one over the front mirror, something being there necessary to protect from wind the semi-transparent plate, which is of course extremely sensitive.

By this time, as described above, the bottom of the square frame had been removed, being attached to the drum instead, and we now found that this floor must have been the cause of most of the trouble. For without it the blast produced no effect on any of the bands, and neither shift or concertina action occurred, up to a speed of 800. No motion of the middle band could be detected, the light going twice round.

It follows that the blast did not take effect on the mirrors, nor directly on the frame, but that by exhausting the air near the middle of the box it must have sucked and bent the floor sufficiently to strain band; all others being readily moved by trivial strains. Always now set the vertical spider line in the centre of the middle white band and watched for its shift.

To see if the irreversible and lagging (very slight) shift still observed could be due to the action of hot air on the glass windows, I arranged to throw into the drum air heated by passing through a metal pipe in a gas flame. The hot air-stream flickered the side bands about, but did not alter the middle one.

Fig. 11.



General view of the whirling machine, and independent support of optical apparatus, in action. The speed indicator is seen on top of front post, ammeter and voltmeter on framework. The long upper girders support the glazed drum which encloses the disks and secures the mirrors from the blast. Telescope, &c., are at back, not seen; boiler-plate screen for observer just visible. Everything independent of floor, and no contact between anything on central altar with anything on gallows framework.

Quantitative Experiments.

41. I now began a series of actual readings, the plan adopted being to set the vertical line of the micrometer in the middle of the white band, and the X of the micrometer on some definite colour of the first bright band on the left: the yellow

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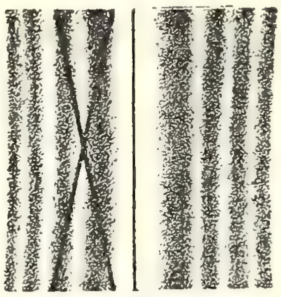
the frame and tilt the mirrors about a horizontal axis; this kind of tilt being well calculated to shift the vertical bands.

I tried also the horizontal set of bands without any floor to the frame (both with and without the drum), and perceived no shift in the middle one of these either, due to the motion of the disks.

being usually selected. Then read their positions, and started a spin. When spinning at a given speed, I reset the wires and read them afresh. Then stopped the motion, reset them, and read once more.

Lastly, shifted the vertical wire till it coincided with the X, and thus obtained the interval between them, or the width of one band corresponding to yellow light.

Fig. 12.



Appearance of the bands with the two micrometer wires set in position.

This process was quite satisfactory, but it is not worth while to quote all the readings; for it was noteworthy that up to 800 revolutions no perceptible shift of the middle band had occurred, but at that speed the first tremor set in, and although there was supposed to be no contact anywhere between machine and frame, yet it was transmitted somehow, through the earth at any rate, and sometimes produced an effect obviously spurious. Often the first spin of a set had the effect of shaking things into place, and subsequent spins were better. This was the case, for instance, on March 16, when the following readings were taken.

March 16. Bands very clear and sharp. The first spin was spoiled by a tremor which set in at 800 revolutions and fogged the bands while it lasted; when they re-appeared they had been shaken aside and broadened. No shift had occurred up to 800. It was afterwards found that the semi-transparent plate was not held quite tightly enough, and that it was affected by tremor with just the above effect. I give only the results of this first spin; not the details.

First Spin.

The width of a band was	when originally at rest	91	divisions.
	when in motion at 1250 revs. . .	170	,,
	when at rest again	144	,,

The middle band . . .	shifted .	37	divisions, to the right.
	returned .	22	,,

This was very bad, but went on with another attempt.

Second Spin.—Direction of motion as before, viz., such as to assist the reflected half-beam.

This time the tremors which still occurred at about 800, 1000, and 1150 revolutions did not seem to produce much effect.

The following were the readings; each wire read on its own micrometer head.

	X-wire, set on yellow of first band to left.	Vertical wire, set in middle of middle band.
At rest	7	60
At 800, no shift of middle band		
At 1000, barely any		
At 1260 revs., set again and read	11·5	68
Stopped. Set again and read	11·5	64

To carry the vertical wire from its position 64 into coincidence with the X, required a motion of 154 divisions (viz., about one revolution and a half of its micrometer screw).

The experiment was satisfactory. Its result was

Width of a band of yellow light	at first	146 divisions.
	when moving . . .	158 ,,
	at last	154 ,,
Displacement of middle band . .	shift to right . . .	8 ,,
	return on stopping .	4 ,,

The highest estimate of this shift is, therefore, $\frac{8}{158}$ths, or say $\frac{1}{20}$th of a band.

The lowest estimate is $\frac{4}{154}$ths, or say $\frac{1}{38}$th of a band.

Then reversed the brushes and immediately took another satisfactory observation.

Third Spin.—Direction reversed, that is, so as to help the transmitted beam.

	X-wire set on yellow of 1st band to left.	Vertical wire set on centre of middle white band.
At rest.	9	69
No shift at 800 or 900. Tremor at 1000		
Set at 1250 and read	15·5	78
Stopped and set again	14	73

To carry the vertical wire from 73 up to the X at 14 needed 165 divisions. The result of this was

Width of band of yellow light	at first	156	divisions.
	when moving . .	$171\frac{1}{2}$	,,
	at last	165	,,
Displacement of middle band .	shift to right . .	9	,,
	return on stopping	5	,,

The highest estimate of this shift is $\frac{9}{172}$nds or say $\frac{1}{19}$th of a band.

The lowest estimate of this shift is $\frac{5}{165}$ths or say $\frac{1}{33}$rd of a band.

And these figures are within the limit of error certainly, the same as those given by the previous experiment with the spin reversed.

Taking this experiment alone, therefore, one may say that reversing the rotation of the disks, from 21 revolutions a second one way, to 21 the other way, does not affect the virtual path of the light between them by so much as the

$$\tfrac{1}{33} - \tfrac{1}{38} = \tfrac{1}{200}\text{th of a wave length,}$$

or even by so much as the

$$\tfrac{1}{19} - \tfrac{1}{20} = \tfrac{1}{400}\text{th of a wave length.}$$

Tested to see how many times the light was on this occasion going round the frame. Found that it was going only twice. Hence the length of path of each half beam was $2 \times 4 \times 2$ feet, since its path was approximately the periphery of a square two feet in the side. The whole path of the two beams is therefore 32 feet. A shift of $\frac{1}{200}$th of a wave length in this length of path means a fraction

$$\frac{\frac{1}{200} \times 6 \times 10^{-5}}{32 \times 30} = 3 \times 10^{-10}.$$

The perpendicular distance of the average light path from the axis of rotation was 1 foot, hence the effective speed of the disks either way was $2\pi \times 21$ feet per second, or say 260 feet a second altogether.

Compared with the speed of light this gives a fraction

$$\frac{260 \times 30}{3 \times 10^{10}} = 2.6 \times 10^{-7}.$$

Comparing these two fractions we may conclude that the meaning of the above experiment is that the ether is not carried forward by the spinning disks with so much as $\frac{1}{800}$th of their velocity.

Modes of Testing Number of Light Journeys.

42. There are many ways of making sure how many times round the frame the light is going. One is to look into the telescope with a low-power eye-piece, or none at all, before the semi-transparent plate is inserted and to give a wide opening to the collimator, often also removing its lens. Three or more images can then be seen in different parts of the large field, and it is easy to see which is the one near the centre. On now tipping the front mirror to and fro, the image which has been only once round (if visible) remains unaffected; the one which has been twice round moves; the three-times-round one moves twice as much; and so on.

By altering adjustments and passing the successive images in review, it is not difficult thus to work up to the high numbers. But on the specification of these it is well to have some check. The best check can be got, with everything in position and bands visible, by passing an opaque strip slowly in front of one of the mirrors and observing the eclipses at the telescope. If the semi-transparent plate is inserted in its holder, these eclipses occur in pairs, with shadows moving oppositely; if not, they occur singly. (A very narrow strip passed not quite close to the mirror may show each single shadow double. Too broad a strip will, of course, merge a pair of shadows into one.)

Three times round naturally gives two pair of eclipses on the front mirror, and three pair on each of the others.

Without the cover-glasses the light can be got many times round, but, when they are on, the faintness of the light which gets through all the surfaces makes it unwise to aim at more excursions, because the definition and visibility of the bands suffers in undue proportion. In fact, the superior sharpness of the twice-round bands perhaps more than compensates for the advantage of the half-as-long-again path belonging to the three-times-round set.

The cover-glasses are, of course, not set absolutely vertical, else the enormous number of reflexions from their surfaces would confuse everything.

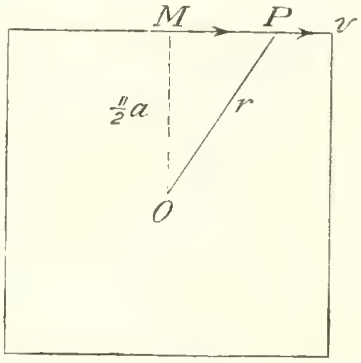
Another way of testing the order of the image in use is to tilt the front mirror so as to broaden the bands by a measured amount, and then to imitate or compensate this by tilting one of the other mirrors. The angle of tilt necessary with one of these will be $n/(n+1)$ of the tilt of the front one.

The three screws against which they each press all have fifty turns to the inch, and hence it is possible to apply this test whenever the eclipse method happens to be inconvenient. For instance, $\frac{1}{10}$th of a revolution of the single screw of the front mirror broadened the bands till one occupied nearly the whole field (with the high-power eye-piece that happened to be on). This meant $\frac{1}{500}$th of an inch advance of one end of a plane of 10 inches base, or an angular tilt of $\frac{1}{5000}$ radian, or 40 seconds of arc. Tilting the back mirror about 20″ gave the same effect, showing that there were twice as many reflexions on it as on the front one, and that the light was therefore going twice round.

General Arithmetic of a Shift.

43. It may be convenient for easy future reference to write down the meaning of any observed reversible shift under given circumstances.

An odd number of reflexions must be used if the light is to be sent many times round, hence triangles and pentagons are excluded. An even number of reflexions has the advantage that it makes the paths of the two half-beams *identical* and not merely parallel (*cf.* fig. 8): but it does not seem readily feasible to get the light round more than once with an even number of reflexions. The square or hexagon are therefore the natural figures for the path of light. Take a square, whose side is a, as the mean path of the light. Then its perpendicular distance from the centre of rotation is $\frac{1}{2}a$; and it is the perpendicular distance which is important, for, since the velocity of light at any point P has to be resolved perpendicular to the radius vector, we get precisely the same tangential component everywhere as exists at the point M.



Let the disks revolve with angular velocity ω, and let the shift of the middle band be x band-widths of a particular wave-length λ. Then, if the light goes n times round each way, with velocity v,

$$\frac{x\lambda}{8na} = \frac{\frac{1}{2}ak\omega}{v}$$

where k is the fraction of the velocity of matter which is imparted to the ether between the disks, the quantity to be determined by observation of x.

Thus

$$\frac{k}{x} = \frac{\lambda v}{4n\omega a^2} = \frac{4 \times 10^5}{n\omega a^2}.$$

The limit of speed of a given material (see § 34) is given by something like

$$\mathrm{T} = \rho v^2 = \rho\left(\omega\frac{a}{\sqrt{2}}\right)^2 = \tfrac{1}{2}\rho\omega^2 a^2,$$

hence the limiting value of k/x, observable by this method, is

$$\frac{4 \times 10^5}{na}\sqrt{\left(\frac{\rho}{2\mathrm{T}}\right)}.$$

Taking 30-ton steel as the material, this gives as the smallest observable value of k/x,

$$\frac{4 \times 10^5}{na} \sqrt{\frac{7}{9 \times 10^9}} = \frac{11 \text{ centims.}}{na}.$$

It is fairly easy to make sure that x is not greater than $\frac{1}{100}$th. In my apparatus a is 2 feet or 60 centims., and the light may go three times round with the cover-glasses on; hence the limiting determination of ether-drag that can be made with it thus arranged is $\frac{1}{1800}$th, unless the setting of the micrometer wires can be relied on more closely than to the $\frac{1}{100}$th of a band, or unless the steel will safely stand more than 30 tons to the square inch.

(Evidently the larger the square the better, and a large enough square might show even the earth's rotation effect, only it is difficult to see how to imitate the effect of stopping and reversing the rotation, at least with the unwieldy size of frame necessary.)

Testing for Cause of Slight Irreversible Shift.

44. The following experiments tend to show (and do distinctly show in my opinion) that such shift as is observed is independent of the width of the bands, and therefore is an absolute shift caused by shake or strain; very likely by a strain caused by a shake, for its effect often dies out slowly.

Spins on 18th March. Light going three times round.

First spin, in direction to help transmitted beam.

Micrometer wires set and moved as before (fig. 12), but only the results of the readings quoted.

Width of yellow band	at rest	114	divisions.
	while revolving 1200 times a minute	159	,,
	at rest again	184	,,
Shift of middle band	shift to the right	6	,,
	return on stopping	5	,,

Found that the front mirror was too near the drum, probably touching it, hence the continuous widening of the band. Moved it back a bit and tightened up the nuts of the semi-transparent plate.

Started again with much narrower bands, and continued spinning and stopping alternately without intermission, all in the same direction.

It may be worth while to quote the actual readings of this set. The wires were set at each stoppage, and likewise during each spin while the speed was kept at 1260 revolutions per minute.

Second spin.

	X-wire.	Vertical wire.
Rest	38½	28
Spinning (at 1260)	32	32½
Rest again	37	26
Spinning again	30	32½
Rest	38	25
Spinning	32½	32
Rest	44	27

To move the vertical wire into coincidence with the X (or *vice versâ*), needed 54 divisions of either micrometer head.

The result of this set may be thus tabulated

	Width of a yellow band.	Shift of the middle band.
Stationary	65½	
		4½ to right.
1260 revolutions	63½	
		6½ return.
Stationary	62	
		6½ to right.
1260 revolutions	61½	
		7½ return.
Stationary	62	
		7 to right.
1260 revolutions	63½	
		5 return.
Stationary	70	
	Average 64	Average 6

Practically, therefore, the absolute shift of the middle band was just the same as in the previous experiment, though the width of the bands was now only about one-third what it then was.

Now reversed the brushes and went on.

Third spin. Reversed. Direction to help reflected beam. Results :—

	Width of yellow band.	Shift of middle band.
Stationary	65	8 to right
1250 revolutions	$63\frac{1}{2}$	7 return
Stationary	$63\frac{1}{2}$	$8\frac{1}{2}$,,
1250 revolutions	63	7 ,,
Stationary	$64\frac{1}{2}$	$8\frac{1}{2}$,,
1250 revolutions	$61\frac{1}{2}$	$7\frac{1}{2}$,,
Stationary	63	
	Average 63·4	Average 7·7

The shift was a trifle greater than before, but so was the tremor. Another shift taken same day gave a shift of 9 and a return of 12. The effect of the tremor seemed increasing.

45. Since such shift as is observed is apparently independent of the width of the bands, it is manifestly well to reduce its apparent significance by having the bands very broad. It might be doubtful how far accuracy of setting could be accomplished with the spider line in the midst of a very broad band. To test this the following observations were made.

Accuracy of Setting of Micrometer Wire.

The bands were broadened, by tilting the back mirror a little, until it took two revolutions of a micrometer head to carry a wire from one to the next. The light was going three times round, and the cover glasses were on. The vertical wire was then carefully set in the centre of the middle band (it is always easy to tell the middle band, even without the colour of the others, by their concertina-like motion to and from it when the corner of the frame is pressed) and the χ was set as near as possible on the yellow of the first band to the left. The position of this colour was not so well defined as when the light only went twice round, especially when the disks were spinning; neither was the middle band quite so clear then as when they were stationary. It is the setting of the vertical wire in the middle band that is really important. Readings of both micrometers being taken, the wires were displaced at random and then re-set and re-read. This was done several times. The following are the results of successive re-settings on the same bands.

	X-wire. (Set and reset in yellow of first band.)	Vertical wire. (Set and reset in centre of middle band.)
Disks stationary	81½	20
	80¼	20
	84	21
	91	21
	83	22
	90	20½
	85	..
	Average 85	Average 20½

The width of this average band was 208½ divisions.

Hence when stationary the error of setting of the vertical wire *each time* did not attain $\frac{1}{100}$th of a band; and the probable error of the average of a series of settings is very small.

Same operation continued, with disks spinning.

	X-wire.	Vertical wire.
Disks revolving 1260 times a minute. (Bands now not so clear. There was too much tremor)	85	26
	90	30
	95	32
	89	34
	89	26
	90	26
	85	27
Average	89	29

The width of this average band was 221 divisions.

Here the error of individual settings of the important wire amounts to $\frac{1}{40}$th of a band in one case, and the probable error is, say, $\frac{1}{70}$th. The error in the average of a series, however, would be less than $\frac{1}{100}$th, even under the above disadvantageous circumstances.

(The average shift caused by this spin was 8·5 divisions, or, say, $\frac{8\cdot5}{215} = \frac{1}{25}$th of a band.)

The effect of the tremor of the whirling machine on the optical part was becoming too conspicuous, and accordingly all the saw cuts and separations between drum and frame had to be examined. A slight contact was discovered.

46. On the 19th March a rather hasty pair of spins were taken, with the following results :—

First, with direction helping the reflected beam.

	X-wire.	Vertical wire.	
At rest	13	78	($\lambda = 169$)
1220 revolutions	5	80	($\lambda = 163$)
At rest	5	81	($\lambda = 164$)

Reversed the brushes, and took a spin in the opposite direction.

Second, direction helping the transmitted beam.

	X-wire.	Vertical wire.	
At rest	13	81	($\lambda = 172$)
1220 revolutions	7	85	($\lambda = 170$)
At rest	—	—	

Something happened that prevented the last readings from being taken. Notice that the yellow band seemed to have slowly returned to its old position in the interval between the two spins. These slow recoveries are frequent. Here the wave-length, or width of band, corresponded to 169 divisions at first, and the first shift was only two divisions; but the shift was spurious, for instead of returning, it went on, or at least stood still. The small shift (4 divisions) in the same direction observed on reversal was also no doubt spurious. I feel sure that I have never observed a genuine reversible shift of the middle band due to rotation.

Now the light was going three times round the frame, and if the ether had been carried round full speed with the disk there would have been a shift of $3\frac{1}{4}$ bands, or 550 divisions at each rotation, or over 1000 divisions in all, whereas not more than two divisions at the outside were seen, and they might readily be spurious.

In saying they were spurious, I mean not that a shift so small as this could not be observed, but that there was amply sufficient tremor to account for it

The entire absence of perceptible shift at about 800 revolutions, before any important tremor has occurred, is to me really the most conclusive fact; and I feel confident that either the ether between the disks is quite unaffected by their motion, or, if affected at all, that it is by something less than the thousandth part. At the same time, so far as rigorous proof is concerned, I should prefer to assert that THE VELOCITY OF LIGHT BETWEEN TWO STEEL PLATES MOVING TOGETHER IN THEIR OWN PLANE AN INCH APART IS NOT INCREASED OR DIMINISHED BY SO MUCH AS $\frac{1}{200}$TH PART OF THEIR VELOCITY.

Air Effect.

47. Of course, there must be an effect due to the air which is whirled with the disks. The index of refraction for air is 1·00029 for yellow light, and so $1 - 1/\mu^2 = {\cdot}00058$. The effect dependent on air amounts, therefore, to $\frac{1}{2000}$th of the speed of the disks, and is smaller than any shift which at present I have been able to observe.

I should like to push the method far enough to detect the air effect.*

A great number of other experiments suggest themselves. It may be objected that the disks were too far apart, or that insufficient time was given for the viscosity of the ether to assert itself, or that the disks had inadequate mass. This last objection is, perhaps, important, and I am proceeding to cope with it, and incidentally with the others, in some new experiments. A positive result could no doubt be obtained by rotating frame and observer instead of the disks.

The apparatus used in the above research was constructed by means of generous aid volunteered by a private friend, to whom I hereby express my grateful thanks.

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* [The effect of centrifugal force on the density of the air between the disks, and the influence of varying density, will be discussed with other matters in a future communication.—July, 1893.]

MORE DETAILED DISCUSSION OF ALLIED PROBLEMS.

Effect of a different Entire Medium upon Aberration.

48. If, instead of air or vacuum, the whole medium contemplated in fig. 4 is changed, the velocity of light is reduced from V to V/μ; wherefore the aberration will change too, unless the telescopic velocity be suitably reduced, or unless the medium is constrained to move in some compensatory manner. If the new medium is just as stationary as the old, and only the receiving telescope, or line of vision, moves, then the aberration angle will become μ times as great as before.

Minute Influence of Motion of Entire Medium on Aberration.

49. But if the medium, instead of being stationary, is drifting in some direction θ, with velocity v, then, perhaps, its motion may have some effect on the aberration. For, though a drifting medium cannot by itself cause aberration, yet it may modify it when otherwise produced. And this we shall find true in the second order of minutiæ. For, in the drifting medium, the rays differ from the wave-normals by the angle ϵ, such that

$$\sin \epsilon = \frac{v}{V} \sin \theta = \alpha \sin \theta,$$

and the velocity of light is

$$V_1 = V \cos \epsilon + v \cos \theta = V (\cos \epsilon + \alpha \cos \theta).$$

Hence an aberration caused by motion of telescope at speed u and angle ϕ, which would naturally be

$$\sin e = \frac{u}{V} \sin \phi = \beta \sin \phi$$

becomes

$$\sin e_1 = \frac{u}{V_1} \sin \phi = \sin e \left(\frac{\cos \epsilon - \alpha \cos \theta}{1 - \alpha^2} \right)$$
$$= \beta \sin \phi - \alpha\beta \sin \phi \cos \theta + \text{higher powers}.$$

The conditions most favourable for observing the second term are when the telescope moves across, and the ether moves along, the ray.

Unless the ethereal velocity near the earth were very great, much greater than the earth's orbital velocity, it would be hopeless to look for this term, as it would require the fixing of a star's position to the five-hundredth of a second, which must be considered quite impossible. The effect is connected with the slight alteration of focal length of the telescope (the difference between sin e and tan e), and may be regarded as a secondary sort of Doppler effect (§ 20).

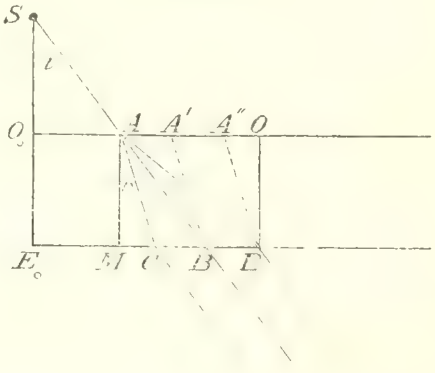
In practice, when the telescope is being carried along by the earth at angle ϕ to the light, it is customary to assume the medium stationary, in which case $\alpha = 0$ and the first term alone exists; or we may assume that the ether is more or less carried along by the earth, in which case $\theta = \phi$, and the second term attains its maximum value, $\frac{1}{2}\alpha\beta$, when the angle is 45°. But even if α is as great as β, the most favourable case, unless the ether has a proper motion of its own, this only means a discrepancy in a star's absolute position of ·001″; so it is hopeless to look for a motion of the medium this way.

50. The experiment of filling a telescope tube with water as suggested by Boscovich, and tried by Airy, and more exactly by Hoek,* aims at a motion quite different from that above contemplated. It aims at moving a portion only of the ether in a partitioned-off region of space. It is easy to show that on Fresnel's theory no different aberrational effect can be thus observed than is observed with an air telescope; in fact, Fresnel himself, in his original letter to Arago (§ 4) contemplated this experiment, and predicted its negative result, but it may be instructive to enter on its consideration in a geometrical fashion.

Effect of Motion combined with Change of a Portion of the Medium.

So long as the whole medium is changed or moved, we have seen what effect there is on aberration. Motion has no effect on it, it drifts the wave-normals but cannot affect the rays; increased μ without motion increases it. But if the ray has to pass through a bounding surface, and if the change or motion occurs only on one side of that surface, then circumstances are different.

Fig. 13.



If, for instance, the source S sends a ray SA, which would have gone to B, a change of medium may carry it to C, while a drift of the medium, carrying A to A′ and B to E, will slant the ray along AE (fig. 13).

It is, therefore, just possible for a drift to neutralize a refraction, and to let a ray enter a dense medium without bending. To this end the drift must equal CB or AA′, and the ray will then be straight.

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* 'Archives Néerlandaises' (1868), vol. 3 p. 180.

Now,

$$\frac{\text{CB}}{\text{AC}} = \frac{\sin(i-r)}{\cos i} = \tan i \cos r - \sin r = \frac{v_1}{\text{V}/\mu},$$

where v_1 is the necessary velocity of drift; so

$$v_1 = \frac{\text{V}}{\mu} \sin i \left(\frac{\cos r}{\cos i} - \frac{1}{\mu}\right).$$

The bending will be usual or unusual according as v_1 is less or greater than this. Ordinarily, of course it is far less.

51. But now suppose the obliquity i has been caused by the aberration necessary to bring a ray to a telescope (moving with velocity u) which except for motion would be looking straight at source; for instance, the telescope whose position was O_0E_0 when the light started from S and has moved to AM when light has reached A; in other words, let i be an *aberration angle;* then

$$\frac{\text{MB}}{\text{AB}} = \frac{u}{\text{V}} = \sin i.$$

But, when a dense medium is inserted in the telescope, or say between the two planes, the time required for the shorter light journey, AC, is longer, and the telescope may get carried as far as OE, where

$$\frac{\text{ME}}{\text{AC}} = \frac{u}{\text{V}/\mu}.$$

To bring the ray to the eye-piece at the right moment, C must drift to E, and the Huyghenian centre A to A″, in the same time. So if v_0 is the ethereal velocity able to undo the effect of the dense medium, and to leave the aberration what it was,

$$\frac{\text{CE}}{\text{AC}} = \frac{v_0}{\text{V}/\mu}.$$

Hence

$$\frac{\text{MC}}{\text{AC}} = \frac{u - v_0}{\text{V}/\mu} = \sin r = \frac{\sin i}{\mu} = \frac{u}{\mu \text{V}}.$$

Therefore

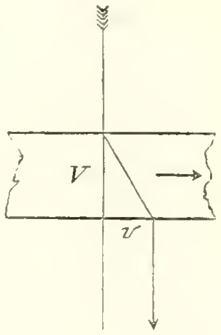
$$\mu^2 (u - v_0) = u$$

or

$$v_0 = \left(1 - \frac{1}{\mu^2}\right) u.$$

If this condition is satisfied, the observed aberration caused by any motion of a telescope will be independent of the medium inside it, although the course of a ray through it is really altered [viz., from AB to AE. The aberration angle in vacuo is MAB; in the medium is A″EO. The diminished velocity of light is exactly compensated by a virtually diminished speed of telescope with respect to the ether inside it; and so a steadily moving telescope sighted on a star can remain parallel to itself, with whatever medium it may be filled.—July, 1893.]

52 But, by moving a portion of medium *relatively to the observer*, say by spinning a glass disk, and looking through it axially near its circumference, where its velocity is u, we are looking through a moving stratum of thickness z, and a parallel shift may in that case be experienced.



The angle is

$$\frac{v}{\mathrm{V}/\mu} = \frac{(1 - 1/\mu^2)\,u}{\mathrm{V}/\mu}, \text{ or for glass about } \frac{\mu u}{2\mathrm{V}};$$

and the shift is z times this angle, viz.,

$$s = \frac{\mu z u}{2\mathrm{V}}.$$

To give a shift of 1 micron = a thousandth of a millimetre = 10^{-4} centims., with a thickness $z = 10$ centims., would require a velocity $u = 2\mathrm{V}s/\mu z = (4 \times 10^{10} \times 10^{-4})/10 = 4 \times 10^5$ centims. per second = 4 kiloms. per second. My machine, at 3000 revolutions a minute, 50 a second, gives a peripheral speed of $50 \times 3 = 150$ metres per second, so the thickness of glass needed to give a shift of 1 micron is $z = 2\mathrm{V}s/\mu u = (4 \times 10^{10} \times 10^{-4})/15000 = 4000/15 = 3$ metres. To-and-fro reflexion may be used to diminish the required thickness.

More detailed discussion of Doppler Effects.

53. There is rather a nice point to be considered in connexion with change (2), § 9, viz., what the pitch, as perceived, really depends on. The coarse statement of examination candidates that it depends on wave-length, or on the frequency of vibration of the source, is of course not true; it depends on the frequency of disturbance reaching the receiver. This fact is suggested by listening or looking through a different medium, wherein the wave-length is quite different; though, indeed, it must be admitted that the medium in ear or eye cannot be changed. It is proved (for the case of sound at

least) by travelling towards the source, when the observed rise of pitch *must* be caused by increased frequency of arrival, the wave-length remaining unaltered.

But when we consider the effect observed in a spectroscope, there might possibly be a difference according as its essential part was a grating or a prism.

For it may be argued that a grating, consisting as it does of a set of apertures of fixed width, *must* deviate and disperse in proportion to wave-length; and hence that if a grating be supplied with crowded waves, either by holding it to an approaching source, or by immersing it in a denser medium, or in a medium flowing from it towards source, it must act as if coarser relatively to the waves, and so deviate and disperse them less.

But although this is a simple and plausible statement it is only half the truth. We had better examine the problem particularly (§ 56), for it is a curious mixture of Doppler effect and aberration, but at present it will suffice to say:

If θ is the deviation caused by a given grating for a given fixed source of frequency 1/T, so that

$$\frac{\mathrm{B}}{\mathrm{N}}\sin\theta = \mathrm{VT},$$

then if the source be approaching at the rate v, the time-interval between successive like phases is diminished in the proportion $(\mathrm{V}-v)/\mathrm{V}$; and accordingly

$$s\sin\theta' = \mathrm{V}(\mathrm{T}-t) = (\mathrm{V}-v)\,\mathrm{T}.$$

If it be the grating that is advancing towards a fixed source, the time interval between the arrival of like phases is likewise diminished, but in the ratio $\mathrm{V}/(\mathrm{V}+v)$; so that

$$s\sin\theta'' = \frac{\mathrm{V}^2}{\mathrm{V}+v}\,\mathrm{T}.$$

It is noteworthy that between θ' and θ'' there is a minute difference of the second order of aberration magnitude;

$$\theta''-\theta' = \frac{v^2}{\mathrm{V}^2}\tan\theta.$$

If the grating be plunged into a different medium, the velocity of advance is changed, and

$$s\sin\theta''' = \frac{\mathrm{V}}{\mu}\,\mathrm{T}.$$

Lastly, if both source and grating are stationary, but the medium flowing from one to the other, or (what is the same thing) if source and grating are moving at the same pace, chasing each other through a stationary medium, the velocity and the wave-length are affected together, and

$$s\sin\theta'''' = (\mathrm{V}+v)\,\mathrm{T}' = (\mathrm{V}+v)\frac{\mathrm{V}}{\mathrm{V}+v}\,\mathrm{T} = s\sin\theta.$$

No Doppler effect, therefore, is produced by a stream of medium flowing past source and receiver if relatively fixed, *i.e.*, if they be moving together through a stationary medium.

We may in short summarise thus :—

Source approaching shortens waves,
Receiver approaching alters virtual velocity,
Medium flowing alters wave-length and velocity together, in an exactly compensatory manner.

Steady Motion of Medium cannot cause any Doppler Effect.

54. Before abandoning the present consideration of the Doppler effect, let us distinctly assure ourselves of the important fact that no steady motion of the medium can change the pitch even infinitesimally, unless source and receiver are moving relatively to each other. Let source recede with velocity v, then the wave-lengths approaching us at their ordinary velocity V are longer than usual,

$$\lambda' = \frac{\mathrm{V} + v}{\mathrm{V}} \lambda.$$

Let receiver approach with same velocity, then it sweeps up per second a number of waves

$$n' = \frac{\mathrm{V} + v}{\lambda'} = \frac{\mathrm{V}}{\lambda} = n.$$

So, without relative motion of source and receiver, there is no Doppler effect, however small.

But the easiest way of assuring ourselves of the impotency of a steady wind on pitch, is to remember that such a wind cannot bring waves at a greater frequency than they are emitted from the source. Gusts will cause wailing, but a *steady* wind has no effect on pitch.

This is true also on a corpuscular theory, though for a slightly different reason. For, consider a machine-gun receding and firing at regular time intervals, it will be seen that, while the distance between the bullets is the same as if it were stationary the speed with which they travel is $\mathrm{V} - v$; and, if a target is chasing the gun at the same pace v, the number caught in a second will be

$$n' = \frac{(\mathrm{V} - v) + v}{\lambda} = n.$$

Effect of Moving Medium on Doppler Effect.

55. But finally examine if wind has any effect on pitch when source and receiver are *not* moving at the same pace, *i.e.*, when a Doppler effect certainly exists. Let source recede with speed u, receiver advance with speed v, and let the medium flow from source towards receiver at speed w (or in any direction at speed $w \sec \theta$; the argument is the same):—

Then speed of wave is

$$V + w.$$

Length of each is

$$\lambda' = \frac{V + w + u}{V} \lambda = \frac{V + w + u}{n}.$$

The number caught per second is

$$n' = \frac{V + w + v}{\lambda'} = \frac{V + w + v}{V + w + u} n,$$

or

$$\frac{dn}{n} = \frac{u - v}{V + w + v}.$$

The medium velocity *does*, therefore, enter into the expression for the ordinary Doppler effect, though in a very subordinate manner, by affecting the velocity of light. It cannot *cause* the effect, but it can *modify* it when otherwise produced.

The simplest plan of detecting this effect of a moving medium, would be by some direct observation of the velocity of light itself; either simultaneously from stars in two opposite directions, or in a given direction at six months' interval. No terrestrial object must be used as source, because it would be moving at practically the same rate as receiver. Hence, for a six-months' experiment, the Jupiter's satellites method of determining the velocity of light would seem the best plan; and, if observations could be exact enough, one could thus get

$$\frac{V - (w + v)}{V + (w + v)},$$

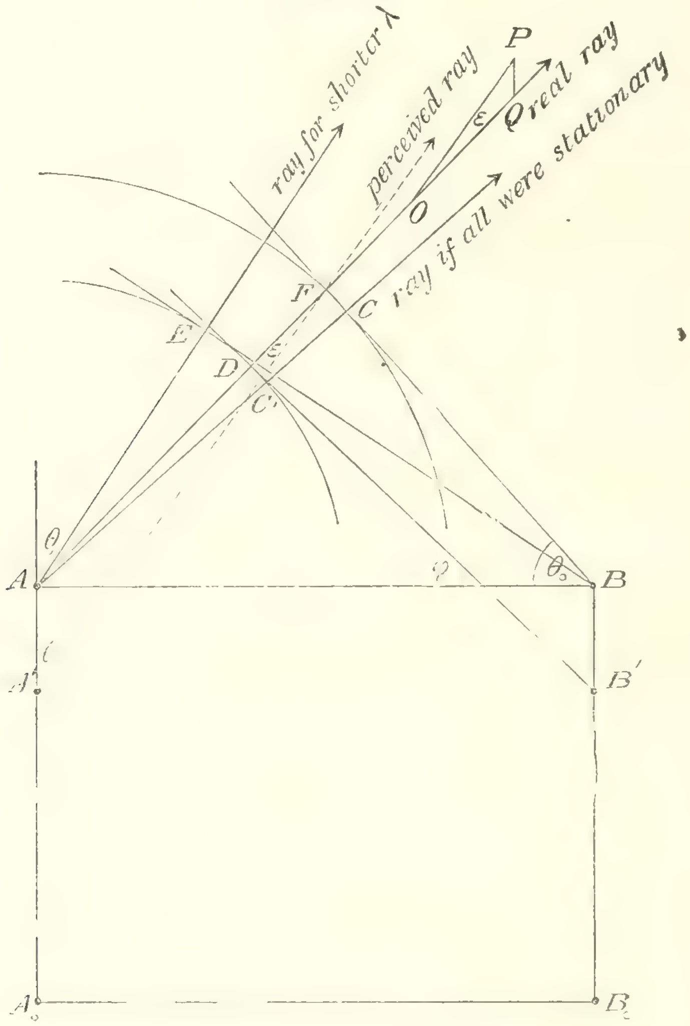
and so determine w the unknown speed of the ether past the solar system.

A simultaneous aberrational method, such as comparing the aberration of two stars 180° apart, will not work well; for if they be at 90° to the apex of earth's motion there will be no discrepancy, while if one be toward that apex and the other away from it there will be no aberration.

Effect of Motion on Diffraction Grating.

56. To avoid any confusion about motion relatively to source, and the alteration of wave-length thus caused, it will be best to abandon our usual convenient plan of letting the ether move, and attend explicitly to the motion of the grating with its telescope and observer; all else being stationary.

Fig. 14.



Details of Doppler effect with moving grating, AB, and telescope OP.

Consider a plane wave, A_0B_0, advancing through a stationary medium with ordinary velocity, V, towards a stationary grating. Let $A_0A = AC = \lambda_0$ be an ordinary wave-length, while $AB = s$ is the width of one complete element of the grating; then BC is a wave-front, and AC is the ray, inclined to the normal to grating at angle $CBA = \theta_0$.

Now let the grating advance with velocity v to meet the wave a distance $BB' = AA' = CC'$ in one period; the disturbance B_0 will only have to go as far as B', and the disturbance A only as far as C'; so drawing a tangent B'D to the sphere of radius AC', we get the wave-front appropriate to moving grating; AD is the ray, inclined to the normal to grating at angle $DB'A' = \phi$.

Now θ_0 and ϕ are very nearly equal; showing that diffraction really does depend on wave-length simply, in spite of motion of grating, so far as minutiæ of the first order are concerned.

But then this direction, ϕ, will not be the direction appreciated by the observer; for the motion of his telescope will cause ordinary aberration, since his motion is partially across the diffracted rays.

Not to confuse the figure, I indicate the telescope OP further along the ray. While the light is travelling along it its eyepiece will have time to move to Q, such that

$$\frac{\mathrm{PQ}}{\mathrm{OQ}} = \frac{v}{\mathrm{V}} = \frac{\mathrm{AA}'}{\mathrm{AF}} = \alpha.$$

Hence, A′F is parallel to the axis of the telescope which receives the ray, and may be called the apparent or perceived ray. The angle at which it is inclined to the grating-normal may be called θ.

Now θ is less than θ_0, and is very nearly the same as if wave-length had been really shortened to AD, instead of AC.

Draw BE a tangent to the AD circle, and we get the wave-front appropriate to this shortened wave and a stationary grating; while AE is the ray belonging thereto, the inclination of which to the grating-normal we may call θ_1. [$\sin\theta_1 = (1 - \alpha)\sin\theta_0$.]

Now, plainly, AE and A′F are very nearly parallel, but not quite; there is a second-order difference between θ and θ_1, which may be readily calculated.

Perhaps the simplest way of displaying the result is to introduce the aberration angle POQ $= \epsilon$ (such that $\sin\epsilon = \alpha\sin\theta$) and to write

$$\sin\theta_1 = \sin\phi - \alpha\cos\phi\sin\theta_0;$$

whereas

$$\sin\theta = \sin\phi\cos\epsilon - \alpha\cos\phi\sin\theta.$$

(Or one might write $\cot\theta = \cot\phi + \alpha\operatorname{cosec}\phi$.)

The difference between the apparent ray and the shortened wave-length ray is approximately

$$(\theta_1 - \theta)\cos\theta = \sin\phi\,(1 - \cos\epsilon) - \alpha^2\cos\phi\sin\theta_0,$$

or

$$\theta - \theta_1 = \alpha^2\tan\theta\,(\cos\phi - \tfrac{1}{2}\sin\theta\sin\phi),$$

and is probably quite too small to be detected.

The point of the whole thing is that a grating has *the same real effect whether moving or stationary*, but that the motion of the observing telescope causes an *aberration* which necessitates very nearly the same alteration of its direction as if the waves were really shortened in simple proportion to the motion. The Doppler effect caused by motion of observer is, therefore, essentially a case of common aberration.

Now, as there is no hypothesis or difficulty whatever about the aberrational effect of a moving telescope, all that has been said of a grating applies, at least broadly, to a prism.

Effect of Motion on the Dispersion of a Prism.

57. The deviating power of a prism depends on its relative refractive index with respect to the surrounding medium; hence, in this sense, its deviation is certainly affected by the length of the waves with which it is supplied.

Its dispersive power, however, is not a superficial, but a deep-seated, phenomenon, depending on its internal structure; and, since no variation of outside medium can affect *internal* wave-length, the dispersive power of a prism may be assumed constant for given waves. It follows that the dispersion caused by a given prism, immersed in different media, is simply proportional to the mean deviation in each case for given kind of light.

But what about the effect of motion?

If only we can assume that the prism interferes with the ether as little as the grating has been supposed to do, then all that has been said of the grating remains true of the prism. If we supposed the prism to modify the free ether inside it, we should have to modify this statement. On the hypothesis of FRESNEL, however, the free ether is not supposed to be affected; and experiments directed to test the matter, by ascertaining the effect of prism chasing a source at the same speed, have resulted in finding this effect zero, in accordance with the above statement. Hence it must be allowed that a Doppler effect observed by a prism depends *really* on wave-length, but *apparently* on frequency, just as is the case with a grating.

It must be noticed that the observation of a Doppler effect by a prism depends entirely on dispersion; *i.e.*, on waves of different length being affected differently. But prisms can be constructed whose dispersion is corrected and neutralized. Such achromatic prisms, if perfectly achromatic, will treat waves of all sizes alike; and, accordingly, the shortening of the waves from a moving source will not produce any effect. Achromatic prisms will behave to terrestrial and to extra-terrestrial sources, *i.e.*, to relatively stationary and relatively moving sources, in the same way.

ARAGO used an achromatic prism on a star when he showed that refractive index was unaffected by motion of the earth.[*] In criticising ARAGO's experiment adversely, MASCART forgets this, and thinks he ought to have perceived a Doppler effect. MASCART used a terrestrial source and an ordinary dispersive prism, when he experienced the same negative result. MAXWELL sent light both ways through his prism, and therefore neutralized all refraction, except what was entirely caused by motion, when he proved that this latter was *nil.*

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[*] BABINET, 'Comptes Rendus,' vol. 9, p. 774 (1839). ARAGO, 'Ann. de Chim. et de Phys.' (3), vol. 37, p. 180 (1853). MAXWELL, 'Phil. Trans.' (1868), vol. 158, p. 532; also 'Ency. Brit.' article "Ether." HOEK, 'Archives Néerlandaises' (1869), vol. 4, p. 443. MASCART, 'Annales de l'Ecole Normale,' vols. 1 and 3 (1872 and 1874); Professor MASCART here describes a large number of negative experiments which he has made as to the effect of motion on most of the phenomena of optics.

Babinet, Hoek, and Mascart, all tried a modified form of the same experiment in an interferential manner, and likewise got a negative result.

58. If we wish to follow out the ether motion through a prism into greater detail we can say :—Let the prism advance with velocity v to meet the waves, and let the ether in it be carried forward with velocity kv, then the virtual velocity of the light towards the prism is $V + v$, and inside the prism is

$$\frac{V+v}{\mu} - kv\,;$$

hence, on ordinary notions of refraction the new index will be

$$\mu' = \frac{(V+v)\,\mu}{(V+v) - k\mu v}\,;$$

or

$$\frac{\mu}{\mu'} = 1 - k\mu\alpha,$$

where

$$\alpha = v/(V+v) = d\lambda/\lambda,$$

or

$$d\mu = k\mu^2\alpha,$$

which, on Fresnel's hypothesis, equals $(\mu^2 - 1)\,\alpha$.

This seems to give a sort of theory of dispersion for the case :—

$$\frac{d\mu}{\mu^2 - 1} = \frac{d\lambda}{\lambda},$$

or

$$\frac{\mu - 1}{\mu + 1} = A\lambda^2,$$

or

$$\mu = \frac{c^2 + \lambda^2}{c^2 - \lambda^2}.$$

Interference Effects with Rays at Different Angles to Ether Drift. Effects of Normal Reflexion. Further Discussion of the Theory of Mr. Michelson's *Experiment.*

59. The experiment of Michelson, already referred to in § 25, has to do with the effect of a plane mirror sending a ray straight back upon itself. Consider the aspect of the mirror necessary to do this ; first, for the case of a moving source in a stationary medium (fig. 15).

Let S_1 be initial position of the source, throwing off a wave-front to M, and itself moving to S, so that

$$S_1S = vT_1\,; \text{ and } S_1M = r_1 = VT_1.$$

Let SM now be reflected at M so as to travel to S_2, and reach it the same time as source, then

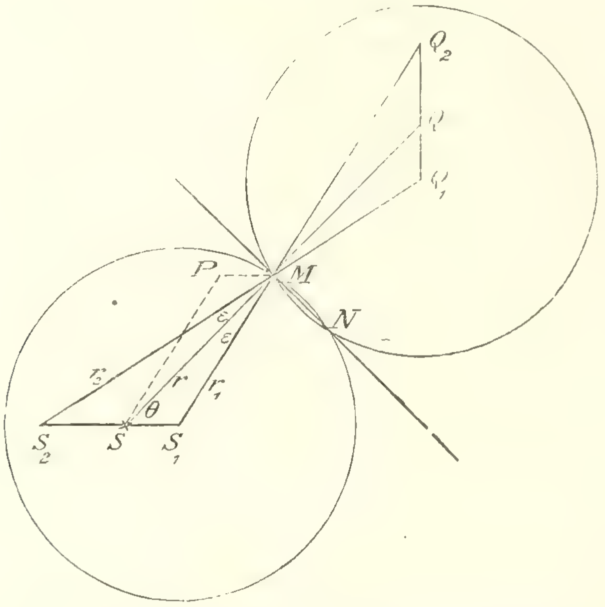
$$SS_2 = vT_2\,; \text{ and } MS_2 = r_2 = VT_2.$$

Hence.

$$\frac{S_1S}{SS_2} = \frac{T_1}{T_2} = \frac{r_1}{r_2},$$

wherefore SM, bisects the angle S_1MS_2, or the angles of incidence and reflexion are precisely equal, and the required mirror is normal to SM, but is not tangential to the wave-front.

Fig. 15.



Details of normal reflexion with moving source or moving medium.

Let Q_1, Q, Q_2 be the images in this mirror of S_1, S, S_2, and with centre Q_1 construct a circle of radius QM cutting the wave-front in N; MN is the mirror able to throw a light ray back on the moving source.

A stationary observing telescope will observe the source along Q_2M; one moving with the source will observe it along QM, that is, in its true direction at moment of vision. The colour will change by the amount log r_1/r, as already said; and, as for any possible interference effect, the fringes will shift by an amount depending on

$$\frac{T_1 + T_2}{2T} = \frac{r_1 + r_2}{2r} = \frac{\cos\epsilon}{1 - \alpha^2} = \frac{\sqrt{(1 - \alpha^2 \sin^2\theta)}}{1 - \alpha^2},$$

which gives very approximately

$$T_1 + T_2 - 2T \approx \alpha^2 T\,(1 + \cos^2\theta).$$

There is, therefore, always a lag of phase caused by the motion, which cannot be made negative, or even zero, but which is a minimum when the motion is across the line of light, and a maximum when along it; being, indeed, twice as great for motion along as it is for motion across. Supplementary angles give the same effect.

One may express the fact by saying that the virtual distance the light has to go is S_1Q_2, or S_2Q_1, instead of SQ.

Case when the Mirror Moves too.

60. It is observed that in this investigation the mirror has been supposed stationary with respect to the medium, it is therefore possible, if the mirror is moving at the same rate as the source, *e.g.*, if they were both fixed and the medium streaming past both, that the circumstances may be a little different; because since the whole of the wave-front does not strike quite simultaneously, there may be time for some effect to occur during its period of contact, short though of course it is. Not even for normal incidence is the time of impact of a finite portion of a wave infinitesimal; for even when the source is infinitely distant (or when a collimating lens is used) it has to be remembered that the waves are not normal to the rays in a moving medium, and that, accordingly, when the incidence of the ray is normal and the medium movement not normal, the wave is inclined to the surface, a minute, but possibly important, angle.

But in MICHELSON'S arrangement the ray is not exactly normal on the tangentially moving mirror, but is inclined so that the mirror is precisely parallel to the wave-front; and so the time of contact is nothing on either mirror.

The statement of theory, therefore, proceeds as follows, without apparent error.

Let S be and remain the position of the source, and let a mirror MN (fig. 15) be arranged normal to SM, so as to send a ray back upon itself, when everything is stationary, in time 2T.

It is required to find if any tilt must be given to the mirror to send the ray back upon itself when the medium is moving, and whether a different time will be taken in the journey.

While light travels from S to M the wave-front's centre has drifted to S_1, and, accordingly, it strikes the mirror obliquely and is reflected as if coming from Q_1; hence it would travel towards S_2 but for the drift. The drift will carry it precisely back to S if $SS_2 = vT_2$, T_2 being the time of the return journey; just as $SS_1 = vT_1$, when T_1 is the time of the outward journey.

Hence, no tilt whatever is required by the mirror, but it reflects the light back upon itself just as when the medium was stationary, and the distance really travelled is exactly the same as it was, viz., $2r$. The velocity of the light is, however, different on the out and in journeys, for

$$V_1 = V \cos \epsilon - v \cos \theta,$$

$$V_2 = V \cos \epsilon + v \cos \theta,$$

so

$$T_1 + T_2 = \frac{r}{V_1} + \frac{r}{V_2} = \frac{2T \cos \epsilon}{1 - \alpha^2} = 2T \frac{\sqrt{(1 - \alpha^2 \sin^2 \theta)}}{1 - \alpha^2},$$

as before (§ 59).

61. In the actual experiment, as performed for instance by MICHELSON, it is natural to use a collimator and plane waves, and since his null result is very surprising and remarkable, it may be as well to examine whether the introduction of the lens pro-

duces any disturbance. At first I thought the lens and glass slabs used by him might possibly be the cause of his failure to get any result, because the ray across the motion travels through the glass obliquely, while along the motion it travels normally. But a little consideration shows that both along and across the motion the effect of the glass would be to increase the lag in a simply proportional manner to the previous lag. And calculation gives as the time of the journey, when a total thickness z of glass is interposed, using FRESNEL's theory that the speed of the ether inside the glass is $1/\mu^2$ of what it is outside,

$$T_1 = \frac{r - z}{V \cos \epsilon + v \cos \theta} + \frac{z}{\frac{V}{\mu} \cos \epsilon + \frac{v}{\mu^2} \cos \theta},$$

and $T_2 =$ corresponding expression with v negative.

So

$$T_1 + T_2 = \frac{2 (r - z) \cos \epsilon}{V (1 - \alpha^2)} + \frac{2 \mu z \cos \epsilon}{V \left(1 - \frac{\alpha^2}{\mu^2}\right)} = \frac{2T \cos \epsilon}{1 - \alpha^2} + \frac{2 \mu z \cos \epsilon}{1 - \alpha^2} \left(\frac{1}{1 - \frac{\alpha^2}{\mu^2}} - \frac{1}{1 - \alpha^2}\right)$$

where

$$T = \frac{r + (\mu - 1) z}{V};$$

wherefore the effect of introducing glass is to increase the lag, but not quite so much as by the equivalent of the extra distance thus virtually added, the second term in the above expression being negative; but the diminution is independent of direction, except when fourth powers of aberration magnitude are attended to. Neglecting these, the effect of the glass is merely to cause, in addition to the lag naturally to be expected, an extra term, independent of direction, of this value:

$$-\frac{2 \mu z}{V} \left(1 - \frac{1}{\mu^2}\right) \alpha^2.$$

MICHELSON'S *Interference Experiment in a Different Medium.*

62. Indeed, the simplest plan would be to consider the effect of immersing the whole arrangement in a different medium. It is merely to change the light velocity V to V/μ, and its mechanical velocity v to v/μ^2 the ethereal velocity inside it. Consequently α becomes α/μ.

The aberration angle ϵ changes to ϵ', such that

$$\sin \epsilon' = \frac{\alpha}{\mu} \sin \theta,$$

and the lag

$$\frac{2r\cos\epsilon}{\mathrm{V}\,(1-\alpha^2)} \quad \text{becomes} \quad \frac{2\mu r\cos\epsilon'}{\mathrm{V}\,(1-\alpha^2/\mu^2)},$$

approximately μ times as great as before. But although this is the case, *the extra lag caused by motion* is not so great inside the medium as it was *in vacuo*, for

$$\frac{2\mu r\cos\epsilon'}{\mathrm{V}\left(1-\dfrac{\alpha^2}{\mu^2}\right)} - \frac{2r\cos\epsilon}{\mathrm{V}\,(1-\alpha^2)} = \frac{2r\cos\epsilon}{\mathrm{V}\,(1-\alpha^2)}\left(\frac{\mu\cos\epsilon'\,(1-\alpha^2)}{\left(1-\dfrac{\alpha^2}{\mu^2}\right)\cos\epsilon} - 1\right),$$

or, approximately,

$$= \frac{2\,(\mu-1)\,r\cos\epsilon}{\mathrm{V}\,(1-\alpha^2)} - \frac{2\mu r}{\mathrm{V}}\left(1-\frac{1}{\mu^2}\right)\alpha^2.$$

[The conclusion here is that whatever may be the effect of a dense medium it is independent of θ, and therefore can have nothing to say to MICHELSON's experiment, which entirely depends on a difference between what can be observed with $\theta = 0$ and $\theta = 90°$.—July, 1893.]

The Laws of Reflexion and Refraction as modified by Motion.

63. It is necessary now to enter on the somewhat thorny question as to the effect of motion upon the laws of reflexion and refraction. FRESNEL by considering some special cases satisfied himself that no discrepancy need be expected on his version of the undulatory theory; and Sir GEORGE STOKES, examining the question in a more general manner in 1846, proved that, at least as far as the first order of minutiæ, the laws were obeyed in spite of any relative motion between mirror and medium (motion of source has obviously nothing to do with it, unless it affects the shape of the incident wave). And the long continued use of artificial horizons by astronomers shows that there has been no practical doubt on the subject, at least as far as reflexion is concerned.

But these statements do not by any means exhaust the subject; the law of reflexion is *not* precisely obeyed in a moving medium, and recently MICHELSON has proposed to utilize the theoretical error (which has never yet been practically realized) as a fresh method of attacking the problem of the relative motion of the ether and the earth.

I propose, therefore, to enter upon it, and I must confess that though the results are easily stated, they have given me much trouble to be sure of, and I have found a good many mare's nests by the way.

The reasoning for reflexion and refraction is much the same, and I attend more pronouncedly to reflexion because without assuming FRESNEL's theory as to the motion of ether inside dense matter we have no guide to what shall happen in refraction;

and although the theory has been to a certain extent, and with fairly high accuracy, verified, yet it can hardly be yet said to have a secure rational basis.

In a drifting medium we must draw a clear distinction between waves and rays; the laws obeyed by one need not be obeyed by the other, for they are inclined to each other, and may become differently inclined after reflexion or refraction.

Now it is pretty plain that if motion is to have any effect upon these aberration angles, the rays must be differently inclined to the direction of drift; and on the other hand, if motion is to affect the reflexion of waves, that it must act during the period of contact of a wave with the reflecting surface; so that if a wave comes down plumb it will rebound as it comes, because its time of contact is then infinitesimal and no finite motion could cause any disturbance. But even in this case of normal incidence the law of reflexion need not be obeyed for rays, for they are not normal to the waves, and will be differently inclined to the direction of drift, unless indeed the latter be either normal or tangential.

64. The following are statements which I will afterwards justify :—

(1) The planes of incidence and reflexion are always the same.

(2) The angles of incidence and reflexion, as measured between rays and normal to surface, in general differ.

(3) If the mirror is stationary and medium moving, they differ by a quantity depending principally on the square of aberration magnitude, *i.e.*, by one part in a hundred million, and a stationary telescope would be able to observe the effect, if it were delicate enough.

(4) If the medium is stationary and mirror moving, the angles differ by a quantity depending principally on the first power of aberration magnitude, *i.e.*, by one part in ten thousand, but a telescope moving with the mirror will not be able to observe this large effect; for the apparent (or commonplace) aberration caused by the motion of the receiver will obliterate the odd powers and leave only the even powers of the aberration, so that the observed effect should be the same as in case 3.

(5) As regards the angles which the reflected and incident waves make with the surface, *i.e.*, as to the obedience to the law of reflexion shown by *waves* instead of by *rays*, in case 3 the angles differ by an amount depending on the first order of aberration, but in case 4 they only differ by the square of this quantity.

(6) At grazing incidence the ordinary laws are accurately obeyed by the rays as observed, and at normal incidence the error is a maximum.

(7) The ordinary laws are obeyed whenever the direction of motion is tangential or normal to the mirror.

(8) In general the shape of the incident waves is not precisely preserved after reflexion, so that, when spherical waves impinge on a mirror in a moving medium, the reflected waves from a plane mirror diverge from a sort of caustic instead of from a point, and the position of the image varies (but almost infinitesimally) with the position of the observer. In other words, such a mirror acts to a parallel beam as if

slightly tilted, to a divergent beam as if slightly curved. But either effect, as observable in the result, is almost hopelessly small.

(9) Similar statements are true for refraction.

65. In considering a plane mirror in a drifting medium it is very tempting to image the direction of drift of successive wave centres (fig. 4); in which case everything will be symmetrical, and the law of reflexion will be obeyed altogether, by both waves and rays, in the simplest possible manner. But a little thought shows that this is illegitimate, for it would make the reflected waves assisted in their progress by the reflected drift just as much as the incident waves are assisted; whereas they are really travelling in the teeth of the wind, their progress being impeded and their wave-length shortened just as much as the incident waves are helped and lengthened (or of course *vice versâ*). Plainly the drift is not reflected, but must be supposed to act on the waves emitted by the image exactly as it acts on the waves emitted by the source.

Another tempting thing to do is to start a system of waves from source and its ordinary image simultaneously, both subject to precisely the same drift velocity, one being the incident, the other the reflected system. But applying this, and taking a pair of waves intersecting at any one point of the mirror, it will be found they have not travelled the same distance to get there, nor have they taken the same time, and the drift of their centres has been different. Moreover, they do not intersect at a second point of the same wave, and, in fact, the system behind the mirror is not in any sense the image of the front set.

The really essential thing is that the phase of the reflected wave shall be identical with that of its incident exciter at the point of contact with the mirror, and accordingly that the time of virtual journey from any point to be considered as an image is to be equal to the time of journey from the corresponding point of the source. Nothing less direct or more geometrical than this seems satisfactory, so it had better be applied in its usual Huyghenian baldness. At the same time a little caution is necessary in using HUYGHENS' construction in a moving medium, for the centre of the elementary waves does not remain at the point of incidence, but drifts away, as in fig. 4, and the construction has to remember this, or it will go wrong.

Laws of Reflexion and Refraction in a Drifting Medium.

66. Since the direction of drift need not be in the plane of incidence, it will be convenient to resolve it into two components, respectively in and perpendicular to that plane, and consider them separately.

Component of Drift Perpendicular to Plane of Incidence.

The perpendicular component is very easily disposed of, as was shown by SIR GEORGE STOKES.* For looking down the normal to the mirror we shall see the incident and reflected rays AO and OB, incident at the point O, affected by the drift in such a way that their direction is A′O and OB′ respectively; but they are still both in a plane normal to the surface, and they still make equal angles with the surface.

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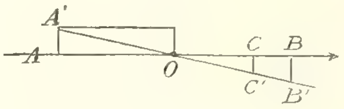
* 'Math. and Phys. Papers,' vol. 1, p. 144.

Fig. 16.



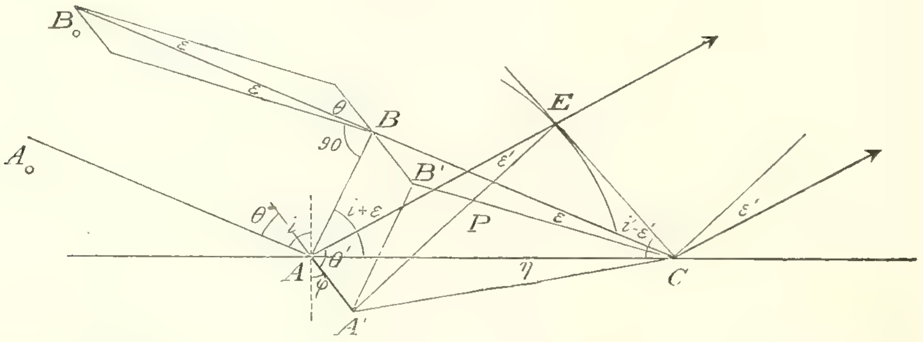
And as regards refraction, an equal length of the refracted ray will be foreshortened into $OC = OB/\mu$; and the drift inside the medium being, according to **Fresnel's** theory, BB'/μ^2, will in the longer time available carry C to C′, such that OC′B′ remains a straight line; accordingly, the plane of the refracted ray is still the plane of incidence. Hence the perpendicular component of the drift is wholly ineffective, and we have only to consider the component in the plane of incidence; call it v.

Component of Drift in Plane of Incidence.

Consider a plane wave A_0B_0 travelling, with an aberration angle ϵ between its normal and its direction of advance caused by a drift in direction AA′, towards a plane mirror, at a pace compounded of the speed of light and of the drift, viz., $V \cos \epsilon + v \cos \theta$.

Let incident and reflected rays make angles i and i' respectively with the mirror-normal, and θ and θ' with the line of drift, so that if ϕ is the inclination of the drift direction to the mirror-normal, $\theta = i - \phi$, $\quad \pi - \theta' = i' + \phi$.

Fig. 17.



The incident and reflected waves will be inclined to the surface by the angles $i + \epsilon$, $i' - \epsilon'$, respectively.

When the wave AB strikes the mirror, A becomes a source of radiation, and B travels on to C, with a velocity compounded of BB′ and B′C. By the time B arrives at C the radiation from A will have spread a distance equal to B′C through the medium, and the centre of a wave with this radius will have had time to drift with the moving medium to A′, a distance equal to BB′; hence, drawing the semicircle A′B′EC, and choosing on it a point E, such that $A'E = B'C$, we have the direction of the reflected ray, viz., AE, and of the reflected wave CE.

67. It is easy to see that the triangle PA′C is isosceles, and accordingly that the angle A′CA is equal to half the difference of the inclinations of incident and reflected waves to the mirror surface ; *i.e.*, calling this angle η,

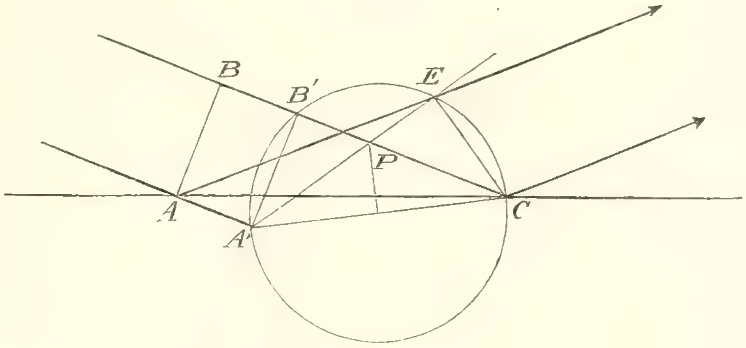
$$2\eta = (i + \epsilon) - (i' - \epsilon'),$$

or

$$\eta = \frac{\epsilon + \epsilon'}{2} + \frac{i - i'}{2},$$

hence the *wave* is reflected precisely as if the mirror were rotated through the angle η and there were no drift ; the angle of virtual rotation being very approximately the mean of the aberration angles.

Fig. 18.



The first approximation to its value is

$$\eta = \alpha \sin i \cos \phi ;$$

it practically vanishes, therefore, for normal incidence and for tangential drift.

Further, as regards the change of width of the beam or distance between the rays, it is apparent that measured along the wave-surface it is the same, because EC = A′B′ = AB ; so measured perpendicularly it changes in the ratio of $\cos \epsilon : \cos \epsilon'$ before and after reflexion.

68. It is not to be supposed that the *ray* is reflected after this manner ; and, in fact, we shall find that the error of ray-reflexion, or difference between angle of incidence and reflexion, $i - i'$, is exceedingly small.

To determine this difference, and the whole circumstances of the problem, we write down the following equations, obvious from figure 17 :

$$\frac{\sin \epsilon}{\sin \theta} = \frac{\mathrm{BB'}}{\mathrm{B'C}} = \frac{v}{\mathrm{V}} = \alpha = \frac{\mathrm{AA'}}{\mathrm{A'E}} = \frac{\sin \epsilon'}{\sin \theta'},$$

$$\theta = i - \phi, \qquad \theta' = \pi - (i' + \phi).$$

Also, for the time of journey of the wave from position AB to EC,

$$t = \frac{\mathrm{BB'}}{v} = \frac{\mathrm{B'C}}{\mathrm{V}} = \frac{\mathrm{BC}}{\mathrm{V} \cos \epsilon + v \cos \theta} = \frac{\mathrm{AE}}{\mathrm{V} \cos \epsilon' + v \cos \theta'} = \frac{\mathrm{A'E}}{\mathrm{V}} = \frac{\mathrm{AA'}}{v}.$$

Lastly

$$\frac{\mathrm{BC}}{\mathrm{AC}} = \frac{\sin(i + \epsilon)}{\cos\epsilon}; \qquad \frac{\mathrm{AE}}{\mathrm{AC}} = \frac{\sin(i' - \epsilon')}{\cos\epsilon'}.$$

These solve the problem, and they may be conveniently worked on the following lines—

$$\sin\epsilon = \alpha\sin(i - \phi); \qquad \sin\epsilon' = \alpha\sin(i' + \phi),$$

$$\frac{\sin i' - \cos i' \tan\epsilon'}{\sin i + \cos i \tan\epsilon} = \frac{\mathrm{AE}}{\mathrm{BC}} = \frac{\mathrm{V}\cos\epsilon' + v\cos\theta'}{\mathrm{V}\cos\epsilon + v\cos\theta} = \frac{\cos\epsilon' - \alpha\cos(i' + \phi)}{\cos\epsilon + \alpha\cos(i - \phi)} = \frac{\cos\epsilon - \alpha\cos(i - \phi)}{\cos\epsilon' + \alpha\cos(i' + \phi)},$$

the last equality being added for convenience, and being true because

$$\cos^2\epsilon - \alpha^2\cos^2\theta = 1 - \alpha^2.$$

Therefore, exactly,

$$\sin i\cos\epsilon - \tfrac{1}{2}\alpha^2\cos i\sin 2(i - \phi)\sec\epsilon = \sin i'\cos\epsilon' - \tfrac{1}{2}\alpha^2\cos i'\sin^2(i' + \phi)\sec\epsilon',$$

whence, expanding $\cos\epsilon$ and neglecting α^4,

$$\sin i - \sin i' = \alpha^2\cos^3 i\sin 2\phi,$$

or

$$i - i' = \alpha^2\cos^2 i\sin 2\phi.$$

The discrepancy between the angles of incidence and reflexion (which I call for brevity the error of reflexion) is therefore exactly expressible in even powers of aberration magnitude, and no part of it reverses with the reversal of the ray. It vanishes for grazing incidence, and is a maximum for normal incidence (at which I am somewhat surprised). It vanishes both for tangential and for normal drift, being a maximum when the medium drifts at 45° to the mirror.

The maximum possible value of the error of reflexion is α^2, or 10^{-8} of a radian, or $0''\cdot00205$, or $\frac{1}{500}$th of a second of arc; an amount which, although equivalent to an error of only 15 inches in the circumference of the earth, it is perhaps possible to detect; especially if, as Mr. MICHELSON suggests, it be increased by multiple reflexion. Indeed, it strikes me as perhaps the simplest way of examining into the motion of the ether near the earth.

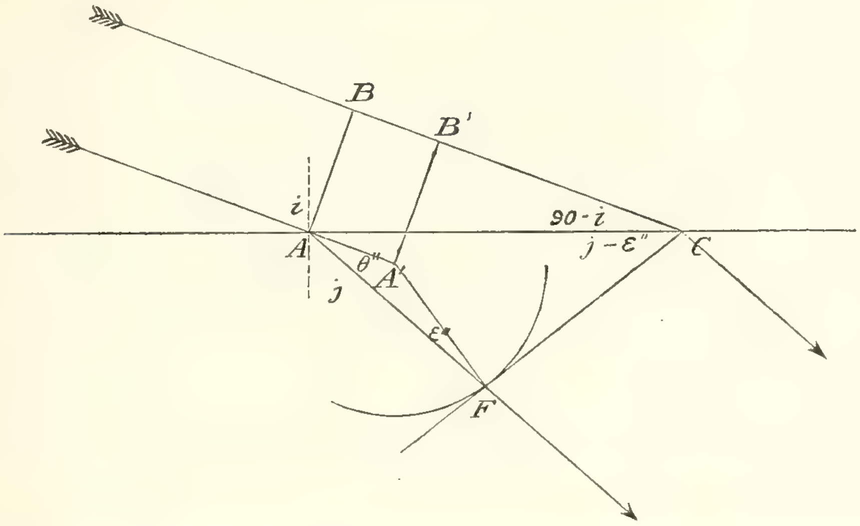
Refraction in a Drifting Medium.

69. The reasoning for refraction is precisely of the same kind, and there needs nothing more but to write down the equations, putting V/μ everywhere instead of V, v/μ^2 instead of v, and consequently α/μ instead of α.

It is best thus to assume FRESNEL's theory, and leave observation to point out any deviation from it that may be existent.

A separate figure may save confusion; and though the general case is easily drawn (like fig. 17) a special case serves better for illustration, and I depict the case of drift along incident ray.

Fig. 19.



θ'' and ϵ'' are the angles between refracted ray and the drift direction and wave-normal respectively; the angle of refraction, defined as usual, may be called j; so that

$$\theta'' = \phi - j,$$

$$\sin \epsilon'' = \frac{\alpha}{\mu} \sin \theta'',$$

$$t = \frac{\mathrm{AA'}}{v/\mu^2} = \frac{\mathrm{A'F}}{\mathrm{V}/\mu} = \frac{\mathrm{AF}}{\mathrm{V}/\mu \cos \epsilon'' + v/\mu^2 \cos \theta''},$$

$$\frac{\mathrm{AF}}{\mathrm{AC}} = \frac{\sin (j - \epsilon'')}{\cos \epsilon''}.$$

These are the equations to be used in conjunction with the previous set, and so it follows that

$$\frac{\sin j - \cos j \tan \epsilon''}{\sin i + \cos i \tan \epsilon} = \frac{\mathrm{AF}}{\mathrm{BC}} = \frac{1}{\mu} \frac{\cos \epsilon'' + \alpha/\mu \cos \theta''}{\cos \epsilon + \alpha \cos \theta} = \frac{1}{\mu} \frac{\cos \epsilon - \alpha \cos (i - \phi)}{\cos \epsilon'' - \alpha/\mu \cos (j - \phi)}.$$

Wherefore

$$\sin i \cos \epsilon - \alpha \sin \phi - \alpha \tan \epsilon \cos i \cos (i - \phi)$$
$$= \mu \left\{ \sin j \cos \epsilon'' - \frac{\alpha}{\mu} \sin \phi + \frac{\alpha}{\mu} \tan \epsilon'' \cos j \cos (j - \phi) \right\}.$$

Or

$$\sin i \cos \epsilon - \mu \sin j \cos \epsilon'' = \tfrac{1}{2} \alpha^2 \left\{ \cos i \sin 2 (i - \phi) - \frac{1}{\mu} \cos j \sin 2 (j - \phi) \right\}.$$

Which shows that the difference between sin i and μ sin j, or the error of refraction, is likewise of the second order of aberration magnitude, *i.e.*, ordinarily speaking, nil;

its value being easily obtainable if ever wanted. The displacement of Fraunhofer lines due to the Sun's rotation is a small thing to detect with a prism spectroscope, but this effect of motion on terrestrial sources, if it is ever to be seen, is 660 times smaller.

70. It may, perhaps, be well to check over our results by the less geometrical method employed by Sir GEORGE STOKES, viz., by expressing the fact that the intersection of the three waves (incident, reflected, and refracted) with the mirror is a joint intersection, and runs along the mirror at a pace which can easily be written down (viz., AC/t); for the wave advances through the medium at a speed V, and the medium helps it along with a component of its drift velocity $v\cos(\theta+\epsilon)$; so the total speed of the joint wave intersection as it runs along the mirror is

$$\frac{V + v\cos(\theta+\epsilon)}{\sin(i+\epsilon)},$$

which it is easy to see is precisely the same as what we should have obtained by attending to rays and to the figure, viz.,

$$\frac{AC}{t} = \frac{V\cos\epsilon + v\cos\theta}{\sin(i+\epsilon)/\cos\epsilon}.$$

So the equations for reflection and refraction can be written down at once, thus:

$$\frac{V + v\cos(i-\phi+\epsilon)}{\sin(i+\epsilon)} = \frac{V - v\cos(i'+\phi-\epsilon')}{\sin(i'-\epsilon')} = \frac{V/\mu - v/\mu^2\cos(j-\phi-\epsilon'')}{\sin(j-\epsilon'')},$$

together with the values of the aberration angles, obtained, say, by resolving the wave and drift velocities perpendicular to the ray, or resultant direction of advance, and expressing the fact that they must neutralize each other;

$$\frac{v}{V} = \frac{\sin\epsilon}{\sin(i-\phi)} = \frac{\sin\epsilon'}{\sin(i'+\phi)} = \frac{\mu\sin\epsilon''}{\sin(j-\phi)}.$$

These two sets of equations contain the entire solution, and of course μ may be written μ_2/μ_1 if it is a question of passage from one medium to another instead of from vacuum to a medium; the V and v then expressing speeds in first medium.

71. In Sir GEORGE AIRY'S* beautifully performed and described experiment of the value of the coefficient of aberration measured by a zenith sector full of water, there should, we see, on FRESNEL'S theory, have been a slight discrepancy, but one wholly too small to observe with the various inaccuracies inseparable from star-light. If it is to be detected it must be with light from a terrestrially fixed source. The obser-

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* 'Phil. Mag.,' iv., vol. 43, p. 310.

vations of HOEK and others, performed with terrestrial light, aimed only at disproving KLINKERFUES' notion that an error proportional to the first power was to be expected, and did not aim at the immense delicacy needed to observe α^2.

Wave-length as altered by Reflexion.

72. Since the laws of reflexion are so closely obeyed, an image in a mirror will practically appear just the same whether the medium is stationary or not, and, accordingly, the image may be treated as the virtual source for all questions relating to wave-length and Doppler effect, and the waves coming from that image will in general be affected by the drift otherwise than are the waves coming from the source, because the direction is different.

For instance, sunshine strikes the earth perpendicularly to its motion, but reflected sunshine may coincide with the direction of motion, and, in that case, will have to travel against (or with) the ether wind precisely as if it came from a terrestrial source, and its wave-length will be affected as already reckoned; in other words, thinking of a mirror moving with the orbital motion of the earth only, considered as circular, the image of the Sun moves as if attached to the mirror (not at twice the rate), and, accordingly, reflected sunshine behaves as regards wave-length precisely as if it were coming from a terrestrial source. [More generally (*i.e.*, including eccentricity of orbit and aberration) reflected light *as seen by an observer moving with the mirror* appears in every respect like direct light.]

For irregular reflexion, *e.g.*, from white paper or from the Moon or a planet, these things can be treated as being themselves the sources.

Change of Phase caused by Reflexion in a Moving Medium.

73. Now consider the phase as affected by reflexion.

Consider the two parallel rays A and B, in fig. 17, distant b from each other, and let B lag initially by an amount $b \tan \epsilon$ behind A (§ 67), then, after reflexion, the distance apart has changed so that $b/\cos \epsilon = b'/\cos \epsilon' = c$ say, and the lag is $b' \tan \epsilon'$.

Hence the gain of lag by reflexion, $b' \tan \epsilon' - b \tan \epsilon$

$$= c\,(\sin \epsilon' - \sin \epsilon),$$

$$= -\,2\alpha c \cos \frac{i + i'}{2} \sin \left(\phi - \frac{i - i'}{2}\right),$$

which, very approximately,

$$= -\,2\alpha c \cos i \sin \phi.$$

For normal incidence and tangential drift it has its maximum value, $2\alpha b$.

Now whatever the initial lag may be, and it may be arbitrary, the final lag will differ from it by this same amount; and if the rays, instead of being parallel, are coincident in path, then no difference in phase is caused by reflexion.

Change of Energy at Reflexion from Moving Mirror

74. When reflexion takes place from a moving (receding) mirror, there is some work done on the mirror by reason of the intrinsic pressure of light. Calling the energy per unit volume $e + e'$, the energy of the incident light per second is Ve, and of the reflected Ve', the pressure is $e + e'$, and the work done per second $(e + e')\,v$.

So

$$Ve - Ve' = (e + e')\,v,$$

or

$$e\,(1 - \alpha) = e'\,(1 + \alpha).$$

Or consider the mirror fixed and medium moving with source away from it, the speed of incident light is $V - v$, of the reflected is $V + v$, but no work is done; so

$$(V - v)\,e = (V + v)\,e'.$$

Wherefore, on either mode of consideration, the energy of the reflected light is from a receding mirror less, from an advancing mirror greater, than that of the incident light, in the ratio

$$\frac{e'}{e} = \frac{1 - \alpha}{1 + \alpha}.$$

Possible Effect of Light Pressure in Astronomy.

75. Light energy per unit volume on Mount Whitney, as determined by LANGLEY, amounted to 67 microbarads, or, say, in outside space, three quarters of 10^{-4} ergs per cubic centimetre; giving a pressure of the same number of dynes per sq. centim.

This pressure on the Moon is withdrawn during eclipses, but, although equal to the ordinary weight of 10,000 tons or so, it is too small to make sensible perturbations, as it could only push the Moon $\frac{1}{40}$th inch in a fortnight.

On a small body, however, it may become comparable with gravitation.*

On a small-enough dust particle, such as may be in tails of comets, the light pressure and gravitative attraction of the Sun might balance. I make the size about 1 micron diameter for a sphere of the density of water, at any distance. Anything smaller than this would be repelled, and would get up an excessive velocity in time.

[* I find that FITZGERALD made a communication years ago to the Royal Dublin Society on this subject.]

Direction of Motion of a place on the Earth.

76. Of all the motions to which the earth is subject its orbital motion is the largest, and is the most important for aberrational effects; but two others must not be overlooked, since they may introduce secular variations into the amount of those effects, viz., the diurnal rotation and the motion of the system through space.

The speed of the motion of the system is only approximately known, but it is estimated at 10·9 miles a second, or 1·75 million C.G.S., and its direction is completely specified by stating a point among the fixed stars.

The speed of the diurnal rotation is $\frac{\text{equatorial circumference}}{\text{1 sidereal day}} \times \cos$ latitude, or very small compared with that of light, and its direction is simply from west to east. It causes a variation in the observed total aberration, amounting to nearly 2 per cent. at the equator.

Both these motions are steady.

The orbital motion is not quite constant in speed, and goes through the whole plane cycle of directions, but its average value may be stated as $\frac{1}{10,000}$ that of light, and its direction is sufficiently expressed for practical purposes by saying that it is in the plane of the ecliptic, and at right angles to the Sun's direction. For instance, a half-moon is roughly in the line of the earth's orbital motion. We are moving as if going away from an increasing half-moon or towards a decreasing half-moon. Another way of putting the matter, is that at midnight the annual and diurnal motions approximately agree in direction, at midday they are opposed. At the epoch of the solstices the agreement is good, *i.e.*, the orbital motion at a solstice is from east to west at noon, from west to east at midnight; and at no time of the year is the error of this statement of very great practical import, for even at the equinoxes 91·7 per cent. of the motion is in the direction stated.

A clock might easily be made to point out the direction of orbital motion. By starlight it is never difficult to realize it, for there are usually planets enough to make the ecliptic manifest, and there is no difficulty in estimating whereabouts the Sun is. Hold a twenty-four hour watch in the plane of the ecliptic with its noon line pointing west, and its hour hand will constantly indicate the direction of the earth's orbital motion. The only difficulty is knowing where the plane of ecliptic is. Consider a terrestrial globe with its axis tilted 23½°, and rotating by internal mechanism once in twenty-four hours. The plane of ecliptic is horizontal, and the direction of motion will be given by a pointer revolving once a year in a horizontal plane, or, more simply, by the appropriate radius of a horizontal card with 365 days of the year written round its circumference. With that alone, however, it would be a little puzzling to compare this slowly changing direction with the position of any given locality on the rotating earth. The whole might be turned by hand till the required locality came to the top, with the axis in the meridian, and then the pointer would agree with the direction of motion; or, more simply, there need be no globe at all, but simply a polar axis revolving once a day opposite to the earth, and carrying with it a dial with the names of the months and days recorded round its circumference, set on the axis at an obliquity of $23\frac{1}{2}$°, and adjusted once for all to coincide with the ecliptic. The date on the card will then point out the line of motion. The clock, if kept to G.M.T., would never give the motion more erroneously than a small correction analogous to the equation of time. By giving the card one step forward every 29th of February, it could be kept right until the whole thing wanted that $\frac{1}{260}$th part of a rotation per century about a vertical axis which precession demands. [My assistant, Mr. E. E. ROBINSON, has connected a clock through a HOOKE's joint with a pointer which moves so as very fairly to indicate the direction of orbital motion at any instant.]

Electrical methods of detecting Motion through Ether.

77. It might perhaps appear possible that electrical methods may succeed in showing a first-order effect of terrestrial motion, since charged bodies in motion repel each other with modified force.

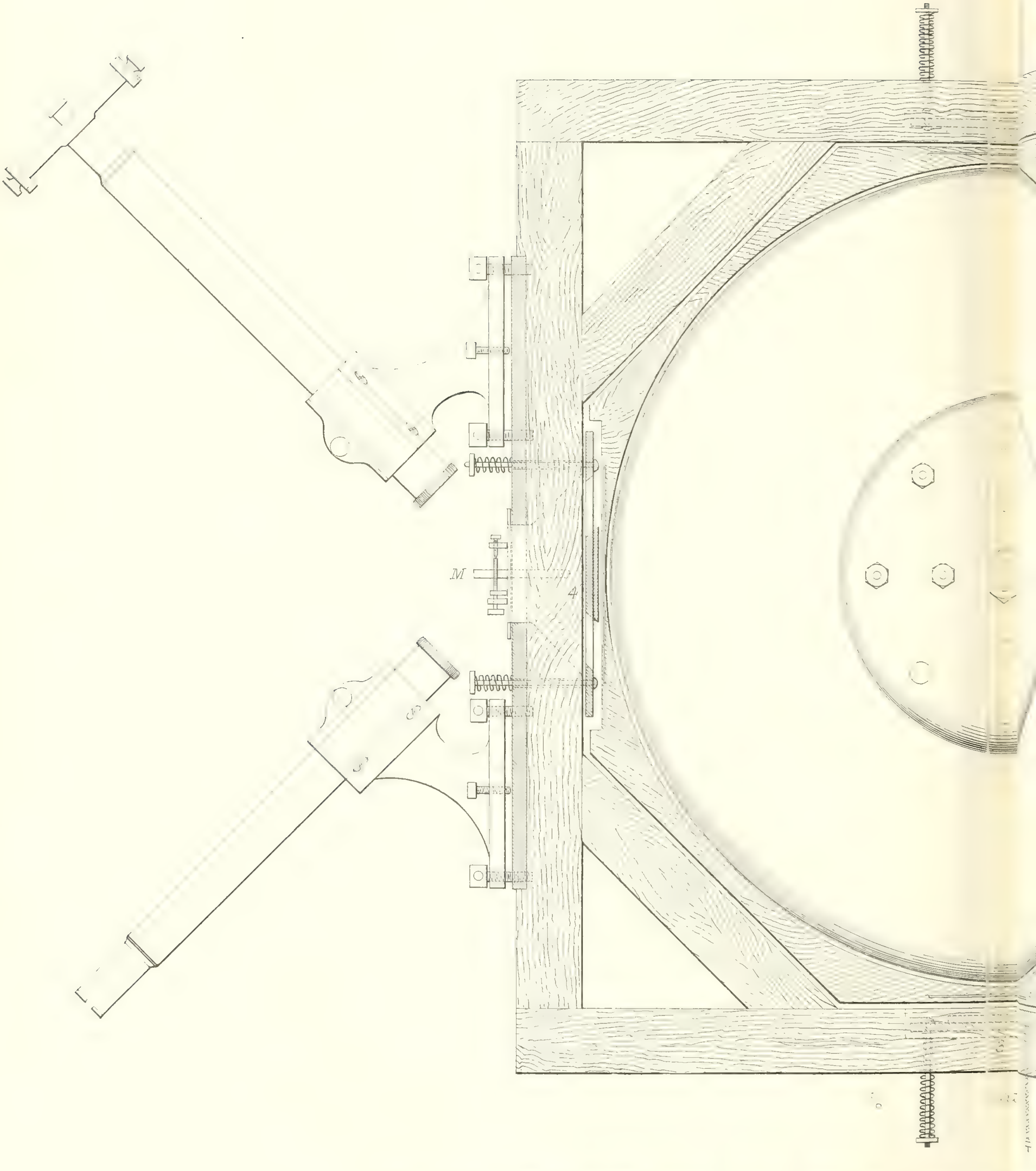
It is not possible to control or vary it except by combining the above several kinds of movement, and FITZGERALD has suggested a plan of observing whatever effect may be caused by the alternate agreement and disagreement between the earth's orbital motion and the solar system's proper motion: say by measuring the attraction of charged parallel plates at intervals of six months.

But, inasmuch as the force between charged bodies is independent of the direction of their motion, or (otherwise) because the electrical attraction between parallel moving charges depends on the product of their velocities, it must be the second-order of aberration magnitude that is really involved.

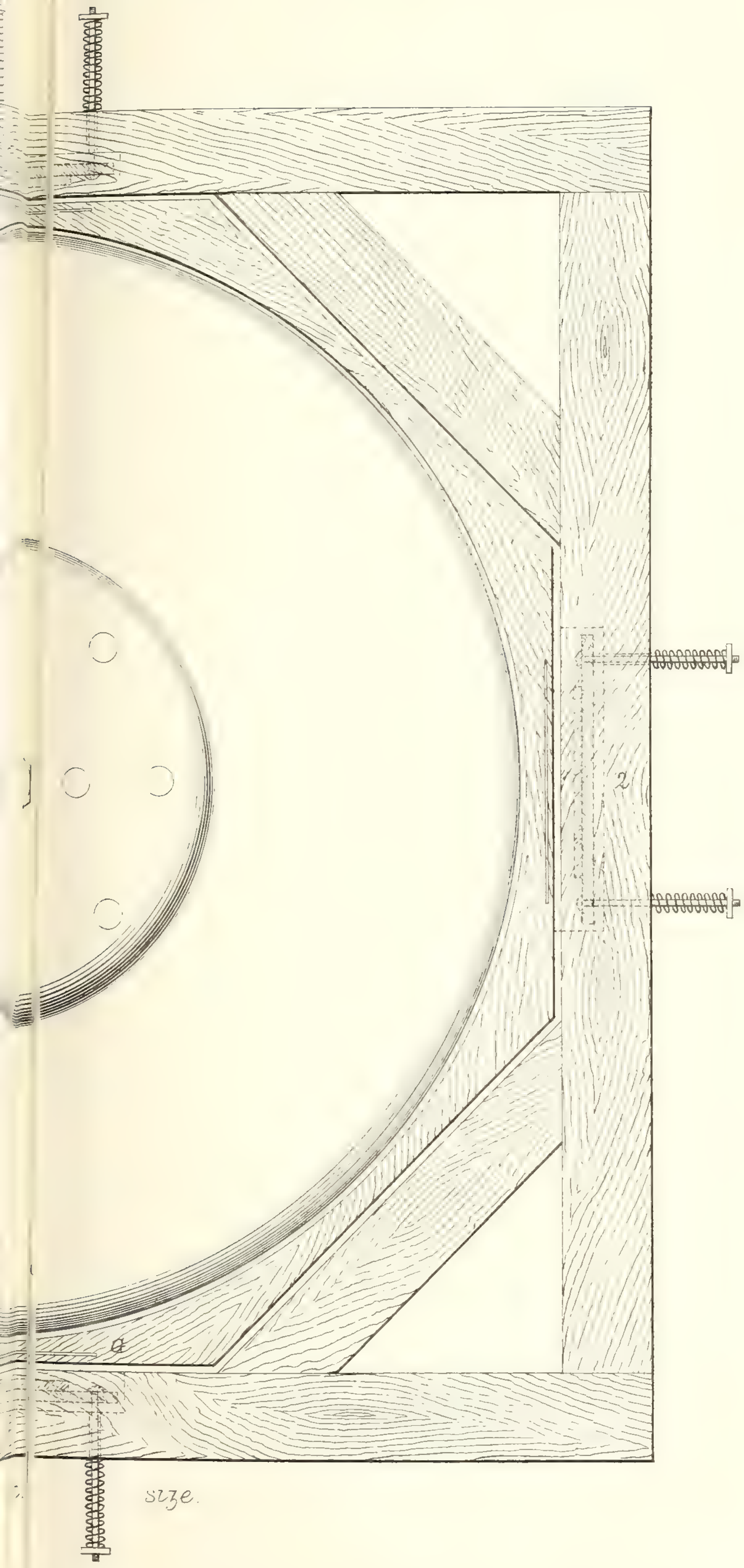
DESCRIPTION OF PLATES 31 AND 32.

Plate 31. Details of optical frame, showing the mode of supporting the mirrors, both the silvered and the semi-transparent.

Plate 32. Details of whirling machine, showing the pair of steel disks, 1 yard in diameter, driven by an electric motor.

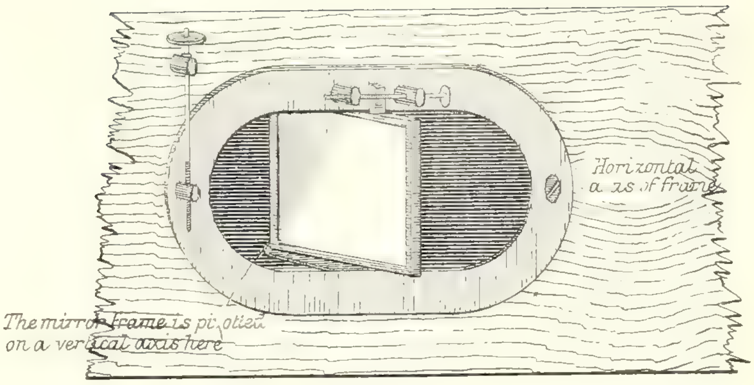


Plan of optical frame with steel disk in positi
frame. G represents one of the panes of optic
shewn, and part of the fixing of the four mirror
frame, each mirror held by a brass plate sup
it is pressed by the spring-bolts shewn



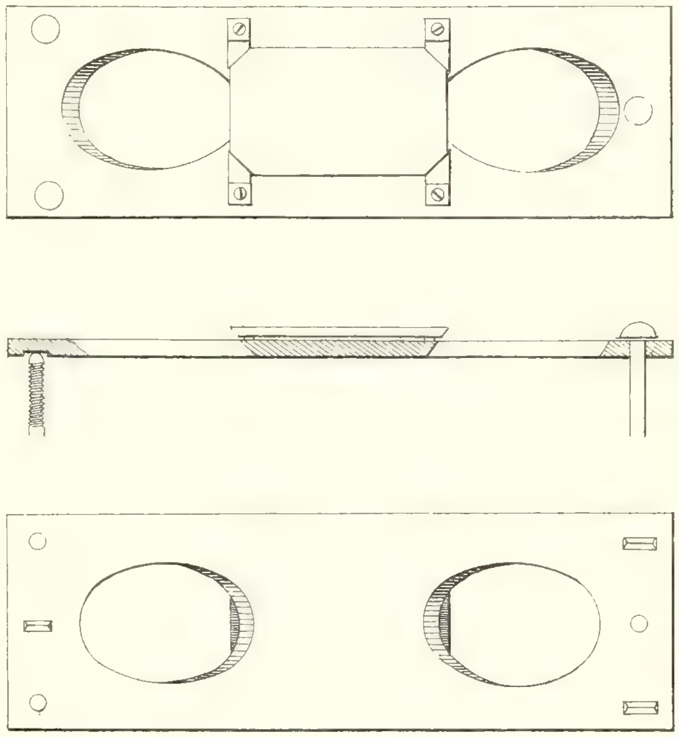
size.

d glazed drum to isolate them from the
ss. Supports of telescope and collimator also
.4., three of them let into recesses in the wooden
by three finely cut screws against which
M is the semi-transparent mirror.



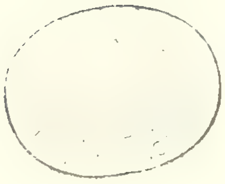
1/4 nat. size

Mode of mounting the semitransparent mirror M so as to give altitude and azimuth movement to the reflected beam.

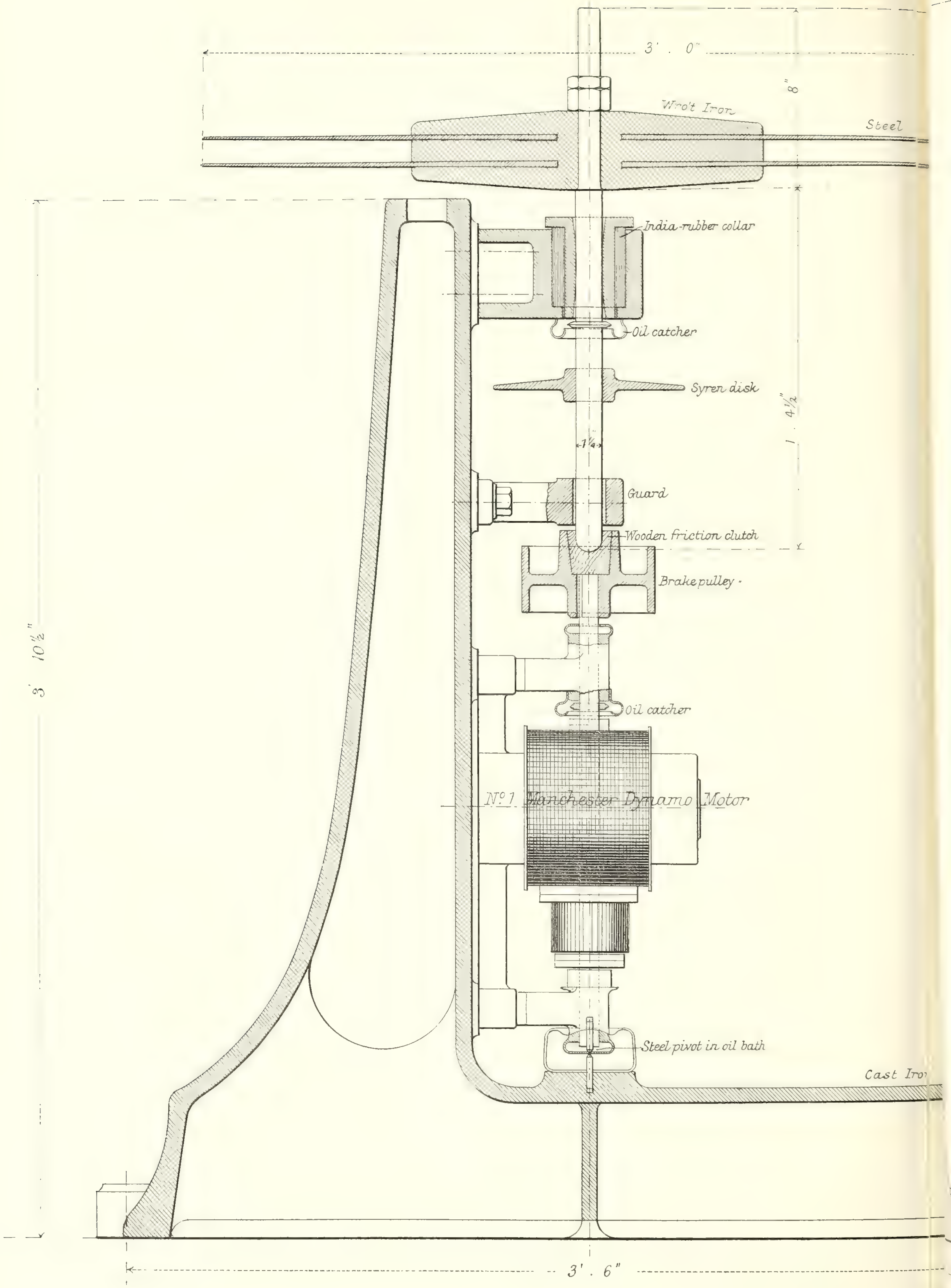


1/4 nat. size

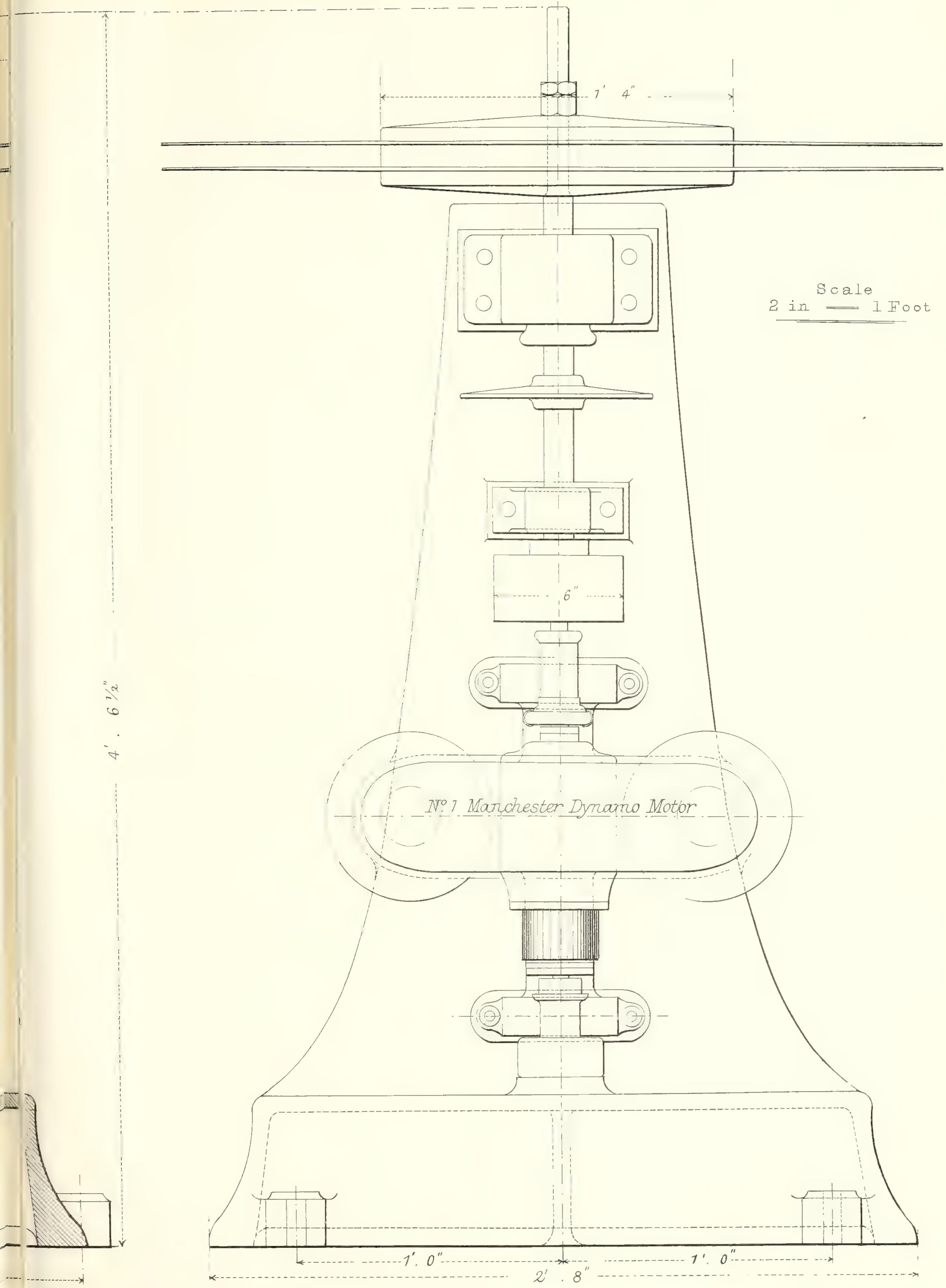
Details of brass plate supporting fourth mirror, front, side and back views.
Back view shows the three slots in which the ends of the supporting screws rest, giving a fine adjustment; the plate being supported by three rigid pushes and three elastic pulls.



Lodge.

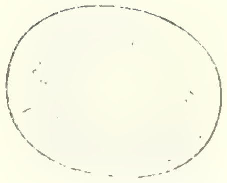


EXPERIMENTAL W



IRLING MACHINE.

West, Newman lith.



XVI. *On the Results of an Examination of the Orientations of a number of Greek Temples with a view to connect these Angles with the Amplitudes of certain Stars at the time the Temples were founded, and an endeavour to derive therefrom the Dates of their Foundation by consideration of the Changes produced upon the Right Ascension and Declination of the Stars by the Precession of the Equinoxes.*

By F. C. PENROSE, *F.R.A.S., Architect.*

Communicated by Professor J. NORMAN LOCKYER, *F.R.S.*

Received December 13, 1892,—Read April 27, 1893.

[*Preliminary Observations.*—MY attention was directed to the above subject by Mr. LOCKYER, who, in the course of his study of the Egyptian temples and the stars which appear to have determined their orientation, was led by a cursory examination of some of the Greek temples to think it likely that the same principle prevailed there also, and, knowing that I was in possession of some measurements which would help in the inquiry, invited me to look into the matter with him. On comparing notes, we found much to promise that a practical correspondence between the Egyptian and the Greek monuments would be found; so that it evidently became worth while to go more fully into the examination of the Greek remains. On the branch of the subject connected with Egypt Mr. LOCKYER has published several articles in 'Nature;'* and in May, 1891, gave a lecture to the Society of Antiquaries, which was held in the rooms of the Royal Society; and there is an article by him bearing on the question in the 'Nineteenth Century' (July, 1892). On the Greek branch I made a preliminary statement to the same Society in February, 1892, an abstract of which was published in 'Nature' on the 25th day of that month.

Unknown to Mr. LOCKYER until after he had been for some time engaged in this research, several articles had been contributed to the 'Rheinisches Museum für Philologie,'† by Herr NISSEN, of Bonn, following an exactly similar line of inquiry, and embracing both Egyptian and Greek monuments. In Herr NISSEN's researches there is some want of exactitude in the measurements which he has used, which interfere with the accuracy of some of the conclusions arrived at. There is, however, much of interest and value in the learning which he has brought to bear on the subject.

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* April 16, May 7 and 21, June 4, and July 2, of 1891; and Jan. 28 and Feb. 18, of 1892.

† Particularly the volumes for 1885 and 1887.

As respects Egypt, there is the strongest possible evidence that, when a temple was built, the direction of the axis was pointed to the place on the horizon where some conspicuous star would rise or set. There is distinct hieroglyphical evidence that such was the case. Mr. LOCKYER has referred to these records at some length in 'Nature' for Jan. 18, 1892. I quote here one passage, being a translation from an hieroglyphical relation of the rebuilding of a temple in the time of Seti I., about 1445 B.C.:—"The living God, the magnificent son of Asti (a name of Thoth), nourished by the sublime goddess in the temple of the sovereign of the country, stretches the rope with joy. With his glance at *Ak* (the middle?) of the Bull's Thigh constellation, he establishes the temple-house of the mistress of Denderah, as took place before." At another place the King says:—"Looking to the sky at the course of the rising stars (and) recognizing the *Ak* of the Bull's Thigh constellation, I establish the corners of the temple of her majesty."

The point being considered as proved that the axis of a temple was pointed to the rising or setting of some particular star, the next step is to discover which star was chosen; and if this can be found, an astronomical clue to the date of the foundation is at once obtained; for, though the *amplitude* of the star—that is, its angular distance from true east or west at rising or setting—by reason of the precession of the equinoxes, would not now coincide with the orientation angle, the date at which it did so can be exactly recovered.

And there is seldom much difficulty in discovering the star, for the stars which could have been observed in a given direction, and bright enough to be seen through an atmosphere always somewhat hazy at low altitudes, are not numerous, so that there is little chance of error in deciding which was the star that suited the orientation.

Besides the hieroglyphical, there is good architectural evidence of stellar orientation. There are instances in Egyptian temples where the doorways have been altered, so as to keep a rising or setting star in view as it deviated from the axis which at first had been directed towards it; and when this plan could succeed no longer, in more than one instance a new temple, following the same cult, has been founded alongside of the earlier one, at a different orientation angle, for the purpose of continuing the observation of the same star.

In the first place, it may be convenient if I explain very briefly how it is that the precession of the equinoxes comes into play in this inquiry.

It is well and generally known that the orbit of the earth is in the form of an ellipse, situate in a plane passing through the Sun and through the centre of the earth, and called the plane of the ecliptic; and that the earth's diurnal rotation takes place upon an axis inclined to the plane of the orbit, at an angle at present measuring about twenty-three degrees and a half, but which fluctuates very slowly between small limits.

Celestial objects are measured upon an imaginary sphere, of indefinite diameter, centrally placed with regard to the earth, in two different ways. The starting points on one system are on the great circle which lies in the plane of the orbit, called the Ecliptic circle, and the measures taken at right angles to it are called latitudes. The coordinate measures round its circumference are called longitudes, exactly analogous to the measures of Geography, excepting that the longitudes are only reckoned in one direction, viz., from west towards east. The other system of measures employs the same imaginary sphere, but the great circle which is used is that which lies in the plane of the earth's Equator, and the measures from it are called declinations instead of latitudes, and the coordinate measures are called right ascensions instead of longitudes, which are reckoned as in the other case, only from west towards east. Thus the place of a star may be given either in latitude and longitude, or in right ascension and declination; and if nothing occurred to disturb the mechanical conditions of the orbit, this alternative reckoning would remain constant, or altered only by the exceedingly slow proper motion of some of the stars; and the summer and winter solstices and the spring and autumn equinoxes would occur in each succeeding year at the exact moment when the sun occupied the same longitude as on the previous one. This, however, is found not to be the case; and the first recorded statement of the fact is attributed to the Greek astronomer, HIPPARCHUS. It must, however, have been practically known to the Egyptians long before his time. The explanation was reserved for NEWTON, who showed that owing to certain gravitational reactions upon our planet, especially on the part of the Moon, the earth's polar axis, whilst remaining nearly constant in its inclination to the plane of the ecliptic, is continually deflected in such a manner that the recurrence of the equinoxes (of which the exact moment can be more easily observed than that of the solstices) is accelerated to the extent of about the twentieth part of a degree annually. This movement necessarily disturbs the relation which exists at any particular epoch between the latitude and longitude, and the right ascension and declination reckonings. During the course of a few years, indeed, the difference is not great, but when years are counted by thousands, the changes in right ascension have to be reckoned in hours. And although the latitude is not much affected, the changes in declination are generally large, sometimes northerly and sometimes southerly, according to the position of the object. It necessarily follows that, together with the declination, the amplitudes of stars at their rising and setting are altered; and thus it has happened that stars once chosen for orientation purposes, after a few hundred years, could by no contrivance be retained in view, but as the law of the variation has been ascertained, the date can be computed when the amplitude of the star and the orientation angle coincided.

In Greece nothing, so far as we know, has been recorded either in history or by inscriptions which offers a parallel to what has been found on this subject in the Egyptian hieroglyphics referred to above; but architectural evidence is not wanting of a character corresponding with that which has been found in Egypt, showing the changes of structure arising out of the precessional movement. The Greek examples, which can be cited, are the two Minerva temples on the Acropolis at Athens, both of which in their orientation can be referred to the Pleiades (a constellation sacred to the goddess) at different epochs ; and at Rhamnus the remains of the two temples of Themis and Nemesis, which evidently followed the same cult, are found side by side, with a difference in their orientation exactly tallying with the precessional movement of Spica.* In the well-preserved temple at Ægina the western wall of the cella has been pierced excentrically by the doorway, and apparently for the object of enabling the setting of Antares to be observed from the adytum. There may have been other examples of the same kind, but the doorways of very few of the earliest temples in the country have been sufficiently well preserved to decide whether the same thing may or may not have been done.

The orientation of temples may be divided into two classes, Solar and Stellar. In the former the orientation lies within the solstitial limits, so that its angular distance from true east and west does not exceed the amplitude of the Sun at the solstices. The stellar orientation exceeds this limit. In Greece there are comparatively few of the latter class.—April 17, 1893.]

The orientation angles given in the lists which follow were obtained from azimuths taken from the Sun or the planet Venus. In almost every case two or more sights were taken. Observations of stars at night were also used from time to time, to test the performance of the theodolite. Magnetic variations were also observed in most instances, which confirmed the opinion I had already arrived at that, owing to local attractions, magnetic bearings were not sufficiently exact for the purposes of this inquiry. The height subtended by the visible horizon opposite the axis of the temple was also in every case observed.

For the present it may be treated as a postulate that in any temple oriented within the solstitial limits of its latitude, the axis was so directed that, on the great festival of the year, the first beam of the rising Sun should fall upon the statue centrally placed in the temple, or on the incense altar in front of it; and as obviously the priests would desire to have due warning of the Sun's approach, it was also arranged that some bright star or constellation should rise or set *heliacally* where it could be seen from the adytum. By heliacal rising or setting, a phrase used in many passages of ancient writers, is meant, that such star should be just visible before the light of the rising Sun should be too powerful. This interval of time would vary according to circumstances, but it would generally require that the depression of the Sun below the horizon should be about ten degrees. Previous to the invention of water clocks or other artificial means of measuring time, the rising or setting of the stars would have been the only reliable chronometers at night.

In the following list the orientation angles are measured from the south and round by way of west, north, and east :—

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* At Tegea also there are divergent foundations, which appear to correspond to changing amplitudes of α Arietis.

Athens. Lat. 37° 38′ 20″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Archaic Temple of Minerva	260° 55′	A	Amplitude of Star or Sun	+7° 51′ 26″	+12° 16′	Pleiades (η Tauri), rising
		B	Corresponding altitude	3° 20′ E.	2° 40 E.	
		C	Declination	+7° 50′	+11° 20′	
		D	Hour angles	$6^h\ 8^m\ 24^s$	$7^h\ 29^m\ 37^s$	
		E	Depression of Sun when Star was heliacal	..	10°	
		F	R.A.	$0^h\ 34^m\ 50^s$	$1^h\ 56^m\ 3^s$	
		G	Approximate date . .	B.C. 1530, April 20, 21		
Jupiter Olympius, older foundation	268° 0′	A	Amplitude of Star or Sun	+3° 8′	+2°	α Arietis, rising
		B	Corresponding altitude	5° 0′ E.	4° 31′ E.	
		C	Declination	+6° 15′	+4° 21′ 10″	
		D	Hour angles	$5^h\ 44^m\ 51^s$	$7^h\ 15^m\ 36^s$	
		E	Depression of Sun when Star was heliacal	..	12°	
		F	R.A.	$23^h\ 19^m\ 0^s$	$0^h\ 39^m\ 45^s$	
		G	Approximate date . .	B.C. 1202, March 30, 31		
Hecatompedon, on site of present Parthenon	257° 7′	A	Amplitude of Star or Sun	+10° 8′	+15° 52′ 41″	Pleiades, rising
		B	Corresponding altitude	2° 45′ E.	2° 25′ E.	
		C	Declination	+9° 58′	+13° 57′	
		D	Hour angles	$6^h\ 17^m\ 18^s$	$7^h\ 37^m\ 56^s$	
		E	Depression of Sun when Star was heliacal	..	10°	
		F	R.A.	$0^h\ 54^m\ 5^s$	$2^h\ 16^m\ 20^s$	
		G	Approximate date . .	B.C. 1150, April 26		
Nike Apteros, or Wingless Victory	275° 43′ 17″	A	Amplitude of Star or Sun	+5° 28′ 59″	−5° 43′ 17″	Spica, setting
		B	Corresponding altitude	3° 0′ W.	5° 22′ E.	
		C	Declination	+6° 10′	−1° 12′ 5″	
		D	Hour angles	$6^h\ 4^m\ 0^s$	$6^h\ 59^m\ 0^s$	
		E	Depression of Sun when Star was heliacal	..	12° 20′	
		F	R.A.	$10^h\ 46^m\ 0^s$	$23^h\ 49^m\ 0^s$	
		G	Approximate date . .	B.C. 1130, March 17		

Athens. Lat. 37° 38′ 20″—(continued).

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Older Erechtheum, foundations under present Temple	251° 39′	A	Amplitude of Star or Sun	−21° 4′	+18° 21′	Antares, setting
		B	Corresponding altitude	3° 0′ W.	0° 50′ E.	
		C	Declination	−14° 31′	+14° 54′	
		D	Hour angles	$4^h\ 59^m\ 0^s$	$7^h\ 54^m\ 0^s$	
		E	Depression of Sun when Star was heliacal	. .	12°	
		F	R.A.	$13^h\ 35^m\ 0^s$	$2^h\ 28^m\ 20^s$	
		G	Approximate date . .	B.C. 1070, April 29		
Earlier Temple of Bacchus	255° 7′ 42″	A	Amplitude of Star or Sun	+10° 42′ 5″	+16° 41′ 26″	Pleiades, rising
		B	Corresponding altitude	3° 30′ E.	3° 3′ E.	
		C	Declination	+10° 35′	+15°	
		D	Hour angles	$6^h\ 15^m\ 24^s$	$7^h\ 43^m\ 43^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$1^h\ 0^m\ 41^s$	$2^h\ 29^m\ 0^s$	
		G	Approximate date . .	B.C. 1030, April 29		
Diana Brauronia .	288° 22′ 49″	A	Amplitude of Star or Sun	−19° 20′ 18″	−18° 22′ 49″	Constellation Aquarius, ζ Aquarii, rising.
		B	Corresponding altitude	6° 15′ E.	5° 59′ E.	
		C	Declination	−11° 6′	−10° 49′ 9″	
		D	Hour angles	$4^h\ 51^m\ 37^s$	$6^h\ 56^m\ 42^s$	
		E	Depression of Sun when Star was heliacal	. .	17°	
		F	R.A.	$20^h\ 13^m\ 0^s$	$22^h\ 18^m\ 0^s$	
		G	Approximate date . .	B.C. 580, Feb. 21, 22		

Olympia. Lat. 37° 38′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
The Heræum, or Temple of Juno	266° 13′ 58″	A	Amplitude of Star or Sun	+7° 35′	+3° 46′	Spica, rising
		B	Corresponding altitude	3° E.	1° 40′ E.	
		C	Declination	+7° 40′	+4° 0′ 1″	
		D	Hour angles	$6^{h}\ 9^{m}\ 2^{s}$	$7^{h}\ 3^{m}\ 10^{s}$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$10^{h}\ 29^{m}\ 4^{s}$	$11^{h}\ 23^{m}\ 12^{s}$	
		G	Approximate date . .	B.C. 1445, September 12		
The Temple of Jupiter	262° 37′ 46″	A	Amplitude of Star or Sun	+8° 38′	+7° 22′ 14″	α Arietis, rising.
		B	Corresponding altitude	3° 0′ E.	1° 42′ E.	
		C	Declination	8° 40′	+6° 52′ 22″	
		D	Hour angles	$6^{h}\ 11^{m}\ 37^{s}$	$7^{h}\ 34^{m}\ 52^{s}$	
		E	Depression of Sun when Star was heliacal	. .	14° 12″	
		F	R.A.	$23^{h}\ 40^{m}\ 0^{s}$	$1^{h}\ 3'\ 5''$	
		G	Approximate date . .	B.C. 790, April 6		
The Metroum, or Temple of Cybele	281° 47′ 2″	A	Amplitude of Star or Sun	+9° 28′ 13″	−9° 47′ 2″	α Arietis, setting
		B	Corresponding altitude over roof of ancient Heræum	5° 45′ W.	2° E.	
		C	Declination	+11° 0′	−6° 30′	
		D	Hour angles	$6^{h}\ 4^{m}\ 50^{s}$	$6^{h}\ 53^{m}\ 10^{s}$	
		E	Depression of Sun when Star was heliacal	. .	14° 6′	
		F	R.A.	$0^{h}\ 2^{m}\ 0^{s}$	$13^{h}\ 0^{m}\ 0^{s}$	
		G	Approximate date . .	B.C. 360, October 9		

The Hiero of Epidaurus. Lat. 37° 35′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Asclepieion or Temple of Esculapius	259° 24′ 50″	A	Amplitude of Star or Sun	+6° 19′	+10° 35′	Pleiades (η Tauri), rising
		B	Corresponding altitude	7° E.	6° 10′ E.	
		C	Declination	+9° 15′	+12° 8′	
		D	Hour angles	$5^h\ 53^m\ 16^s$	$7^h\ 31^m\ 41^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$0^h\ 47^m\ 35^s$	$2^h\ 26^m\ 0^s$	
		G	Approximate date . .	B.C. 1275, April 28, 29		
Temple of Diana .	255° 49′	A	Amplitude of Star or Sun	−13° 7′ 43	+14° 11′ 0″	β Scorpii, setting
		B	Corresponding altitude	3° W.	6° 10′ E.	
		C	Declination	−8° 30′	+14° 59′ 5″	
		D	Hour angles	$4^h\ 57^m\ 0^s$	$7^h\ 59^m\ 0^s$	
		E	Depression of Sun when Star was heliacal	. .	13°	
		F	R.A.	$13^h\ 32^m\ 0^s$	$2^h\ 29^m\ 0^s$	
		G	Approximate date . .	B.C. 780, April 29		

Tegea. Lat. 37° 27′ 45″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Minerva	This I could only obtain approximately from walls, not of the Temple itself, which are shown on the German plan in 'Mittheilungen.' 267° 12′ 30″	A	Amplitude of Star or Sun	+5° 3′ 45″	+0°	Spica, rising
		B	Corresponding altitude	+3° 0′ E.	+3° 5′ E.	
		C	Declination	+5° 51′	+1° 52′ 12″	
		D	Hour angles	$6^h\ 2^m\ 45^s$	$6^h\ 56^m\ 32^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$10^h\ 48^m\ 13^s$	$11^h\ 43^m$	
		G	Approximate date . .	B.C 1075, September 18		

The above can only be regarded as a possible solution.
Should the Temple be re-examined by excavation more exact data could be obtained.
(See note on this Temple at p. 833.)

Rhamnus. Lat. 38° 13′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Themis	268° 30′ 14″	A	Amplitude of Star or Sun	+5° 16′ 36″	+1° 29′ 46″	Spica, rising
		B	Corresponding altitude	3° 0′ E.	1° 22′ E.	
		C	Declination	+6° 0′	+2° 1′ 15″	
		D	Hour angles	$6^h\ 3^m\ 37^s$	$6^h\ 57^m\ 39^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$10^h\ 47^m\ 53^s$	$11^h\ 41^m\ 55^s$	
		G	Approximate date . .	B.C. 1092, September 17		
Temple of Nemesis	271° 24′ 50″	A	Amplitude of Star or Sun	+2° 45′ 30″	−1° 24′ 50″	Spica, rising
		B	Corresponding altitude	3° 0′ E.	2° 0′ E.	
		C	Declination	+4° 5′	+0° 7′ 37″	
		D	Hour angles	$5^h\ 57^m\ 19''$	$6^h\ 51^m\ 32^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$11^h\ 4^m\ 52^s$	$11^h\ 58^m\ 50^s$	
		G	Approximate date . .	B.C. 747, September 22		

Nemea. Lat. 37° 49′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Jupiter	250° 39′ 18″	A	Amplitude of Star or Sun	−21° 56′	+19° 20′ 42″	Constellation Aquarius, ζ Aquarii, setting
		B	Corresponding altitude	7° 30′ W.	6° 35′ E.	
		C	Declination	−12° 16′ 20″	+19° 18′ 39″	
		D	Hour angles	$4^h\ 41^m\ 3^s$	$8^h\ 1^m\ 7^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$19^\circ\ 47^m\ 40^s$	$8^h\ 30^m$	
		G	Approximate date . .	B.C. 1040, July 27, 28		

Corfu. Lat. 39° 36′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple at Kardaki, near the city	274° 39′ 35″	A	Amplitude of Star or Sun	−1° 45′ 42″	−4° 39′ 35″	γ Pegasi, rising
		B	Corresponding altitude	3° 0′ E.	0° E.	
		C	Declination	0	−3° 35′ 20″	
		D	Hour angles	$5^h\,43^m\,19^s$	$7^h\,21^m\,56^s$	
		E	Depression of Sun when Star was heliacal	. .	17° 15′	
		F	R.A.	$21^h\,48^m\,23^s$	$23^h\,27^m\,0^s$	
		G	Approximate date . .	B.C. 875, March 10, 11		

Sunium. Lat. 37° 38′ 48″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Minerva	284° 9′ 7″	A	Amplitude of Star or Sun	+12° 27′	−14° 9′	Pleiades (η Tauri), setting
		B	Corresponding altitude	3° W.	0° E.	
		C	Declination	+11° 40′	−11° 9′ 39″	
		D	Hour angles	$6^h\,21^m\,6^s$	$6^h\,15^m\,29^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R. A.	$1^h\,9^m\,52^s$	$13^h\,46^m\,27^s$	
		G	Approximate date . .	B.C. 845, October 21		

Corinth. Lat. 37° 55′ 0″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple. Dedication unknown	249° 10′	A	Amplitude of Star or Sun	−23° 26′	+20° 50′	Antares, setting
		B	Corresponding altitude	3° 15′ W.	55′ E.	
		C	Declination	−16°	+16° 53′	
		D	Hour angles	$4^h\,50^m\,37^s$	$8^h\,13^m\,0^s$	
		E	Depression of Sun when Star was heliacal	. .	13° 40′	
		F	R.A.	$13^h\,50^m\,0^s$	$2^h\,54^m$	
		G	Approximate date . .	B.C. 770, May 6		

Bassæ. Lat. 37° 25′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Apollo Epicurius	The axis of temple 108° 26′ 6″ But there was an Eastern doorway 270° 26′ 6″	A	Amplitude of Star or Sun	+3° 11′	−0° 26′ 6″	Spica, rising
		B	Corresponding altitude	3° 0′ E.	0° 45′ E.	
		C	Declination	+3° 57′	+0° 6′ 37″	
		D	Hour angles	$5^h\ 57^m$	$6^h\ 50^m\ 8^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A..	$11^h\ 5^m\ 50^s$	$11^h\ 59^m\ 0^s$	
		G	Approximate date . .	B.C. 728, September 22		

Platæa. Lat. 38° 13′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Unnamed temple. Possibly Juno	280° 38′ 10″	A	Amplitude of Star or Sun	+9° 50′	−10° 38′ 10″	α Arietis, setting
		B	Corresponding altitude	3° 0′ W.	3° 1′ E.	
		C	Declination	+9° 31′	−6° 19′ 35″	
		D	Hour angles	$6^h\ 14^m\ 46^s$	$6^h\ 56^m\ 12^s$	
		E	Depression of Sun when Star was heliacal	. .	15°	
		F	R.A..	$23^h\ 48^m$	$12^h\ 59^m$	
		G	Approximate date . .	B.C. 650, October 9		

Lycosura. Lat. 37° 23′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Despoina	272° 11′ 4″	A	Amplitude of Star or Sun	−1° 2′	−2° 11′ 4″	γ Pegasi, rising
		B	Corresponding altitude	+3° 0′ E.	0° 35′ E.	
		C	Declination	+1° 0′	−1° 25′	
		D	Hour angles	$5^h\ 48^m\ 56^s$	$7^h\ 35^m\ 56^s$	
		E	Depression of Sun when Star was heliacal	. .	19° 40′	
		F	R.A..	$22^h\ 0^m$	$23^h\ 47^m$	
		G	Approximate date . .	B.C. 650, March 16		

Ephesus. Lat. 37° 56′ 30″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Diana. Old foundation, found by Mr. Wood, beneath the fourth century temple*	275° 21′ 5″	A	Amplitude of Star or Sun	+0° 25′ 0″	−5° 21′ 5″	Spica, rising
		B	Corresponding altitude	6° 0′ E.	4° 55′ E.	
		C	Declination	+3° 57′	−1° 10′ 43″	
		D	Hour angles	$5^h\ 41^m\ 49^s$	$6^h\ 47^m\ 2^s$	
		E	Depression of Sun when Star was heliacal	. .	10°	
		F	R.A.	$11^h\ 5^m\ 48^s$	$12^h\ 11^m\ 0^s$	
		G	Approximate date . .	B.C. 715, September 25		

Ægina. (1) Lat. 37° 45′. (2) Lat. 37° 44′ 30″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
(1) Temple of Minerva, on the mountain called Jupiter Panhellenius	249° 0′ 27″	A	Amplitude of Star or Sun	−23° 55′	+20° 59′ 33″	Antares, setting
		B	Corresponding altitude	3° 0′ W.	1° 20′ E.	
		C	Declination	−16° 45′	+17° 0′ 33″	
		D	Hour angles	$4^h\ 49^m\ 20^s$	$8^h\ 11^m\ 30''$	
		E	Depression of Sun when Star was heliacal	. .	14° 50′	
		F	R.A.	$13^h\ 57^m$	$2^h\ 57^m\ 50^s$	
		G	Approximate date . .	B.C. 630, May 7		
(2) Temple at the harbour	280° 16′ 3″	A	Amplitude of Star or Sun	+10° 19′ 40″	−10° 16′ 3″	α Arietis, setting
		B	Corresponding altitude	3° 0′ W.	2° 40′ E.	
		C	Declination	+10° 0′	−6° 27′	
		D	Hour angles	$6^h\ 15^m\ 53^s$	$6^h\ 32^m\ 34^s$	
		E	Depression of Sun when Star was heliacal	. .	14° 25′	
		F	R.A.	$23^h\ 53^m\ 0^s$	$13^h\ 1^m\ 15^s$	
		G	Approximate date . .	B.C. 550, October 9		

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* It will be observed that the orientation of this temple at Ephesus is nearly the same as that of Nike Apteros at Athens, and there is not much difference either in the latitude or in the apparent height of the mountain eastwards; but in this case the rising, and in the other the setting of Spica has been adopted. This choice has been influenced by archæological considerations; and in this case the large amount of margin allowed for the star's amplitude appears to be justified by the very wide central intercolumnations of this temple.

Megalopolis. Lat. 37° 25′.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Temple of Jupiter Soter	279° 42′ 28″	A	Amplitude of Star or Sun	+9° 42′ 28″	−9° 42′ 28″	α Arietis, setting
		B	Corresponding altitude	3° 20′ W.	3° 10′ E.	
		C	Declination	+9° 43′ 53″	−5° 26′ 40″	
		D	Hour angles	$6^h\ 13^m\ 8^s$	$6^h\ 46^m\ 37^s$	
		E	Depression of Sun when Star was heliacal	. .	12° 30′	
		F	R.A.	$23^h\ 50^m$	$12^h\ 49^m\ 45^s$	
		G	Approximate date . .	B.C. 605, October 6		

Argos. Lat. 37° 41′ 13″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Heræum Temple of Juno.—The later Temple close to the site of the earlier (For latter, see p. 833).	285° 59′ 20″	A	Amplitude of Star or Sun	−15° 41′ 13″	−15° 59′ 20″	Constellation Aquarius, ζ Aquarii, rising
		B	Corresponding altitude	3° E.	2° 30′ E.	
		C	Declination	−10° 28′	−11° 1′ 12″	
		D	Hour angles	$5^h\ 11^m\ 39^s$	$7^h\ 5^m\ 15^s$	
		E	Depression of Sun when Star was heliacal	. .	19° 34′	
		F	R.A.	$20^h\ 21^m\ 58^s$	$22^h\ 15^m\ 34^s$	
		G	Approximate date . .	B.C. 425, February 21		

It will be noticed that the amplitudes given amongst the solar elements of the above list in the majority of cases agree exactly with the temple's orientation, but that in a few cases—and in Athenian temples only*—a somewhat more northerly amplitude has been taken. This has been adopted from finding that if the exact orientation angle had been used the Sun's depression would not have been sufficient to allow the stars to be sufficiently well observed, but that if the Sun had risen in a line with the northern jamb of the eastern doorway or columnar opening of the peristyle (a deviation of which the amount can be fairly well defined), not only could the solar and stellar elements be harmonized, but it would have had the advantage also of giving a longer arc of solar illumination on the statue.

As an example of the way by which the elements given in the list have been arrived at, I will give a sketch of the method used in one particular case, viz., that of the great Temple of Jupiter at Olympia. On the list it is shown that this temple has an orientation angle of 262° 37′ 46″, which gives the eastern axis an amplitude of 7° 22′ 14″ north. The eastern mountains subtend an angle of 2° 4′.

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* That is not reckoning the late example at Olympia (the Metroum), and the very doubtful case of Tegea when, as in its place in the list, referred to Spica.

I assume that in the climate of Greece no star with which we are concerned in this inquiry, excepting Sirius, could be expected to be seen under ordinary circumstances heliacally at an angle less than 3° (independent of refraction) above the true horizon. I have indeed myself seen Rigel heliacally, and in the same direction as the Sun, at a lower elevation;* but allowing for variation of weather and the glow which skirts the horizon when the sun is about 10° below it, I consider the altitude of 3°, as above stated, the proper angle to take for the stars generally. When there is a mountain high enough to exclude the glow referred to, it is favourable for the detection of the star, but that is not the case at Olympia. The Sun, however, is differently situated, and in the climate of Greece can be reckoned upon to throw a strong illumination as soon as a quarter of his refracted orb is clear of the visible horizon. This at moderate altitudes allows the reduction of the apparent height of the mountain by 0° 22′.

We have then to determine the Sun's place in the case before us, having the Sun's amplitude 7° 22′ 14″, an altitude of 1° 42′, and the terrestrial latitude. Applying the formula

$$\text{Sin } \delta = \text{cos zenith dist} \times \text{cos colat} + \text{sin zenith dist} \times \text{sin colat} \times \text{sin ampl},$$

we obtain for the Sun's declination + 6° 52′ 22″.

This, at the value of the obliquity at, say, 800 years B.C., gives for the Sun's R.A., $1^h\ 3^m\ 15^s$.

Applying the same formula for a star having the same amplitude, but with an altitude of 3°, we obtain for its declination 7° 40′.

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* I made, with such opportunities as presented themselves, naked-eye observations of heliacal stars. The following appear to be the most valuable.

	Name of Star.	Magnitude.	Altitude of Star.		Sun's depression.		Difference of azimuth Sun and Star.
			°	′	°	′	°
Sunset, April 19. At sea. Latitude about 37° 50′	ζ Argus . .	2	9	47	10	49	102
	γ Andromedæ	2	9	0	12	0	17
Sunrise, April 26. Livadia. Latitude 38° 26′	20 Libræ . .	3	9	0	10	17	163
Sunset, May 1. At Sea. Latitude about 39° 50′	ε Canis Major	1·5	7	20	8	0	76
	Rigel . . .	1	2	40	9	48	41

The stars on the occasions above recorded were distinctly seen, and were found in the open sky—in most cases with the distraction of ship-lights about. These observations by no means show the limiting angles of visibility of heliacal stars by younger eyes looking from a darkened chamber through a narrow opening and towards points in the horizon where it would be known that the stars must rise. Three degrees appears to have been about the angle considered necessary by PTOLEMY. (See BIOT, 'Recherches sur l'Année Vague des Égyptiens.')

It is now necessary to inquire if there be any bright star or star group which, at a date consistent with archæological possibilities, would have had a declination near the above, and also such right ascension that it could also have been heliacal. At this stage of the inquiry we may find by an approximate method the time that such a star would take to pass from altitude 3° to the meridian, and this would be about $6^h\ 8^m$, and that it would require the Sun to rise from $1\frac{1}{4}$ to $1\frac{1}{2}$ hours later (that is, the Sun must be so much further from the meridian in R.A.) to enable the star to be seen.

We have therefore now to find a star of which the R.A. is approximately $23^h\ 40^m$ and the declination $+ 7° 40'$.

The search for the star may be made in several ways. From the R.A. and declination given above, a celestial latitude and longitude might be computed for some probable archæological epoch and recomputed into modern right ascensions and declinations, or with the latitude so obtained and the longitude adjusted for the amount of precessional movement we might refer to maps, such as those of the British Association, which show both the R.A. and declination, and the latitudes and longitudes of stars. A ready approximate method is afforded by the globe described by Mr. LOCKYER* on which the pivot can be shifted so as to suit the movements of the pole for different epochs. I have found that on a stereographic projection of the sphere, taken on the pole of the ecliptic, but showing R.A. hours and parallels of declination, the approximate places of stars as affected by precession can readily be found by marking the point under consideration, together with the straight line which coincides with the solstitial colure and the pole of the ecliptic, on tracing paper, and, keeping the latter mark superimposed, making the tracing paper revolve round the point on the projection which represents the pole of the ecliptic; it may thus be made to indicate both the R.A. and declination at a different date, which latter also is measured by the angle through which the colure line has revolved.

In the case before us, the place representing $23^h\ 40^m$ R.A. and $+ 7° 40'$ declination on being turned about till it reached an angle of polar movement due to about 2650 years, reckoned back from 1850 A.D., rested upon the modern place of α Arietis and pointed out this star for more rigid calculation. It should be mentioned that there are in every case of intra-solstitial orientations four possible solutions of this step. The Sun's amplitude may be due either to the spring or the autumnal place, and the star might be heliacal either at rising or setting. I have tried these four possible solutions in the case of every temple of which I had the requisite data by the approximate method above described, and have never failed in one of early date to find one solution, and *in no case more than one.*

When, as in this case, we have found the star (and it should be noticed that this star—the brightest of the first sign of the zodiac and therefore most fitting for a temple of Jupiter—is also the time star to the Olympieum at Athens, and apparently to other temples of the same deity) it becomes necessary to find for it exact places for several epochs by the formulæ proper for precessional movement, and then it suffices to draw upon paper curves passing through the places so found ; from which curves those due to intermediate dates may be taken with sufficient accuracy. This has been done on the following figure which represents the precessional movements of α Arietis.

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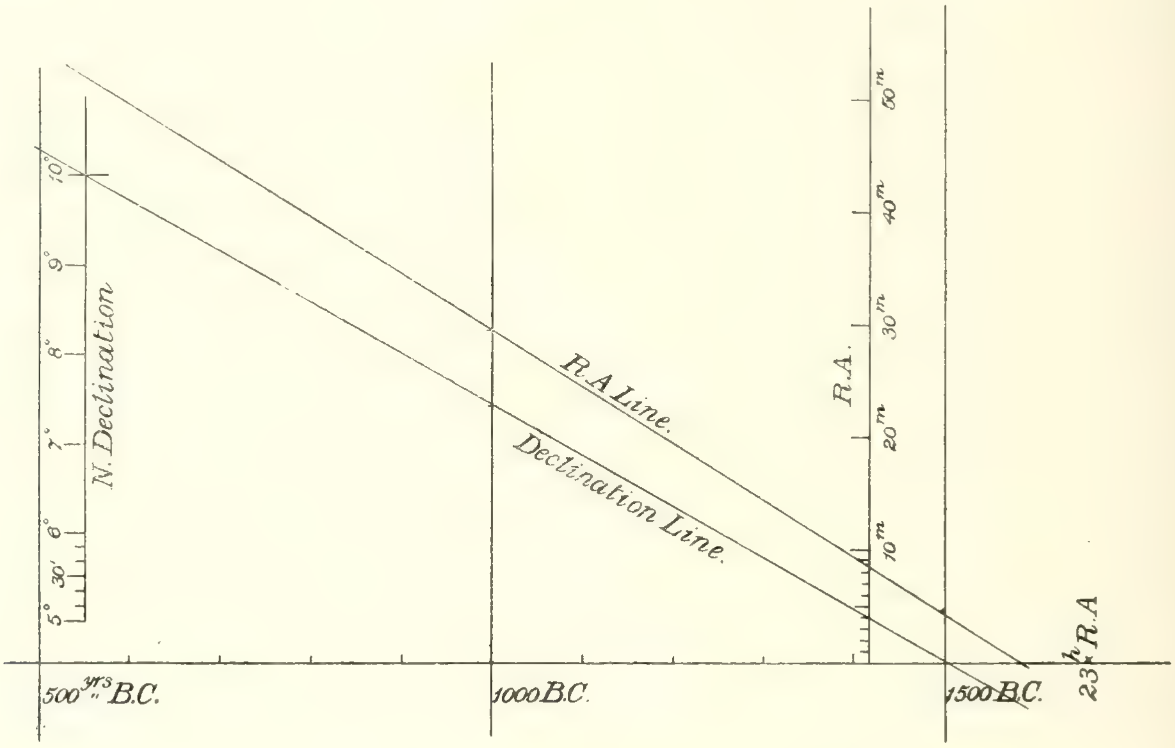
* 'Nature,' January 28, 1892.



In this diagram, years B.C. are measured horizontally from left to right. The R.A. and declination measure vertically.

Let now the hour angle of the star be taken with the approximate declination as stated above, + 7° 40′, and let also that of the Sun be computed, combining its declination of + 6° 52′ 22″ with an assumed depression below the horizon which on trial may be reasonably taken at 12°, as the star, though a bright one, is not of the first magnitude, and the horizon is comparatively low. When the difference between the two hour angles has been deducted from the Sun's R.A. to give that of the star, let the latter be compared with the places on the diagram and it will be found that the correspondences are close, though not exact—and further, that a few trials of small variations of declination point out that a very near approach can be made by making the star's declination + 8° 40′ which implies an amplitude of + 8° 38′ (not much more than 1° from the axis) and therefore well within the allowable distance. Nothing more is now required than a small addition to the Sun's depression, and a complete correspondence will be the result between the star's R.A. and its declination. 790 B.C. is the date due to that particular place on the diagram, and this has to be taken as the approximate date of the first foundation of the temple, and this coincides extremely well with the chronology of the development of Olympia. The temple now extant, however, would probably have been a rebuilding on the same lines, and 100 years or thereabouts would not have rendered the same star unfit for use as a time warner.

As respects the amount of solar depression in the above given list, I have used even with the brightest stars a minimum of 10° as the proper measure for heliacal observation.* I have ranked the Pleiades amongst the brightest stars on account of the brilliant effect upon the eye produced by concentration: but in all cases of secondary or smaller magnitudes they will be found to be combined with a greater degree of solar depression. This does not come from arbitrary assumption in the calculation. The computer has no choice in the matter beyond some slight liberty that may be taken with the amplitudes. The Sun's right ascension and the place of the star govern the solution.

Up to this point the connection has been in great measure assumed between the orientation of the temple, the sunrise, and the heliacal star. It is proposed in the remarks which follow to endeavour to justify this confidence.

I may consider it as sufficiently shown by M. BURNOUF ('Légende Athénienne'), and by Herr NISSEN in various articles published in the 'Rheinisches Museum,' and by Mr. LOCKYER in 'Nature,' that with the ancients the most approved time for adoration was the moment of sunrise. The principle prevailed not only with the Egyptians and the Greeks but also with the Romans. 'VITRUVIUS,' IV., 5, is very precise on the point; and it may also be gathered from the following passages:—

> Surgit, et ætherii spectans orientia solis
> Lumina, rite cavis undam de flumine palmis
> Sustulit, &c. VIRG. 'Æn.,' VIII.

and

> Antequam stantes repetat paludes
> Imbrium divina avis imminentum,
> Oscinem corvum prece suscitabo
> Solis ab ortu. HOR. 'Od.,' III., 27.

It is obvious that the priests would desire to have warning of the Sun's approach in time to make preparations for special functions on the great festival of the year belonging to any particular temple. Before the invention of the water-clock, or other artificial way of measuring time, the heavenly bodies at their rising or setting—the only points in their course when their positions could be accurately observed—were the only reliable time markers. The heliacal rising or setting of particular stars is referred to in many passages of ancient writers—meaning the time when the star could just be seen as it rose or set before the light of the rising Sun should be too powerful. And different periods of the year are referred to by the mention of these phenomena. In the 'Agamemnon' of ÆSCHYLUS the fall of Troy is said to have taken place at the setting of the Pleiades. Two unhealthy periods of the year are intended by HORACE in the passage, 'Od.' iii., 1,

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* PTOLEMY appears to have adopted 11° of solar depression for Egyptian heliacal observation. But in Greece there are almost always mountains obstructing the true horizon—a circumstance greatly in favour of the observer as already noticed. (See BIOT, 'Recherches sur l'Année Vague des Égyptiens,' p. 58.)

> "Nec sævus Arcturi cadentis
> Impetus aut orientis Hædi."

That the first beam of sunrise should fall upon the statue centrally placed in the adytum of a temple, or on the incense altar in front of it, on a particular day, it would be requisite that the orientation of the temple should coincide with the amplitude of the Sun when it rose above the visible horizon, be it mountain or plain.

That a star should act as time warner, it was necessary that it should have so nearly the same amplitude as the Sun that it could be seen from the adytum through the eastern door, if it was to give warning at its rising, or to have a similar but reversed amplitude towards the west if its heliacal setting was to be observed, and it follows that in the choice of the festival day and the corresponding orientation of the temple on these principles, both the amplitude of the Sun at its rising, and that of the star eastwards or westwards, as the case might be, would have to be considered in connection with one another.

From what has been said it is obvious that in the intra-solstitial temples the list of available bright stars and constellations is, in the first place, limited to those which lie within a few degrees of the ecliptic, and it will be found that in the list above given, and those which follow (if we omit Eleusis, where the conditions were exceptional), all but one of the stars are found in the zodiacal constellations. A very great limitation is imposed, in the second place, by one of the conditions being the heliacal rising or setting of those stars from which the selection has to be made, so that when both these combined limitations are taken into account it becomes improbable to the greatest degree that in every instance of intra-solstitial temples of early foundation of which I have accurate particulars—being twenty-eight in number, and varying in their orientation from 21° north to 18° south of the true east—there should be found a bright heliacal star or constellation in the right position, at dates not in themselves improbable, unless the temples had been so oriented as to secure this combination.

Besides these there are on the list of those I have examined seven temples, evidently of comparatively late foundation, of which two only admit of stellar time warning. The others, of which the sites were in the great cities of Athens and Ephesus, were probably at the time of their foundation within a region of artificial time measures. The use of the stars, too, may at last have become discredited on account of the discovery of the continuity of the precessional movement.

We see, then, that there is strong ground for considering the orientation of a temple with reference both to the Sun and to some time-warning star, and, especially as it will be shown, that in later temples, respecting which alone we have any information available, the solar coincidences correspond in several cases with what is known historically of the dates of the principal festivals.

We may feel satisfied that the axis of the temple shows in general the line of the Sun's approach. In the lists given above there are but three clear exceptions to this rule—exceptions already referred to.

The star, however, in the case of any temple would answer the purpose of a time-warner, provided it showed itself from the adytum at any point within the eastern opening, if rising, or the western if setting, and some amount of uncertainty may arise in settling the date of the temple's foundation from this cause—but when the solar and stellar elements are properly combined, the margin of possible error is not great, because the right ascension of the Sun can be calculated rigidly, and this determines that of the star very closely, by deducting from the Sun's R.A. the interval of time required for the heliacal observation of the star, and when once the star's right ascension is established, its declination, and the date corresponding to its place, become known.

In the majority of cases above given—about two-thirds of the number—the dates are clearly earlier than any remains at present visible, but it is not so altogether. At Athens, in the Archaic temple, there are parts of the foundations which cannot but go back to a very early date, and at the temple of Jupiter Olympius, there is a wall of rude workmanship of a date much anterior to the works of PISISTRATUS, which are themselves intermediate in date between this wall and the present Cossutian temple. The angle, also, is different from that of the remains at present visible. The remains of the earlier temple of Bacchus must also be very old. At Rhamnus, the temple of Themis has the appearance of great antiquity, and there are traces at Sunium of a structure underneath the existing temple which seem of a date not inconsistent with that which has been deduced from the orientation. At Ephesus there are foundations of three temples lying one over the other. The middle one was the work of CRŒSUS. The lowest of the three may quite well have been as early as 715 B.C., and in five examples where we have architectural remains standing, namely, the temple at Corinth, both those at Ægina, the later Heræum at Argos, and the Metroum at Olympia, I see no reason for dissenting from the dates derived from the orientation. And even in the case of the Heræum at Olympia, it may be noticed that although the date of the existing structure (unquestionably the most ancient example of temple architecture in Greece, of which any remains are standing above ground) would scarcely, consistently with architectural analogy, be placed so early as the middle of the fifteenth century B.C., yet it might have been almost coëval with the establishment of the Dorian supremacy in the Peloponnesus in the middle of the eleventh century, and if at that date it had been built parallel to the lines provided for a more ancient shrine of the orientation date, the statue might have been still illuminated by the rising Sun, preceded by the same time-warning star—Spica. Both sun and star, indeed, would have demanded a certain amount of change of amplitude, but still within the limits of the eastern opening.

In addition to the above list, is the great temple of Eleusis. Its orientation lies just within the solstitial limits. There could scarcely have been any observation of stars towards the west, as in that direction the temple is completely blocked by the terraced rock; but it may have had openings to the north and south as well as the east. I do not find any heliacal star eastwards bright enough to be of any service, but a very conspicuous one, Fomalhaut, would set towards the south in the direction of a cross axis of the temple at an epoch of about 1300 B.C., but neither is this heliacal.

There can be little doubt, however, that the star which was peculiarly connected with the worship of this temple was the great dog-star, Sirius—a star which has been shown by Mr. LOCKYER to have played a great part in the orientation of some of the Egyptian temples. The date corresponding to the rising of this star at an amplitude identical with that of the temple, would be 2100 B.C., and as the hour would be near to midnight at the time of year when the Eleusinian mysteries (as known by later records) were celebrated, we may use this circumstance with great probability to assist in the inquiry. As respects the amplitude of the star, it is not unlikely that when the temple was built, the axis, like some of the temples at Athens, was so directed that the star should first show itself towards the northern jamb of the eastern opening, as this would allow it to traverse a more extensive arc than if it rose exactly on the axis. If we assign two degrees for this deviation, the date would be 1400 B.C., and the day of the month for midnight rising September 13.

The elements so assigned are as follows:—

Eleusis. Lat. 38° 2′ 15″.

Name of Temple.	Orientation angle.			Elements.	Name of Star.
Temple of Ceres	296° 51′	A	Amplitude	24° 51′ S.	Sirius, rising
		B	Corresponding altitude	2°	
		C	Declination	−18° 0′ 56″	
		D	Hour angles	$4^h\ 49^m\ 52^s$	
		E	R.A.	$4^h\ 15^m$	
		F	Approximate date	1400 B.C., September 13	

It should be observed that Sirius was not at the period under discussion favourably situated for determination of date, as derived from its precessional movement, because a small variation of amplitude and its corresponding effect upon the declination would produce a very large alteration in respect of time, and we have here no help from the sun's R.A. I think, however, we may fairly conclude that the date must have been somewhere between 1300 B.C. and 1500.

The day of the month on which the mysteries commenced in historical times is considered to have been September 16. This it will be seen is very close to the calculation derived from the midnight rising of Sirius as above given. The dates when the sun would rise on the axis of the temple would be either January 21 or November 21, which days do not seem to have had any association with the mysteries.

There remain also amongst the temples which I examined, five, of which the orientation lies between the solstitial limits, all of them apparently of late foundation, which do not appear to have any connection with heliacal stars; namely, at Athens, the Theseum, the later Erechtheum, the later Temple of Bacchus, and the existing Jupiter Olympius; and at Ephesus the latest of the three great temples of Diana. The orientation of these temples can supply no data for pointing out the year of their foundation, but it may throw light upon the month and day of their principal festivals.

The Theseum, with orientation angle 283° 6′ 2″, had the sun rising over Hymettus in the direction of its axis on March 2 or October 10. The festival of the Thesea is considered to have been held on October 8 or 9.

The axis of the later Erechtheum—orientation angle 265° 9′ 22″—was visited by the rising sunbeam on April 4 or September 7. Or, if the Sun at its rising was made to appear towards the northern door jamb, which as before observed seems to have been the arrangement in some other of the Athenian temples, and if the amplitude be increased by 2°, the days would be either April 8 or September 3. The third of September is the assigned date of a great festival—the Niceteria—held in this temple.

The later Temple of Bacchus, at Athens, of which the orientation angle is 255° 49′ 30″, would have received the first sunbeam along its axis on April 23 or August 18. That of Jupiter Olympius is 270° 5′ 2″. The sunrise dates would be March 26 or September 15. The axis of the later temple at Ephesus, of which the orientation angle is 284° 35′ 17″, would have coincided with sunrise on February 28 or October 14.

Lastly, amongst the temples which I examined I met with five examples outside the solstitial limits, temples, namely, which at no time of the year could have had a sunbeam coinciding with their axes—and with these I may associate two others which I did not visit and of which I can give no exact measures, and a site which I did visit, but where no antique remains are actually visible, namely, that of the former Temple of Venus, at Ancona.* If, as may be presumed, the present cathedral is built on the old foundations, the orientation would in this case also be extra-solstitial.

These temples were very probably arranged, as many of those in Egypt appear to have been, so as to coincide with the rising or setting of some star. In five cases of those referred to, the peculiarity of the site must have influenced the direction of the axis. I have not yet subjected any of these to sidereal discussion; the list is as follows:—

1. The great temple at Delphi, which has not yet been excavated, and where con-

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* Ante domum Veneris quam Dorica sustinet Ancon. JUV., 'Sat.' IV.

sequently I was only enabled to take the bearings of certain ancient walls, which appear to have had some connection with the temple, and which cannot fail to be carefully co-related to it when the excavations under the French Archæological School are sufficiently advanced to show how the temple actually stood.

The well-known wall of the inscriptions, or rather the stylobate of the Stoa of the Athenians in front of it, has an orientation angle of 231° 34′ 31″, and the altitude of the mountain at right angles to it is 3° 10′.

2. The Doric temple at Mycenæ, which rests partly on the ruins of an archaic palace, has an orientation of 173° 20′. The mountain to the north is very near and high; that to the east I found to be 7° 44′, and that to the west 2° 40′. To the south it was not measured.

3. The Cabeirion temple, near Thebes, has an orientation angle of 186° 27′ 45″. In this temple there is a cross wall at a different angle to what it would be if square with the main axis, viz., 257° 57′ 35″, looking east. There is a clear view to the north, where the mountain's altitude is 1° 37′, but the temple is inclosed on the three other sides by near hills. The lowest is that to the east, with an altitude of 8° 38′.

4. In their search for the Athenian Agora, the German Archæological School discovered a small temple lying in the direction of the valley, between the Areopagus and the Pnyx. Its orientation angle is 317° 28′ 21″. I did not ascertain the altitude of the hills around. The clearest view would have been towards the north-west, where the altitude of Mt. Parnes would be less than 3°. It is a very small temple, and its position could not but have been determined by the valley in which it is situated.

5. The temple of Diana Propylæa at Eleusis has an orientation angle of 313° 43′ 13″. The mountain heights are much the same as those opposite the great temple already given.

6 and 7. The other two temples referred to above are those of Elateia, in Bœotia, and Vakklia, in Arcadia, near where the Ladon falls into the Alpheus. The latter temple is described as built upon the ridge of a hill.

8. The walls of the cathedral at Ancona, which, as already observed, it is reasonable to suppose rest upon the foundations of the Temple of Venus, have an orientation angle of 223° 11′ 23″. The east and west views would have been practically unobstructed.

It is quite possible that, in the case of 2, 3, 6, and 7, the Sun, at its rising, could have been admitted by an eastern doorway, as at Bassæ.

If the temples be arranged according to the heliacal stars which have been shown in the above pages to have coincided within close limits with their axes, they will be grouped as follows, in order of the dates which have been assigned to the different temples :—

		B.C.	
Spica	The Heræum of Olympia	1445	Rising star.
	Nike Apteros, Athens	1130	Setting ,,
	T. of Themis, Rhamnus	1092	Rising ,,
	,, Minerva, Tegea	1075 uncertain*	Rising ,,
	,, Nemesis, Rhamnus	747	Rising ,,
	,, Apollo, Bassæ	728	Rising ,,
	,, Diana, Ephesus	715	Rising ,,
α Arietis	Jupiter Olympius, Athens	1202	Rising ,,
	T. of Jupiter, Olympia	790	Rising ,,
	T. at Platæa	650	Setting ,,
	T. of Jupiter, Megalopolis	605	Setting ,,
	T. at the harbour, Ægina	550	Setting ,,
	The Metroum, Olympia	360	Setting ,,
The Pleiades (η Tauri)	The archaic Temple of Minerva, Athens	1530	Rising ,,
	The Asclepieion, Epidaurus	1275	Rising ,,
	The Hecatompedon, Athens	1150	Rising ,,
	The earlier T. of Bacchus, Athens	1030	Rising ,,
	T. of Minerva, Sunium	845	Setting ,,
Antares	Earlier Erechtheum, Athens	1070	Setting ,,
	T. at Corinth	770	Setting ,,
	T. on the Mountain, Ægina	630	Setting ,,
Constellation Aquarius (ζ Aquarii)	T. at Nemea (Jupiter)	1040	Setting ,,
	Diana Brauronia, Athens	580	Rising ,,
	Later Heræum, Argos	425	Rising ,,
γ Pegasi	T. at Kardaki, Corfu	875	Rising ,,
	T. of Despoina, Lycosura	650	Rising ,,
β Scorpii	T. of Diana, Epidaurus	780	Setting ,,
Sirius	T. of Ceres, Eleusis	1400	Rising ,,

If the list be arranged according to the time of year on which the rising Sun coincided with the orientation, they would group as follows :—

Feb. 21.	Diana Brauronia, Athens.
,, ,,	Later T. at Argos.
Mar. 10.	Kardaki, Corfu.
,, 16.	Despoina at Lycosura.
,, 17.	Nike Apteros, Athens.
,, 30.	Jupiter Olympius, Athens.
April 6.	T. of Jupiter at Olympia.
,, 20.	Archaic T. at Athens.
,, 26.	Hecatompedon, Athens.
,, 29.	Earlier T. of Bacchus, Athens.
,, ,,	Older Erechtheum, Athens.
,, ,,	Asclepieion, Epidaurus.
,, ,,	T. of Diana, Epidaurus.
May 6.	T. at Corinth.
May 7.	T. on Mountain, Ægina.
July 27.	T. of Jupiter, Nemea.
Sept. 12.	The Heræum, Olympia.
,, 13.	T. of Ceres, Eleusis.
,, 17.	,, Themis, Rhamnus.
,, 18.	,, Minerva, Tegea (?).*
,, 22.	,, Nemesis, Rhamnus.
,, 25.	,, Diana, Ephesus.
,, ,,	,, Apollo, Bassæ.
Oct. 6.	,, Jupiter, Megalopolis
,, 9.	T. at Platæa.
,, ,,	T. near harbour, Ægina.
,, ,,	The Metroum, Olympia.
,, 21.	T. of Minerva, Sunium.

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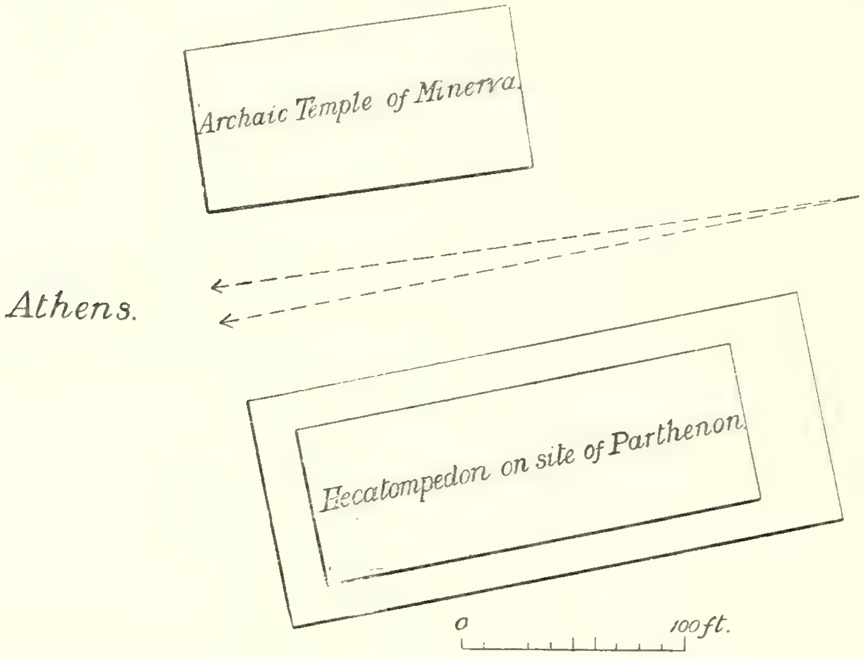
* See note, p. 817.

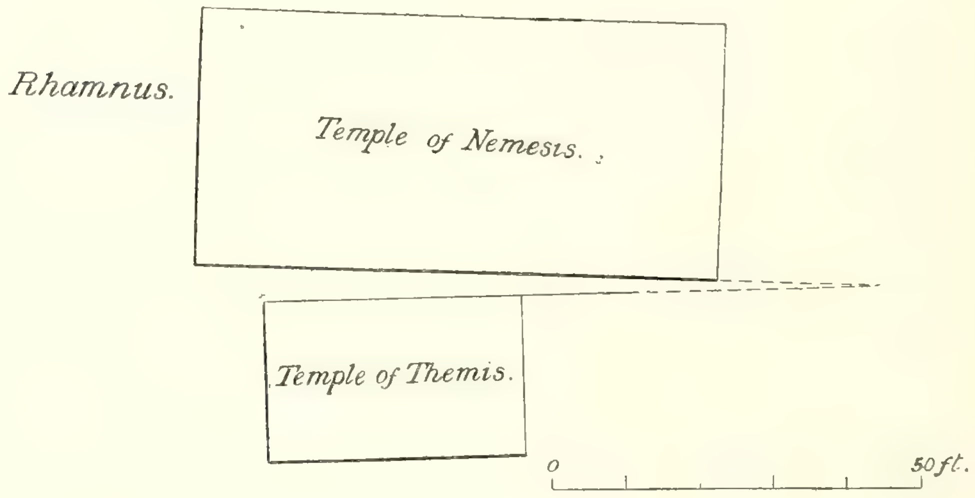
It will be seen from the above list that the dates resolve themselves almost entirely into spring and autumn festivals, as if for intercession for a favourable harvest to come, and for thanks for the harvest granted.

Fig. 1.


Archaic Temple of Minerva.
Athens.
Hecatompedon on site of Parthenon.
0
100 ft.

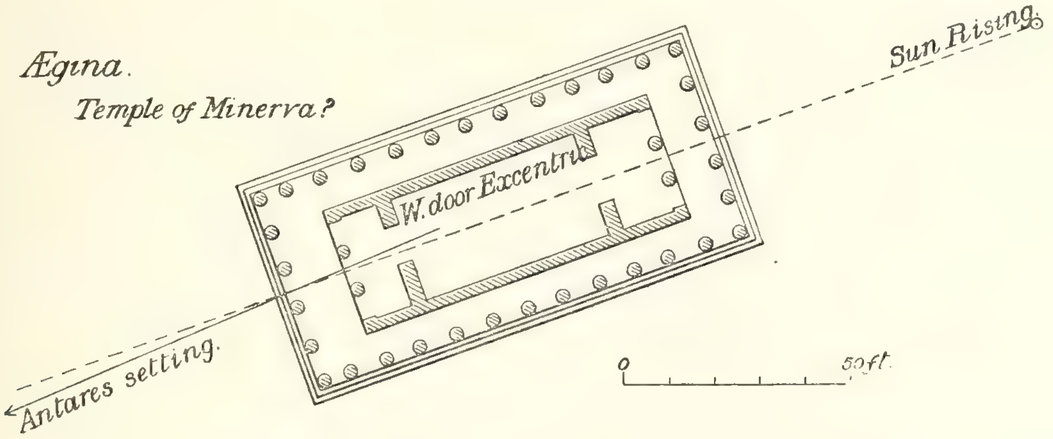
Block plan of the two principal temples of Minerva on the Athenian Acropolis. The northern, which was the more ancient of the two, appears to have been oriented so as to observe the Pleiades heliacally about B.C. 1500. The southern temple, which occupied part of the site of the present Parthenon, had a more northerly orientation so as to follow the more northerly declination of the same star group,

Fig. 2.



The two temples at Rhamnus, having the same cult, showing that the northern of the two, which was built later than the other, had the more southerly orientation. In this case, a star, Spica, which was changing its declination from north to south, had to be followed.

Fig. 3.



Plan of the temple on the mountain at Ægina. In this case, a setting star having a more southerly amplitude than the orientation had to be observed. The western door of the cella of the temple is built considerably out of the centre so as to enable the star to be observed from the adytum.

[Since this paper was first communicated to the Society, I have examined the five cases of extra-solstitial temples, of which I had taken particulars, and have found in them much to confirm the general theory.

To begin with two that stand nearly north and south—one being the Doric temple, built on the Acropolis of Mycenæ, at some period much subsequent to the great Cyclopean walls (for it is built on the ruins of the palace of the ancient kings) and the other, the Cabeirion temple to the westward of Thebes, and about three miles distant from it.

It may be presumed that both these temples had eastern doorways for the admission of the Sun at its rising, to the interior of the cella, as was the case at Bassæ, although the evidence which was preserved at Bassæ is wanting in these two examples; but at Mycenæ, at any rate, it may be inferred from the position of the mountains round about the temple which would have blocked any stars in the directions of the axis, both north and south, whilst the eastern flank is much less obstructed. Working upon that theory, we may obtain from the orientation the following solution, which agrees very well with what is known respecting the history of the place.

Mycenæ. Lat. 37° 43′ 20″.

Orientation angle of assumed eastern doorway.			Stellar elements.	Solar elements.
263° 19′ 40″	A	Amplitude	+ 6° 40′ 20″	+ 6° 40′ 20″
	B	Corresponding altitude . .	8°	7° 22′
	C	Declination	+ 10° 8′ 44″	+9°·45′ 51″
	D	Hour angle	$5^h\ 50^m\ 25^s$	$7^h\ 28^m\ 49^s$
	E	Depression of Sun	..	11°
	F	R.A.	$23^h\ 53^m\ 43^s$	$1^h\ 32^m\ 7^s$
	G	Approximate date	B.C. 540	
			α Arietis rising	

The Cabeirion temple, near Thebes, is surrounded by hills, except towards the north. Of the other three sides, the eastern is least obstructed, so that the Sun could enter the temple by an eastern door about half-an-hour after sunrise. In this direction, however, I can find no suitable star which could be combined with the Sun heliacally at any permissible epoch—and this, whether it be sought at right angles to the axis of the temple, or in the direction of the cross wall which has been mentioned. But a time-warning star, northwards on the true axis of the temple, can be found, and a very significant one it is, namely, γ Draconis. This star seems to have been a favourite at the Egyptian Thebes—as well as other places in that country—as has been shown by Mr. LOCKYER, in 'Nature,' for February 18, 1892, and in the 'Nineteenth Century,' No. 185, p. 46. It is well-known that the Bœotian Thebans called their city "The City of the Dragon," claiming their descent from the fabled dragon's teeth sown by their founder. In this temple, therefore, we have astronomic confirmation of the tradition that CADMUS introduced the worship of the Egyptian as well as that of the Phœnician deities.

As γ Draconis is only a second magnitude star, it is more likely to have been used as a time-warner, than for such purposes as the more brilliant stars may be supposed to have been used at dead of night for producing a mysterious glow by reflection from polished surfaces according to the hint given by HERODOTUS when speaking of a temple at Tyre :* "Καὶ ἐν ἀυτῶ ἦσαν στῆλαι δυό, ἡ μὲν χρυσοῦ ἀπέφθου ἡ δὲ σμαρὰγδου λίθου, λάμποντος τὰς νύκτας μέγαθος." But, although this might have succeeded with the light from Sirius, Arcturus, or Capella, γ Draconis is more likely to have waited on the Sun as a time-warner. If so, we may base our calculations on its first appearance at the western limit of the north opening, thus anticipating the actual orientation angle by about two degrees. This would take place shortly after the lower transit of the star—below the pole—for at the epoch we are considering, it would not have dipped below the horizon, as it does now. The elements would be as follows :—

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* 'HEROD.,' Book II., p. 44.

Thebes. Lat. 38° 19′ 30″ N.

Orientation angle of assumed eastern doorway.			Stellar elements.	Solar elements.
276° 27′ 45″	A	Amplitude	+ 85° 50′	− 6° 27′ 45″
	B	Corresponding altitude . .	3°	8° 20″
	C	Declination	+ 54° 28′ 13″	+ 0° 21′ 12″
	D	Hour angle	..	$6^{h}\ 41^{m}\ 28^{s}$
	D′	In case of star reckoned from lower transit	$0^{h}\ 28^{m}\ 41^{s}$	..
	E	Sun's depression	..	11°
	F	R.A.	$16^{h}\ 46^{m}\ 50^{s}$	$11^{h}\ 56^{m}\ 45^{s}$
	G	Approximate date	1160 B.C., September 20	
			γ Draconis rising after transit below the pole	

As respects the time of year, the date is suggestive, since it agrees almost exactly with that of the celebration of the Eleusinian mysteries.

The remaining three of the five temples in this group cannot have been solar temples at all, as their orientation lies outside the solstitial limits; neither could the sunrise have been admitted at an eastern doorway. The question to be considered is whether there is reasonable probability of a bright star having been used in the manner suggested above and explained by the passage from HERODOTUS there quoted and further confirmed by parallel cases in Egypt. It seems essential to a solution of this character that the star which is found to fit the orientation should be among the most brilliant.

At Eleusis, besides the Temple of the Great Mysteries, there is the Temple of Diana Propylæa, of which the orientation angle is 133° 43′ 13″, or 313° 43′ 13″, according as we take the axis at its north-west or south-east direction. To the south-east no suitable star offers itself. In the other direction there are two, both first magnitude stars, namely, Arcturus and Capella. I give the elements of both ; but, for a reason to be mentioned, I think Capella, which almost equals Arcturus in brightness is the most probable.

Eleusis. Lat. 38° 2′ 15″.

Orientation angle.					
133° 43′ 13″ or 313° 43′ 13″	A	Amplitude	+ 43° 43′ 13″	A	+ 43° 43′ 13″
	B	Corresponding altitude .	4° 30′	B	4° 30′
	C	Declination	+ 36° 13′ 39″	C	+ 36° 13′ 39″
	D	Hour angle	$7^h\ 46^m\ 33^s$	D	$7^h\ 54^m\ 49^s$
	E	R.A.	$12^h\ 9^m\ 0^s$	E	$1^h\ 54^m\ 49^s$
	F	Approximate date . . .	770 B.C., July 16–17	F	1010 B.C., Feb. 18-19
			Arcturus setting midnight		Capella setting midnight

The *Little Mysteries,* Eleusinian, are considered to have been celebrated about the 19th February, which exactly agrees with the appearance of Capella at midnight on the axis of this temple. The orientation day of the solar Temple of *Diana Brauronia* at Athens has been shown to have been the 21st of the same month.

At Athens there is a small temple having this kind of orientation lately discovered by the German archæologists in searching for the ancient Agora. It also points nearly north-west and south-east. This orientation appears to admit of a solution similar to that of the temple at Eleusis last mentioned, through Arcturus setting about B.C. 700, but I reserve the list of the elements as the local altitudes require confirmation.

The last temple on the list is that of the presumed Sanctuary of Venus at Ancona. The commanding site occupied by the mediæval Duomo is stated to have been that of the ancient temple, and there seems no reason to doubt this tradition. In that case the walls of the church would probably rest upon the old foundations. It seemed, therefore, worth while to examine their orientation, which gave for this angle 223° 11′ 23″, which is equivalent to a northern amplitude of 46° 48′ 37″, and, combining this with the latitude and the usual altitude for a star, elements are obtained as below.

Ancona. Lat. 43° 37′ 7″.

Orientation angle.				
223° 11′ 23″	A	Amplitude	46° 48′ 37″	Arcturus rising
	B	Corresponding altitude .	3°	
	C	Declination	+ 34° 16′ 38″	
	D	Hour angle	$8^h\ 16^m\ 50^s$	
	E	R.A.	$12^h\ 15^m\ 20^s$	
	F	Approximate date . . .	620 B.C.; if midnight, Nov. 16–17	

Besides the temples already mentioned, I have examined, with the help of approximate data, several other examples, especially from Sicily and Naucratis. The indications appear to be favourable, but, as I have not sufficiently exact material to work upon, it seems better not to make any further reference to them here.—April 17, 1893.]

ADDENDUM.—August 4, 1893.

Since the date on which I brought the above list before the Royal Society, I have received, through the kindness of Dr. WALDSTEIN, the relative position of the earlier and later Heræum at Argos, from which the following elements have been deduced for the former :—

Argos. Lat. 37° 41′ 16″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Ancient Argive Heræum	287° 13′ 20″	A	Amplitude of Star or Sun	−15° 4′ 58″	−17° 13′ 20″	Antares, rising
		B	Corresponding altitude	3° 0′ E.	2° 30′ E.	
		C	Declination	−10°	−11° 58′ 24″	
		D	Hour angles	$5^h\ 13^m\ 4^s$	$6^h\ 15^m\ 40^s$	
		E	Depression of Sun when Star was heliacal	..	10°	
		F	R.A.	$12^h\ 51^m\ 48^s$	$13^h\ 54^m\ 24^s$	
		G	Approximate date . .	B.C. 1760		

Instead of the elements given above (p. 812) for Tegea, I propose an amendment based on the following considerations, namely :—In the plan published by the German archæologists ('Mittheilungen des Archaeol. Instituts,' 1883) are shown some foundations, named "*Antikes Fundament*," close to the eastern front of the main temple, which make an angle—by measurement, 4° 11′—in a more southerly direction than the other. Architectural fragments, also, of a very ancient character, were found. Judging from this, the later temple appears to have been built so as to follow the same star as the earlier, the elements being as follows :—

Tegea. Lat. 37° 27′ 45″.

Name of Temple.	Orientation angle.			Stellar elements.	Solar elements.	Name of Star.
Old Temple of Minerva	271° 23′ 30″	A	Amplitude of Star or Sun	+3° 0′	−1° 23′ 30″	α Arietis, rising
		B	Corresponding altitude	3° 0′	4° 0′ 0″	
		C	Declination	+4° 12′ 15″	+1° 19′ 52″	
		D	Hour angles	$5^{h}\ 57^{m}\ 45^{s}$	$7^{h}\ 9^{m}\ 23^{s}$	
		E	Depression of Sun when Star was heliacal	. .	12° 12′	
		F	R.A.	$23^{h}\ 0^{m}\ 22^{s}$	$24^{h}\ 12^{m}\ 0^{s}$	
		G	Approximate date . .	B.C. 1580,	March 22-23	
Later Temple of Minerva	267° 12′ 30″	A	Amplitude of Star or Sun	+6° 31′ 6″	+2° 47′ 30″	α Arietis, rising
		B	Corresponding altitude	3° 0′ E.	2° 22′ E.	
		C	Declination	+7° 0′ 0″	3° 39′ 12″	
		D	Hour angles	$6^{h}\ 6^{m}\ 21^{s}$	$7^{h}\ 14^{m}\ 13^{s}$	
		E	Depression of Sun when Star was heliacal	. .	12° 30′	
		F	R.A.	$23^{h}\ 25^{m}\ 20^{s}$	$0^{h}\ 12^{m}\ 30^{s}$	
		G	Approximate date . .	B.C. 1080		

XVII. *Memoir on the Theory of the Compositions of Numbers.*

By P. A. MacMahon, *Major R.A., F.R.S.*

Received November 17—Read November 24, 1892.

§ 1. *Unipartite Numbers.*

1. Compositions are merely partitions in which the order of occurrence of the parts is essential; thus, while the partitions of the number 3 are (3), (21), (111), the compositions are (3), (21), (12), (111).

The enumerations of the compositions of a number n into p parts, zeros excluded, is given by the coefficient of x^n in the expansion of

$$(x + x^2 + x^3 + \ldots)^p;$$

this expression may be written

$$\left(\frac{x}{1-x}\right)^p,$$

and the coefficient of x^n is seen to be

$$\binom{n-1}{p-1}.^*$$

The generating function of the total number of compositions of n is

$$\sum_1^\infty (x + x^2 + x^3 + \ldots)^p = \frac{x}{1-2x},$$

hence the number in question is

$$2^{n-1}.$$

2. If the parts of the compositions are limited not to exceed s in magnitude, the generating function of the number into p parts is

$$(x + x^2 + x^3 + \ldots + x^s)^p = x^p\left(\frac{1-x^s}{1-x}\right)^p,$$

---

* In the continental notation I write $\binom{n}{p}$ for $\frac{n!}{p!(n-p)!}$.

and herein the coefficient of x^n is

$$\binom{n-1}{p-1} - \binom{p}{1}\binom{n-s-1}{n-s-p} + \binom{p}{2}\binom{n-2s-1}{n-2s-p} - \ldots$$

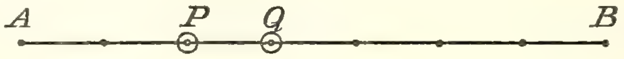
The number of parts being unrestricted the generating function is

$$\sum_p x^p \left(\frac{1-x^s}{1-x}\right)^p = \frac{x(1-x^s)}{1-2x+x^{s+1}}.$$

The expression $\binom{n-1}{p-1}$ is unchanged by the substitution of $n-p+1$ for p; hence the numbers of compositions of n into p parts and into $n-p+1$ parts are identical.

3. The graph of a number n is taken to be a straight line divided at $n-1$ points into n equal segments.

The graph of a composition of the number n is obtained by placing nodes at certain of these $n-1$ points of division.



AB being the graph of the number 7, for the representation of the composition (214), nodes are placed at the points P, Q, so that in moving from A to B by steps proceeding from node to node, 2, 1, and 4 segments of the line are passed over in succession. Although strictly speaking the initial and final points A, B are nodes on the graphs of all the compositions, it is only the inter-terminal nodes that will be considered in what follows, as appertaining to the graph.

The number of parts in the composition exceeds by unity the number of nodes on the graph.

For a composition of n into p parts we can place nodes at any $p-1$ out of the $n-1$ points of the graph of the number. The number of such composition graphs is at once seen to be

$$\binom{n-1}{p-1},$$

and further, since each of the $n-1$ points of the number graph is or is not the position of a node, the total number of composition graphs is

$$2^{n-1}.$$

4. Associated with any one graph, there is another graph obtained by obliterating the nodes and placing nodes at the points not previously occupied.

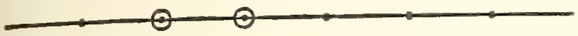
These graphs are said to be conjugate.

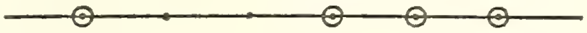
If a graph denotes a composition of n into p parts, the conjugate graph denotes a composition of n into $n-p+1$ parts.

This notion supplies, in consequence, a graphical proof of the theorem of Art. 2.

Compositions of a number are conjugate when their graphs are conjugate.

E.g. The conjugate graphs





yield the conjugate compositions

(214) (13111)

The composition conjugate to a given composition may be written down, without constructing the graph, by the rules about to be explained.

The composition must first be prepared—

(i.) By writing successions of units in the power symbolism; for example, s successive units must be written 1^s;

(ii.) By intercalating 1^0 between each successive pair of non-unitary parts; thus, when aa or ab occur (a and b being superior to unity) we have to write $a1^0a$, $a1^0b$ respectively.

For the moment, call the non-unitary parts and the symbolic powers of unity the "elements" of the composition.

When an element does not occur at either end of the composition it is called "non-terminal," when at one end only "terminal," and when at both ends (thus constituting the entire composition), "doubly terminal."

The rules for procession to the conjugate are :—

I. If m or 1^m be doubly terminal, substitute 1^m for m or m for 1^m.

II. If m or 1^m be terminal, substitute 1^{m-1} for m or $m+1$ for 1^m.

III. If m or 1^m be non-terminal, substitute 1^{m-2} for m or $m+2$ for 1^m.

The composition thus obtained is in the "prepared" form and can be transformed to the ordinary form.

E.g. To find the conjugate of (231141), take the "prepared" form

$$(21^{0}31^{2}41),$$

and, beginning from the left, by

Rule	II.	For	2	substitute	1,
,,	III.	,,	1^0	,,	2,
,,	III.	,,	3	,,	1,
,,	III.	,,	1^2	,,	4,
,,	III.	,,	4	,,	1^2,
,,	II.	,,	1	,,	2,

resulting in the conjugate composition,

$(12141^{2}2)$, or non-symbolically (1214112)

An examination of the rules shows that they are reversible, and that the process gives a one-to-one correspondence between compositions of n into p and $n - p + 1$ parts.

A composition in general has, when prepared for conjugation, four different forms, viz. :—

$$(1)\ a_1 1^{\alpha_1} a_2 1^{\alpha_2} \dots a_{s-1} 1^{\alpha_{s-1}} a_s.$$

$$(2)\ a_1 1^{\alpha_1} a_2 1^{\alpha_2} \dots a_{s-1} 1^{\alpha_{s-1}} a_s 1^{\alpha_s}$$

$$(3)\ 1^{\alpha_1} a_2 1^{\alpha_2} \dots a_{s-1} 1^{\alpha_{s-1}} a_s$$

$$(4)\ 1^{\alpha_1} a_2 1^{\alpha_2} \dots a_{s-1} 1^{\alpha_{s-1}} a_s 1^{\alpha_s};$$

in all the four forms $a_1, a_2, a_3, \dots a_s$ may have any positive integral values superior to unity. The numbers, $\alpha_1, \alpha_2, \dots \alpha_s$, may have any positive integral values, including zero, with the exceptions,

In form (2) α_s cannot be zero.
(3) α_1 ,,
(4) α_1 and α_s ,,

The conjugates of the forms are

$$(1)\ 1^{a_1-1} \,.\, \alpha_1 + 2 \,.\, 1^{a_2-2} \,.\, \alpha_2 + 2 \dots 1^{a_{s-1}-2} \,.\, \alpha_{s-1} + 2 \,.\, 1^{a_{s-1}-1},$$

$$(2)\ 1^{a_1-1} \,.\, \alpha_1 + 2 \,.\, 1^{a_2-2} \,.\, \alpha_2 + 2 \dots 1^{a_{s-1}-2} \,.\, \alpha_{s-1} + 2 \,.\, 1^{a_s-2} \,.\, \alpha_s + 1,$$

$$(3)\ \alpha_1 + 1 \,.\, 1^{a_2-2} \,.\, \alpha_2 + 2 \dots 1^{a_{s-1}-2} \,.\, \alpha_{s-1} + 2 \,.\, 1^{a_s-1},$$

$$(4)\ \alpha_1 + 1 \,.\, 1^{a_2-2} \,.\, \alpha_2 + 2 \dots 1^{a_{s-1}-2} \,.\, \alpha_{s-1} + 2 \,.\, 1^{a_s-2} \,.\, \alpha_3 + 1.$$

5. Two compositions are said to be inverse (the one of the other) when the parts of the one, read from left to right, are identical with those of the other when read from right to left.

A composition may therefore be self-inverse.

In the graph of a self-inverse composition, the nodes must be symmetrically placed with respect to the extremities of the graph. If the number be even, the number of segments of the graph is even, and the two central nodes (nodes nearest to the centre of the graph) may be coincident, or they may include 2, 4, or any even number of segments. A self-inverse composition of an even number, say $2m$, into an even number, say $2p$, of parts, can only occur when the two central nodes of the graph are coincident and, attending to one side only of this node, we find that the number of self-inverse compositions of the number $2m$, composed of $2p$ parts, is equal to the number of compositions of m composed of p parts. In a notation, which is self-explanatory, we may write

$$\text{SIC}\,(2m, 2p) = \text{C}\,(m, p) = \binom{m-1}{p-1}.$$

Next consider the self-inverse compositions of $2m$ into an uneven number, $2p-1$, of parts. The two central nodes must be distinct, and may include any even number of segments. If this even number be 2κ the corresponding number of self-inverse compositions is equal to the number of compositions of $m-\kappa$ into $p-1$ parts.

Hence

$$\text{SIC}\,(2m, 2p-1) = \text{C}\,(m-1, p-1) + \text{C}\,(m-2, p-2) + \ldots + \text{C}\,(p-1, p-1),$$

or

$$\text{SIC}\,(2m, 2p-1) = \text{C}\,(m, p) = \binom{m-1}{p-1}.$$

Self-inverse compositions of uneven numbers occur only when the number of parts is uneven, and it is easy to prove that

$$\text{SIC}\,(2m-1, 2p-1) = \text{C}\,(m, p) = \binom{m-1}{p-1}.$$

Hence, without restriction of the number of parts,

$$\text{SIC}\,(2m) = \text{SIC}\,(2m+1) = \text{C}\,(m+1) = 2^m.$$

This completes the enumeration of the self-inverse compositions.

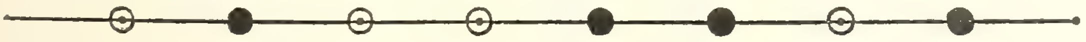
6. Two compositions which are at once conjugate and inverse, may be termed "inverse conjugates."

A composition whose conjugate is its own inverse is said to be "inversely conjugate."

Inversely conjugate compositions of a number n which have p parts, can occur only when $p = n - p + 1$, or $n = 2p - 1$, an uneven number.

The inversely conjugate compositions of $2m+1$ are composed of $m+1$ parts.

Consider a graph in which white and black nodes have reference to the two inverse conjugates respectively.



The black nodes are placed to the right and left of the centre of the graph in a manner similar to the white nodes to the left and right. If on the right there are

s black nodes and $m - s$ white nodes; to the left there are similarly placed s white nodes and $m - s$ black nodes.

Of the graph of the number there are m points to the right of the centre at which white and black nodes can be placed in 2^m distinct ways. Hence it at once follows that the number $2m + 1$ possesses in all 2^m inversely conjugate compositions. Otherwise, we may say that the number of inversely conjugate compositions of $2m + 1$ is equal to the number of compositions of $m + 1$.

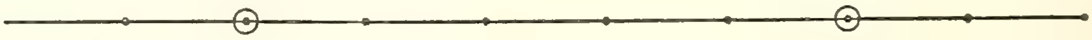
It will be observed that the number $2m + 1$ has precisely the same number, 2^m, of self-inverse compositions.

There is, in fact, a one-to-one correspondence between the compositions of $2m + 1$, which are inversely conjugate, and those which are self-inverse.

To explain this, take the graph last represented. Read according to the black nodes we obtain the inversely conjugate composition

(23121) of the number 9.

To proceed to the corresponding self-inverse composition obliterate the black nodes to the right of the centre, and also the white nodes to the left of the centre Substituting white nodes for the black nodes then remaining we have the graph



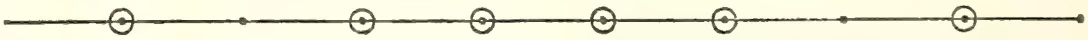
of the self-inverse composition

(252).

Again, reading the original graph according to white nodes, we have the inversely-conjugate composition

(12132),

and proceeding, as before, with the exception that black and white nodes are obliterated on the left and right of the centre respectively, we obtain the graph



of the self-inverse composition

(1211121).

The process is a general one, and shows that we can always pass from a composition which is inversely conjugate to one which is self-inverse.

E.g., of the number 7 we have the correspondence

Inversely conjugate,	Self inverse.
(4111)	(7)
(3211)	(313)
(2221)	(232)
(2131)	(21112)
(1312)	(151)
(1222)	(12121)
(1123)	(11311)
(1114)	(1111111)

The general form of an inversely conjugate composition is

$$a_1 1^{a_1} a_2 1^{a_2} a_3 1^{a_3} \ldots \alpha_3 + 2 \ . \ 1^{a_3-2} \ . \ \alpha_2 + 2 \ . \ 1^{a_2-2} \ . \ \alpha_1 + 2 \ . \ 1^{a_1-1}$$

in its form prepared for conjugation.

7. The compositions of a number, m, give rise to the compositions of $m + 1$ by rules somewhat similar to those in ARBOGAST'S method of derivations. The rules are obvious as soon as stated.

Each composition of the number m gives rise to two compositions of the number $m + 1$.

I. By prefixing the part unity.

II. By increasing the magnitude of the first part by unity.

All the compositions thus obtained are necessarily distinct. As an example see the subjoined scheme for passing from the compositions of 3 to those of 4.

111 12 21 3.

1111 211 112 22 121 31 13 4.

If the conjugates of these two lines be taken, the result is the same as the two lines inverted. We have the theorem, easily proved: "The conjugate of the $\substack{\text{first}\\\text{second}}$ derivative of a composition is the $\substack{\text{second}\\\text{first}}$ derivative of the conjugate composition."

8. The theory of the compositions of numbers is closely connected with the theory of the *perfect partitions* of numbers.[*] The connection is between the compositions of all multipartite numbers and the perfect partitions of unipartite numbers. The enumeration of the compositions of a single multipartite number enumerates also the perfect partitions of an infinite number of unipartite numbers. There is, moreover,

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[*] "The Theory of Perfect Partitions of Numbers and the Compositions of Multipartite Numbers," The Author, 'Messenger of Mathematics.' New Series, No. 235. November, 1890.

as was shown, *loc. cit.*, a one-to-one correspondence between the compositions of the unipartite number m and the perfect partitions, comprising m parts, of the whole assemblage of unipartite numbers.

Defining a perfect partition of a number to be one which contains one, and only one, partition of every lower number, it was shown that if

$$(\alpha, \beta, \gamma, \delta, \ldots)$$

be any composition of the number

$$\alpha + \beta + \gamma + \delta + \ldots,$$

the partition,

$$\{1^{\alpha} \,.\, (1+\alpha)^{\beta} \,.\, (1+\alpha \,.\, 1+\beta)^{\gamma} \,.\, (1+\alpha \,.\, 1+\beta \,.\, 1+\gamma)^{\delta} \ldots\},$$

the exponents being symbolic, denoting repetitions of parts, is a perfect partition of the number

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)\ldots - 1.$$

§ 2. *Multipartite Numbers.*

9. The multipartite number $\overline{\alpha\beta\gamma\ldots}$ may be regarded as specifying $\alpha+\beta+\gamma+\ldots$ things, α of one sort, β of a second, γ of a third, and so forth.

To illustrate partitions and compositions of such numbers, I write down those appertaining to the bipartite number $\overline{21}$.

Partitions.	Compositions.
$(\overline{21})$	$(\overline{21})$
$(\overline{20}\,\overline{01})$	$(\overline{20}\,\overline{01})$, $(\overline{01}\,\overline{20})$
$(\overline{11}\,\overline{10})$	$(\overline{11}\,\overline{10})$, $(\overline{10}\,\overline{11})$
$(\overline{10}^2\,\overline{01})$	$(\overline{10}^2\,\overline{01})$, $(\overline{10}\,\overline{01}\,\overline{10})$, $(\overline{01}\,\overline{10}^2)$

I speak of the parts of the partition or composition, and observe that each part is a multipartite number of the same nature, or order of multiplicity, as the number partitioned.

There is (*see* Art. 8, *loc. cit.*) a one-to-one correspondence between the compositions of the multipartite

$$\overline{\alpha\beta\gamma\ldots}$$

and the perfect partitions of the unipartite number

$$a^{\alpha}b^{\beta}c^{\gamma}\ldots - 1,$$

where $a, b, c, \ldots$ are any different prime numbers.

This correspondence is entirely distinct from that alluded to in Art. 12.

10. Taking for the present the general multipartite number to be

$$\overline{p_1p_2p_3\ldots}$$

it is easily seen that the enumeration of the compositions into r parts is the same problem as the enumeration of the distributions of $p_1+p_2+p_3+\ldots$ things, of which p_1 are of one sort, p_2 of a second, p_3 of a third, and so forth into r different parcels.

This number* is the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots$ in the expansion of

$$(h_1+h_2+h_3+\ldots)^r$$

wherein h_s denotes the sum of the homogeneous products, of degree s, of the quantities

$$\alpha_1, \alpha_2, \alpha_3 \ldots$$

Hence, the generating function of the total number of compositions is

$$\frac{h_1+h_2+h_3+\ldots}{1-h_1-h_2-h_3-\ldots}.$$

The coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}\ldots$ in the expansion of $(h_1+h_2+h_3+\ldots)^r$ is readily found to be

$$\begin{aligned}
&\binom{p_1+r-1}{p_1}\binom{p_2+r-1}{p_2}\binom{p_3+r-1}{p_3}\ldots\\
-&\binom{r}{1}\binom{p_1+r-2}{p_1}\binom{p_2+r-2}{p_2}\binom{p_3+r-2}{p_3}\ldots\\
+&\binom{r}{2}\binom{p_1+r-3}{p_1}\binom{p_2+r-3}{p_2}\binom{p_3+r-3}{p_3}\ldots\\
-&\ldots \text{ to } r \text{ terms.}
\end{aligned}$$

and the enumeration is analytically complete.

11. This method is laborious in the case of high multipartite numbers since r may have all values from unity to $p_1+p_2+p_3+\ldots$; but fortunately the series, above written, possesses some remarkable properties which can be utilized so as greatly to abridge the necessary labour.

Let $F(p_1p_2p_3\ldots)$ and $f(p_1p_2p_3\ldots, r)$ denote, respectively, the total number of compositions and the number comprised of r parts of the multipartite number $\overline{p_1p_2p_3\ldots}$; so that,

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* See the Author, "Symmetric Functions and the Theory of Distributions," 'Proceedings of the London Mathematical Society,' vol. 19, Nos. 318–320.

$$F(p_1p_2p_3\ldots) = \sum_1^{\Sigma p} f(p_1p_2p_0\ldots, r).$$

Writing

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots = 1 - a_1 + a_2 - a_3 + \ldots$$

$$\frac{h_1+h_2+h_3+\ldots}{1-h_1-h_2-h_3-\ldots} = \frac{a_1-a_2+a_3-\ldots}{1-2(a_1-a_2+a_3-\ldots)}.$$

This new form of the generating function gives a relation connecting the number of compositions of any multipartite with numbers related to lower multipartites.

For unipartite numbers

$$\frac{\alpha_1}{1-2\alpha_1} = \Sigma\, F(p_1)\,\alpha_1^{p_1}$$

yielding

$$F(1) = 1$$

and

$$F(p_1) = 2F(p_1 - 1) \text{ when } p_1 > 1.$$

For bipartite numbers

$$\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{1-2(\alpha_1+\alpha_2-\alpha_1\alpha_2)} = \Sigma\, F(p_1p_2)\,\alpha_1^{p_1}\alpha_2^{p_2}$$

giving

$$F(p_1p_2) = 2F(p_1-1, p_2) + 2F(p_1, p_2-1) - 2F(p_1-1, p_2-1),$$

and similarly for multipartite numbers

$$\begin{aligned} F(p_1p_2p_3\ldots) = {} & 2\{F(p_1-1, p_2, p_3, \ldots) + \ldots\} \\ & - 2\{F(p_1-1, p_2-1, p_3, \ldots) + \ldots\} \\ & + 2\{F(p_1-1, p_2-1\ p_3-1, \ldots) + \ldots\} \\ & - \ldots \end{aligned}$$

a formula absolutely true for all multipartites superior to $\overline{111\ldots}$, and universally true if $F(000\ldots)$ when it occurs be interpreted to mean $\frac{1}{2}$.[*]

12. A simple expression is obtainable for $F(p_1p_2)$. We have to find the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}$ in

$$\frac{\alpha_1+\alpha_2-\alpha_1\alpha_2}{1-2(\alpha_1+\alpha_2-\alpha_1\alpha_2)}$$

or, this is the coefficient of $\alpha_1^{p_1}$ in

$$2^{p_2-1}\frac{(1-\alpha_1)}{(1-2\alpha_1)^{p_2+1}},$$

---

[*] See *post*, Art. 39.

and thence

$$F(p_1p_2) = 2^{p_1+p_2-1}\frac{(p_1+p_2)!}{p_1!\,p_2!} - 2^{p_1+p_2-2}\frac{(p_1+p_2-1)!}{1!\,(p_1-1)!\,(p_2-1)!}$$
$$+\,2^{p_1+p_2-3}\frac{(p_1+p_2-2)!}{2!\,(p_1-2)!\,(p_2-2)!} - \ldots$$

until one of the denominator factorials becomes zero.*

13. There is another method by which the value of $F(p_1p_2p_3\ldots\mu)$ can be easily obtained from the numbers $f(p_1p_2p_3\ldots, r)$ which compose the value of $F(p_1p_2p_3\ldots)$. Writing $h_1 + h_2 + h_3 + \ldots = H$, we may write the generating function

$$\frac{H}{1-H} = \Sigma\,F(p_1p_2p_3\ldots).(p_1p_2p_3\ldots)$$

where $(p_1p_2p_3\ldots)$ denotes the symmetric function

$$\Sigma\,\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$$

and

$$H^r = \Sigma f(p_1p_2p_3\ldots, r).(p_1p_2p_3\ldots).$$

Let

$$d_r = \partial_{a_r} + a_1\partial_{a_{r+1}} + a_2\partial_{a_{r+2}} + \ldots$$
$$D_r = \frac{1}{r!}(\partial_{a_1} + a_1\,\partial_{a_2} + a_2\,\partial_{a_3} + \ldots)^r,$$

D_r being an operator of the r^{th} order obtained by symbolical multinomial expansion, as in TAYLOR's theorem of the Differential Calculus, and not denoting r successive performances of linear operations. The effect of the operation of D_r upon a symmetric function of the quantities $\alpha_1, \alpha_2, \alpha_3\ldots$ expressed in the notation of partitions, is well known; it obliterates one part r from every symmetric function partition which possesses such a part, and causes every other symmetric function to vanish.

Also $D_r(r) = 1$.

Hence

$$D_\mu H^r = \overset{p}{\Sigma} f(p_1p_2p_3\ldots\mu, r).(p_1p_2p_3\ldots).$$

To evaluate $D_\mu H^r$ we require the well-known formulæ

$$d_\mu h_s = (-)^{\mu+1}h_{s-\mu},$$

leading to

$$d_\mu H = (-)^{\mu+1}(1 + H).$$

When operating upon a function of H, d_μ is equivalent to d_1, when μ is uneven, and equivalent to $-d_1$ when μ is even. Hence, with such an operand, we have the equivalences

---

* Tables for the verification of formulæ will be found *post* pp. 898–900.

$$D_1 \equiv d_1,$$

$$D_2 \equiv \frac{1}{2!} d_1 (d_1 + 1),$$

$$D_3 \equiv \frac{1}{3!} d_1 (d_1 + 1)(d_1 + 2),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D_\mu \equiv \frac{1}{\mu!} d_1 (d_1 + 1) \dots (d_1 + \mu - 1),$$

the products on the right denoting successive operations.*

To evaluate $D_\mu H^r$ as an algebraic function of H, it is necessary to operate successively with d_1, $d_1 + 1, \dots d_1 + \mu - 1$ and to divide the result by μ!

We have

$$D_1 H^r = r H^{r-1} + r H^r,$$

$$D_2 H^r = \binom{r}{2} H^{r-2} + \binom{r}{1}^2 H^{r-1} + \binom{r+1}{2} H^r.$$

Suppose that generally

$$D_\mu H^r = \sum_{s=0}^{s=\mu} \binom{r+s-1}{s} \binom{r}{\mu - s} H^{r-\mu+s}.$$

Operation with

$$\frac{1}{\mu+1}(d_1 + \mu)$$

gives

$$D_{\mu+1} H^r = \sum_{s=0}^{s=\mu+1} \binom{r+s-1}{s} \binom{r}{\mu+1-s} H^{r-(\mu+1)+s}.$$

so that the assumed law is established inductively.

---

* The truth of this law is seen by comparison with the corresponding algebraic relations. Denoting by $s_1, s_2, \dots$ the sums of powers of $\alpha_1, \alpha_2, \alpha_3, \dots$

$$1 - a_1 x + a_2 x^2 - \dots = \exp. - (s_1 x + \tfrac{1}{2} s_2 x^2 + \tfrac{1}{3} s_3 x^3 + \dots),$$

and putting

$$s_{2\kappa+1} = s_1, \quad s_{2\kappa} = - s_1,$$

$$1 - a_1 x + a_2 x^2 - \dots = (1 + x)^{-s_1}.$$

Hence

$$a_\mu = \frac{1}{\mu!} s_1 (s_1 + 1) \dots (s_1 + \mu - 1),$$

from which we pass to the operator relation

$$D_\mu = \frac{1}{\mu!} d_1 (d_1 + 1) \dots (d_1 + \mu - 1).$$

Observe also the relation

$$h_\mu = \frac{1}{\mu!} s_1 (s_1 - 1) \dots (s_1 - \mu + 1).$$

Substituting in a previous identity

$$\sum_{s=0}^{s=\mu}\binom{r+s-1}{s}\binom{r}{\mu-s}\mathrm{H}^{r-\mu+s}=\sum^{p} f(p_1p_2p_3\ldots\mu, r)\,.\,(p_1p_2p_3\ldots),$$

therefore,

$$\sum_{s=0}^{s=\mu}\binom{r+s-1}{s}\binom{r}{\mu-s}\sum^{p} f(p_1p_2p_3\ldots, r-\mu+s)\,.\,(p_1p_2p_3\ldots)$$
$$=\sum^{p} f(p_1p_2p_3\ldots\mu, r)\,.\,(p_1p_2p_3\ldots).$$

This is an absolute identity and, equating coefficients after writing

$$\binom{r+s-1}{s}\binom{r}{\mu-s}=\phi(r, s),$$

we obtain

$$\phi(r,\mu)f(p_1p_2p_3\ldots, r)+\phi(r,\mu-1)f(p_1p_2p_3\ldots, r-1)+\ldots$$
$$+\phi(r,\mu-r+1)f(p_1p_2p_3\ldots, 1)=f(p_1p_2p_3\ldots\mu, r).$$

This formula enables the calculation of the number $f(p_1p_2p_3\ldots\mu, r)$ from the successive numbers

$$f(p_1p_2p_3\ldots, r),\quad f(p_1p_2p_3\ldots, r-1),\ldots f(p_1p_2p_3\ldots, 1).$$

14. A more useful result is obtained by summing each side of this identity for a values of r from 1 to $\Sigma p+\mu$. This result is

$$\sum^{r}\{\phi(r,\mu)+\phi(r+1,\mu-1)+\ldots+\phi(r+\mu, 0)\}f(p_1p_2p_3\ldots r)=\mathrm{F}(p_1p_2p_3\ldots\mu).$$

It can be shown that the expression in brackets { } to the left has the values

$$2^{\mu-1}\binom{r+\mu-1}{r}\frac{2r+\mu}{\mu};$$

therefore,

$$\mathrm{F}(p_1p_2p_3\ldots\mu)=\sum^{r} 2^{\mu-1}\binom{r+\mu-1}{r}\frac{2r+\mu}{\mu}f(p_1p_2p_3\ldots, r)$$
$$=2^{\mu-1}(\mu+2)f(p_1p_2p_3\ldots, 1)+2^{\mu-1}\frac{(\mu+1)(\mu+4)}{2!}f(p_1p_2p_3\ldots, 2)$$
$$+2^{\mu-1}\frac{(\mu+1)(\mu+2)(\mu+6)}{3!}f(p_1p_2p_3\ldots, 3)+\ldots,$$

a formula of great service when μ is large, as the number of arithmetical operations is comparatively small.

As an example we can find another series for $F(p_1p_2)$.

Since

$$F(p_1) = 1 + \binom{p_1 - 1}{1} + \binom{p_1 - 1}{2} + \ldots + 1$$

wherein

$$\binom{p_1 - 1}{r} \text{ is the value of } f(p_1, r),$$

we find

$$F(p_1p_2) = 2^{p_2-1}(p_2 + 2) + 2^{p_2-1}\frac{(p_2+1)(p_2+4)}{2!}(p_1 - 1)$$
$$+ 2^{p_2-1}\frac{(p_2+1)(p_2+2)(p_2+6)}{3!}\binom{p_1 - 1}{2} + \ldots,$$

a series which it is easy to identify with that previously given (Art. 12).

15. Useful formulæ of verification are obtainable.

It has been shown in Art. 13 that the operation $(-)^{\mu+1} d_\mu$ is equivalent to d_1 when the operand is a function of H only.

Now

$$\frac{(-)^{\mu+1}}{\mu} d_\mu = \Sigma (-)^{\alpha+\beta+\ldots-1} \frac{(\alpha+\beta+\ldots-1)!}{\alpha!\,\beta!\ldots} D_1^{\alpha} D_2^{\beta} \ldots,$$

the summation having reference to all positive integer solutions of the equation,

$$\alpha + 2\beta + 3\gamma + \ldots = \mu.$$

Operating on the expression

$$\Sigma f(p_1p_2p_3 \ldots, r) \,.\, (p_1p_2p_3 \ldots)$$

with $(-)^{\mu+1} d_\mu$ for successive positive integral values of μ, and equating the coefficients of the symmetric function $(p_1p_2p_3 \ldots)$ we find the relations—

$$
\begin{aligned}
& f(p_1p_2p_3 \ldots 1, r) \\
&\quad = - \{f(p_1p_2p_3 \ldots 1^2, r) - 2f(p_1p_2p_3 \ldots 2, r)\} \\
&\quad = + \{f(p_1p_2p_3 \ldots 1^3, r) - 3f(p_1p_2p_3 \ldots 21, r) + 3f(p_1p_2p_3 \ldots 3, r)\} \\
&\quad = - \ldots \\
&\quad = (-)^{s+1} \Sigma (-)^{\alpha+\beta+\ldots-1} \frac{(\alpha+\beta+\ldots-1)!\, s}{\alpha!\, \beta! \ldots} f(p_1p_2p_3 \ldots 2^{\beta} 1^{\alpha}, r),
\end{aligned}
$$

the condition of summation being

$$\alpha + 2\beta + 3\gamma + \ldots = s.$$

Either summing these identities with respect to r, or operating as before upon the expression

$$\Sigma\, \mathrm{F}(p_1p_2p_3\ldots).(p_1p_2p_3\ldots),$$

we obtain the identities

$$\begin{aligned}
&\mathrm{F}(p_1p_2p_3\ldots 1)\\
&\quad = -\{\mathrm{F}(p_1p_2p_3\ldots 1^2) - 2\mathrm{F}(p_1p_2p_3\ldots 2)\}\\
&\quad = +\{\mathrm{F}(p_1p_2p_3\ldots 1^3) - 3\mathrm{F}(p_1p_2p_3\ldots 21) + 3\mathrm{F}(p_1p_2p_3\ldots 3)\}\\
&\quad = \ldots\\
&\quad = (-)^{s+1}\,\Sigma\,(-)^{\alpha+\beta+\ldots-1}\frac{(\alpha+\beta+\ldots-1)!\,s}{\alpha!\,\beta!\ldots}\mathrm{F}(p_1p_2p_3\ldots 2^\beta 1^\alpha).
\end{aligned}$$

These relations are readily verified in the particular case

$$(p_1p_2p_3\ldots 1) = (1).$$

16. Another very useful result is derived from the algebraical result noticed in the foot-note to Art. 13.

Since the supposition $(-)^{\mu-1}s_\mu = s_1$ leads to the formula

$$\frac{1}{\mu!}s_1(s_1-1)\ldots(s_1-\mu+1) = h_\mu$$

and

$$h_\mu = \Sigma\,(-)^{\mu+v_1+v_2+\ldots}\frac{(v_1+v_2+\ldots)!}{v_1!\,v_2!\ldots}a_1^{v_1}a_2^{v_2}\ldots$$

we reach the operator relation

$$\frac{1}{\mu!}d_1(d_1-1)\ldots(d_1-\mu+1) = \Sigma\,(-)^{\mu+v_1+v_2+\ldots}\frac{(v_1+v_2+\ldots)!}{v_1!\,v_2!\ldots}\mathrm{D}_1^{v_1}\mathrm{D}_2^{v_2}\ldots$$

whenever, as in the present case, the operand is such that $(-)^{\mu-1}d_\mu = d_1$.

Assuming

$$\frac{1}{\mu!}d_1(d_1-1)\ldots(d_1-\mu+1)\,\mathrm{H}^r = \binom{r}{\mu}\mathrm{H}^{r-\mu}(1+\mathrm{H})^\mu$$

and operating with $\frac{1}{\mu+1}(d_1-\mu)$ we find

$$\frac{1}{(\mu+1)!}d_1(d_1-1)\ldots(d_1-\mu)\,\mathrm{H}^r = \binom{r}{\mu+1}\mathrm{H}^{r-\mu-1}(1+\mathrm{H})^{\mu+1}$$

verifying the assumption.

Hence operation on the relation

$$H^r = \Sigma f(p_1 p_2 p_3 \ldots, r) . (p_1 p_2 p_3 \ldots)$$

yields the relation

$$\binom{r}{\mu} H^{r-\mu} (1 + H)^{\mu}$$
$$= \Sigma (-)^{\mu + v_1 + v_2 + \ldots} \frac{(v_1 + v_2 + \ldots)!}{v_1 ! \, v_2 ! \ldots} D_1^{v_1} D_2^{v_2} \ldots \{\Sigma f(p_1 p_2 p_3 \ldots, r) . (p_1 p_2 p_3 \ldots)\},$$

or

$$\binom{r}{\mu} \Sigma \left\{ f(p_1 p_2 p_3 \ldots, r - \mu) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, r - \mu + 1) \right.$$
$$\left. + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, r - \mu + 2) + \ldots \right\} . (p_1 p_2 p_3 \ldots)$$
$$= \Sigma \left\{ f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r) \right.$$
$$+ \binom{\mu - 2}{1} f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \binom{\mu - 2}{2} f(p_1 p_2 p_3 \ldots 2^2 1^{\mu-4}, r)$$
$$\left. + \ldots \right\} . (p_1 p_2 p_3 \ldots),$$

or, equating coefficients and inverting

$$f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r)$$
$$+ (\mu - 2) f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \ldots$$
$$= \binom{r}{\mu} \left\{ f(p_1 p_2 p_3 \ldots, r - \mu) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, r - \mu + 1) \right.$$
$$\left. + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, r - \mu + 2) + \ldots \right\}.$$

If $r < \mu$ the dexter vanishes and

$$f(p_1 p_2 p_3 \ldots 1^{\mu}, r) - (\mu - 1) f(p_1 p_2 p_3 \ldots 21^{\mu-2}, r)$$
$$+ (\mu - 2) f(p_1 p_2 p_3 \ldots 31^{\mu-3}, r) + \ldots = 0.$$

If $r = \mu$ the dexter is

$$f(p_1 p_2 p_3 \ldots, 0) + \binom{\mu}{1} f(p_1 p_2 p_3 \ldots, 1) + \binom{\mu}{2} f(p_1 p_2 p_3 \ldots, 2) + \ldots$$

where $f(p_1 p_2 p_3 \ldots, 0)$ in general vanishes, but

$$f(0, 0) = 1.$$

Performing the summation in regard to r,

$$\begin{aligned} & F(p_1 p_2 p_3 \ldots 1^{\mu}) - (\mu - 1) F(p_1 p_2 p_3 \ldots 21^{\mu-2}) + \ldots \\ & \quad + (-)^{\mu+v_1+v_2+\ldots} \frac{(v_1 + v_2 + \ldots)!}{v_1!\, v_2! \ldots} F(p_1 p_2 p_3 \ldots 2^{v_2} 1^{v_1}) + \ldots \\ & = \binom{\mu}{0} f(p_1 p_2 p_3 \ldots, 0) + \left\{ \binom{\mu}{1} + \binom{\mu+1}{1} \binom{\mu}{0} \right\} f(p_1 p_2 p_3 \ldots, 1) + \ldots \\ & \quad + \theta_{\mu,s} f(p_1 p_2 p_3 \ldots, s) + \ldots, \end{aligned}$$

where

$$\theta_{\mu,s} = \binom{\mu}{s} + \binom{\mu+1}{1}\binom{\mu}{s-1} + \binom{\mu+2}{2}\binom{\mu}{s-2} + \ldots + \binom{\mu+s}{s}$$

that is to say

$$(1 + x)^{\mu} (1 - x)^{-\mu-1} = \Sigma\, \theta_{\mu,s}\, x^s.$$

E.g., $r < \mu$,

$$f(21^3, 2) - 2f(2^2 1, 2) + f(32, 2) = 0,$$

verified by

$$22 - 2 \times 16 + 10 = 0. \quad \text{(See Tables, pp. 898–900.)}$$

$r = \mu$,

$$f(21^3, 3) - 2f(2^2 1, 3) + f(32, 3) = f(2, 0) + 3f(2, 1) + 3f(2, 2)$$

verified by

$$93 - 2 \times 57 + 27 = 0 + 3 \times 1 + 3 \times 1.$$

For the summation formula

$$\begin{aligned} & F(1), \\ = {} & F(1^2) - F(2), \\ = {} & F(1^3) - 2F(21) + F(3), \\ = {} & F(1^4) - 3F(21^2) + F(2^2) + 2F(31) - F(4), \\ = {} & \ldots\ldots \\ = {} & 1, \end{aligned}$$

verified by

$$\begin{aligned} & 1, \\ = {} & 3 - 2, \\ = {} & 13 - 2 \times 8 + 4, \\ = {} & 75 - 3 \times 44 + 26 + 2 \times 20 - 8, \\ = {} & \ldots\ldots \\ = {} & 1; \end{aligned}$$

also

$$\begin{aligned}&F(41^3) - 2F(421) + F(43)\\ &= f(4, 0) + (3 + 4 \times 1) f(4, 1) + (3 + 4 \times 3 + 10 \times 1) f(4, 2)\\ &\quad + (1 + 4 \times 3 + 10 \times 3 + 20 \times 1) f(4, 3)\\ &\quad + (4 \times 1 + 10 \times 3 + 20 \times 3 + 35 \times 1) f(4, 4),\end{aligned}$$

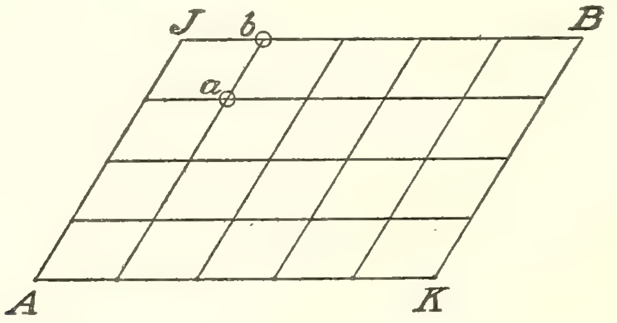
verified by

$$\begin{aligned}&3408 - 3776 + 768\\ &= 0 + 7 \times 1 + 25 \times 3 + 63 \times 3 + 129 \times 1,\\ &= 400.\end{aligned}$$

§ 3. *The Graphical Representation of the Compositions of Bipartite Numbers.*

17. The graphical method, that has been employed in the case of unipartite compositions, can be extended so as to meet the cases of bipartite, tripartite, and multipartite numbers in general. For the present the bipartite case alone is under consideration. The graph of a bipartite number $(\overline{pq})$ is derived directly from the graphs of the unipartite numbers (p), (q).

Take $q + 1$ exactly similar graphs of the number p and place them parallel to one another, at equal distances apart, and so that their left hand extremities lie on a right line; corresponding points of the $q + 1$ graphs can then be joined by right lines and a reticulation will be formed which is the graph of the bipartite number $\overline{pq}$.



We have AK a graph of the number p, and $q + 1$ such graphs parallel to one another; and AJ a graph of the number q, and $p + 1$ such graphs parallel to one another.

The angle between AK and AJ is immaterial.

The points A, B, are the "extremities" or the "initial" and "final" points of the graph.

The remaining intersections are termed the "points" of the graph.

The lines of the graph have either the "direction" AK or the direction AJ. These will be called the α and β directions respectively. Through each point of the graph pass lines in each of these directions. Each line is made up of segments, and we speak of α segments and of β segments indicating that the corresponding lines (on which lie the segments) are in the α and β directions.

Suppose a traveller to proceed from A to B by successive steps. A step is performed by moving over a certain number of α segments and subsequently moving over a certain number of β segments. A step is thus made up of two figures—say an α figure and a β figure. The number of segments moved over may be zero in either, but not in both of these two figures of the step.

A step may be taken from A to any point a of the graph; a second step may be taken from a to any point of the graph aB which has a and B for its initial and final points; subsequent steps are taken on a similar principle, and the last step terminates at the point B and completes the procession from the point A to the point B. A step which takes x, α segments followed by y, β segments, is taken to be the representation of a bipartite part, $\overline{xy}$. A procession from the initial to the final point (from A to B) of the graph thus represents a sequence of bipartite parts which constitutes a composition of the bipartite number $\overline{pq}$. To every procession from A to B corresponds a composition of the bipartite $\overline{pq}$, and the enumeration of the different processions is identical with the enumeration of the total number of different compositions.

The steps of a procession are marked out by nodes placed at the points of the reticulation, which terminate the first, second, third, &c., and penultimate steps When nodes are thus placed, we have the graph of a composition. If nodes be placed at the points a, b of the graph, we obtain the graph of the composition

$$(\overline{13}\ \overline{01}\ \overline{40})$$

of the bipartite $\overline{54}$.

The number of parts in a composition is always one greater than the number of nodes in its graph.

Considering the number of compositions of $\overline{pq}$, of two parts, it is clear that we may place the defining node at any one of $(p+1)(q+1)-2$ points of the reticulation. Hence (and this may be verified by previous work) the number of two-part compositions is

$$(p+1)(q+1)-2.$$

18. The graph of a composition, traced from A to B, passes over certain segments, and may be said to follow a certain line of route through the reticulation. Other compositions in general follow the same line of route; they all have their defining nodes upon the line of route, and a certain number of defining nodes will, in general, be common to all of them.

Consider the path $AcbB$ in the graph given below. All compositions whose graphs follow this line of route must have the node b; b, in fact, is an essential node along this line of route.

Essential nodes occur on a line of route at all points where the course changes from the β to the α direction.

We may regard a line of route as defined by these essential nodes since the line of route is completely given by these nodes.

Every composition-graph involves nodes of two kinds—

(1) Those which are essential to its line of route;

(2) Those which, in this respect, are not essential.

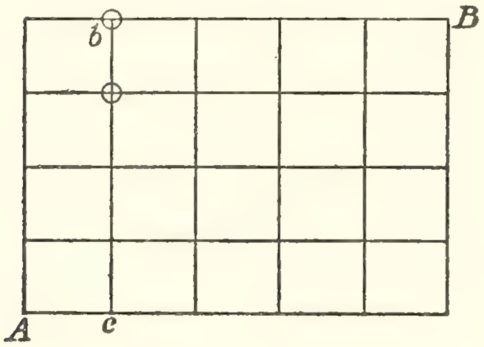
There are $p + q - 1$ "points" along every line of route.

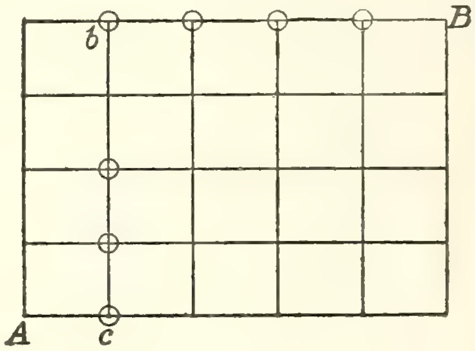
We have now to discuss the compositions whose graphs follow a given line of route, and we find that they naturally arrange themselves in pairs.

19. Associated with any one graph is another obtained by obliterating those nodes which are not essential to its line of route, and by then placing nodes at those points on the line of route not previously occupied by nodes.

These two graphs are said to be conjugate.

The compositions, represented by these graphs, are likewise said to be conjugate.





Conjugate graphs are shown above; $AcbB$ is the line of route, and b is the essential node.

The corresponding conjugate compositions of the bipartite number $\overline{54}$ are

$$(\overline{13}\ \overline{01}\ \overline{40}) \text{ and } (\overline{10}\ \overline{01}\ \overline{01}\ \overline{02}\ \overline{10}\ \overline{10}\ \overline{10}\ \overline{10}).$$

If the graph of a composition of $\overline{pq}$, of r parts, possesses s essential nodes, it is clear that the conjugate composition has $p + q - r + s + 1$ parts.

Of compositions, of the bipartite $\overline{pq}$, whose graphs possess s essential nodes, there is a one-to-one correspondence between those of r parts and those of $p + q - r + s + 1$ parts.

Corresponding to an essential node in the graph of a composition there exist in the composition itself adjacent parts $\ldots \overline{p_1q_1}\ \overline{p_2q_2} \ldots$ possessing the property of q_1 and p_2, being both superior to zero. Thus, from inspection of a composition, we are enabled to determine the number of essential nodes in its graph.

It is useful to recognise four species of contact between adjacent parts of a composition

$$\text{in} \ldots \overline{p_1q_1}\;\overline{p_2q_2} \ldots$$

if q_1 is zero and p_2 zero we have a zero-zero contact
q_1 ,, p_2 positive ,, zero-positive ,,
q_1 positive p_2 zero ,, positive-zero ,,
q_1 ,, p_2 positive ,, positive-positive ,,

In this nomenclature we may say that the graph of a composition possesses as many essential nodes as the composition itself possesses positive-positive contacts.

20. The theorem arrived at may be stated as follows:—

"Of compositions of $\overline{pq}$ possessing s positive-positive contacts, there is a one-to-one correspondence between those of r parts and those of $p + q - r + s + 1$ parts."

An essential node corresponding to a positive-positive contact in the composition occurs at a point of the graph where there is a change from a β direction to an α direction; we may say that this is a $\beta\alpha$ point on the line of route. Similarly to positive-zero, zero-positive, zero-zero contacts in the composition correspond $\beta\beta$, $\alpha\alpha$, $\alpha\beta$ points respectively on the line of route.

The number of different lines of route that can be traced on the graph of the bipartite number is the number of permutations of p symbols α, and q symbols β, for this is the number of ways in which the p α-segments and the q β-segments, which make up a line of route, can form a succession.

Hence the number of lines of route through the reticulation is

$$\binom{p+q}{p}.$$

The whole of the compositions of $\overline{pq}$ can be arranged in conjugate pairs.

E.g., The correspondence in regard to the compositions of the bipartite $\overline{22}$ is shown in parallel columns.

$$(\overline{22}) \qquad s = 0, \quad r = 1 \qquad\qquad (\overline{10}\;\overline{10}\;\overline{01}\;\overline{01}) \qquad s = 0, \quad r = 4$$

$$\left.\begin{array}{l}(\overline{20}\;\overline{02})\\(\overline{21}\;\overline{01})\\(\overline{10}\;\overline{12})\end{array}\right\} \; s = 0, \quad r = 2 \qquad\qquad \left.\begin{array}{l}(\overline{10}\;\overline{11}\;\overline{01})\\(\overline{10}\;\overline{10}\;\overline{02})\\(\overline{20}\;\overline{01}\;\overline{01})\end{array}\right\} \; s = 0, \quad r = 3$$

$$\left.\begin{array}{l}(\overline{02}\ \overline{20})\\(\overline{11}\ \overline{11})\\(\overline{01}\ \overline{21})\\(\overline{12}\ \overline{10})\end{array}\right\}\ s=1,\quad r=2 \qquad \left.\begin{array}{l}(\overline{01}\ \overline{01}\ \overline{10}\ \overline{10})\\(\overline{10}\ \overline{01}\ \overline{10}\ \overline{01})\\(\overline{01}\ \overline{10}\ \overline{10}\ \overline{01})\\(\overline{10}\ \overline{01}\ \overline{01}\ \overline{10})\end{array}\right\}\ s=1,\quad r=4$$

$$\left.\begin{array}{l}(\overline{01}\ \overline{20}\ \overline{01})\\(\overline{01}\ \overline{01}\ \overline{20})\\(\overline{10}\ \overline{02}\ \overline{10})\\(\overline{11}\ \overline{10}\ \overline{01})\end{array}\right\}\ s=1,\quad r=3 \qquad \left.\begin{array}{l}(\overline{01}\ \overline{10}\ \overline{11})\\(\overline{02}\ \overline{10}\ \overline{10})\\(\overline{11}\ \overline{01}\ \overline{10})\\(\overline{10}\ \overline{01}\ \overline{11})\end{array}\right\}\ s=1,\quad r=1$$

$$(\overline{01}\ \overline{11}\ \overline{10})\quad s=2,\quad r=3 \qquad (\overline{01}\ \overline{10}\ \overline{01}\ \overline{10})\quad s=2,\quad r=4.$$

The compositions present themselves in pairs of conjugate groups according to the several values of s and r. In the above example for $s=1$, $r=3$, there is a self-conjugate group. This happens when $2r = p+q+s+1$.

A self-conjugate group exists for even or uneven values of s, according as $p+q$ is uneven or even.

21. The enquiry now is in regard to the number of lines of route through the reticulation which possess exactly s essential nodes.

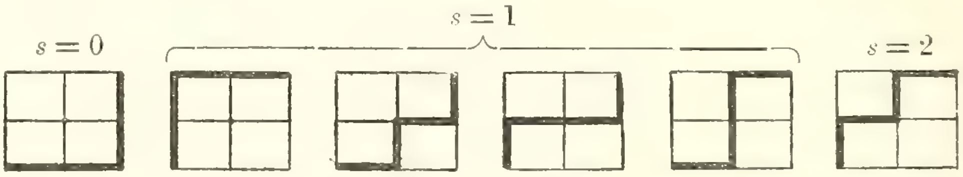
It may be observed, in passing, that from any line of route may be derived a composition whose graph exhibits only essential nodes and no others. This may be called the principal composition along the line of route; it will have $s+1$ parts, and each of the s contacts of its parts will be positive-positive.

There is a one-to-one correspondence between the lines of route having s essential nodes and the compositions of $s+1$ parts, all of whose contacts are positive-positive.

Also the number of compositions, number of parts unrestricted, all of whose contacts are positive-positive is equal to the number of different lines of route.

E.g., for the number $\overline{22}$,

Lines of route



Compositions $(\overline{22})$ $(\overline{02}\ \overline{20})$ $(\overline{11}\ \overline{11})$ $(\overline{01}\ \overline{21})$ $(\overline{12}\ \overline{10})$ $(\overline{01}\ \overline{11}\ \overline{10})$

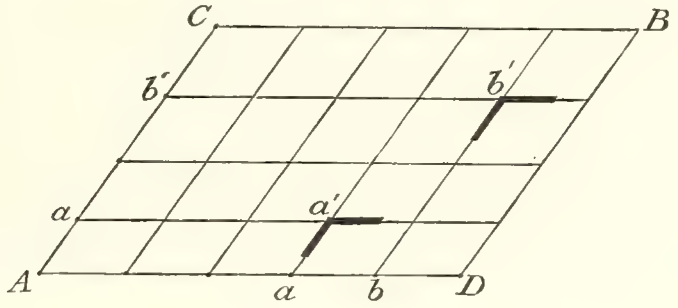
and there are no other compositions having all contacts positive-positive.

22. The number of different lines of route with exactly s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

Of this theorem I give three proofs, because its thorough examination is necessary for the purpose of leading up to the more difficult theories connected with tripartite and higher multipartite numbers.

First Proof.



In each of the adjacent sides AD, AC of the graph of $\overline{pq}$, select any s "points" (see definition of "point") $a, b, c, \ldots$ in order from the point A. The two points a, a are seen to determine an essential node a'; the two points b, b an essential node b', &c., and a line of route necessarily exists which possesses these essential nodes and no others. The points along AD and AC, from which s points may be selected, are in number p and q respectively; along AD s points can be selected in $\binom{p}{s}$ ways, and along AC in $\binom{q}{s}$ ways; any selection on AD can be taken with any selection on AC. Hence the number of lines of route having exactly s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

23. *Second Proof.*

We determined the total number of lines of route by considering the permutations of p, α-segments, and q, β-segments. Whenever in any such permutation there is a sequence $\beta\alpha$ there must be an essential node upon the corresponding line route. We have simply to find the number of permutations of the $p + q$ symbols in $\alpha^p\beta^q$ which possess exactly s, $\beta\alpha$-contacts.

Write down the s, $\beta\alpha$-sequences

$$\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots\beta\alpha\ldots$$

and the $s + 1$ intervals between them. In these intervals we have to distribute the letters in

$$\alpha^{p-s}\ \beta^{q-s}$$

in such manner as to introduce no fresh $\beta\alpha$-contacts. For each of these $p + q - 2s$ letters there is a choice of $s + 1$ intervals. The $p - s$ letters α may thus be distributed in $\binom{p}{s}$ ways, and the $q - s$ letters β in $\binom{q}{s}$, and each distribution of the letters α may occur with each distribution of the letters β. Hence the total number of permutations is

$$\binom{p}{s}\binom{q}{s}.$$

24. *Third Proof.**

I now come to a method which is valuable in the theory of multipartite numbers in general, and also intrinsically of great interest.

I consider, as in the second proof, the permutations of the $p+q$ letters $\alpha^p\beta^q$, which exhibit exactly s, $\beta\alpha$-contacts. The proof depends upon showing there exists a one-to-one correspondence between these permutations, and those in which the letter β occurs exactly s times in the first p places counted from the left.

Suppose the permutation with s, $\beta\alpha$-contacts to be

$$\alpha^{x_1}\beta^{y_1}\quad\beta\alpha\quad\alpha^{x_2}\beta^{y_2}\quad\beta\alpha\quad\alpha^{x_3}\beta^{y_3}\quad\beta\alpha\quad\alpha^{x_4}\beta^{y_4}\quad\beta\alpha\quad\alpha^{x_5}\beta^{y_5}\quad\beta\alpha\quad\alpha^{x_6}\beta^{y_6}$$

where, for convenience, s has the special value 5.

Any of the indices x, y may be zero.

Observe that

$$x_1+x_2+x_3+x_4+x_5+x_6+5=p,$$
$$y_1+y_2+y_3+y_4+y_5+y_6+5=q.$$

Obliterate the letters β, which do not occur in $\beta\alpha$-contacts, and the letters α, which do occur in $\beta\alpha$-contacts, there remains a succession

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4}\beta\alpha^{x_5}\beta\alpha^{x_6}$$

of p letters, β occurring s times.

Next obliterate in the original permutation the letters α, which do not occur in $\beta\alpha$-contacts, and the letters β, which do so occur; there remains a succession

$$\beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4}\alpha\beta^{y_5}\alpha\beta^{y_6}$$

of q letters.

Take these two successions for the left and right portions of a new permutation, viz.:—

$$\alpha^{x_1}\beta\alpha^{x_2}\beta\alpha^{x_3}\beta\alpha^{x_4}\beta\alpha^{x_5}\beta\alpha^{x_6}\ .\ \beta^{y_1}\alpha\beta^{y_2}\alpha\beta^{y_3}\alpha\beta^{y_4}\alpha\beta^{y_5}\alpha\beta^{y_6}\alpha,$$

and we have made a perfectly definite transformation of a permutation involving exactly s $\beta\alpha$-contacts into another possessing the property that the letter β occurs s times in the first p places.

E.g., to transform

$$\beta^4\alpha^2\beta^3\alpha,$$

write it

$$\alpha^0\beta^3\quad\beta\alpha\quad\alpha\beta^2\quad\beta\alpha;$$

---

* This proof is of fundamental importance in the succeeding part of this investigation.

thence the successions

$$\alpha^0\beta\alpha\beta,$$

$$\beta^3\alpha\beta^2\alpha,$$

and the permutation

$$\beta\alpha\beta^4\alpha\beta^2\alpha,$$

in which β occurs twice in the first three places.

Now, it happens that these transformed permutations are very easily enumerated.

The number of permutations of the letters in $\alpha^p\beta^q$, which possess the property that the letter β occurs exactly s times in the first p places is the coefficient of $\mu^s\alpha^p\beta^q$ in the development of

$$(\alpha+\mu\beta)^p(\alpha+\beta)^q.$$

This is evident from a consideration of the actual multiplication process.

Hence, in regard to the bipartite $\overline{pq}$,

$$(\alpha+\mu\beta)^p(\alpha+\beta)^q$$

is the generating function for the number of lines of route possessing a given number of essential nodes. The complete coefficient of $\alpha^p\beta^q$ is found to be

$$1+\binom{p}{1}\binom{q}{1}\mu+\binom{p}{2}\binom{q}{2}\mu^2+\ldots+\binom{p}{s}\binom{q}{s}\mu^s+\ldots+\binom{p}{q}\binom{q}{q}\mu^q,$$

generality not being lost by the supposition $p \nless q$. Hence the number of lines of route possessing s essential nodes is

$$\binom{p}{s}\binom{q}{s}.$$

Observe that the known formula

$$\binom{p}{0}\binom{q}{0}+\binom{p}{1}\binom{q}{1}+\ldots+\binom{p}{s}\binom{q}{s}+\ldots+\binom{p}{q}\binom{q}{q}=\binom{p+q}{q}$$

supplies a verification of the result.

It has also been proved that the number of compositions of $s+1$ parts having all contacts positive-positive is $\binom{p}{s}\binom{q}{s}$.

25. On each of the $\binom{p}{s}\binom{q}{s}$ lines of route, which have s essential nodes, may be represented

$$2^{p+q-s-1}$$

compositions, because the $p + q - s - 1$ non-essential nodes on each line of route may be selected as composition-nodes in this number of ways.

Hence, the total number of compositions is

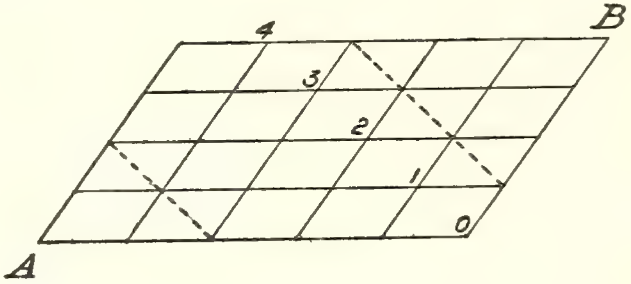
$$F(pq) = \overset{s}{\Sigma} \binom{p}{s} \binom{q}{s} 2^{p+q-s-1}$$

(*cf.* the expressions for this number obtained in Arts. 12 and 14).

Analytically, this result is equivalent to the algebraic expansion

$$\frac{1}{1-2(x+y-xy)} = \frac{1}{(1-2x)(1-2y)} + \frac{2xy}{(1-2x)^2(1-2y)^2} + \frac{2^2x^2y^2}{(1-2x)^3(1-2y)^3} + \ldots$$

26. The important transformation of permutations established above is interesting when viewed graphically.



A line of route is traced by a certain succession of α and β segments. That portion of a line of route traced by the initial p segments terminates in one of the points 0, 1, 2, 3, 4 in the diagram. All these points lie on a straight line through the right-hand lower corner. If one of these points be marked s, s of the p segments will be β segments, and every line of route passing through the point s has the property that s symbols β occur in the first p places of the corresponding permutation of the symbols α and β. Hence, there is a one-to-one correspondence between the lines of route possessing 0, 1, 2, 3, 4, . . . s, . . . essential nodes and the lines of route passing through the points marked 0, 1, 2, 3, 4, . . . s, . . .

Observe that the number of lines of route in the graph, of which A and s are the initial and final points, is $\binom{p}{s}$, and in the graph of which s and B are the initial and final points, is $\binom{q}{s}$.

Hence of the graph AB the number of lines of route passing through the point s is

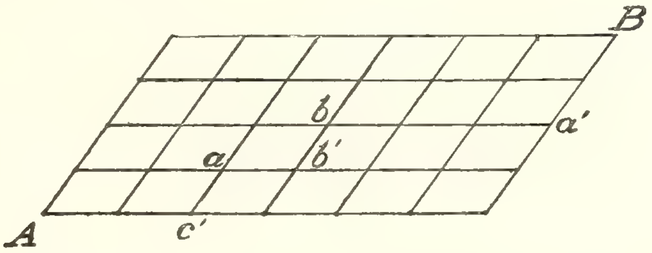
$$\binom{p}{s} \binom{q}{s},$$

and since every line of route must pass through one of the points 0, 1, 2, 3, 4, . . . s, . . . we have

$$\sum^{s}\binom{p}{s}\binom{q}{s}=\binom{p+q}{p}.*$$

Inverse Bipartite Compositions.

27. A line of route being marked out on a reticulation from A to B, the inverse line of route is obtained by rotating the reticulation through two right angles and interchanging the letters A and B.



Consider a line of route from A to B having essential nodes a, b. On the inverse line of route a', b', c' are the essential nodes.

The principal compositions along these lines of route are—

$$(\overline{21}\ \overline{11}\ \overline{32}) \text{ from } A \text{ to } B.$$

$$(\overline{02}\ \overline{31}\ \overline{11}\ \overline{20}) \text{ from } B \text{ to } A.$$

---

* Consider also the points of the graph distant t segments from A. The number of such points is the coefficient of x^t in the product

$$(1+x+x^2+\ldots+x^p)\ (1+x+x^2+\ldots+x^q),$$

and if

$$\begin{array}{lll} t<q+1 & \text{is equal to} & t+1, \\ \;>q<p+1 & ,, & q+1, \\ \;>p & ,, & p+q-t+1. \end{array}$$

These points lie on lines parallel to the line 01234 . . . s . . . and we obtain a graphical proof of the identities,

$$\binom{t}{0}\binom{p+q-t}{q}+\binom{t}{1}\binom{p+q-t}{q-1}+\ldots+\binom{t}{t}\binom{p+q-t}{q-t}=\binom{p+q}{p} \text{ for } t<q+1;$$

$$\binom{t}{0}\binom{p+q-t}{q}+\binom{t}{1}\binom{p+q-t}{q-1}+\ldots+\binom{t}{q}\binom{p+q-t}{0}=\binom{p+q}{p} \text{ for } p+1>t>q;$$

$$\binom{t}{p}\binom{p+q-t}{0}+\binom{t}{p-1}\binom{p+q-t}{1}+\ldots+\binom{t}{t-q}\binom{p+q-t}{p+q-t}=\binom{p+q}{p} \text{ for } t>p$$

where the total number of terms in all these identities is

$$\sum_{0}^{q}(t+1)+(p-q)\,(q+1)+\sum_{t=p+1}^{t=p+q}(p+q-t+1)=(p+1)\,(q+1),$$

which is the total number of points in the graph (including A and B), and is therefore right.

All the contacts in these compositions are necessarily positive-positive; hence the leading and ending parts are the only ones that can involve a zero element; in the leading part the leading element may be zero, and in the ending part the ending element.

Every part, of a principal composition, which does not possess a zero element necessitates an essential node on the graph of the principal composition on the inverse line of route. Thus, if the one principal composition has s nodes, $s + 1$ parts and t parts without a zero element, the other principal composition has t nodes, $t + 1$ parts and s parts without a zero element.

These may be called "inverse principal compositions."

The number of pairs of inverse principal compositions is equal to half the number of distinct lines of route through the reticulation. The number of pairs is thus—

$$\tfrac{1}{2}\binom{p+q}{p}.$$

Otherwise, we may say that in regard to compositions, all of whose contacts are positive-positive, there is a one-to-one correspondence between those having $s + 1$ parts and t parts without a zero element and those having $t + 1$ parts and s parts without a zero element.

The number $s + 1 - t$ is either 0, 1, or 2.

E.g., for the bipartite $\overline{22}$ the correspondence is

$$\begin{array}{cc}
s = 0,\ t = 1 & s = 1,\ t = 0 \\
(\overline{22}) & (\overline{02}\ \overline{20}) \\
\\
s = 1,\ t = 1 & s = 1,\ t = 1 \\
(01\ \overline{21}) & (\overline{12}\ \overline{10}) \\
\\
s = 1,\ t = 2 & s = 2,\ t = 1 \\
(\overline{11}\ \overline{11}) & (\overline{01}\ \overline{11}\ \overline{10}).
\end{array}$$

28. This particular case of inversion leads easily to the general idea. Suppose a line route traced on a reticulation, and suppose marked the s essential nodes on the line drawn from A to B, and also the t essential nodes of the inverse line of route. These $s + t$ nodes are distinct. Along the line we may place an additional node at an unoccupied point and interpret the new compositions as read before and after rotation of the reticulation. The new compositions have each acquired an additional part; they each have the same number of parts without zero elements as before; the added node has either introduced a zero-positive contact into each or a positive-zero contact into each, according as the node is on an α or on a β line between adjacent nodes.

Altogether there are $p+q-1-s-t$ points at disposal, which may be selected as positions for nodes in $2^{p+q-1-s-t}$ ways. Suppose j_1+j_2 additional nodes taken, such that j_1 zero-positive and j_2 positive-zero contacts are introduced into each composition; the number of parts in each composition will be increased by j_1+j_2.

We have a one-to-one correspondence between the compositions, having

$s+1+j_1+j_2$ parts,
t parts without a zero element,
j_1 zero-positive contacts,
j_2 positive-zero contacts;

and those having

$t+1+j_1+j_2$ parts,
s parts without a zero element,
j_1 zero-positive contacts,
j_2 positive-zero contacts.

We have thus pairs of inverse compositions; the correspondence for the bipartite $\overline{22}$ is

$(s, t, j_1, j_2) =$ $(0, 1, 1, 0)$, $(1, 0, 1, 0)$, $(1, 1, 0, 1)$,
$(\overline{10}\ \overline{12})$ and $(\overline{02}\ \overline{10}\ \overline{10})$; $(\overline{11}\ \overline{01}\ \overline{10})$;

$(0, 1, 0, 1)$, $(1, 0, 0, 1)$, $(1, 1, 1, 0)$,
$(\overline{21}\ \overline{01})$ and $(\overline{01}\ \overline{01}\ \overline{20})$; $(\overline{01}\ \overline{10}\ \overline{11})$.

$(0, 1, 1, 1)$, $(1, 0, 1, 1)$,
$(\overline{10}\ \overline{11}\ \overline{01})$ and $(\overline{01}\ \overline{01}\ \overline{10}\ \overline{10})$;

Of the above, two compositions $(\overline{11}\ \overline{01}\ \overline{10})$, $(\overline{01}\ \overline{10}\ \overline{11})$ may be termed self-inverse.

29. There are twelve compositions, viz., those with zero-zero contacts, which do not appear in the theory.

To enumerate these compositions, consider the lines of route with s essential nodes or bends ┌; every such line must have either $s-1$, s, or $s+1$ bends ┘. If the allied $\alpha\beta$ permutation neither commences with α nor ends with β it is $s-1$; if either α commences or β ends it is s; if both α commences and β ends it is $s+1$.

The enumeration of the lines of route gives

$$\binom{p-1}{s-1}\binom{q-1}{s-1} \text{ for } s-1 \text{ bends } ┘\,;$$

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s} \text{ for } s \text{ bends } ┘\,;$$

$$\binom{p-1}{s}\binom{q-1}{s} \text{ for } s+1 \text{ bends } ┘\,.$$

To obtain compositions along these lines of route which have not zero-zero contacts, we have at disposal

$$\begin{gathered} p+q-2s, \\ p+q-2s-1, \\ p+q-2s-2 \end{gathered}$$

points respectively at which non-essential nodes may be placed. Hence the number of compositions which have s positive-positive contacts (see Art. 19) and no zero-zero contacts is—

$$\begin{aligned} & 2^{p+q-2s}\binom{p-1}{s-1}\binom{q-1}{s-1} \\ & + 2^{p+q-2s-1}\left\{\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s}\right\} \\ & + 2^{p+q-2s-2}\binom{p-1}{s}\binom{q-1}{s}, \end{aligned}$$

which is

$$\frac{(p+s)(q+s)}{pq}\,2^{p+q-2s-2}\binom{p}{s}\binom{q}{s}.$$

Hence the total number of compositions which have no zero-zero contacts is

$$2^{p+q-2}+\frac{(p+1)(q+1)}{pq}\,2^{p+q-4}\binom{p}{1}\binom{q}{1}+\frac{(p+2)(q+2)}{pq}\,2^{p+q-6}\binom{p}{2}\binom{q}{2}+\ldots,$$

and, since the total number of compositions which have s positive-positive contacts is

$$2^{p+q-1-s}\binom{p}{s}\binom{q}{s},$$

we find that the number of compositions having s positive-positive contacts and also zero-zero contacts is

$$2^{p+q-2s-2}\left\{2^{s+1}-\frac{(p+s)(q+s)}{pq}\right\}\binom{p}{s}\binom{q}{s},$$

and the total number of compositions having zero-zero contacts is

$$2^{p+q-2}+2^{p+q-4}\left(2^2-\frac{(p+1)(q+1)}{pq}\right)\binom{p}{1}\binom{q}{1}+2^{p+q-6}\left\{2^3-\frac{(p+2)(q+2)}{pq}\right\}\binom{p}{2}\binom{q}{2} \\ +\ldots$$

E.g., putting $\overline{pq}=\overline{22}$, we thus verify

$$4+(4-\tfrac{9}{4})\,4+\tfrac{1}{4}(8-4)=12$$

that the number is 12.

30. Self-inverse compositions can only occur upon self-inverse lines of route. For the existence of such a line, one at least of the elements of the bipartite number must be even. If both elements be even there is a central point in the reticulation, and the number of self-inverse lines of route for the bipartite $\overline{2p', 2q'}$ is equal to the number of lines of route of the bipartite $\overline{p'q'}$; that is to say, the number is

$$\binom{p'+q'}{p'}.$$

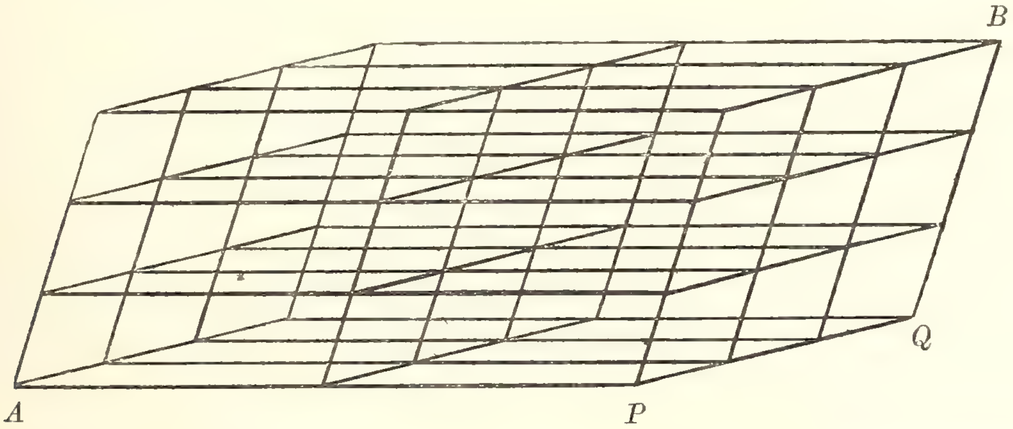
If the bipartite be $\overline{2p'+1, 2q'}$ it is easy to see that the number is also

$$\binom{p'+q'}{p'}.$$

The number of self-inverse compositions in both cases is evidently equal to the total number of compositions of the bipartite $\overline{p'q'}$.

§ 4. *The Graphical Representation of the Compositions of Tripartite and Multipartite Numbers.*

31. The graph of a tripartite number may be in either two or three dimensions. It may be derived from a bipartite graph in a manner similar to that in which the bipartite has been derived from the unipartite graph. In the tripartite number $\overline{pqr}$ we take $r+1$, exactly similar graphs of the bipartite $\overline{pq}$, and place them similarly with corresponding lines parallel, and like points lying on straight lines; when these straight lines are drawn the graph is complete. The $r+1$ bipartite graphs may be in the same plane or in parallel planes according as the tripartite graph is required to be in two or three dimensions.



The figure depicts the graph of the tripartite $\overline{233}$. The points of the reticulation are identical with the points of the $r+1$ reticulations of the bipartites $\overline{pq}$. Observe that in two dimensions there are intersections of lines which are not points of the reticulation. On the other hand in three dimensions all intersections are also points.

Other than the initial and final points A and B there are $(p+1)(q+1)(r+1)-2$ points.

The graph involves lines in three different directions; say an α, a β, and a γ direction. These are parallel to AP, PQ, and QB respectively.

Through each point of the graph pass lines in all three directions, and a segment joining two adjacent points is called an α segment when it is in the α direction.

A line of route proceeds from A to B, from point to point of the reticulation. The number of lines of route is the number of permutations of the symbols in $\alpha^p\beta^q\gamma^r$, and is, therefore,

$$\binom{p+q+r}{p,\, q,\quad}.$$

A step along a line of route traverses in succession any number of α, β, and γ segments, and in any one step the segments must be taken in the order, α, β, γ. The number of segments traversed may be zero in one or two, but not in three of these directions.

A step is represented by a tripartite number $\overline{p_1q_1r_1}$ and a succession of steps, the first starting at A, and the last terminating at B, is represented by a succession of tripartite numbers constituting a composition of the tripartite $\overline{pqr}$.*

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, &c., and penultimate steps.

Essential nodes occur upon lines of route whenever the direction at a point changes from β to α, from γ to α, or from γ to β. These will be alluded to briefly as $\beta\alpha$, $\gamma\alpha$, or $\gamma\beta$ essential nodes. To these essential nodes on the graph of a composition correspond respectively zero-positive, positive-positive, and positive-zero contacts in the composition itself. It is convenient to call these collectively essential contacts. The theory of conjugate composition exists as in the bipartite theory. On every line of route there are $p+q+r-1$ points, and if there be s essential nodes, $2^{p+q+r-1-s}$ distinct composition graphs can be delineated along the line of route. Each of these has s essential contacts, and, as in the bipartite case, we establish a one-to-one correspondence between the compositions having s essential contacts, and $s+t+1$ parts, and those having s essential contacts, and $p+q+r-t$ parts. The graph of a quadripartite number is derived from the tripartite graph, and generally the graphs of the multipartite numbers of order n from those of the multipartite numbers of order $n-1$, on the same principle as the tripartite from the bipartite; and, moreover, in two dimensions.

32. For the multipartite number

$$\overline{p_1p_2\ldots p_{n-1}p_n}$$

---

* Observe that the thirteen compositions of the tripartite $\overline{111}$ are elegantly represented on the edges of a single cube, the six lines of route lying between opposite corners.

we take $p_n + 1$ exactly similar graphs of the multipartite number

$$\overline{p_1 p_2 \ldots p_{n-1}}$$

and place them similarly with corresponding lines parallel and like points lying on straight lines. When these straight lines are drawn the graph is complete. Other than the initial and final points A, B there are $(p_1 + 1)(p_2 + 1) \ldots (p_n + 1) - 2$ *points* in the graph.

There are lines in n different directions passing through or meeting at each point (including A and B). Call these directions the α_1 direction, the α_2, &c., the α_n direction. A segment joining two adjacent points is called an α segment when it is in the α direction. A step through the reticulation traverses in succession any number of $\alpha_1, \alpha_2, \ldots \alpha_n$ segments; but *in any one step* the segments must be taken in the order $\alpha_1, \alpha_2, \ldots \alpha_n$. In a step the number of steps traversed may be zero in any one, any two, &c., any $n - 1$ of the n different directions. If a step involve p'_1 segments in the α_1 direction, p'_2 segments in the α_2 direction and so on, it may be represented by the multipartite number

$$\overline{p'_1 p'_2 \ldots p'_n}.$$

A succession of steps, the first starting at A and the last terminating at B, is represented by a succession of multipartite numbers, constituting a composition of the multipartite

$$\overline{p_1 p_2 p_3 \ldots p_n}.$$

Any composition follows a certain line of route through the reticulation. The number of distinct lines of route is the number of permutations of the letters in

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n},$$

and is therefore

$$\binom{p_1 + p_2 + \ldots + p_n}{p_1, p_2, \ldots p_{n-1}},$$

employing an obvious extension of notation.

The graph of a composition is obtained by placing nodes at the points which terminate the first, second, &c., and penultimate steps. Essential nodes occur upon a line of route whenever the direction at a point changes from α_u to α_t where $u > t$. At this point the contact between the adjacent parts of the composition is such that a part terminating with $n - u$ zero elements precedes a part commencing with $t - 1$ zero elements. We may speak of this as a contact of $n - u$ zeros with $t - 1$ zeros. The number of zeros in contact, being

$$n - 1 - (u - t),$$

may be any number from 0 to $n - 2$, according to the magnitude of $u - t$; and there are $j + 1$ different contacts for which the number of zeros in contact is j.

33. I now enquire, in respect of the tripartite reticulation, into the number of lines of route which possess exactly s_{21}, $\beta\alpha$ essential nodes, s_{32}, $\gamma\beta$ and s_{31}, $\gamma\alpha$ essential nodes. The second method of investigation adopted in the bipartite case will be employed.

Consider the permutations of the symbols in $\alpha^p\beta^q\gamma^r$ which possess s_{21}, $\beta\alpha$; s_{32}, $\gamma\beta$; and s_{31}, $\gamma\alpha$ contacts, and suppose

$$s_{21}+s_{32}+s_{31}=s.$$

It will be shown that the number of such permutations is equal to the number of permutations in which

β occurs s_{21} times in the first p places,
γ ,, s_{31} ,, ,, ,, ,,
γ ,, s_{32} ,, in the q places succeeding the first p.

A permutation of the former kind involves successions of letters

$$\beta\alpha,\ \gamma\beta,\ \gamma\alpha,\ \gamma\beta\alpha\,;$$

letters between consecutive successions being in alphabetical order.

As regards these successions, and attending to them alone, the permutation exhibits some permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma}\ (\gamma\beta)^{s_{32}-\sigma}\ (\gamma\beta\alpha)^{\sigma}\ (\gamma\alpha)^{s_{31}}.$$

These permutations are

$$\frac{s_{21}+s_{32}+s_{31}-\sigma)\,!}{(s_{21}-\sigma)\,!\,(s_{32}-\sigma)\,!\,\sigma\,!\,s_{31}\,!}\ \text{in number.}$$

Selecting any one of these there are, with reference to the terms,

$$s_{21}+s_{32}+s_{31}-\sigma+1$$

different positions in which other letters may be placed. It is clear that α cannot be placed after a term $(\gamma\beta)$, or γ before a term $(\beta\alpha)$, for these placings would lead to additional $(\beta\alpha)$ and $(\gamma\beta)$ contacts. The letter α may be placed before any of the terms $(\beta\alpha)$, $(\gamma\beta)$, $(\gamma\beta\alpha)$, $(\gamma\alpha)$, and after any of the terms $(\beta\alpha)$, $(\gamma\beta\alpha)$, $(\gamma\alpha)$. Hence out of the $s_{21}+s_{32}+s_{31}-\sigma+1$ different positions (relative to the terms) $s_{21}+s_{31}+1$ positions may be occupied by a letter α. The letter β may be placed in all the $s_{21}+s_{32}+s_{31}-\sigma+1$ positions without the introduction of additional $\beta\alpha$, $\gamma\beta$, $\gamma\alpha$ terms. The letter γ must not be in a place preceding $(\beta\alpha)$, and hence it may occupy $s_{32}+s_{31}+1$ different positions.

Besides the letters occurring in the terms

$$(\beta\alpha)^{s_{21}-\sigma}\ (\gamma\beta)^{s_{32}-\sigma}\ (\gamma\beta\alpha)^{\sigma}\ (\gamma\alpha)^{s_{31}},$$

α occurs $p-s_{21}-s_{31}$ times,
β ,, $q-s_{21}-s_{32}+\sigma$ times,
γ ,, $r-s_{32}-s_{31}$ times.

The $p - s_{21} - s_{31}$ letters α may be distributed in $s_{21} + s_{31} + 1$ positions in

$$\binom{p}{s_{21} + s_{31}} \text{ ways.}$$

The $q - s_{21} - s_{32} + \sigma$ letters β may be distributed in $s_{21} + s_{32} + s_{31} - \sigma + 1$ positions in

$$\binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma} \text{ ways.}$$

The $r - s_{32} - s_{31}$ letters γ may be distributed in $s_{32} + s_{31} + 1$ positions in

$$\binom{r}{s_{32} + s_{31}} \text{ ways.}$$

Hence, for any given permutation of the terms in

$$(\beta\alpha)^{s_{21}-\sigma}\,(\gamma\beta)^{s_{32}-\sigma}\,(\gamma\beta\alpha)^{\sigma}\,(\gamma\alpha)^{s_{31}}$$

there are

$$\binom{p}{s_{21} + s_{31}}\binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma}\binom{r}{s_{32} + s_{31}}$$

arrangements of the remaining letters which do not introduce additional $\beta\alpha$, $\gamma\beta$ or $\gamma\alpha$ contacts.

Hence for a given value of σ there are

$$\binom{p}{s_{21} + s_{31}}\binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma}\binom{r}{s_{32} + s_{31}}\frac{(s_{21} + s_{32} + s_{31} - \sigma)!}{(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!}$$

permutations.

To complete the enumeration we have to sum this expression in regard to σ.

We have

$$\begin{aligned}
&\Sigma\binom{q + s_{31}}{s_{21} + s_{32} + s_{31} - \sigma}\frac{(s_{21} + s_{32} + s_{31} - \sigma)!}{(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!}\\
&= \Sigma\,\frac{(q + s_{31})!}{(q - s_{21} - s_{32} + \sigma)!\,(s_{21} - \sigma)!\,(s_{32} - \sigma)!\,\sigma!\,s_{31}!}\\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!}\,\Sigma\,\frac{s_{32}!}{\sigma!\,(s_{32} - \sigma)!}\cdot\frac{(q - s_{32})!}{(s_{21} - \sigma)!\,(q - s_{21} - s_{32} + \sigma)!}\\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!}\,\Sigma\binom{s_{32}}{\sigma}\binom{q - s_{32}}{s_{21} - \sigma}\\
&= \frac{(q + s_{31})!}{s_{31}!\,s_{32}!\,(q - s_{32})!}\binom{q}{s_{21}}\\
&= \binom{s_{21} + s_{31}}{s_{21}}\binom{q}{s_{32}}\binom{q + s_{31}}{s_{21} + s_{31}}.
\end{aligned}$$

Hence the number of permutations sought is

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}}.$$

34. It can be shown by direct expansion that this number is the coefficient of $\lambda_{21}{}^{s_{21}}\lambda_{31}{}^{s_{31}}\lambda_{32}{}^{s_{32}}\alpha^p\beta^q\gamma^r$ in the development of

$$(\alpha+\lambda_{21}\beta+\lambda_{31}\gamma)^p\,(\alpha+\beta+\lambda_{32}\gamma)^q\,(\alpha+\beta+\gamma)^r,$$

which expresses the number of permutations in which

β occurs s_{21} times in the first p places.
γ ,, s_{31} ,, ,,
γ ,, s_{32} times in the q places succeeding the first p.

This remarkable identity of numbers suggests a one-to-one correspondence between the permutations of the two kinds, but I have not as yet been able to determine the law of the transformation. It would appear to be a very difficult problem. The number of lines of route in the tripartite reticulation which possess

s_{21}, $\beta\alpha$ essential nodes,
s_{32}, $\gamma\beta$,, ,,
s_{31}, $\gamma\alpha$,, ,,

is thus

$$\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+31}\binom{r}{s_{32}+s_{31}}.$$

35. Hence the identity

$$\frac{p+q+r)!}{p!\,q!\,r!}=\Sigma\binom{s_{21}+s_{31}}{s_{21}}\binom{p}{s_{21}+s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{21}+s_{31}}\binom{r}{s_{32}+s_{31}},$$

the summation being for all positive integral, including zero, values of s_{21}, s_{31}, s_{32}, which yield positive terms.

36. The generating function for the number of lines of route having s essential nodes; that is $s_{21}+s_{31}+s_{32}=s$; is

$$(\alpha+\lambda\beta+\lambda\gamma)^p\,(\alpha+\beta+\lambda\gamma)^q\,(\alpha+\beta+\gamma)^r,$$

the number in question being the coefficient of $\lambda^s\alpha^p\beta^q\gamma^r$.

37. The whole coefficient of $\alpha^p\beta^q\gamma^r$ being

$$C_0 + C_1\lambda + C_2\lambda^2 + \ldots + C_s\lambda^s + \ldots$$

$$C_s = \Sigma \binom{s_{21}+s_{31}}{s_{21}} \binom{p}{s_{21}+s_{31}} \binom{q}{s_{32}} \binom{q+s_{31}}{s_{21}+s_{31}} \binom{r}{s_{32}+s_{31}}$$

the summation being subject to the condition

$$s_{21} + s_{31} + s_{32} = s.$$

Moreover, denoting the whole number of compositions of the tripartite $\overline{pqr}$ by $F(pqr)$, we have

$$\begin{aligned} F(pqr) &= \overset{s}{\Sigma}\, C_s\, 2^{p+q+r-s-1} \\ &= 2^{p+q+r-1} \overset{s}{\Sigma}\, C_s\, (\tfrac{1}{2})^s. \end{aligned}$$

38. Hence $F(pqr)$ is the coefficient of $\alpha^p\beta^q\gamma^r$ in the development of the product

$$2^{p+q+r-1}(\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma)^p (\alpha + \beta + \tfrac{1}{2}\gamma)^q (\alpha + \beta + \gamma)^r,$$

which is more conveniently written

$$\tfrac{1}{2}(2\alpha + \beta + \gamma)^p (2\alpha + 2\beta + \gamma)^q (2\alpha + 2\beta + 2\gamma)^r.$$

39. This product is a generating function which enumerates the compositions of the single multipartite number $\overline{pqr}$, but the generating function of all tripartite numbers can be at once derived from it.

It is

$$\frac{1}{2} \frac{1}{\{1 - s(2\alpha + \beta + \gamma)\}\{1 - t(2\alpha + 2\beta + \gamma)\}\{1 - u(2\alpha + 2\beta + 2\gamma)\}},$$

in which, when expanded, the coefficient of $(s\alpha)^p(t\beta)^q(u\gamma)^r$ is the number of compositions of the tripartite number $\overline{pqr}$.

The generating function previously obtained for tripartite numbers from the analytical theory was

$$\frac{\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma}{1 - 2(\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma)},$$

the number of compositions being given by the coefficient of $\alpha^p\beta^q\gamma^r$. The addition of $\frac{1}{2}$ to this fraction brings it to the better form

$$\frac{1}{2} \frac{1}{1 - 2(\alpha + \beta + \gamma - \beta\gamma - \gamma\alpha - \alpha\beta + \alpha\beta\gamma)},$$

which is consistent also with the circumstance that it was found convenient analytically to regard the number of compositions of multipartite zero as being the fraction $\frac{1}{2}$.

40. The number may be stated as the coefficient of $(s\alpha)^p(t\beta)^q(u\gamma)^r$ in the expansion of

$$\frac{1}{2}\frac{1}{1-2(s\alpha+t\beta+u\gamma-tu\beta\gamma-us\gamma\alpha-st\alpha\beta+stu\alpha\beta\gamma)},$$

and we have shown that this fraction is equivalent to that portion of the expansion of the fraction

$$\frac{1}{2}\frac{1}{\{1-s(2\alpha+\beta+\gamma)\}\{1-t(2\alpha+2\beta+\gamma)\}\{1-u(2\alpha+2\beta+2\gamma)\}}$$

which is a function only of $s\alpha$, $t\beta$, and $u\gamma$.

41. It will now be shown that the fraction

$$\frac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

is, in fact, a generating function which enumerates the compositions of multipartite numbers of order n.

This important theorem will be demonstrated by showing that the aggregate of terms in the expansion of the last written fraction, which is composed entirely of powers of $s_1\alpha_1$, $s_2\alpha_2$, . . . $s_n\alpha_n$, is correctly represented by the fraction

$$\frac{1}{2}\frac{1}{1-2(\Sigma s_1\alpha_1-\Sigma s_1s_2\alpha_1\alpha_2+\ldots(-)^{n+1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n)},$$

which has been already shown to be a true generating function.

42. For brevity put

$$S_\kappa = s_\kappa(2\alpha_1+2\alpha_2+\ldots+2\alpha_\kappa+\alpha_{\kappa+1}+\ldots+\alpha_n) = s_\kappa(A_\kappa+2\alpha_\kappa)$$
$$b_1 = \Sigma s_1\alpha_1,\ b_2 = \Sigma s_1s_2\alpha_1\alpha_2,\ \&c.\ \ldots$$
$$M = (1-2s_1\alpha_1)(1-2s_2\alpha_2)\ldots(1-2s_n\alpha_n).$$

The two fractions under comparison are

$$\frac{1}{2}\frac{1}{-1+2(1-s_1\alpha_1)(1-s_2\alpha_2)\ldots(1-s_n\alpha_n)}=\frac{1}{2N},$$

and

$$\frac{1}{2}\frac{1}{(1-S_1)(1-S_2)\ldots(1-S_n)}=\frac{1}{2D}.$$

Since $\mathrm{M} - \mathrm{N} = (2^2 - 2)\, b_2 - (2^3 - 2)\, b_3 + \ldots + (-)^n (2^n - 2) b_n$,

$$\frac{\mathrm{N}}{\mathrm{D}} = \frac{\mathrm{M}}{\mathrm{D}} - \frac{\mathrm{M} - \mathrm{N}}{\mathrm{D}} = \prod_1^n \frac{1 - 2s_\kappa \alpha_\kappa}{1 - \mathrm{S}_\kappa} - \frac{(2^2 - 2)\, b_2 - (2^3 - 2)\, b_3 + \ldots + (-)^n (2^n - 2)\, b_n}{(1 - \mathrm{S}_1)(1 - \mathrm{S}_2) \ldots (1 - \mathrm{S}_n)}$$

$$= \prod_1^n \left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right) - \frac{(2^2 - 2)\, b_2 - \ldots + (-)^n (2^n - 2)\, b_n}{(1 - \mathrm{S}_1)(1 - \mathrm{S}_2) \ldots (1 - \mathrm{S}_n)};$$

the second member of the right-hand side of this identity has now to be transformed into a series of partial fractions of the same form as those which arise from the product

$$\prod_1^n \left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right).$$

43. Let $\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_t$ be any t different numbers selected from the first n integers $1, 2, \ldots n$, arranged in ascending order of magnitude.

Let

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2} = 2 s_{\kappa_1} \alpha_{\kappa_1} s_{\kappa_2} \alpha_{\kappa_2}$$

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3} = \mathrm{B}^{(1)}_{\kappa_2\kappa_3} \mathrm{S}_{\kappa_1} + \mathrm{B}^{(1)}_{\kappa_3\kappa_1} \mathrm{S}_{\kappa_2} + \mathrm{B}^{(1)}_{\kappa_1\kappa_2} \mathrm{S}_{\kappa_3}$$

. . . .

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2 \ldots \kappa_t} = \Sigma\, \mathrm{B}^{(1)}_{\kappa_1\kappa_2} \mathrm{S}_{\kappa_3} \ldots \mathrm{S}_{\kappa_t},$$

the summation being for the $\binom{t}{2}$ terms obtained from the $\binom{t}{2}$ expressions $\mathrm{B}_{\kappa_1\kappa_2}{}^{(1)}$ that it is possible to construct from the t integers $\kappa_1, \kappa_2, \ldots \kappa_t$.

Further, let

$$\mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_{j+1}} = (2^{j+1} - 2)\, s_{\kappa_1} \alpha_{\kappa_1} s_{\kappa_2} \alpha_{\kappa_2} \ldots s_{\kappa_{j+1}} \alpha_{\kappa_{j+1}}$$

$$\mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_t} = \Sigma\, \mathrm{B}^{(j)}_{\kappa_1\kappa_3 \ldots \kappa_{j+1}} \mathrm{S}_{j+2} \ldots \mathrm{S}_t$$

the summation being for the $\binom{t}{j+1}$ terms obtained from the $\binom{t}{j+1}$ expressions $\mathrm{B}_{\kappa_1\kappa_2 \ldots \kappa_{j+1}}{}^{(j)}$ that it is possible to construct from the t integers $\kappa_1, \kappa_2, \ldots \kappa_t$.

44. The following lemma is required :—

"*Lemma.*

$$\Sigma\, \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_{t-1}} \mathrm{S}_{\kappa_t} = \binom{t - j - 1}{1} \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_t},$$

$$\Sigma\, \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_{t-2}} \mathrm{S}_{\kappa_{t-1}} \mathrm{S}_{\kappa_t} = \binom{t - j - 1}{2} \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_t},$$

. . . .

$$\Sigma\, \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_{t-s}} \mathrm{S}_{\kappa_{t-s+1}} \ldots \mathrm{S}_{\kappa_t} = \binom{t - j - 1}{s} \mathrm{B}^{(j)}_{\kappa_1\kappa_2 \ldots \kappa_t}$$

the summations being subject to the same conditions as before. The truth of these relations is obvious from the elementary theory of permutations."

45. Consider now the series of functions—

$$\mathrm{P}_{\kappa_1\kappa_2} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2},$$
$$\mathrm{P}_{\kappa_1\kappa_2\kappa_3} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\kappa_3},$$
$$\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\kappa_3\kappa_4} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\kappa_3\kappa_4} + \mathrm{B}^{(3)}_{\kappa_1\kappa_2\kappa_3\kappa_4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^t\,\mathrm{B}^{(t-1)}_{\kappa_1\kappa_2\quad\kappa_t}$$

From the foregoing lemma are derived the relations—

$$\Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_t} = \mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t},$$
$$\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}\mathrm{S}_{\kappa_4}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{1}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t},$$
$$\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4}\mathrm{S}_{\kappa_5}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{2}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \binom{t-3}{1}\mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \mathrm{B}^{(3)}_{\kappa_1\kappa_2\ldots\kappa_t},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_s}\mathrm{S}_{\kappa_{s+1}}\ldots\mathrm{S}_{\kappa_t} = \binom{t-2}{s-2}\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \binom{t-3}{s-3}\mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^s\,\mathrm{B}^{(s-1)}_{\kappa_1\kappa_2\ldots\kappa_t}.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence, by addition with alternate signs,

$$(-)^t\,\mathrm{B}_{\kappa_1\kappa_2\ldots\kappa_t}{}^{(t-1)} = \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} - \Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-1}}\mathrm{S}_{\kappa_t} + \Sigma\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-2}}\mathrm{S}_{\kappa_{t-1}}\mathrm{S}_{\kappa_t} - \ldots$$
$$+ (-)^t\,\Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4}\ldots\mathrm{S}_{\kappa_t}.$$

Here the summations are in respect of the t numbers $\kappa_1, \kappa_2, \ldots \kappa_t$.

46. A similar identity may be established for every selection of t integers out of the numbers 1, 2, 3, . . . n.

Summing all these identities, and remembering that

$$\overset{n}{\Sigma}\,\mathrm{B}_{\kappa_1\kappa_2\ldots\kappa_t}{}^{(t-1)} = (2^t - 2)\,b_t,$$

we obtain

$$(-)^t(2^t - 2)\,b_t = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{t-1}}\mathrm{S}_{\kappa_t} + \ldots + (-)^t\,\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_t},$$

or, attributing to t all integer values from 2 to n inclusive, we have the series of relations :—

$$(2^2-2)\,b_2 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}$$

$$-(2^3-2)\,b_3 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3} - \Sigma\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}$$

$$+(2^4-2)\,b_4 = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3\kappa_4} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}\mathrm{S}_{\kappa_4} + \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4}$$

. . . .

$$(-)^n(2^n-2)\,b_n = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n} - \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{n-1}}\mathrm{S}_{\kappa_n} + \ldots + (-)^n\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\ldots\mathrm{S}_{\kappa_n}.$$

Now

$$\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3} = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}(\mathrm{S}_{\kappa_3}+\mathrm{S}_{\kappa_4}+\ldots+\mathrm{S}_{\kappa_n})$$

$$\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4} = \overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}(\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_4}+\mathrm{S}_{\kappa_3}\mathrm{S}_{\kappa_5}+\ldots+\mathrm{S}_{\kappa_{n-1}}\mathrm{S}_{\kappa_n})$$

with similar relations in the case of the other symmetric functions.

47. Hence, by addition,

$$\begin{aligned}
&(2^2-2)\,b_2-(2^3-2)\,b_3+\ldots+(-)^n(2^n-2)\,b_n\\
&\quad=\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2}(1-\mathrm{S}_{\kappa_3})(1-\mathrm{S}_{\kappa_4})\ldots(1-\mathrm{S}_{\kappa_n})\\
&\qquad+\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\kappa_3}(1-\mathrm{S}_{\kappa_4})(1-\mathrm{S}_{\kappa_5})\ldots(1-\mathrm{S}_{\kappa_n})\\
&\qquad+\ldots\\
&\qquad+\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_1\ldots\kappa_{n-1}}(1-\mathrm{S}_{\kappa_n})\\
&\qquad+\overset{n}{\Sigma}\,\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n},
\end{aligned}$$

and thence,

$$\begin{aligned}
&\frac{(2^2-2)\,b_2-(2^3-2)\,b_3+\ldots+(-)^n(2^n-2)\,b_n}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}\\
&\quad=\overset{n}{\Sigma}\frac{\mathrm{P}_{\kappa_1\kappa_2}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_2})}\\
&\qquad+\overset{n}{\Sigma}\frac{\mathrm{P}_{\kappa_1\kappa_2\kappa_3}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_2})(1-\mathrm{S}_{\kappa_3})}\\
&\qquad+\ldots\\
&\qquad+\overset{n}{\Sigma}\frac{\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t}}{(1-\mathrm{S}_{\kappa_1})(1-\mathrm{S}_{\kappa_2})\ldots(1-\mathrm{S}_{\kappa_t})}\\
&\qquad+\ldots\\
&\qquad+\frac{\mathrm{P}_{123\ldots n}}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}.
\end{aligned}$$

The relation

$$\frac{\mathrm{N}}{\mathrm{D}} = \prod_{1}^{n}\left(1 + \frac{s_\kappa \mathrm{A}_\kappa}{1 - \mathrm{S}_\kappa}\right) - \frac{(2^2 - 2)\, b_2 - (2^3 - 2)\, b_3 + \ldots + (-)^n (2^n - 2)\, b_n}{(1 - \mathrm{S}_1)(1 - \mathrm{S}_2)\ldots(1 - \mathrm{S}_n)}$$

now becomes

$$\frac{\mathrm{N}}{\mathrm{D}} = 1 + \sum^{n} \frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1 - \mathrm{S}_{\kappa_1}} + \sum^{n} \frac{s_{\kappa_1}s_{\kappa_2}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - \mathrm{P}_{\kappa_1\kappa_2}}{(1 - \mathrm{S}_{\kappa_1})(1 - \mathrm{S}_{\kappa_2})} + \ldots + \sum^{n} \frac{s_{\kappa_1}\ldots s_{\kappa_{n-1}}\mathrm{A}_{\kappa_1}\ldots\mathrm{A}_{\kappa_{n-1}} - \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_{n-1}}}{(1 - \mathrm{S}_{\kappa_1})(1 - \mathrm{S}_{\kappa_1})\ldots(1 - \mathrm{S}_{\kappa_{n-1}})},$$

the final fraction

$$\frac{s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_n}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2}\ldots\mathrm{A}_{\kappa_{n-1}} - \mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_n}}{(1 - \mathrm{S}_1)(1 - \mathrm{S}_2)\ldots(1 - \mathrm{S}_n)}$$

vanishing as will be seen presently.

48. This form of the fraction $\frac{\mathrm{N}}{\mathrm{D}}$ will be employed in order to show that the expansions of $\frac{1}{\mathrm{D}}$ and $\frac{1}{\mathrm{N}}$ are effectively identical. This arises from the circumstance that the right-hand side of the above identity when expanded contains no single term expressible as a function of $s_1\alpha_1$, $s_2\alpha_2$, $s_3\alpha_3$, ... $s_n\alpha_n$.

49. Of the fractions $\Sigma\,\frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1 - \mathrm{S}_{\kappa_1}}$ consider the typical fraction

$$\frac{s_{\kappa_1}\mathrm{A}_{\kappa_1}}{1 - \mathrm{S}_{\kappa_1}} = \frac{s_{\kappa_1}(2\alpha_1 + \ldots + 2\alpha_{\kappa_1-1} + \alpha_{\kappa_1+1} + \ldots + \alpha_n)}{1 - s_{\kappa_1}(2\alpha_1 + \ldots + 2\alpha_{\kappa_1} + \alpha_{\kappa_1+1} + \ldots + \alpha_n)};$$

since the numerator contains a factor s_{κ_1} and no quantity α_{κ_1} the fraction when expanded can contain no term which is a function of $s_1\alpha_1$, $s_2\alpha_2$, ... $s_n\alpha_n$ alone.

Consider next the typical fraction

$$\frac{s_{\kappa_1}s_{\kappa_2}\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - \mathrm{P}_{\kappa_1\kappa_2}}{(1 - \mathrm{S}_{\kappa_1})(1 - \mathrm{S}_{\kappa_2})};$$

the numerator is

$$s_{\kappa_1}s_{\kappa_2}(\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} - 2\alpha_{\kappa_1}\alpha_{\kappa_2})$$

and

$$\mathrm{A}_{\kappa_1}\mathrm{A}_{\kappa_2} = (2\alpha_1 + \ldots + 2\alpha_{\kappa_1-1} + \ldots + \alpha_{\kappa_2} + \ldots + \alpha_n)(2\alpha_1 + \ldots + 2\alpha_{\kappa_1} + \ldots + 2\alpha_{\kappa_2-1} + \ldots + \alpha_n),$$

and, therefore, contains a term

$$2\alpha_{\kappa_1}\alpha_{\kappa_2};$$

hence the numerator is free from such a term and every term in it contains one of the quantities

$$\alpha_{\kappa_3}, \alpha_{\kappa_4}, \ldots \alpha_{\kappa_n};$$

but the whole fraction contains no quantities

$$s_{\kappa_3}, \alpha_{\kappa_4}, \ldots s_{\kappa_n}$$

and thus it is manifest that the fraction can contribute no term which is a function of

$$s_1\alpha_1, \ldots s_n\alpha_n.$$

50. To simplify the discussion of the remaining fractions it is necessary to consider some particular properties of the typical numerator.

The function

$$\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t}$$

may be expressed as

$$\mathrm{B}^{(1)}_{\kappa_1\kappa_2\ldots\kappa_t} - \mathrm{B}^{(2)}_{\kappa_1\kappa_2\ldots\kappa_t} + \ldots + (-)^t\,\mathrm{B}^{(t-1)}_{\kappa_1\kappa_2\ldots\kappa_t},$$

or as

$$\begin{aligned}\{&\overset{t}{\Sigma}\,(2^2-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}(\mathrm{A}_{\kappa_3}+2\alpha_{\kappa_3})\ldots(\mathrm{A}_{\kappa_t}+2\alpha_{\kappa_t})\\ &-\overset{t}{\Sigma}\,(2^3-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\alpha_{\kappa_3}(\mathrm{A}_{\kappa_4}+2\alpha_{\kappa_4})\ldots(\mathrm{A}_{\kappa_t}+2\alpha_{\kappa_t})\\ &+\ldots\\ &+(-)^t\,(2^t-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa_t}\}\,s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_t}\end{aligned}$$

the summations being in respect of the t numbers

$$\kappa_1, \kappa_2, \ldots \kappa_t.$$

Writing

$$\mathrm{P}_{\kappa_1\kappa_2\ldots\kappa_t} = \mathrm{P}'_{\kappa_1\kappa_2\ldots\kappa}\; s_{\kappa_1}s_{\kappa_2}\ldots s_{\kappa_t},$$

we have now the identity

$$\begin{aligned}\mathrm{P}'_{\kappa_1\kappa_2\ldots\kappa_t} = \overset{t}{\Sigma}\,(2^2-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\mathrm{A}_{\kappa_3}\mathrm{A}_{\kappa_4}\ldots\mathrm{A}_{\kappa_t} + \Sigma\,(2^3-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\alpha_{\kappa_3}\mathrm{A}_{\kappa_4}\ldots\mathrm{A}_{\kappa_t}\\ + \ldots + (2^t-2)\,\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa}.\end{aligned}$$

51. To establish this it is sufficient to observe that the coefficient of

$$\alpha_{\kappa_1}\alpha_{\kappa_2}\ldots\alpha_{\kappa_s}\,\mathrm{A}_{\kappa_{s+1}}\ldots\mathrm{A}_{\kappa_t}$$

in the former series is

$$\binom{s}{2} 2^{s-2}(2^2-2) - \binom{s}{3} 2^{s-3}(2^3-2) + \ldots + (-)^{s+1}\binom{s}{s}(2^s-2),$$

which has the value $2^s - 2$.

52. Hence the numerator of the typical fraction may, to a factor $s_{\kappa_1} s_{\kappa_2} \ldots s_{\kappa_t}$ *près* be thrown into the form :—

$$2(\mathrm{A}_{\kappa_1} + \alpha_{\kappa_1})(\mathrm{A}_{\kappa_2} + \alpha_{\kappa_2}) \ldots (\mathrm{A}_{\kappa} + \alpha_{\kappa_t}) - (\mathrm{A}_{\kappa_1} + 2\alpha_{\kappa_1})(\mathrm{A}_{\kappa_2} + 2\alpha_{\kappa_2}) \ldots (\mathrm{A}_{\kappa_t} + 2\alpha_{\kappa_t}),$$

which is of great service.

Of the series of quantities

$$s_1, s_2, \ldots s_n,$$

the typical fraction contains only the t quantities

$$s_{\kappa_1}, s_{\kappa_2}, \ldots s_{\kappa_t}.$$

If it can be shown that the numerator consists of terms each of which contains quantities of the series

$$\alpha_1, \alpha_2, \ldots \alpha_n,$$

which are not included in the set

$$\alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots \alpha_{\kappa_t},$$

it will follow at once that the fraction can, when expanded, contain no term which is a function of

$$s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n.$$

It suffices to show that the numerator vanishes when all the quantities of the set

$$\alpha_1, \alpha_2, \ldots \alpha_n,$$

not included in the set

$$\alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots \alpha_{\kappa_t},$$

are put equal to zero.

Under these circumstances, remembering that $\kappa_1 \, \kappa_2 \ldots \kappa_n$ are in ascending order of magnitude, we have

$$\mathrm{A}_{\kappa_1} = \alpha_{\kappa_2} + \alpha_{\kappa_3} + \ldots + \alpha_{\kappa_t}$$

$$\mathrm{A}_{\kappa_2} = 2\alpha_{\kappa_1} + \alpha_{\kappa_3} + \ldots + \alpha_{\kappa_t}$$

$$\mathrm{A}_{\kappa_3} = 2\alpha_{\kappa_1} + 2\alpha_{\kappa_2} + \alpha_{\kappa_4} + \ldots + \alpha_{\kappa_t}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathrm{A}_{\kappa_{t-1}} = 2(\alpha_{\kappa_1} + \alpha_{\kappa_2} + \ldots + \alpha_{\kappa_{t-2}}) + \alpha_{\kappa_t}$$

$$\mathrm{A}_{\kappa_t} = 2(\alpha_{\kappa_1} + \alpha_{\kappa_2} + \ldots + \alpha_{\kappa_{t-1}})$$

and the relations

$$\begin{aligned} A_{\kappa_1} + 2\alpha_{\kappa_1} &= A_{\kappa_2} + \alpha_{\kappa_2} \\ A_{\kappa_2} + 2\alpha_{\kappa_2} &= A_{\kappa_3} + \alpha_{\kappa_3} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ A_{\kappa_{t-1}} + 2\alpha_{\kappa_{t-1}} &= A_{\kappa_t} + \alpha_{\kappa_t} \\ A_{\kappa_t} + 2\alpha_{\kappa_t} &= 2\,(A_{\kappa_1} + \alpha_{\kappa_1}) \end{aligned}$$

hence the numerator factor becomes

$$2\,(A_{\kappa_1} + \alpha_{\kappa_1})\,(A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t}) - (A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t})\, 2\,(A_{\kappa_1} + \alpha_{\kappa_1}) = 0.$$

53. This result proves incidentally, as stated above, that the final fraction

$$\frac{s_{\kappa_1}\, s_{\kappa_2} \ldots s_{\kappa_n}\, A_{\kappa_1}\, A_{\kappa_2} \ldots A_{\kappa_n} - P_{\kappa_1\, \kappa_2 \ldots \kappa_n}}{(1 - S_1)\,(1 - S_2) \ldots (1 - S_n)}$$

vanishes identically.

It also terminates the proof which it has been the object of this portion of the investigation to set forth.

54. The analytical result may be stated as follows :—

The fraction

$$\tfrac{1}{2}\, \frac{1}{\{1 - s_1\,(2\alpha_1 + \alpha_2 + \ldots + \alpha_n)\}\, \{1 - s_2\,(2\alpha_1 + 2\alpha_2 + \ldots + \alpha_n)\} \ldots \{1 - s_n\,(2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n)\}}$$

is equal to the product of the fraction

$$\tfrac{1}{2}\, \frac{1}{1 - 2\,(\Sigma\, s_1\alpha_1 - \Sigma\, s_1 s_2 \alpha_1 \alpha_2 + \ldots + (-)^{n+1}\, s_1 s_2 \ldots s_n\, \alpha_1 \alpha_2 \ldots \alpha_n)}$$

and the series

$$1 + \Sigma\, \frac{2\,(A_{\kappa_1} + \alpha_{\kappa_1})\,(A_{\kappa_2} + \alpha_{\kappa_2}) \ldots (A_{\kappa_t} + \alpha_{\kappa_t}) - (A_{\kappa_1} + 2\alpha_{\kappa_1})\,(A_{\kappa_2} + 2\alpha_{\kappa_2}) \ldots (A_{\kappa_t} + 2\alpha_{\kappa_t})}{(1 - S_{\kappa_1})\,(1 - S_{\kappa_2}) \ldots (1 - S_{\kappa_t})}$$

where

$$S_\kappa = s_\kappa\,(2\alpha_1 + \ldots + 2\alpha_\kappa + \alpha_{\kappa+1} + \ldots + \alpha_n) = s_\kappa\,(A_\kappa + 2\alpha_\kappa)$$

and the summation is in regard to every selection of t integers from the series $1, 2, 3, \ldots n$, and t takes all values from 1 to $n - 1$.

55. It may be interesting to give the simplest cases at length.

Order 2.

$$\tfrac{1}{2}\, \frac{1}{\{1 - s_1\,(2\alpha_1 + \alpha_2)\}\, \{1 - s_2\,(2\alpha_1 + 2\alpha_2)\}}$$

$$= \tfrac{1}{2}\, \frac{1}{1 - 2\,(s_1\alpha_1 + s_2\alpha_2 - s_1 s_2 \alpha_1 \alpha_2)} \times \left[1 + \frac{s_1\alpha_2}{1 - s_1\,(2\alpha_1 + \alpha_2)} + \frac{2\, s_2\alpha_1}{1 - s_2\,(2\alpha_1 + 2\alpha_2)}\right],$$

56. *Order* 3.

$$\tfrac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}$$

$$=\tfrac{1}{2}\frac{1}{1-2(s_1\alpha_1+s_2\alpha_2+s_3\alpha_3-s_2s_3\alpha_2\alpha_3-s_3s_1\alpha_3\alpha_1-s_1s_2\alpha_1\alpha_2+s_1s_2s_3\alpha_1\alpha_2\alpha_3)}$$

$$\times\left[1+\frac{s_1(\alpha_2+\alpha_3)}{1-s_1(2\alpha_1+\alpha_2+\alpha_3)}+\frac{s_2(2\alpha_1+\alpha_3)}{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)}+\frac{s_3(2\alpha_1+2\alpha_2)}{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)}\right.$$

$$+\frac{s_1s_2\alpha_3(2\alpha_1+\alpha_2+\alpha_3)}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}}$$

$$+\frac{2s_1s_3\alpha_2(\alpha_1+\alpha_2+\alpha_3)}{\{1-s_1(2\alpha_1+\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}$$

$$\left.+\frac{2s_2s_3\alpha_1(2\alpha_1+2\alpha_2+\alpha_3)}{\{1-s_2(2\alpha_1+2\alpha_2+\alpha_3)\}\{1-s_3(2\alpha_1+2\alpha_2+2\alpha_3)\}}\right].$$

57. The general algebraical result, interpreted arithmetically, shows that

$$\tfrac{1}{2}\frac{1}{\{1-s_1(2\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(2\alpha_1+2\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)\}}$$

is a generating function for the enumeration of the compositions of multipartite numbers.

58. The original generating function of the earlier sections has by the addition of ineffective terms become factorized, and can thus be dissected for detailed examination.

The process seems to be analogous to the chemical operation by which the addition of a flux causes an element to be the more easily melted.

As a direct consequence of the geometrical representation of the compositions of multipartite numbers on a reticulation it is of great interest.

59. To resume. The number of compositions of the multipartite number $\overline{p_1p_2\ldots p_n}$ is the coefficient of

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the expanded product

$$\tfrac{1}{2}(2\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(2\alpha_1+2\alpha_2+\ldots+\alpha_n)^{p_2}\ldots(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)^{p_n},$$

or we may say that it is the coefficient of the symmetric function

$$\Sigma\,\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the development of the symmetric function

$$\tfrac{1}{2}\Sigma\,(2\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(2\alpha_1+2\alpha_2+\ldots+\alpha_n)^{p_2}\ldots(2\alpha_1+2\alpha_2+\ldots+2\alpha_n)^{p_n}$$

in a sequence of monomial symmetric functions.

60. The unsymmetrical form gives a direct connection between the numbers of the compositions and of the permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

In the unsymmetrical product the term $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$, attached to some numerical coefficient, arises in connection with every permutation of the letters in the term. One factor of such coefficient is manifestly

$$2^{p_1-1};$$

if the letter α_2, in a particular permutation, occur q_2 times in the last $p_2+p_3+\ldots+p_n$ places of the permutation there will be a factor

$$2^{q_2};$$

further, if the letter α_s, in the same permutation, occur q_s times in the last $p_s+p_{s+1}+\ldots+p_n$ places of the permutation there will be a factor

$$2^{q_s}.$$

Hence for the permutation considered there arises a term

$$2^{p_1-1+q_2+q_3+\ldots+q_n},$$

and the number of compositions must therefore be

$$2^{p_1-1}\Sigma\, 2^{q_2+q_3+\ldots+q_n},$$

the summation being taken in regard to every permutation of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

E.g., to find in this way the number of compositions of the bipartite $\overline{22}$, we have the following scheme :—

			q_2
$p_1=2$	$\alpha_1\alpha_1$	$\alpha_2\alpha_2$	2
	$\alpha_1\alpha_2$	$\alpha_1\alpha_2$	1
	$\alpha_1\alpha_2$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_1$	$\alpha_1\alpha_2$	1
	$\alpha_2\alpha_1$	$\alpha_2\alpha_1$	1
	$\alpha_2\alpha_2$	$\alpha_1\alpha_1$	0

Therefore

$$\mathrm{F}\,(22)=2^{2-1}\,(2^2+2^1+2^1+2^1+2^1+2^0)=26.$$

61. The unsymmetrical product may be written in the form

$$2^{p_1+p_2+\ldots+p_n-1}\{\alpha_1+\tfrac{1}{2}(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\tfrac{1}{2}(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots$$
$$\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n}$$

and herein the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ may be written

$$2^{\Sigma p-1}\overset{s}{\Sigma}\,C_s(\tfrac{1}{2})^s=\overset{s}{\Sigma}C_s2^{\Sigma p-s-1},$$

where C_0, C_1, C_2, ... are certain integers.

Observe that C_s is the coefficient of $\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ in the product

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n}$$

Considering the reticulation of the multipartite number $\overline{p_1p_2\ldots p_n}$, suppose there to be D_s lines of route which possess exactly s essential nodes. Upon these lines of route are represented

$$D_s2^{\Sigma p-s-1}$$

distinct compositions, and the whole number of compositions is manifestly

$$\Sigma\,D_s2^{\Sigma p-s-1}.$$

Hence

$$\Sigma\,D_s2^{\Sigma p-s-1}=\Sigma C_s2^{\Sigma p-s-1},$$

or $D_s=C_s$ presumably (see *post*, Sec. 5, Art. 71), that is to say, the number of distinct lines of route through the reticulation which possess exactly s essential nodes is the coefficient of

$$\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the product

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n}.$$

62. To interpret the product arithmetically we find that for a particular permutation of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

the factor λ^s will occur if

α_2 occurring r_2 times in the first p_1 places of the permutation
α_3 ,, r_3 ,, p_1+p_2 ,, ,,
. . . .
α_n ,, r_n ,, $p_1+p_2+\ldots+p_{n-1}$,, ,,

the sum $r_2+r_3+\ldots+r_n=s$.

63. Hence the number of lines of route through the reticulation which possess exactly s essential nodes is equal to the number of permutations of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

for which

$$r_2+r_3+\ldots+r_n=s;$$

r_t denoting the number of times that the letter α_t occurs in the first

$$p_1+p_2+\ldots+p_{t-1}$$

places of permutation.

64. We are at once led to an important identity of enumeration in the pure theory of permutations. Following the nomenclature of a previous section of the investigation, a line of route through the reticulation is traced out by a succession of steps, each step being an α_1 step or an α_2, &c., or an α_n step.

The whole length of the line of route there are altogether p_1, α_1 steps, p_2, α_2 steps, &c., . . . p_n, α_n steps.

Without regard to the characteristic of lines of route in respect of essential nodes, the whole number of lines of route is equal to the number of different orders in which these steps can be taken, viz., equal to the whole number of permutations of the letters in

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}.$$

An essential node occurs whenever a step α_u *immediately* precedes a step α_t, where $u>t$.

In the corresponding permutation there is a contact

$$\alpha_u\alpha_t,\quad u>t,$$

which may be called a *major* contact.

Hence a line of route with s essential nodes is represented by a permutation with s major contacts. We have thus the theorem :—

"In the reticulation of the multipartite number $\overline{p_1p_2\ldots p_n}$ the number of lines of route which possess exactly s essential nodes is equal to the number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

which possess exactly s major contacts."

"Calling a contact $\alpha_u\alpha_t$ a major contact when $u>t$ the number of permutations of the letters in the product

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n},$$

which possess exactly s major contacts, is given by the coefficient of

$$\lambda^s \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in the product

$$\{\alpha_1 + \lambda(\alpha_2 + \ldots + \alpha_n)\}^{p_1} \{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \ldots + \alpha_n)\}^{p_2} \ldots \{\alpha_1 + \alpha_2 + \ldots + \alpha_n\}^{p_n}."$$

I am not aware if this problem has been previously solved in this form, or, indeed, if it has ever been attacked before.*

65. "The number of permutations of the letters in the product

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n},$$

which possess exactly s major contacts, is equal to the number of permutations for which

$$r_2 + r_3 + \ldots + r_n = s,$$

r_t denoting the number of times that the latter α_t occurs in the first

$$p_1 + p_2 + \ldots + p_{t-1}$$

places of the permutation."

66. It is easy to obtain a refinement of these theorems which has been foreshadowed by the detailed bipartite and tripartite cases which have preceded. Major contacts in a permutation, and, consequently, also the essential nodes along a line of route, are of $\binom{n}{2}$ different kinds.

Consider the product

$$(\alpha_1 + \lambda_{21}\alpha_2 + \lambda_{31}\alpha_3 + \ldots + \lambda_{n1}\alpha_n)^{p_1} (\alpha_1 + \alpha_2 + \lambda_{32}\alpha_3 + \lambda_{42}\alpha_4 + \ldots + \lambda_{n2}\alpha_n)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \lambda_{n,n-1}\alpha_n)^{p_{n-1}} (\alpha_1 + \alpha_2 + \ldots + \alpha_n)^{p_n}.$$

The coefficient of $\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$ consists of a number of monomial products of the quantities λ, each attached to a numerical coefficient. Of these products a certain number are of a definite degree s. The sum of the coefficients of these products gives the number of permutations of the letters in

$$\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

which possess exactly s major contacts.

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* See a paper by H. FORTY, M.A., "On Contact and Isolation, a Problem in Permutations," 'Proceedings London Mathematical Society,' vol. 15.

Let one such product with its attached numerical coefficient be

$$\kappa\lambda_{x_1 1}{}^{\xi_1}\lambda_{x_2 1}{}^{\xi_2} \ldots \lambda_{y_1 2}{}^{\eta_1}\lambda_{y_2}{}^{\eta_2} \ldots \lambda_{z_1 3}{}^{\zeta_1}\lambda_{z_2 3}{}^{\zeta_2} \ldots$$

The number κ enumerates a certain number of the permutations which have s major contacts, and $\Sigma\kappa$ enumerates the entire number.

The problem is to determine the particular permutations enumerated by the number κ.

If p_1 and p_2 be both greater than zero, the contact $\alpha_2\alpha_1$ must occur in one or more of the enumerated permutations.

If p_1 be zero, λ_{21} is absent, while if p_2 be zero, α_2 has no existence, and also λ_{21} does not appear.

Hence, λ_{21} only appears when both p_1 and p_2 are greater than zero. It follows that the products which involve λ_{21} only enumerate by their coefficients those permutations which possess $\alpha_2\alpha_1$ contacts. In fact, the absence of either α_1 or α_2, by causing λ_{21} to vanish, diminishes the total number of permutations which have a definite number of major contacts; this would not be the case if the terms comprising λ_{21} as a factor did not enumerate $\alpha_2\alpha_1$ contacts; if $\alpha_2\alpha_1$ contacts were not enumerated, the vanishing of α_1 or α_2 would not alter the enumeration of the permutations possessing a given number of contacts, for this given number would be complete without $\alpha_2\alpha_1$ contacts at all.

It is next to be shown that a product involving $\lambda_{21}{}^{s_{21}}$ comprises exactly s_{21}, $\alpha_2\alpha_1$ contacts.

Suppose $p_1 \nless s_{21}$, $p_2 < s_{21}$.

Then α_2 occurs less than s_{21} times.

And, therefore, $\lambda_{21}\alpha_2$ cannot be raised to the power s_{21}.

And, therefore, λ_{21} cannot occur in any term to so high a power as s_{21}.

Similarly if $p_1 < s_{21}$ it is evident that λ_{21} cannot occur to so high a power as s_{21}. Therefore, unless both p_1 and p_2 are at least as great as s_{21}, λ_{21} cannot occur in any term to so high a power as s_{21}. It follows that the products which involve $\lambda_{21}{}^{s_{21}}$ cannot enumerate permutations possessing fewer than s_{21}, $\alpha_2\alpha_1$ contacts. For suppose that the permutations enumerated possessed only σ_{21}, $\alpha_2\alpha_1$ contacts where $\sigma_{21} < s_{21}$. If p_1 or p_2 be diminished so as to be less than s_{21} the permutations possessing σ_{21}, $\alpha_2\alpha_1$ contacts would not *necessarily* be diminished in number, whilst those possessing s_{21} such contacts as well as the products involving $\lambda_{21}{}^{s_{21}}$ would certainly vanish. Hence the products involving $\lambda_{21}{}^{s_{21}}$ cannot, through their coefficients, enumerate permutations possessing fewer than s_{21}, $\alpha_2\alpha_1$ contacts. Hence the number κ enumerates permutations having at least

ξ_1, $\alpha_{x_1}\alpha_1$ contacts
ξ_2, $\alpha_{x_2}\alpha_1$,,
. . . .
η_1, $\alpha_{y_1}\alpha_2$,,
η_2, $\alpha_{y_2}\alpha_2$,,
. . . . . .
ζ_1, $\alpha_{z_1}\alpha_3$,,
ζ_2, $\alpha_{z_2}\alpha_3$,,
. . . . .

and these numbers must represent also the exact numbers of the contacts in question because the sum

$$\Sigma\,\xi + \Sigma\,\eta + \Sigma\,\zeta + \ldots$$

gives the whole number of contacts under consideration.

The number κ obviously denotes the number of lines of route through the reticulation of the multipartite

$$\overline{p_1 p_2 \ldots p_n}$$

which possess

ξ_1 essential nodes of the kinds $\alpha_{x_1}\alpha_1$
ξ_2 ,, ,, ,, $\alpha_{x_2}\alpha_1$

and so forth.

67. By an easy arithmetical interpretation the number κ also denotes the number of permutations in which

α_{x_1} occurs ξ_1 times in the first p_1 places.
α_{x_2} ,, ξ_2 ,, ,, ,, ,,
. . . .
α_{y_1} ,, η_1 ,, between the p_1th and p_2th places.
α_{y_2} ,, η_2 ,, ,, ,, ,, ,,
. . . .
α_{z_1} ,, ζ_1 ,, ,, ,, p_2th ,, p_3th ,,
α_{z_2} ,, ζ_2 ,, ,, ,, ,, ,, ,, ,,
. . . .

§ 5. *Extension of the idea of Composition.*

68. The idea of composition is capable of enlargement from a particular point of view.

In regard to unipartite numbers, consider p units placed in a row

$$1\ 1\ 1\ 1\ 1\ 1\ldots,$$

there are $p - 1$ spaces between them which may be occupied by algebraic symbols at pleasure. We may select a definite number of different symbols, and choose any one to occupy any one of the $p - 1$ spaces. If this definite number be k, we can, by filling each space at pleasure, arrive at k^{p-1}, different expressions involving the p units.

Clearly we may take as one of these symbols the simple unoccupied blank space,* since such space left between any two numbers, quantities, or expressions of any kinds has, in every case, a well-understood signification, not always, however, the same, in mathematical work.

If we restrict ourselves to a single symbol, only one expression involving the units is possible; choosing this symbol to be that which indicates addition we merely get

$$1 + 1 + 1 + 1 + 1 + 1 + \ldots,$$

which is p, or the number which enumerates the units. Had the chosen symbol been that denoting subtraction, the expression would have denoted $-(p - 2)$; a blank space would have yielded a succession of p units; the symbol of multiplication, unity, and so forth.

All the modes of obtaining expressions from the p units may be called combinations of the first order in respect of the p units.

Passing to the case of two different symbols we may choose to employ the sign of addition and the blank space. We thus obtain 2^{p-1} different expressions which are the several compositions of the number p.

In general, expressions obtained by choosing from any k different symbols may be called combinations of order k in respect of the p fundamental units.

69. There is a one-to-one correspondence between these combinations and the "Trees" which have an altitude k and p terminal knots.

As an example take k and p each equal to 3, and, further, take as symbols, the sign of addition, the blank space and the symbol $|$ of unspecified signification.

The correspondence between the nine trees and the nine combinations of order three is shown below :—

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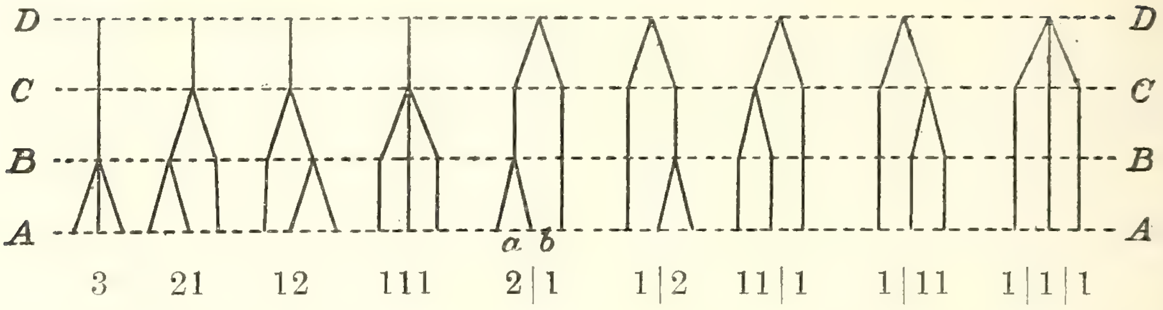
* The unoccupied blank space might be represented by a definite symbol such as O, as suggested to me by one of the referees.



The process consists in writing down a unit for each terminal knot. The $p-1$ intervals between the units correspond to the $p-1$ inter-terminal-knot spaces. If a space leads to a bifurcation in row B the symbol is $+$; if in row C it is a blank space; if in row D it is $|$.

Thus, take the fifth tree above, we have

$$1+1|1$$

for the space a leads to a bifurcation in row B, giving the symbol $+$ and the space b leads to a bifurcation in row D giving the symbol $|$. Hence the corresponding combination

$$2|1.$$

On the same principle a tree can always be drawn to represent any combination of order k in respect of p units.

70. Passing to the case of multipartite numbers, consider tripartite numbers as representative of the general case. Arrange a row of multipartite units of the three kinds, viz. :—

$$100 \quad 100 \quad 100 \ldots 010 \quad 010 \quad 010 \ldots 001 \quad 001 \quad 001 \ldots$$

and employing two different symbols, viz., the sign of addition and the blank space, we arrive at a certain composition of the tripartite $\overline{p_1p_2p_3}$, supposing the numbers of the units 100, 010, 001 that appear in the row to be p_1, p_2, and p_3 respectively. Without altering the positions of the symbols introduced we may change the order of the units 100, 010, 001, and thus obtain other compositions. Permutations of these units between contiguous blank spaces are not permissible, so that for fixed positions of the pluses and blank spaces we do not obtain in general

$$\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!}$$

compositions, but some lesser number.

This arises from the commutative nature of the symbol $+$.

Had neither of the symbols employed been such as obey the commutative law of algebra, the whole number of combinations of the second order would have been simply

$$2^{p_1+p_2+p_3-1}\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!}.$$

Thus, in this case, the symbol + introduces a complexity into the theory of compositions.

Choosing similarly from k different symbols, none of which are commutative, there are

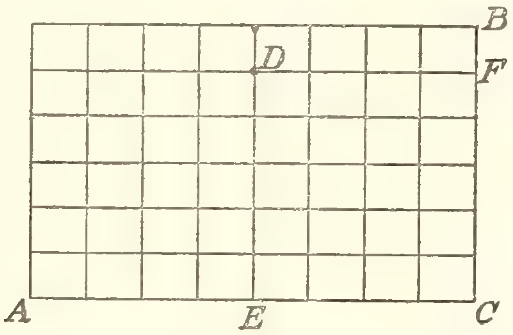
$$k^{p_1+p_2+p_3-1}\frac{(p_1+p_2+p_3)!}{p_1!\,p_2!\,p_3!},$$

combinations of order k, and the general generating function may clearly be written

$$\frac{1}{k}\cdot\frac{1}{1-k(\alpha_1+\alpha_2+\ldots+\alpha_n)};$$

but there is a lesser number of combinations if one or more of the symbols be commutative.

71. For the purpose of this investigation the most interesting case to consider is that in which one symbol, viz., the blank space, is non-commutative, and the remaining $k-1$ symbols commutative. This species of combination is not brought under view merely for the purpose of adding and discussing new complexity. Its introduction is absolutely vital to the investigation as clinching and confirming a conclusion which was momentarily assumed during the consideration of the compositions of multipartite numbers. (See *ante*, Art. 61.) Consider the reticulation of a multipartite number. That of a bipartite number will suffice as indicative of the general case.



A combination of order k (of the nature under view) is regarded as having *m parts* when $m-1$ blank space symbols occur in the combination.

Let us first enquire how many combinations possess only a single part. We require nodes of k different kinds, viz., a blank space node and $k-1$ other different nodes. A blank space node may be either essential or non-essential, but an essential node must be also a blank space node. The only line of route through the reticulation which does not involve an essential (that is a blank space) node is ACB. Consequently the graphs of all the combinations having but one part must be along the line of route ACB. Similarly in the reticulation of the multipartite number

$$\overline{p_1 p_2 \ldots p_n},$$

the graphs of all the one-part combinations will be along the line of route traced out by p_1, α_1 segments, p_2, α_2 segments, and so on in succession. The $k-1$ different symbols at disposal may be placed at pleasure at $p_1 + p_2 + \ldots + p_n - 1$ points along this line of route. Hence

$$(k-1)^{p_1 + p_2 + \ldots + p_n - 1}$$

different one-part combinations are obtainable.

In other words the generating function for such combinations is

$$h_1 + (k-1)\,h_2 + (k-1)^2\,h_3 + \ldots + (k-1)^{n'-1}h_{n'} \quad (n' = \infty,$$

where in a previous notation h_n is the homogeneous product sum, degree n, of the quantities

$$\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n.$$

72. Next as to the combinations which have two parts. At any point D of the reticulation place a blank space node. All two-part combinations whose graphs pass through D must follow the line of route AEDFB, for otherwise an additional essential (and blank space) node would be introduced. The whole combination may be split up into a one-part combination along the line of route AED, followed first by a blank space, and then by a one-part combination along the line of route DFB. All the two-part combinations whose graphs pass through the point D are obtained by associating every one-part combination in the reticulation AD with every one-part combination in the reticulation DB.

Hence the whole number of combinations, having two parts, of the multipartite number

$$\overline{p_1 p_2 \ldots p_n}$$

is

$$\Sigma\,(k-1)^{p'_1 + p'_2 + \ldots + p'_n - 1}(k-1)^{p''_1 + p''_2 + \ldots + p''_n - 1}$$

where

$$p'_1 + p''_1 = p_1\,;\ \ p'_2 + p''_2 = p_2,\ \ldots\ p'_n + p''_n = p_n.$$

Hence, the generating function for two-part combinations is

$$\{h_1 + (k-1)\,h_2 + (k-1)^2\,h_3 + \ldots + (k-1)^{n'-1}\,h_{n'}\}^2. \qquad n' = \infty.$$

73. Pursuing this chain of reasoning it is completely manifest that the generating function of m-part combinations is

$$\{h_1 + (k-1)\,h_2 + (k-1)^2\,h_3 + \ldots + (k-1)^{n'-1}h_{n'}\}^m \qquad (n' = \infty\,,$$

and hence the complete generating function of combinations of the multipartite number

$$\overline{p_1 p_2 \cdot \quad \cdot p_n}$$

is

$$\frac{h_1 + (k-1)\,h_2 + \ldots + (k-1)^{n'-1}h_{n'}}{1 - h_1 - (k-1)\,h_2 - \ldots - (k-1)^{n'-1}h_{n'}} \qquad (n' = \infty\,,$$

which is effectively the same as

$$\frac{1}{k}\cdot\frac{1}{1 - k\,\Sigma\,\alpha_1 + k\,(k-1)\,\Sigma\,\alpha_1\alpha_2 - k\,(k-1)^2\,\Sigma\,\alpha_1\alpha_2\alpha_3 + \ldots + (-)^n k\,(k-1)^{n-1}\alpha_1\alpha_2\ldots\alpha_n};$$

for the latter is obtained by adding the fraction $1/k$ to the former and then transforming from homogeneous product sums to elementary symmetric functions.

Just as in the case of $k = 2$, corresponding to compositions, this generating function admits of an important transformation to a factorized redundant form.

74. In the above fraction put $s_1\alpha_1, s_2\alpha_2, \ldots s_n\alpha_n$ for $\alpha_1, \alpha_2, \ldots \alpha_n$ respectively; it may then be written

$$\frac{k-1}{k}\cdot\frac{1}{-1 + k\,\{1-(k-1)\,s_1\alpha_1\}\,\{1-(k-1)\,s_2\alpha_2\}\ldots\{1-(k-1)\,s_n\alpha_n\}} = \frac{1}{k\mathrm{N}}.$$

For brevity put

$$\mathrm{S}_t = s_t\,(k\alpha_1 + k\alpha_2 + \ldots + k\alpha_t + \alpha_{t+1} + \ldots + \alpha_n) = s_t\,(\mathrm{A}_t + k\alpha_t)$$

$$b_1 = \Sigma\,s_1\alpha_1, \qquad b_2 = \Sigma\,s_1s_2\alpha_1\alpha_2, \qquad \&\text{c.},$$

$$\mathrm{M} = (1 - ks_1\alpha_1)\,(1 - ks_2\alpha_2)\ldots(1 - ks_n\alpha_n),$$

and

$$\frac{1}{k}\cdot\frac{1}{(1-\mathrm{S}_1)\,(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)} = \frac{1}{k\mathrm{D}}.$$

It will be shown that $\frac{1}{k\mathrm{D}}$ is a generating function equivalent to the former in regard to terms which are products of powers of

$$s_1\alpha_1,\ s_2\alpha_2,\ \ldots\ s_n\alpha_n\,;$$

for

$$\mathrm{M}-\mathrm{N}=\{k^2-k(k-1)\}b_2-\{k^3-k(k-1)^2\}b_3+\ldots+(-)^n\{k^n-k(k-1)^{n-1}\}b_n,$$

and thence

$$\frac{\mathrm{N}}{\mathrm{D}}=\frac{\mathrm{M}}{\mathrm{D}}-\frac{(\mathrm{M}-\mathrm{N})}{\mathrm{D}}=\prod_1^n\left(1+\frac{s_t\mathrm{A}_t}{1-\mathrm{S}_t}\right)-\frac{\{k^2-k(k-1)\}b_2-\ldots+(-)^n\{k^n-k(k-1)^{n-1}\}b_n}{(1-\mathrm{S}_1)(1-\mathrm{S}_2)\ldots(1-\mathrm{S}_n)}.$$

75. The investigation proceeds precisely as in the case of $k=2$ with the following result. The fraction

$$\frac{1}{k}\cdot\frac{1}{\{1-s_1(k\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(k\alpha_1+k\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(k\alpha_1+k\alpha_2+\ldots+k\alpha_n)\}}$$

is equal to the product of the fraction

$$\frac{1}{k}\cdot\frac{1}{1-k\Sigma s_1\alpha_1+k(k-1)\Sigma s_1s_2\alpha_1\alpha_2-\ldots+(-)^nk(k-1)^{n-1}s_1s_2\ldots s_n\alpha_1\alpha_2\ldots\alpha_n},$$

and the series

$$1+\Sigma\frac{k(\mathrm{A}_{t_1}+\alpha_{t_1})(\mathrm{A}_{t_2}+\alpha_{t_2})\ldots(\mathrm{A}_{t_u}+\alpha_{t_u})-(\mathrm{A}_{t_1}+k\alpha_{t_1})(\mathrm{A}_{t_2}+k\alpha_{t_2})\ldots(\mathrm{A}_{t_u}+k\alpha_{t_u})}{(k-1)(1-\mathrm{S}_{t_1})(1-\mathrm{S}_{t_2})\ldots(1-\mathrm{S}_{t_u})},$$

the summation having regard to every selection of u integers from the series $1, 2, 3, \ldots n$, and u takes all values from 1 to $n-1$.

As in the former case the relation

$$\mathrm{A}_t+k\alpha_t=\mathrm{A}_{t+1}+\alpha_{t+1},$$

which becomes

$$\mathrm{A}_{t_u}+k\alpha_{t_u}=k(\mathrm{A}_{t_1}+\alpha_{t_1})$$

where t_u and t_1 are the highest and lowest suffixes present, shows that the terms under the summation sign do not involve any products of $s_1\alpha_1$, $s_2\alpha_2$, $\ldots$ $s_n\alpha_n$ only, and therefore as far as concerns the generating function may be put equal to zero.

76. Hence the number of combinations of order k of the multipartite number

$$\overline{p_1p_2\ldots p_n}$$

is the coefficient of $(s_1\alpha_1)^{p_1}(s_2\alpha_2)^{p_2}\ldots(s_n\alpha_n)^{p_n}$ in the expansion of the generating function

$$\frac{1}{k}\cdot\frac{1}{\{1-s_1(k\alpha_1+\alpha_2+\ldots+\alpha_n)\}\{1-s_2(k\alpha_1+k\alpha_2+\ldots+\alpha_n)\}\ldots\{1-s_n(k\alpha_1+\ldots+k\alpha_n)\}}$$

that is to say it is the coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ in the product

$$\frac{1}{k}(k\alpha_1+\alpha_2+\ldots+\alpha_n)^{p_1}(k\alpha_1+k\alpha_2+\alpha_3+\ldots+\alpha_n)^{p_2}\ldots(k\alpha_1+k\alpha_2+\ldots+k\alpha_n)^{p_n},$$

which may be written

$$k^{p_1+p_2+\ldots+p_n-1}\{\alpha_1+\frac{1}{k}(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\frac{1}{k}(\alpha_3+\ldots+\alpha_n)\}^{p_2}$$
$$\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n}.$$

The coefficient of $\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$ is

$$\overset{s}{\Sigma}\, C_s\, k^{\Sigma p-s-1},$$

where C_s is the coefficient of

$$\lambda^s\,\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the product

$$\{\alpha_1+\lambda(\alpha_2+\ldots+\alpha_n)\}^{p_1}\{\alpha_1+\alpha_2+\lambda(\alpha_3+\ldots+\alpha_n)\}^{p_2}\ldots\{\alpha_1+\alpha_2+\ldots+\alpha_n\}^{p_n}.$$

If in the reticulation of the multipartite number there be D_s lines of route which possess exactly s essential nodes,

$$D_s\, k^{\Sigma p-s-1}$$

combinations of order k may be represented upon these lines. Hence the whole number of combinations is

$$\overset{s}{\Sigma}\, D_s\, k^{\Sigma p-s-1}.$$

Hence

$$\overset{s}{\Sigma}\, D_s\, k^{\Sigma p-s-1}=\overset{s}{\Sigma}\, C_s\, k^{\Sigma p-s-1}$$

a relation which is true for *all positive integral values of k.*

77. Hence

$$D_s=C_s$$

the important relation temporarily assumed in the investigation concerning compositions. (Art. 61.)

78. The theorem thus established, viz., that the number of distinct lines of route through the reticulation of the multipartite

$$\overline{p_1p_2\ldots p_n},$$

which possess exactly s essential nodes, is given by the coefficient of

$$\lambda^s\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in the product

$$\{\alpha_1 + \lambda(\alpha_2 + \ldots + \alpha_n)\}^{p_1} \{\alpha_1 + \alpha_2 + \lambda(\alpha_3 + \ldots + \alpha_n)\}^{p_2} \ldots \{\alpha_1 + \alpha_2 + \ldots + \alpha_n\}^{p_n}$$

gives the theorem in compositions :—

"The number of compositions of the multipartite

$$\overline{p_1 p_2 \cdots p_n}$$

which possess exactly s contacts of

$$n - u \text{ zeros with } \quad - 1 \text{ zeros,}$$

subject to the condition $u > t$, is

$$\mathrm{C}_s 2^{p_1 + p_2 + \ldots + p_n - s - 1}$$

where C_s is the coefficient of

$$\lambda^s \alpha_1^{p_1} \alpha_2^{p_2}, \ldots \alpha_n^{p_n}$$

in the above-mentioned product."

79. Also the further theorem :—

"The number of compositions of the multipartite

$$\overline{p_1 p_2 \cdots p_n}$$

which possess exactly

$s_{u_1 t_1}$ contacts of $n - u_1$ zeros with $t_1 - 1$ zeros.
$s_{u_2 t_2}$,, $n - u_2$,, $t_2 - 1$,,
. . . .

is the product of

$$2^{p_1 + p_2 + \ldots + p_n - s - 1} \qquad [s = \Sigma s_{ut}]$$

and the coefficient of

$$\lambda_{u_1 t_1}^{s_{u_1 t_1}} \lambda_{u_2 t_2}^{s_{u_2 t_2}} \ldots \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_n^{p_n}$$

in the product

$$(\alpha_1 + \lambda_{21}\alpha_2 + \ldots + \lambda_{n1}\alpha_n)^{p_1} (\alpha_1 + \alpha_2 + \lambda_{32}\alpha_3 + \ldots + \lambda_{n2}\alpha_n)^{p_2} \ldots (\alpha_1 + \alpha_2 + \ldots + \alpha_n)^{p_n}."$$

80. The generating function which enumerates the combinations of order k of multipartite numbers may be written

$$\frac{1}{k} + \frac{\mathrm{H}'}{k-1} + \frac{\mathrm{H}'^2}{(k-1)^2} + \frac{\mathrm{H}'^3}{(k-1)^3} + \ldots + \frac{\mathrm{H}'^m}{(k-1)^m} + \ldots$$

where

$$\alpha'_1 = (k-1)\,\alpha_1, \qquad \alpha'_2 = (k-1)\,\alpha_2, \text{ \&c.}$$

h'_m is the homogeneous product sum of degree m of the quantities

$$\alpha'_1, \alpha'_2, \ldots \alpha'_n,$$

and

$$H' = h'_1 + h'_2 + h'_3 + \ldots$$

The coefficient of

$$\alpha_1^{p_1}\alpha_2^{p_2}\ldots\alpha_n^{p_n}$$

in

$$\frac{H'^m}{(k-1)^m}$$

enumerates, in respect of the multipartite $\overline{p_1p_2\ldots p_n}$, the number of combinations having m parts.

From previous work this coefficient is

$$(k-1)^{p_1+p_2+\ldots+p_n-m} f(p_1p_2\ldots p_n, m),$$

where

$$f(p_1p_2\ldots p_n, m) = \binom{p_1+m-1}{p_1}\binom{p_2+m-1}{p_2}\ldots\binom{p_n+m-1}{p_n}$$
$$-\binom{m}{1}\binom{p_1+m-2}{p_1}\binom{p_2+m-2}{p_2}\ldots\binom{p_2+m-2}{p_n}$$
$$+ \text{\&c.},$$

to m terms.

81. Hence the whole number of combinations is

$$(k-1)^{\Sigma p-1} f(p_1p_2\ldots p_n, 1) + (k-1)^{\Sigma p-2} f(p_1p_2\ldots p_n, 2) + .$$

a result which is immediately obtainable from the reticulation.

82. There exists a very interesting correspondence between the compositions of the multipartite

$$\overline{1^{n-1}}$$

into k parts, zeros not excluded, and the combinations of order k of the unipartite number

$$n,$$

zeros excluded.

The generating function for the compositions of multipartite numbers into k parts zeros not excluded, is

$$(1 + h_1 + h_2 + \ldots)^k = (1-\alpha_1)^{-k}(1-\alpha_2)^{-k}\ldots(1-\alpha_n)^{-k},$$

hence the number in the case of $\overline{p_1p_2\ldots p_n}$ is

$$\binom{k+p_1-1}{p_1}\binom{k+p_2-1}{p_2}\binom{k+p_3-1}{p_3}\ldots\binom{k+p_n-1}{p_n},$$

which for the multipartite $\overline{1^{n-1}}$ is

$$k^{n-1}.$$

This expression also gives (*ante*) the number of combinations of order k of the unipartite number

$$n,$$

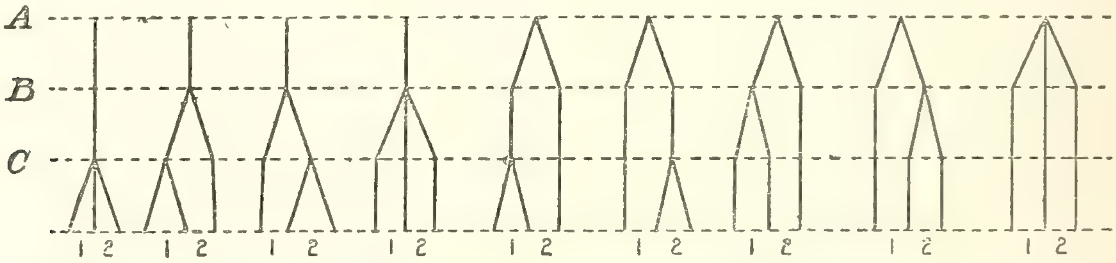
zeros excluded.

83. The correspondence may be shown by reference to the "Theory of Trees."

The trees to be considered are of altitude k and have n terminal knots.

For simplicity take $n = k = 3$.

The trees are



which, as shown above, represent combinations of order 3 of the unipartite number 3. These are

$$3 \quad 21 \quad 12 \quad 111 \quad 2|1 \quad 1|2 \quad 11|1 \quad 1|11 \quad 1|1|1,$$

the number of parts (observe that blank space symbols are between adjacent parts) being equal to the number of bifurcations in the second row from the top, increased by unity.

Now these trees may be interpreted so as to represent the compositions of the multipartite $(\overline{1^2})$ into three parts, zero parts not excluded, by attending to the connection between the two bifurcations of each tree and the two inter-terminal-knot spaces.

In any row of knots in a tree we have or we have not bifurcations, and each bifurcation communicates either with the first or with the second inter-terminal-knot space.

Beginning with the row marked A, if we find no bifurcation we write $\overline{00}$; if we find a bifurcation communicating with the first space but not with the second we write $\overline{10}$; if with the second space and not with the first $\overline{01}$; if there be two bifurcations, which necessarily in the present case communicate with both spaces, we write $\overline{11}$; and proceeding in the same way with the rows B and C in succession we will have finally written down three bipartite parts constituting a composition of the multipartite $\overline{11}$.

We thus obtain, beginning with the left hand tree,

$$(\overline{00}\;\overline{00}\;\overline{11}),\ (\overline{00}\;\overline{01}\;\overline{10}),\ (\overline{00}\;\overline{10}\;\overline{01}),\ (\overline{00}\;\overline{11}\;\overline{00}),\ (\overline{01}\;\overline{00}\;\overline{10}),\ (\overline{10}\;\overline{00}\;\overline{01}),$$
$$(\overline{01}\;\overline{10}\;\overline{00}),\ (\overline{10}\;\overline{01}\;\overline{00}),\ (\overline{11}\;\overline{00}\;\overline{00}),$$

in order of correspondence with the above written combinations, of order 3, of the unipartite number 3.

84. The process is perfectly general. Every tree of altitude k is representative alike of a combination of order k of a unipartite number, zeros excluded, and of a composition of a unitary multipartite, zeros not excluded, and each combination or composition of the nature considered is uniquely represented by a tree.

The number of compositions of the multipartite $\overline{1^{n-1}}$, zeros not excluded, into k or fewer parts is

$$1^{n-1} + 2^{n-1} + \ldots + k^{n-1}$$

a number which also represents the number of the aggregate of the combinations of the number n, of orders 1, 2, . . . k, zero parts not excluded.

The interesting fact here brought to light is the connection between the unipartite numbers and the unitary multipartite numbers.

85. We have seen that k^{m-1} expresses the number of combinations of order k possessed by the unipartite number m. Each combination involves a certain number of the k different symbols and we may inquire the number of combinations which involve exactly p out of the k symbols. It is clear that one combination can be formed which involves any one symbol and none of the others; hence, k combinations involve but a single symbol. Two out of k symbols may be selected in $\binom{k}{2}$ different ways; for each such selection we must take the number of combinations of the number m, of order 2, and subtract the number of these in which but a single symbol appears. Hence, the number of combinations involving exactly two symbols is

$$\binom{k}{2}(2^{m-1} - 2).$$

Similarly three out of k symbols may be selected in $\binom{k}{3}$ ways, and for each selection we take the whole number of combinations of order 3 and subtract those of them which involve exactly two symbols or exactly one.

Hence we arrive at the number

$$\binom{k}{3}\{3^{m-1} - 3\,(2^{m-1} - 2) - 3\}$$
$$= \binom{k}{3}\{3^{m-1} - 3\,.\,2^{m-1} + 3\,.\,1^{m-1}\}.$$

In this way it is easy to see that the number of combinations involving exactly p symbols is

$$\binom{k}{p}\{p^{m-1} - p\,(p-1)^{m-1} + \binom{p}{2}(p-2)^{m-1} - \ldots + (-)^{p+1}p\}$$
$$= \binom{k}{p}\Delta^p\,(0^{m-1})$$

in the notation of finite differences.

We have now the well known identity

$$k^{m-1} = \binom{k}{1}\Delta\,(0^{m-1}) + \binom{k}{2}\Delta^2\,(0^{m-1}) + \ldots + \binom{k}{p}\Delta^p\,(0^{m-1}) + \ldots + \binom{k}{k}\Delta^k\,(0^{m-1}),$$

and we have seen its interpretation in the theory of the combinations of order k of a given unipartite number m.

Tables of Compositions of Multipartite Numbers.

No.	1	part	Total
1	1	. . .	1

No.	1	2	parts	Total
2	1	1	. . .	2
$\overline{1^2}$	1	2	. . .	3

No.	1	2	3	parts	Total
3	1	2	1	. . .	4
$\overline{21}$	1	4	3	. . .	8
$\overline{1^3}$	1	6	6	. . .	13

No.	1	2	3	4	parts	Total
4	1	3	3	1	. . .	8
$\overline{31}$	1	6	9	4	. . .	20
$\overline{2^2}$	1	7	12	6	. . .	26
$\overline{21^2}$	1	10	21	12	. . .	44
$\overline{1^4}$	1	14	36	24	. . .	75

No.	1	2	3	4	5	parts	Total
5	1	4	6	4	1	. .	16
$\overline{41}$	1	8	18	16	5	. .	48
$\overline{32}$	1	10	27	28	10	. .	76
$\overline{31^2}$	1	14	45	52	20	. .	132
$\overline{2^21}$	1	16	57	72	30	. .	176
$\overline{21^3}$	1	22	93	132	60	. .	308
$\overline{1^4}$	1	30	150	240	120	. .	541

No.	1	2	3	4	5	6	parts	Total
6	1	5	10	10	5	1	. .	32
$\overline{51}$	1	10	30	40	25	6	. .	112
$\overline{42}$	1	13	48	76	55	15	. .	208
$\overline{41^2}$	1	18	78	136	105	30	. .	368
$\overline{3^2}$	1	14	55	92	70	20	. .	252
$\overline{321}$	1	22	111	220	190	60	. .	604
$\overline{31^3}$	1	30	177	388	360	120	. .	1076
$\overline{2^3}$	1	25	138	294	270	90	. .	818
$\overline{2^21^2}$	1	34	219	516	510	180	. .	1460
$\overline{21^4}$	1	46	345	900	960	360	. .	2612
$\overline{1^6}$	1	62	540	1560	1800	720	. .	4683

No.	1	2	3	4	5	6	7	parts	Total
7	1	6	15	20	15	6	1	. .	64
$\overline{61}$	1	12	45	80	75	36	7	. .	256
$\overline{52}$	1	16	75	160	175	96	21	. .	544
$\overline{51^2}$	1	22	120	280	325	186	42	. .	976
$\overline{43}$	1	18	93	216	255	150	35	. .	768
$\overline{421}$	1	28	183	496	655	420	105	. .	1888
$\overline{41^3}$	1	38	288	856	1205	810	210	. .	3408
$\overline{3^21}$	1	30	207	588	810	540	140	. .	2316
$\overline{32^2}$	1	34	255	772	1120	780	210	. .	3172
$\overline{321^2}$	1	46	399	1324	2050	1500	420	. .	5740
$\overline{31^4}$	1	62	621	2260	3740	2880	840	. .	10404
$\overline{2^31}$	1	52	489	1728	2820	2160	630	. .	7880
$\overline{2^21^3}$	1	70	759	2940	5130	4140	1260	. .	14300
$\overline{21^5}$	1	94	1173	4980	9300	7920	2520	. .	25988
$\overline{1^7}$	1	126	1806	8400	16800	15120	5040	. .	47293

[During the considerable time that has elapsed since this paper was read I have discovered the general theory of the transformations of Arts. 41 and 75.

Let X_1, X_2, X_3 be general linear functions of x_1, x_2, x_3 as exhibited in the Notation of the Theory of Matrices

$$(X_1, X_2, X_3) = \left(\begin{array}{ccc} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{array}\right)(x_1, x_2, x_3),$$

and consider the algebraic fraction

$$\frac{1}{(1 - s_1X_1)(1 - s_2X_2)(1 - s_3X_3)}.$$

I have established that that portion of the expansion of this fraction, which is a function of products of powers of s_1x_1, s_2x_2, s_3x_3 only, is represented by the fraction

$$\frac{1}{\mathrm{N}},$$

where

$$\begin{aligned} \mathrm{N} = 1 &- a_1s_1x_1 - b_2s_2x_2 - c_3s_3x_3 \\ &+ | a_1b_2 | s_1s_2x_1x_2 + | a_1c_3 | s_1s_3x_1x_3 + | b_2c_3 | s_2s_3x_2x_3 \\ &- | a_1b_2c_3 | s_1s_2s_3x_1x_2x_3 ; \end{aligned}$$

the notation being that in use in the Theory of Determinants.

The coefficients of N are the several co-axial minors of the determinant of the matrix defining X_1, X_2, and X_3, viz.:—

$$\left| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right| .$$

The result is immediately deducible from the identity

$$\left| \begin{matrix} \dfrac{s_1(a_1x_1 - \mathrm{X}_1)}{1 - s_1\mathrm{X}_1} - 1, & \dfrac{a_2s_1x_1}{1 - s_1\mathrm{X}_1}, & \dfrac{a_3s_1x_1}{1 - s_1\mathrm{X}_1} \\ \dfrac{b_1s_2x_2}{1 - s_2\mathrm{X}_2}, & \dfrac{s_2(b_2x_2 - \mathrm{X}_2)}{1 - s_2\mathrm{X}_2} - 1, & \dfrac{b_3s_2x_2}{1 - s_2\mathrm{X}_2} \\ \dfrac{c_1s_3x_3}{1 - s_3\mathrm{X}_3}, & \dfrac{c_2s_3x_3}{1 - s_3\mathrm{X}_3}, & \dfrac{s_3(c_3x_3 - \mathrm{X}_3)}{1 - s_3\mathrm{X}_3} - 1 \end{matrix} \right|$$

$$\times \left| \begin{matrix} 1 - s_1\mathrm{X}_1, & 0, & 0 \\ 0, & 1 - s_2\mathrm{X}_2, & 0 \\ 0, & 0, & 1 - s_3\mathrm{X}_3 \end{matrix} \right|$$

$$= \left| \begin{matrix} a_1s_1x_1 - 1, & a_2s_1x_1, & a_3s_1x_1 \\ b_1s_2x_2, & b_2s_2x_2 - 1, & b_3s_2x_2 \\ c_1s_3x_3, & c_2s_3x_3, & c_3s_3x_3 - 1 \end{matrix} \right| .$$

the determinant last written being, with changed sign, the value of N.

The theorem for the case of n variables $x_1, x_2, \ldots x_n$ will be completely manifest from the above.

It appears to be one of considerable importance with regard to the generating functions which present themselves in this domain of the Theory of Numbers.

The results of its further investigation I hope to bring before the Royal Society in the near future.—Added August 25, 1893. P. A. M.]

XVIII. *On the Evidences of a Submergence of Western Europe, and of the Mediterranean Coasts, at the Close of the Glacial or so-called Post-glacial Period, and immediately preceding the Neolithic or Recent Period.*[*]

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Received December 15, 1892,—Read March 9, 1893.

[PLATE 33.]

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[*] I briefly broached this subject at the meeting of the British Association at Swansea in 1880: an abstract is given in *Reports*, p. 581.

28.11.93

Introductory Remarks.

In a paper read before the Geological Society* early this year, I gave the evidence—the result of personal observation—which led me to conclude that the South of England had been submerged to the depth of not less than about 1000 feet between the Glacial (or Post-glacial) and the recent or Neolithic periods. That evidence was based upon the characters, physical and palæontological, of a peculiar superficial drift, for which I proposed the term of "Rubble-drift," to distinguish it from the valley, marine, and glacial drifts of the same districts.† Under this term I include various detrital deposits to which different designations have been attached. Amongst the more important of these are the drift called "head" over the Raised Beaches of the Channel and the Ossiferous Fissures of South Devon.

Various explanations have been suggested to account for the "head," such as, 1st, an excessive rainfall, accompanied by great cold; 2nd, the sliding of masses of snow and ice over slopes; 3rd, waves of translation; 4th, torrential fluviatile action during a period of great cold. I have stated in the paper referred to the objections to these several explanations.‡ Some of them, no doubt, would suffice to produce a portion of the observed effects, but they fail to embrace the whole, and they all involve consequences which are incompatible with the general facts. They all, also, with one exception, depend on subaërial agencies, to which there is the general objection that these agencies involve a certain amount of friction and weathering which are conspicuously wanting—or, if present, it is in a very slight degree—in the deposits under review. There is the further objection that some of the phenomena indicate the exercise of a propelling force for which the suggested causes are manifestly inadequate. There are other points, apparently inconsistent with such agencies, connected more especially with the Ossiferous fissures and the Loess of the continental area, which will be considered more fully in the following pages.

In the present paper my object will be to show that the phenomena on which I relied as proofs of submergence in England, extend likewise over large continental areas. Owing to the extent of the subject, it will be necessary to confine the inquiry to the more prominent and typical cases. Nor will it be necessary to give full geological details, as they are to be found in the works of the several geologists referred to. I shall, therefore, give only such particulars as are needed for the discussion of their views, and for the purpose of generalizing the phenomena they describe.*

---

* 'Quart. Journ. Geol. Soc.,' vol. 48, p. 263, 1892. I have there entered into more details, and specified the objections to previous explanations, which it is therefore unnecessary to repeat here.

† Both Sir R. Murchison ('Quart. Journ. Geol. Soc.,' vol. 7, p. 349), and Professor J. Geikie ('Prehistoric Europe,' p. 140), have contended that certain detrital deposits, spread widely over the South of England, could not be referred to ordinary marine or fluviatile agencies. The former attribute them to a wave of translation: the latter (which I unknowingly overlooked in the above-named paper) to the agency of frozen snow-drifts during the Glacial period.

‡ *Ibid.*, pp. 326–328.

Though this rubble-drift in some of its phases sometimes simulates the characters of the other drift deposits, I found it impossible to reduce it to the terms of any of those others. Whereas these latter are spread out in beds more or less horizontal, and keep to definite lines, this drift drapes the hills, follows divergent directions, and ends only when it reaches the floor of the valleys. The faunal débris of this drift also differs essentially in its main characteristics from that of the other drift deposits. There is an entire absence both of marine and fluviatile shells ; the remains found in it are those of *land animals* and *land shells* alone, with traces of land plants, such remains in fact as could have been derived from a *land surface*, and from a land surface only. Owing to their extreme friability, the shells are of rare occurrence, and the most common one, the *Pupa marginata*, is so minute, that it often escapes observation. Another feature to be noted, is that the bones of the Mammalia (which belong to the ordinary Quaternary group), are distinguished by their very *fragmentary state* and by the *absence of wear*, whether of the broken fragments or of the

* [The notices of the several detrital beds I have included under the term of "Rubble-drift" have, with the exception of the Loess, generally been incidental and limited to some one phase. With respect to that form of it represented by the Ossiferous fissures, an impression would appear to have prevailed that the subject was exhausted by the early researches of CUVIER and MARCEL DE SERRES, for though so much has been done of late years in explanation of the contents of the remarkable Prehistoric Caves in France and elsewhere on the continent, there have been no detailed descriptions of those Ossiferous fissures, with the exception of those of Gibraltar (1878), and slight notices of those of Nice (1887), since that of DE LA BECHE in 1828 ; nor, with the exception of the short notice by M. LANZA in 1855, has anything been added to our knowledge of the numerous Dalmatian fissures since the Abbé FORTIS wrote in 1778. In Italy, the latest contribution I can find is the one by Professor CAPELLINI in 1879. The recent papers of STEFANI in the 'Annales de la Soc. Géol. de Belgique' (1891), on the Upper Tertiary and Post-Pliocene beds of the Mediterranean Basin barely touch on any of those which form the subject of this inquiry.

It is the same with the rubble-drift on slopes. Besides the instances I have given in France (1860–71), and at Gibraltar (1878), and of the *Argile à blocaux* in Belgium (1872), I am not aware of any more recent memoirs on this particular subject. The later more general works on the geology of the South of Europe and the North of Africa, have added nothing to the facts described by the earlier writers I have referred to.

The contemporaneous deposit of Loess forms an exception to this neglect. The recent literature on this subject is copious and varied, but it is mainly confined to that section of the Loess which has a fluviatile origin, and is not applicable to that other section which extends beyond the reach of river-floods and to the greater heights of Central Europe. It is to the latter alone that my observations are at present confined, and they only relate to a few points.[1]

I should add that I have been obliged, for various reasons, to confine myself in each section to a limited number of the more typical instances. I have not, however, I believe, omitted any cases but such as would confirm the evidence of those which I have adduced.—J. P., *June*, 1893.]

---

[1] I have discussed the subject of the fluviatile Loess in a paper which appeared in the 'Phil. Trans.' for 1864, p. 247.

entire bones, as well as by *their freedom from all traces of gnawing;* conditions in marked contrast with those presented by the bones of the caves, which have commonly been gnawed by predaceous animals, and with those of the fluviatile deposits which are usually more or less worn.

The physical characters of this drift are equally well marked. The detritus of which it is composed, is always of *local origin* and *angular.* The component fragments retain their *sharp angles* uninjured. It is without stratification, and yet not without a certain order such as might result from successive masses of débris sliding down slopes and shot into hollows. In many respects, such as its wide distribution, and its want of stratification, it resembles a subaërial glacial detritus, but its entire want of wear, the presence of and condition of the bones, and especially the presence of delicate land shells, are incompatible with such an origin. The absence of foreign transported materials or boulders, and of marine shells, is equally incompatible with a submarine glacial origin.

Taking all these facts into consideration, the only agent which appears to me capable to have produced such results, is that of an upheaval of a submerged land following upon a wide-spread submergence. This upheaval by displacing the superincumbent body of water, would give rise, as shown in a paper by the late Mr. W. HOPKINS, of Cambridge,* to divergent effluent currents, which would sweep down from the higher to the lower levels the débris of the submerged surface. Such would have happened, if after a temporary submergence the land had again been upheaved, and the former levels approximately restored. All the phenomena presented by this Rubble-drift are explicable upon this hypothesis, and upon none other that I can see. Of the submergence we can only infer that it was slow and gradual, but of the upheaval we have evidence to show that it was by stages alternately slow and more or less rapid. That evidence is best exhibited, not in districts where the rocks are of an uniform character, as then, in the absence of any difference in the débris, the rubble has the appearance of a talus without divisional planes, but where the rocks are composed of beds of different textures and degrees of hardness then the débris shows sorting and a kind of rough bedding, and this gives a clue to the manner in which the accumulation took place. These conditions obtain best in chalk districts, where layers of light earthy chalk débris alternate with others of heavy splintery flints, as in the case of the mass of Rubble-drift (or *head*) overlying the Raised beaches of Brighton and Sangatte.

---

* 'Quart. Journ. Geol. Soc.,' vol. 4, p. 90, Mr. HOPKINS, shows that if a considerable area at the bottom of the sea were *suddenly* elevated, currents, the velocity of which would depend principally upon the depth of the sea, would diverge in all directions from the central disturbance. "Calculations," he says, "prove beyond all doubt that paroxysmal elevations, beneath the sea, varying from 50 to 100 feet in height, may produce currents of which the velocities shall vary from at least 5 or 6 to 15 or 20 miles an hour, provided the depth of the sea do not exceed 800 or 1000 feet." In considering the magnitude of the blocks which might be moved, he found that the force exerted on a surface of given magnitude *increases as the square of the velocity*, and that it "varies as *the sixth power of the velocity of the current.*" But the movements must be repeated for large blocks to travel beyond short distances. In this inquiry, however, we have to deal with smaller quantities and less paroxysmal movements.

The Brighton beach abuts, like the recent one, against a cliff, over which the drift was shot from the uplands above, not in one sweep, but in a succession of throws, some of which were of sufficient force to move large masses of flints and great blocks of Tertiary sandstone from a distance of two to three miles inland, whilst others had only force enough to carry down the lighter chalk rubble, and impalpable marly sediment often finely laminated, showing that it was deposited in water, and in water, at the time, comparatively still. We thus have, as it were, a measure of the forces in operation during the upheaval, for as the divergent currents, caused by the upheaval of the land through the superincumbent body of water, would vary in velocity and power according to the rapidity of the upheaval, it is evident that the weight and quantity of material moved will be in accordance with the strength and length of those currents. The variations in the structure of the "head" show the movements to have been alternately slow and more rapid, or by fits and starts, though, as there is no break in the continuity of the deposit, it is probable that they were continuous or nearly so. It is also obvious that a body of water 1000 feet or more deep, moving even slowly off the land, would have formed an engine of enormous power, and as I have before remarked, capable, like NASMYTH's hammer, of exerting that power with the gentlest touches or with almost irresistible force. For the fuller discussion of these points, I must, however, refer to the paper before mentioned, in which these points are described more fully.*

In that paper I have shown that the Rubble-drift is very widely, though generally very sparsely, spread over the South of England. I now propose to show that it may be traced over much of Western Europe and the Mediterranean coasts (see Plate). My own personal observations are limited to parts of France and Italy. I have, however, the less cause to regret this circumstance, as some of the principal exhibitions of it have been explored by other geologists in a way which would have been difficult for a single individual to have accomplished, and their evidence is free from the bias which might be supposed to attach to the advocate of a theory. At the same time there is the disadvantage, that without attention being specially directed to this particular drift, some of its phases may easily be overlooked and passed by, so that further research may prove it to be far more general than here described. It is also to be observed that in continental areas we meet with fresh phases of the rubble-drift, which, while differing from what we find in England, equally bear testimony in the same direction.

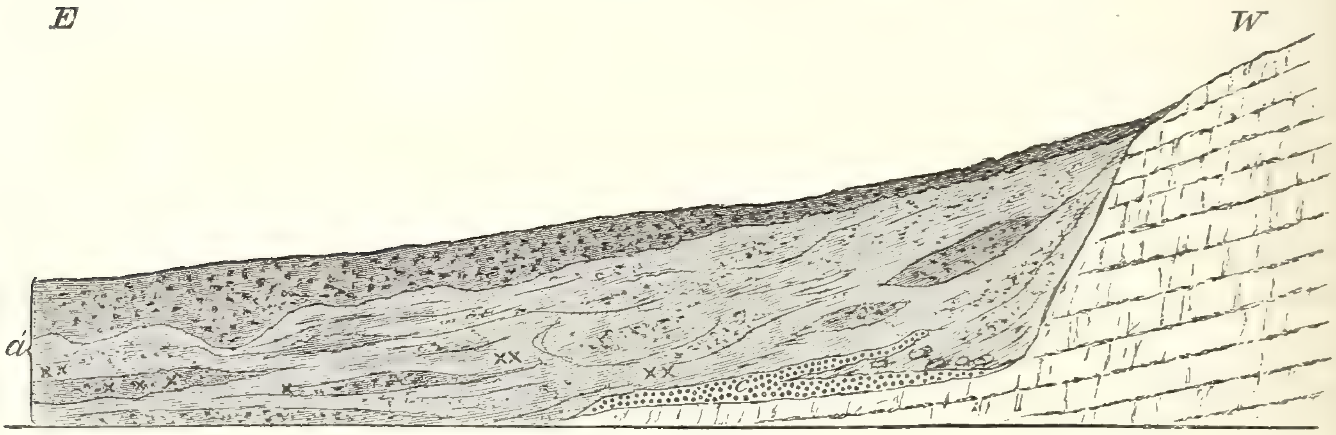
---

* 'Quart. Journ. Geol. Soc.,' vol. 48, p. 331.

The Coast Sections of France.

The phenomena on the north coast of France are very similar to those on the south coast of England. The section of the Raised Beach and overlying Rubble-drift (or *head*) at Sangatte on the northern slope of Cape Blanc Nez is identical with that at Brighton, but the former is more accessible and better exposed. The raised beach (*c*), which rises about 10 feet above the level of the present beach, abuts, as at Brighton, against an old cliff. It contains very few shells, and those mostly in fragments. The species are the same as those commonly met with in the English raised beaches, and consist of*—

Fig. 1.—*Section of the west end of Sangatte Cliff, near Calais.*



a'. Alternating irregular masses, often contorted, of coarse and fine flint-and-chalk rubble and marl, with layers of light coloured loam (or Loess). The top bed has lost much of its chalky matter, and passes at Sangatte into mere flint gravel. Mammalian remains have been found at ××× , land shells at ××, and a *palæolithic* flint implement at × (others have been picked up on the shore.)

c. Raised beach. A portion of the beach has been caught and turned up in the rubble. At the foot of the old cliff lie large blocks of chalk. *C*. Chalk.

Purpura lapillus.	*Tellina balthica.*
Littorina littorea.	*Mytilus edulis.*
,, *obusata.*	*Cardium edule.*
Modiola modiolus.	

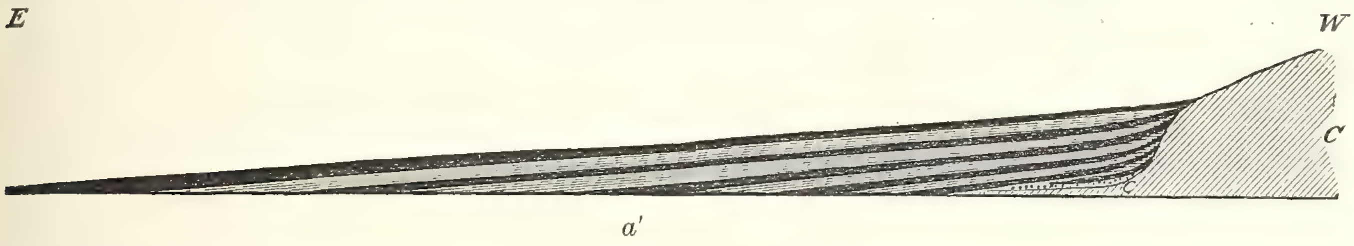
This beach is buried under a *head* consisting of chalk and flint rubble not regularly stratified, but spread out in irregular lenticular masses. In some of these the detritus is coarse and the bedding tumultuous; in others the débris is fine, and even laminated in places. These alternations are repeated several times, but it is not possible to say how often, as the beds overshoot one another, and the lines of demarcation are obscure. The last or upper bed is, however, as at Brighton, the most massive and important, and is the one which has been propelled to the greatest distance. A view of part of this fine section* is given in fig. 1. The section, however, varies with every fall of the cliff. The partial uprooting of a portion of the beach (*c*) is a noticeable feature.

---

* M. C. Barrois, in 'Ann. Soc. Géol. du Nord,' vol. 7, p. 182, 1880.

The following diagram will show the successive increments of growth,† supposing them to have been regular and defined.

Fig. 2.—*Diagram to illustrate the mode of formation of the 'Head.'*



The dark bands represent coarse débris, and the light, fine rubble and loam, or derived Loess. The uppermost bed has lost much of its calcareous matter through filtration of the surface waters.

Except at Folkestone and Portland, land Mollusca are very scarce in the English coast sections. At Sangatte they are numerous in places, though limited to the few following species, and to the finer portions of the rubble :—

Helix concinna.
„ *pulchella.*
Succinea oblonga.
Pupa marginata.
Arion ater.
Limax agrestis.

On the other hand, Mammalian remains are scarcer than with us. I have found a few indeterminable fragments of bone, and M. Robbe had in his collection part of the lower jaw of *Elephas primigenius.* Of palæolithic flint implements, several specimens have been found on the shore, and one *in situ* in the rubble.

Another point calling for notice is the distance to which the flint rubble spreads out into the plain, which extends from the foot of the chalk hills to near Calais and Guines. Here, as in the coast-plain between Brighton and Selsea, the calcareous portion of the rubble decreases in amount as the drift ranges from its base on the slope of the hills to the more distant points to which it has been carried. The chalky portion is lost either through solution or broken up by friction and incorporated with the loess or brick-earth. In a pit near Hames I found in the thin beds of ochreous flint, gravel, and brick-earth, to which the rubble-drift is there reduced, fragments of a tooth of the *Mammoth,* and a few specimens of *Pupa marginata,* as in the analogous beds at Upchurch, in Kent.

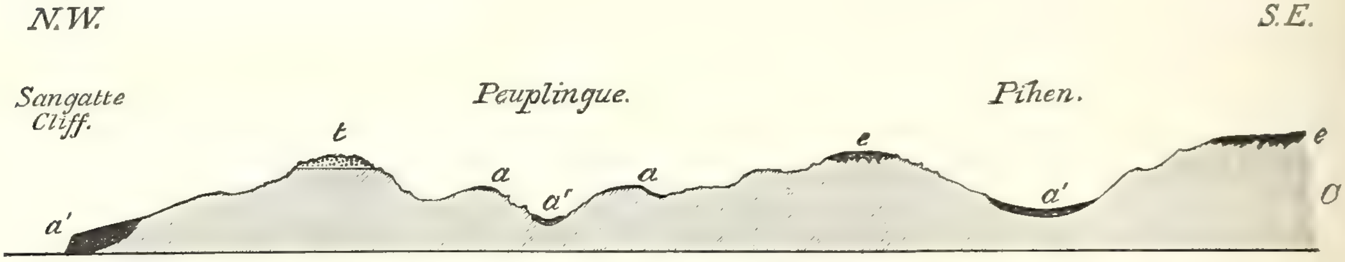
The dry chalk valleys at the back of Blanc Nez exhibit that phase of this drift, so common in the chalk hills of Kent, consisting of trails of ochreous gravel. These are usually attributed to the action of former streams, but they seem to me to be more intimately connected with the rubble-drift, into the main body of which these gravel-streams debouche in the plains below.* Such, for instance, as in the case of the narrow stream of unstratified ochreous gravel from 3 to 10 feet in thickness, which descends the dry chalk valley from near Pihen to Hames, where it enters the broad alluvial plain near Guines. Midway in this stream of gravel M. SAUVAGE found a jaw of *Rhinoceros tichorhinus*.†

---

* I have described this section with its land shells in detail in 'Quart. Journ. Geol. Soc.,' vol. 7, p. 274, 1851, and vol. 21, p. 440, 1865.

† It is the same in the Brighton Cliff (see 'Quart. Journ. Geol. Soc.,' vol. 48, p. 263).

Fig. 3.—*Section from Sangatte Cliff to above La Queenvacherie.*



a. Loess. *a'.* Angular rubble. *t.* Lower Tertiary strata.
e. Red clay with flints. *C.* Chalk.

The relation of these streams of gravel to the *head* at Sangatte is similar to that which exists between the *head* (Elephant Bed) at Brighton and the drift in the bed of the valleys on the coast of Rottingdean and other places thence to Eastbourne.‡ They all consist of débris from the surrounding hills, except that in the one case only a trail has been left, whilst in the *head* it has accumulated *en masse*.

There is a small counterpart of the chalk-rubble on the Wissant side of Blanc Nez; and at a short distance out of Wissant on the road to Tardinghen is a hillock of gravel, containing a remarkably large proportion of chert fragments of the Lower Greensand, referable probably to the Rubble-drift.

Thence to Boulogne there is little to notice, the encroachment of the sea having removed all the later coast deposits, only leaving here and there slight traces of angular rubble. To this also belongs a pocket of drift with remains of the Mammoth exposed in carrying the railway through the Kimmeridge clay on the high slopes between the old town of Boulogne and the coast.

Abbeville.—The sections here are of much interest, as they show the relation of the Rubble-drift to the fluviatile deposits of the old river, and likewise to the Raised Beaches. The marine or estuarine bed at Menchecourt is 24 feet above the sea-level, which, allowing for its distance inland, corresponds with that of the Raised Beaches of the Channel. It is here, however, overlaid by the old river beds with fluviatile shells and Mammalian remains together with palæolithic flint implements, while the bed marked *b* in my original section,* and which, after descending the hill slope, passes beneath the alluvium in the valley, represents the rubble-drift such as is met with on slopes.

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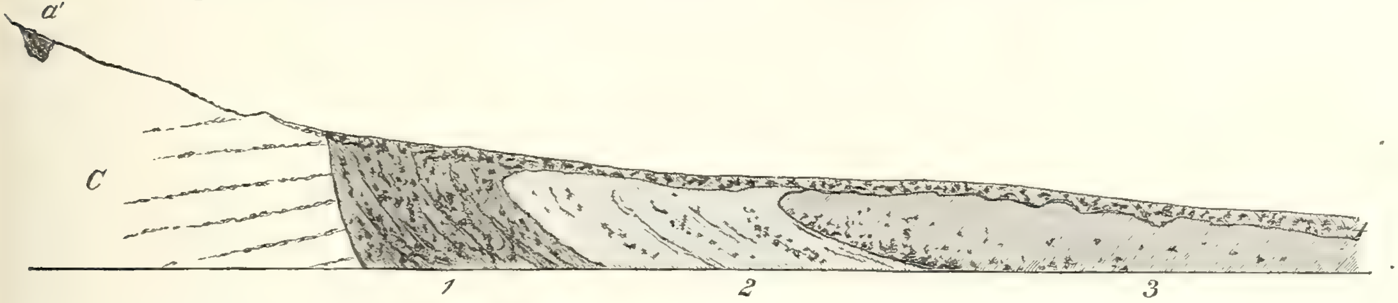
* In some instances streams may have preceded these later gravel trails.

† The specimen is in the Boulogne Museum.

‡ See section in 'Quart. Journ. Geol. Soc., vol. 48, Plate 7, fig. 1.

Since that section was taken another (fig. 4) very illustrative section, corresponding in its main features with the *head* on the Channel coast, was for a short time to be seen in a road-cutting in the suburb of Mercadé, north of Abbeville. This section showed the same remarkable reversal of the out-cropping edges of the beds, so conspicuous at Brighton and Portland, with additional features of some importance.

Fig. 4.—*Section of Rubble-Drift on the slope of the hill to the north of Abbeville.*



a' {
1. Coarse angular flint- and chalk-rubble.
2. Chalk-rubble with seams of Loess and flint gravel. Land shells (*Helix*—several species, *Pupa marginata*, &c.) abounded in this bed.†
3. Coarse brown loam or brick-earth, with dispersed flint gravel and a seam of it at base.
4. Mixed rubble derived from the underlying beds.

The dip of the base line should correspond more closely with that of the surface.

I saw no Mammalian remains nor any shells other than land shells. The slope above is bare with the exception that about half-way up there is a cavity, *a'*, 10 to 15 feet deep and filled with chalk débris and brick-earth. The summit of the hill is capped by plateau Loess. The total thickness of the beds Nos. 1 to 4 (fig. 4) may be estimated at about 40 feet. This rubble possibly forms, at Porte Mercadé, the bed of ochreous flint gravel in which M. BOUCHER DE PERTHES found a considerable number of palæolithic flint implements.

Besides the unusual abundance of land shells, this section is of interest from the circumstance that whereas at Brighton and Sangatte the rubble is piled up without distinct divisional lines, here, on the contrary, there are well-marked main divisions. To whatsoever cause this may be due—whether to the lesser force of the effluent currents than on the coast where the successive layers are more intermingled and the divisional lines rendered more obscure, or to the position being more sheltered, I cannot say. In any case, we have here three major divisions corresponding with three main upheavals, besides some minor ones in bed No. 2. This agrees, as a whole, perfectly well with the Brighton and Sangatte sections, where fine marly beds predominate in the central division, with coarse beds below and above, the last one evidently resulting from a stronger propulsion and exercising greater erosive action.

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* 'Phil. Trans.' for 1860, figs. 1, 2, p. 284, and Plate 10, section 1.

† On my first visit I had not time to make a complete collection of these shells, and when I returned the section was covered over.

This section strengthens the conclusion forced upon us by the coast sections that the upheaval was not one continuous and uniform movement, but a succession of movements of greater or lesser rapidity, irregular in their action and not long prolonged. By prolonged, I mean continuous movements but of variable rapidity. If there were rests, they did not last long enough to leave marks of shore-wear at these successive stages on the upraised land. That there were pauses is shown by the division of the beds in fig. 4, but they could not have been of long duration.

The Normandy Coast.—Between the Somme valley and Hâvre the coast presents a line of chalk cliffs, closely resembling in all their features those between Brighton and Eastbourne. Like as on those parts of our coast, where the encroachment of the sea has removed the raised beach and left only detached trails of the rubble-drift in the valleys, so on the French coast the erosion of the cliffs, which may be estimated at 1 to 2 feet annually,* has removed all traces of the beach; but of the rubble-drift which overlaid the beaches and ascended higher inland, considerable portions remain in the bed and at the mouth of the valleys of Tréport, Fécamp, and Etretat. This drift also contains at places remains of the extinct Mammalia. At Mers it consists of a mass of loam with chalk-and-flint rubble, as at Freshwater. It is probable also that the great bank of flint shingle or gravel from 12 to 18 feet thick which lies off the Pointe du Houdel, near St. Valéry, and extends along the coast for a distance of about 10 miles, at a height of 16 feet above high tide, originated with the rubble-drift of the Somme valley. At St. Adresse, near Hâvre, the rubble-drift appears to be represented by a bed of slightly subangular flint gravel forming a low cliff extending to the foot of the chalk hills, and connected by a thin trail with the chalk of the hills and with the red clay with flints which caps them.

Six miles to the westward of Cherbourg a "head," 25 feet thick, of large angular fragments of the local granitic rocks in a matrix of clay, sand, and Loess, fronts the cliffs. Under it are traces of a raised beach in the form of rolled and water-worn pebbles, but there is no clear section, nor could I find any fossils. Between Granville and St. Paire, is a section showing a small quantity of Rubble-drift on a slope, and under it the broken edges of the rock are bent back in a manner similar to the case I have recorded in Devonshire.

The Channel Islands afford evidence of submergence and upheaval of a character different from that we have hitherto noticed, but no less striking. Both Guernsey and Jersey have been surrounded by a Raised beach* overlaid by a "head" of Rubble-drift, though it is only at intervals that remnants of the beach are now to be seen. In Guernsey, sections more or less perfect are shown on the cliff south of St. Peter's, at St. Martin's Point, in Saints Bay, La Pezerie,† Creux des Fées,† Hâvre, Bordeaux,† at Fort Gray, Lihou Fort, and Cobo.‡ Also at a few places on the north side of the island. Amongst the best preserved are those at Lihou passage and Firman Bay, of which latter the following (fig. 5) is a section.

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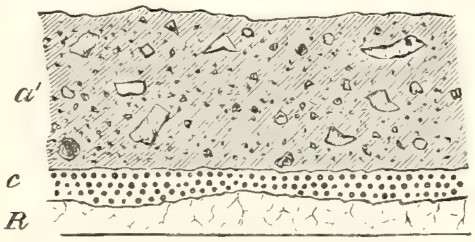
* LAMBARDIE estimated the loss on the Normandy coast at 1 "pied de roi" annually. M. BAUDE mentions later observations continued from 1800 to 1847, which show that at the Cape d'Ailly, near Dieppe, the loss was 0·80 mètre yearly, but at the Cape La Hêve, near Hâvre, it was only 0·30 mètre.

Fig. 5.—*Raised Beach, Firman Bay.*



a'. Head of angular fragments of the local rocks—some of the fragments of large size—in a matrix of loam or brick-earth, 20 feet thick.

c. Raised beach of well-rolled pebbles and subangular blocks of granite, &c., 6 feet thick.

R. Granite (decomposed), 5 feet.

The height of the beach is less than on the English coast, being generally only 5 to 8 feet above the high tide level. No shells are recorded.

The greater part of the Island (as also Jersey) forms a plateau 300 to 350 feet high, of granitic and metamorphic rocks, and is without any commanding heights. This plateau is covered very generally by a deposit of Loess or brick-earth from 5 to 10 feet thick, extending over the highest points of the surface. The Loess is identical with that on the mainland, and that it is not to be confounded with the decomposed granite or other rocks which it overlies, is shown in the following section (fig. 6).

To whichever of the generally assigned causes the origin of "Loess" has been attributed—whether to river floods, or to glacial inundations, or to rainwash—it is impossible to admit that the Loess of Guernsey and Jersey can be attributed to any of them. There are no rivers in either island, and the watercourses are mere small brooks that could scarcely flood the lowest ground, and certainly could never, in present nor past times, have reached the plateau on which the Loess occurs. Nor are there any hills, rising above the general level of the plateaux, the wash from which could have been spread over those plateaux. Nor can it be admitted that it was formed when the island was connected with the mainland, and that the Loess is due to the extension of the land flood-waters, over what was then part of the continental area; for, unless the Loess were older than the raised beaches, it is obvious, as those beaches extended all round the islands, that at the time of the deposition of the Loess, the islands were then, as now, detached from the mainland. The Loess in fact is closely connected with the "head" and not infrequently associated with it. A thin layer of an angular rubble similar to that which forms the "head," is also often to be found at the base of the Loess, and as the rubble is newer than the beaches, so must the Loess likewise be newer, and subsequent therefore to the severance of the islands from the mainland. Further, if the formation of this brick-earth cannot be attributed to floods or to a rain-wash from higher ground, it must have an origin independent of those to which the Loess is ordinarily assigned.

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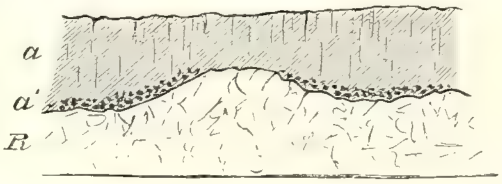
* For notices of the raised beaches and drift beds of these islands, see the papers by Sir W. C. TREVELYAN, 'Proc. Geol. Soc.,' vol. 2, p. 577, 1837; by R. A. C. GODWIN-AUSTEN, 'Quart. Journ. Geol. Soc.' vol. 7, p. 118, 1851; ANSTED'S 'Channel Islands,' p. 280–296, 1862.

† These places I give from sketches in my possession made by Mr. DE LA CONDAMINE.

‡ At Cobo there is a second beach about 30 to 40 feet above the lower one, which latter is the one I have taken for my base line.

Fig. 6.—*Section on the hill near St. Andrews.*



a. Light brown Loess with a few angular rock fragments (*a'*) at base, 4 to 6 feet.
R. Decomposed granite, 4 to 8 feet. The solid rock does not show here.

On the other hand—that a uniform sediment of that character should be formed during such a submergence as we have described, is, owing to the waste of the softer surface beds and decomposed rocks by the advancing waters, what we might expect. This waste was general over all the area submerged, and the waters must have been rendered turbid to a considerable distance from the coast,* so that not only the mainland but the adjacent islands also were covered with a mantle of sedimentary matter deposited during those periods of comparative quiet or lulls, which are shown to have occurred in the formation of the *Head.* The absence of marine remains is readily accounted for by the temporary nature of the occupation of the land by the sea waters, as well as by the circumstance that the waters would be rendered for a time unfit for the habitation of marine life. We shall revert to this subject again when we have to speak of the continental Loess.

If we suppose that the Loess in these islands was deposited during and after submergence, it follows that as the land rose, it would be removed where it was in the way of the effluent currents, and carried with the angular rubble down to lower levels, or to a distance. That this was the case is shown by the fact that the "head," which covers the beaches, consists of angular local rubble, with Loess or brick-earth (derived from the plateaux) as a matrix and forming occasional seams and overlying beds. The following diagram (fig. 7) will illustrate my meaning.

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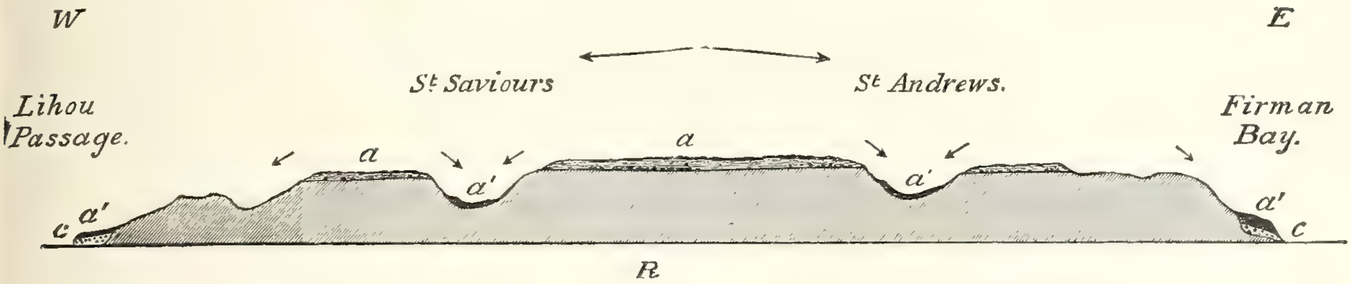
* On the coast of China the sea is coloured yellow to a distance of 100 miles from land by the fine Loess-mud carried down by the rivers.

Fig. 7.—*Diagram Section across the Island of Guernsey.*

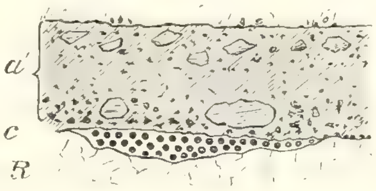


a. Brick-earth or loess.
a'. Rubble-drift or "head."
c. Raised beaches.
R. Slates and granite rocks.

The arrows represent the direction of the effluent currents as the land emerged from beneath the waters, leaving portions of the fine sediment on the plateaux, but sweeping it off the slopes and down the valleys.

Jersey.—In this island there are the remains of a low-level Raised beach at Fort Regent, St. Clements, La Motte, Mont Orgueil, Ann's Port, St. Catherine's Bay, Rozel Bay, and Boulay Bay.* They are covered as in Guernsey by a rubble-drift or "Head," and are about 5 to 15 feet above the sea-level.† They are well seen at Ann's Port and St. Catherine's Bay; the section at the former place is as under.

Fig. 8.—*Section of the Raised Beach and Head on the north side of Port Ann Bay.*



a'. "Head," largely composed of loam roughly bedded, with fragments, and a few angular blocks, of the rocks from the hills above, 12 feet.
c. Raised beach, chiefly of diorite pebbles, 2 feet.
R. Diorite rock.

In none of these beaches have there been found any pebbles of foreign origin, with the exception of fragments of chalk flints, as in the beaches on the Devon and Cornish coasts; nor did I see any shells, nor have any been recorded, except by Mr. T. W. DANBY, who discovered in a Raised Beach on the Hermitage Rock, St. Aubyn's Bay, "an abundance of shells of species now flourishing on the adjacent shore." The Beach, which was about 6 feet above high water mark, has since been concealed by the harbour works.‡ In Jersey, as in Guernsey, brick-earth or Loess is widely spread and covers the highest parts of the island. It likewise has been referred to disintegration of the rock *in situ,* or to rain-wash. But as I have explained in describing Guernsey, the height at which it occurs, precludes the possibility of its being rain-wash, while though superposed on the disintegrated rock surfaces, the absence of the quartz grit of decomposed granite or syenite, and the presence of angular rock fragments or rubble at its base,* equally show its special and independent origin. This was the opinion of Professor ANSTED,† who says "In addition to the soil derived from the decomposition and disintegration of the rocks in all the island, there are occasional deposits of some extent of brick-earth and potter's clay." It is also the opinion of Dr. A. DUNLOP,‡ who found that in one place the brick-earth had a thickness of 50 feet. No Mammalian remains have been found in these drifts, which tends to confirm the early separation of the islands from the mainland.

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* Professor NOURY mentions besides, Portelet Bay, Crabbé, Petit Port (Vicard): he assigns a different origin to the beaches, 'Géologie de Jersey,' p. 162, 1886.

† Dr. DUNLOP informs me that there are some beaches considerably higher, but these I have not seen.

‡ 'Geol. Mag.,' for 1876, p. 143.

Similar phenomena are exhibited on the French coast. M. TRIBOLET§ says that, on the small island of Bréhat on the north coast of Brittany, there are deposits of loam about 2 feet thick, overlying the granite, "completely identical with those which are known in Switzerland and Germany, under the name of *Loess.*" It there contained a few land shells (*Pupa, Helix, Succinea,* &c.), with the small concretions termed *race,* so common in Loess. M. TRIBOLET found the same Loess on the adjacent mainland, and supposed that it might have been deposited by the water descending from small glaciers on the inland hills, but these are of small height and show no sign of glacial action. The other reasons I have applied to the Channel Islands with respect to its special and local origin, apply equally to the island of Bréhat.

One feature that I failed to notice in Guernsey occurs in marked distinctness in Jersey. This is the distance to which the "head" has been propelled from its base. The section at the islet of La Motte is even more illustrative than that at Godrevy in Cornwall, which I have before described.‖ This islet lies 1½ mile south-east of St. Heliers, on a part of the coast where the shore is low, but rising gradually inland to a height, at Mont Ubé, of 149, and at Prince's Tower, 2½ miles inland, of 200 feet. A rubble-drift descends the slopes of Mont Ubé and St. Clements (160 feet) and forms a small low cliff on the coast, while at the distance of about 1000 feet from the shore, and accessible at low water, is the small, flat islet of La Motte. It is only a few acres in extent, and consists of a base of diorite, capped by the remains of an old beach, overlaid by a mass of rubble-drift or *head,* the section of which is very similar to that represented in fig. 8. It consists of—

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* At COPP's brickfield I found one to three feet of angular rubble under several feet of brick-earth, and at the base decomposed granite *in situ.*

† 'The Channel Islands,' p. 296; see also pp. 279–295.

‡ 'Quart. Journ. Geol. Soc.,' vol. 45, p. 118, 1889.

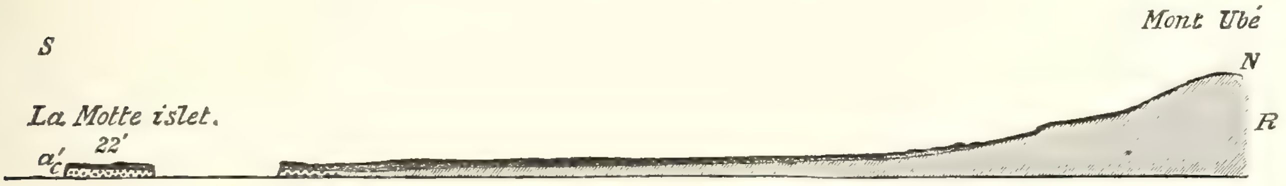
§ 'Ann. Soc. Géol. du Nord,' vol. 5, p. 100, 1878.

‖ 'Quart. Journ. Geol. Soc.,' vol. 48, p. 281.

1. "Head" of angular fragments and small blocks of diorite and syenite, passing down into débris of decomposed syenite or granite (chiefly at its base), mixed with more or less loam or brick-earth, and containing small flat calcareous concretions (*race*) and a few rolled pebbles derived from the beach.	6 to 8 feet.
2. Raised beach, formed of pebbles of diorite and red granite or syenite. There were no shells to be seen. Remains only in places.	6 to 12 inches.
3. Diorite	2 to 3 feet.

The syenite and diorite fragments in the rubble are derived from the hills of the adjacent coast, while the brick-earth which caps those hills forms the chief portion of the rubble matrix. Notwithstanding the slight difference of level and the very small gradient of the slope from the inland hills to La Motte, a considerable spread of rubble-drift has been propelled thus far out (fig. 9), which, I conceive, could only have been effected by a strong effluent current, passing from the mainland sea-ward during upheaval of the land. The hills are so low and distant that no snow-slide could possibly have effected this transport.

Fig. 9.—*Section from La Motte to Mont Ubé.*



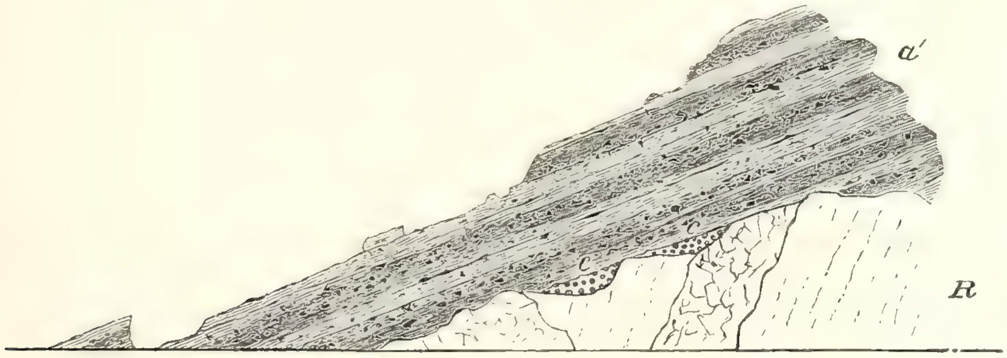
a'. Rubble-drift, composed of granitic and diorite débris in a brick-earth or Loess, covered by a sandy earth and soil.

c. Raised beach—only portions of this remain.

The phenomena, however, are readily explicable on the assumption that, as with the *head* at Brighton and Sangatte, the driving force was that of a superincumbent body of water, flowing outwards. As at Sangatte, the forcible impact of the *head* on the beach has led to the incorporation of fragments of the beach in the *head.*

Alderney.—I did not visit this island, but an unpublished section, taken many years

Fig. 10.—*Section on the Coast of Alderney* (De La Condamine).



a', Rubble-drift or Head; c, Raised Beach. No description is attached to Mr. De La Condamine's sketch, but this drawing, compared with his other drawings, leaves me in no doubt of its meaning. The rubble seems partly consolidated.

ago by the Rev. H. M. De La Condamine, gives what seems to me a very correct representation of a rubble-drift on a slope, covering the remnants of a raised beach.

We have seen that after the deposition of the angular rubble the land on our Western Coasts was left considerably above the level at which it now stands, so that the land area was much enlarged. Such likewise appears to have been the case with the Channel Islands, which are fringed by submerged forests, like those rooted on the rubble-drift of Devon and Cornwall. On this surface, flint flakes and Neolithic implements, together with the remains of Deer, &c., are found, but no traces of Palæolithic Man have been discovered on or around the Islands.

Brittany.—M. Ch. Barrois has shown[*] that in Finisterre, the coast is fringed in many places by Raised Beaches at a height of from 10 to 30 feet above the present beach, and he describes one in particular in the Bay of Kerguillé, not far from Brest, which may be taken as a type of the rest. It lies against Silurian rocks and contains pebbles (elsewhere there are boulders) chiefly of Palæozoic and Plutonic rocks, with numerous pebbles of white quartz and some chalk flints. M Barrois considers that the pebbles and boulders of these Raised Beaches are derived in part from the local rocks, whilst others, which comprise a great variety of porphyries, are from rocks in the hydrographical basins of the adjacent rivers, which include the Loire, and he attributes their transport to the action of coast and river ice at the commencement of the Quaternary period.

I am not, however, satisfied that the drift of the shingle was altogether from that direction, for besides the pebbles of porphyry and of basalt foreign to the district, and which M. Barrois supposes may have come from the central plateau of France, there are upper chalk (Senonian) flints, showing a transport from the Coast of Picardy, or from the eastward, which would agree with that which then prevailed on the English coast. A current from the eastward would account also for the foreign boulders like those on the Sussex coast. Another point of resemblance between the French and English beaches, is the often enormous size of the pebbles, which form true boulder beaches.

M. Barrois traces the extension of these beaches, and of pebble banks derived from them, along the coast of South Brittany.[†] M. Durocher[‡] has also figured a well-marked Raised Beach, 12 to 15 feet above the sea level, but apparently without any overlying detrital matter, and without shells, at the mouth of the River Vilain.

Thus far there can be no doubt of the synchronism of the land movements at the later Quaternary period, on both sides of the English Channel, but as we proceed further south, the breaks in the continuity of the beaches become longer, and the evidence less conclusive. This may be partly due to the fact that the coasts on the Bay of Biscay and of Portugal being more exposed to the severe westerly gales, the old beaches have generally been destroyed, and remnants only of detrital *head* or Rubble-drift left.

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[*] 'Soc. Géol. du Nord,' vol. 4, p. 186, 1877.

[†] A short notice by M. T. Héna ('C. Rend.,' p. 1370, 1874) indicates the presence of a Raised Beach and *head* on the shores of the Côtes du Nord.

[‡] 'Bull. Soc. Géol. de France,' 2^{e} sér., vol. 6, p. 212, 1849.

The Coast south of Brittany.—In the Island of Noirmoutier,* on the coast of La Vendée, the "Quartzite du Cobe" forms a low cliff, 15 to 20 feet high, against which abuts what from description seems to be a mass of angular rubble-drift rising to the height of 10 to 25 feet above the sea-level, but without a Raised Beach. On *the coast between Biarritz and St. Jean de Luz*, I saw no Raised Beaches, though at places there are traces of a Rubble-drift. This is well marked at the entrance of the valley at Bidart, where it forms a slope of angular débris derived from the local strata and from the plateau gravel. There is another section at the mouth of a small river south of Bidart where the débris forms a thick roughly stratified mass of gravel and loam (or Loess) inclined on the hill side.

The evidence, though slight, is sufficient to show that on the south-western coasts of France, the Rubble-drift is present, thereby indicating the operation of the same common cause which affected the coasts of the Channel. More striking evidence, however, to the same effect, but of a different class, exists in the interior of the country, and on the south coast, some of which we will now describe.

Inland Forms of the Rubble-drift in France and Belgium.

These embrace, 1st, the high-level or plateau Loess; 2nd, the angular drift or breccia on slopes; and 3rd, the ossiferous fissures.

The Loess.—Few questions in Quaternary Geology have given rise to more frequent discussion than the origin of the Loess.† It was suggestively considered by LYELL,‡ and a critical review of the various opinions concerning it has been given by Professor JAMES GEIKIE.§ The more general expressions of Continental opinion will be found in the works of M.M. DE LAPPARENT|| and CREDNER,¶ and others. Here we need only consider its origin, and must refer for details to the original memoirs.

There is a general agreement amongst geologists that the Loess of the great river valleys is, up to the height to which the old rivers rose, due to floods connected with the melting of the ice and snow of the Glacial period. The Loess, however, is not

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[*] 'Mém. Soc. Géol. de France,' vol. 1, p. 325.

[†] M. D'ARCHIAC has given an excellent summary up to the year 1848 in his 'Histoire des Progrès de la Géologie,' vol. 2, chapters iii.–vi.

[‡] 'Antiquity of Man,' 4th edit., chapter xvi.

[§] 'Prehistoric Europe,' chapters ix. and xi.

[||] 'Traité de Géologie,' p. 1084.

[¶] 'Traité de Géologie et de Paléontologie,' French Translation, pp. 639–641.

confined to the limits of those valleys, but occurs on the watersheds which separate them, as, for example, on the pass above Ham, between the basins of the Somme and the Oise, at a height of 184 feet above the sea-level. Not only so, but deposits of Loess cover the plateaux between these and other river-valleys of the north of France and Belgium which rise to the height of 400 to 650 feet, whilst in central France the Loess on the hills and plateaux attains a height of 700 feet, and in the neighbourhood of Lyons of some 1300 feet (about 400 metres).*

But this is far from showing the limits of height which it attains in Central Europe. In the upper valleys of the Rhine and Danube it rises to altitudes of not less than 1500 feet, and even higher in some parts of Central and Eastern Europe. The Odenwald, the Taunus, and other upland tracts, are cloaked with Loess up to a considerable height. The Danube and many of its tributaries flow through vast tracts of it. Lower Bavaria is thickly coated with Loess, and it attains a great development in Bohemia, Austria, and Moravia—in the latter country rising to an elevation of 1300 feet. It is equally abundant in Hungary, Galicia, and Transylvania, and in the valleys of the Carpathians it stretches up to heights of 800 and 2000 feet.

That the great rivers of Europe were, during the Glacial period, laden in times of flood with fine sediment which they deposited to certain heights in the valleys through which they flowed, admits of little doubt. It is the view which, with other geologists, I have myself advocated.† It is impossible, however, to suppose that that Loess which lies spread in thick sheets over whole countries, covering them as though with a mantle, and rising to great heights, can, with the hydrography of those countries as it now exists, or as it existed in Glacial times, be the result of any ordinary river floods. The whole question is too large to be discussed here, but I would draw attention to some of the objections which attend its reference to river floods, with the river valleys and the land gradients in the form they now exist.

To meet the difficulties, two suggestions have been made. The one is that at the time of the deposition of the Loess " the amount of depression and re-elevation in the central (mountain) region was considerably in excess of that experienced in the lower countries or those nearer the sea, and that the rate of subsidence in the latter was never so considerable as to cause submergence or the admission of the sea into the interior of the continent by the valleys of the principal rivers."‡ It was supposed that the depression might have been at the rate of five feet in a century in the mountains, and only as many inches in the same time nearer the coast, and that the Loess was accumulated during subsidence and removed during the upheaval of the land. But there is nothing to corroborate this view, and even supposing that these movements might to some extent have equalized the levels, the height to which the waters would still have been heaped up whilst wide valleys remained open with rivers offering ready channels of escape, is not easy to understand. Other objections to this view are urged by Professor J. GEIKIE.

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* 'Bull. Soc. Géol. Franc.,' 2nd ser., vol. 16, p. 1057.

† 'Phil. Trans.,' 1864, p. 247.

‡ 'The Antiquity of Man,' p. 383.

The other suggestion is that the great rivers of Europe had been dammed back for a time by the advance* of the great northern ice sheet, or, as others have suggested, by the block caused by great masses of ice carried down by these rivers at the break up of the ice in spring, as now of frequent occurrence in Arctic regions. The former suggestion is inadmissible, because, whatever the cause was, it was one that affected the whole central area, whereas a northern ice sheet would only have blocked the rivers flowing north, and not those flowing south. The other suggestion of ice-dams in the rivers might answer to some of the conditions, but would scarcely meet the case where the areas covered are so vast and high. We should have expected also to find greater traces of destructive and transporting action. If the Loess had been confined to narrower valleys this cause might have been available, but the extensive plains and high hills over which it extends renders it difficult to imagine that river floods alone could have spread such a mantle over large portions of Europe. For these reasons, amongst others, I do not think that any land-flood hypothesis will satisfactorily account for all the phenomena.

[More lately the Eolian theory, proposed by the Baron F. VON RICHTHOFEN† to account for the origin of the loamy deposit which covers large surfaces in China and some adjacent districts, has been thought applicable to the European area. This deposit is described by the Baron as perfectly similar to the Loess of Europe in composition, structure, and mode of occurrence. It extends from the alluvial plains, only a few feet above the sea-level, to heights of 8000 feet and more; and, while thin in some places, it attains in others a vertical thickness of 500 to 1500 feet. It abounds in land shells, and contains also the bones of animals of the same genera, and mostly of the same species, as those that now live in steppes or on grassy plains. He ascribes this remarkable deposit to the action of the high winds that sweep over these parts of Asia and drive before them clouds of fine dust which in a few hours form thick dust-drifts, burying in them the land shells of the district and the bones of the animals left on the grassy surface. From this, I infer that the bones of the skeletons should be found not far apart, and sometimes in close relation to the skeleton, but whether or not this is the case, or whether any of them belong to extinct species, I cannot gather.

Of the power of these winds and the height of the country devastated by them, there is ample confirmatory evidence. A recent traveller in these regions‡ says that on the high plateaux of Thibet and North-Western China, the winds are often fearful, and when the storm bursts, the valleys viewed from a height, disappear in the dust which forms into waves that the tempest hurls in all directions, building up mountains of sand. These vast sandy tracts are dotted with tamarisk trees, which serve to stay the moving surface of the desert. M. BONVALOT attributes these dust-storms which were experienced up to heights of 14,000 feet or more, moving enormous waves of the sand eastward, to the erosive power of the winds on low chains of crumbling marls.

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* T. BELT, 'Quart. Journ. Geol. Soc.,' vol. 30, p. 490, 1874; 'Quart. Journ. of Science' for January, 1877.

† 'Brit. Assoc. Rep.,' 1873, *Sect.* p. 86; 'China,' vol. 7, p. 97, 1877; 'Geological Magazine' for 1882, p. 293. See also R. PUMPELLY in 'Amer. Journ. Science and Arts,' 3rd series, vol. 17, p. 133, 1879.

‡ 'Across Thibet,' by M. BONVALOT, English translation, 1891.

The great height of the land, the vast thickness and the irregularity of the deposit, and the absence of river- or sea-shells, preclude, as RICHTHOFEN justly observes, our looking upon this deposit as due either to marine or fluviatile sedimentation, whilst the irregularity in its dimensions, the variable and great heights at which it occurs, and the character and position of its entombed organisms, are all such as might ensue from the action of devastating winds and overwhelming dust clouds. If we could suppose snowdrifts to be consolidated and remain in permanence, the annual increments would in a period comparatively short, fill up valleys and accumulate in vast masses in the more sheltered places, and it is in manner like this that I presume we are to understand that the so-called Loess of China has been formed.

But there are many conditions at variance with those of the European Loess. Amongst them, it may be observed, that the latter is generally but from 10 to 50 feet thick, and only occasionally attains a thickness exceeding 100 feet or attains the wind-devastating heights of the other. The European Loess also exhibits not infrequent traces of water action, and the organic remains are not in their original position and entirety, but are scattered and dispersed. The bones of the animals especially are almost always found single, often in isolated fragments, and are very few in number, whereas if buried by the dust, as we presume they were in China, on or near the spot where they died, the whole skeleton, more or less entire, should be found *in situ*. Further, the climatal conditions under which the two deposits were formed, seem to have been entirely distinct.

While, therefore, I would accept Baron VON RICHTHOFEN's explanation for the remarkable wind-formed deposit of China and Central Asia, it does not seem applicable to the Loess of Europe.*

Dr. A. NEHRING† describes a Loess in Northern Germany which claims to have some points of analogy with that of RICHTHOFEN, but the evidence seems only of local application, and not to be generally conclusive. The Mammalia he names are mostly the ordinary species of the Quaternary fauna in Western Europe.—J.P., *April*, 1893.]

The plateau Loess in France‡ and Belgium is often divisible into two parts. In some cases this feature (which may be also occasionally seen in England) is due to secondary causes and not to original structure. It is where a homogeneous bed of Loess, say from 10 to 15 feet thick, has, by the long continued action of the rainfall, been so modified in its upper part by the removal of all soluble matter, that the residual insoluble argillaceous and siliceous matter, with the iron oxide in an altered state, assumes the appearance of a separate bed. This upper division is known as the *terre à briques*, and the lower is the *ergeron* of the Belgian geologists. There are, however, cases in which the distinction is real, and the lower bed seems to be of a different age from the upper and may be due to an earlier glacial drift.

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* Other objections are stated by Professor JAMES GEIKIE in "Prehistoric Europe," 2nd edit., pp. 165, 244.

† 'Geol. Mag.' for 1882, p. 570, and various previous papers in German periodicals.

‡ Amongst the many other papers on this subject those of MM. HÉBERT and DE MERCEY for the north of France should be consulted.

M. A. BRIART[*] also points out that in Belgium, besides the ordinary river-valley Loess, there is a mid-level Loess, which he names *Limon des plaines moyennes,* as well as a high-level Loess, or the *Limon des hauts plateaux;* and, though he sees in these a great lithological analogy, he considers them distinct in point of time. He says that, whereas the plateau Loess is unfossiliferous, the Loess of the valleys and slopes contains abundant remains of the ordinary Quaternary Mammalia, together with land shells (*Helix, Pupa,* and *Succinea*[†]). He further states that this mid-level Loess frequently rests on an angular and sandy drift of débris derived from the adjacent local rocks. As these are the characters of one phase of the Rubble-drift, it is not improbable that the deposit to which M. BRIART refers may be the equivalent of the Faversham, Upchurch, and similar beds in the Thames Valley.

I take this mid-level rubbly Loess to have been formed, like the "head" and the angular rubble on slopes, during the emergence of the land. The results during submergence were different. As the waters then gradually crept up the valleys they would have abraded portions of the softer strata and the older drift beds, including the river-valley Loess, and so become charged with sediment, which, as the turbid waters rose, would be carried to higher grounds, and there be deposited in intervals of rest and where the currents were weakest.

South of the Alps Professor F. SACCO[‡] has also recognized similar divisions of the Loess in Piedmont. He describes (1) a Loess of the plains of fluviatile origin; (2) a Loess of the hills of meteorological origin: the latter, which attains a height of 400 metres and contains sixty-two species of land shells, he assigns to the close of the Glacial Period; (3) a high-level Loess of the mountains with remains of the Mammoth. In Spain, south of the Eastern Pyrenees, there is also a large development of Loess. The general conditions seem, in many cases, to point to some connection between the high-level Loess and Alpine ranges.

Let us now see what the consequences of this gradual submergence would be. The mountains of Europe were still loaded with the ice of the Glacial period, and the rivers surcharged with the silt from the great glaciers and abraded lands. The volume of sediment with which the surrounding waters were charged became, as the flood waters advanced, gradually greater; while the height which the Loess reaches in the valleys of the great Alpine-born rivers would indicate that the submergence was probably not less than from 1500 to 2000 feet. It was, in any case, sufficient to cover all the lesser inequalities of surface over which the Loess spreads, and accords with the assumption that, whatever the cause was, it acted equally in all directions round certain central areas. It is also in accordance with the singularly homogeneous character of the Loess, which is such as would be due to the agency of a common fluid medium.

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[*] 'Ann. Soc. Géol. de Belgique,' vol. 18, Mémoires, 1892

[†] M. BRIART adds *Limnea* and *Planorbis*, on the authority of M. LADRIÈRE, but M. LADRIÈRE'S correlations have yet to be confirmed.

[‡] 'Bull. Soc. Géol. France,' 3rd ser., vol. 16, p. 229, 1888.

As the waters, on the upheaval of the land, retreated, they carried with them a portion of the sediment deposited on the heights, which again would be re-deposited at lower levels on the slopes and plains that were least under the influence of the effluent currents. I do not imagine that at any time this deposition of this Loess was sufficent to fill the great valleys of the Rhine and Danube, but only to drape their contours, while the scour of the effluent waters kept the main channels free. Not only have great valleys been thus scoured, but large tracts in the midst of Loess-laden countries have been swept bare by the great volume and more rapid flow of the retreating waters. For wherever the velocity of the effluent currents was less than 8 feet per second, the precipitation of fine sediments may have taken place, but where the channels were more contracted, and the currents exceeded 8 feet per second, no such deposit, or but little, need have been left on the land, so that under similar flood conditions, these deposits might cover certain large tracts, whilst adjacent tracts might remain bare of them.

It may be objected that the deposit of Loess is so great that it could not have been formed in the short time I would allow for the submergence. But it must be understood that I look upon the Loess, as a whole, as belonging to more than one period, in the sense that it was, in the main, primarily a valley deposit in time of floods; and, secondly, as just described, a reconstruction in great part of a more general character —the one the accumulation of long ages, and the other of brief time. The first is a case of ordinary sedimentation; the other, one of wider sweep and mainly of redistribution. It is well known how one heavy rainfall will remove and re-deposit many feet of an alluvial deposit like the Loess. Striking evidence of this is afforded by the changes in the valley of the Ganges and its tributaries, described by Mr. JAMES FERGUSSON.* Old channels have there been filled up and new ones formed with extraordinary rapidity; and, on the banks of the Brahmapootra, "hundreds of miles" of alluvial land are swept away every year and re-deposited lower down the river course. A few seasons suffice to form deposits 40 feet thick, whilst little addition is made in other places. We may suppose, therefore, that as the ocean waters, charged with silt, advanced up the great river valleys, and met the river waters laden with glacial mud, a vast homogeneous mass of sediment would be the result, and this would necessarily lodge over the areas most free from the scour of the varying currents and retreating waters, or in the more sheltered places.

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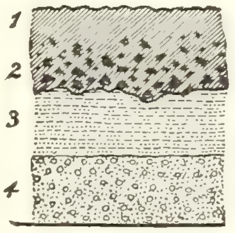
* 'Quart. Journ. Geol. Soc.,' vol. 1,9 p. 321.

The cause here assigned for the origin of this section of the Loess accords with that to which I have attributed other phenomena of a different class on heights in France and elsewhere, and with the facts we have noted in the Channel Islands. Like also these other divisions of the Rubble-drift this Loess contains only the débris of a land surface, consisting of *land shells* and the remains of the *later Quaternary Mammalia*, together with which there have been occasionally found a few fragments of the human skeleton.

Rubble-drift on Slopes.

The North of France and Belgium.—I have before referred to some of the more striking sections of this drift on the coast-line in the North of France. We may notice a few of the inland exposures and also one that occurs in a conterminous district of Belgium. The character and position of some of the Quaternary beds in the province of Namur, described by M. Dupont, agree exactly with those of the Rubble-drift in the South of England. His "*Argile à blocaux*" and "*Limon homogène*" are counterparts of our angular drift and brick-earth, and, as with us, they vary accordingly as they are spread out in valleys and plains, or are massed on the slopes of the hills.* The following section (fig. 11) gives the succession in the valleys.

Fig. 11.



1. Loam, not stratified.
2. The same with angular rock-fragments.
3. Loam, stratified, fluviatile.
4. Bed of rolled pebbles, fluviatile.

M. Dupont says of the bed No. 2, which constitutes his "*Argile à blocaux,*" that it has deeply eroded the underlying beds, and when it has caught up pebbles from those beds, a great number are broken, and the fractured edges remain sharp. When overlying hard rocks these are unaffected by it. There is an absence of stratification, though in places there is a sort of rude (*torrentielle*) bedding. The angular fragments (sometimes including blocks of large size) are all of local origin, and there are none transported from a distance. The loamy matrix varies in colour and substance according to the nature of the local rocks from the surface of which it is derived. The loam No. 1 forms a brick-earth, and is intimately connected with No. 2, into which it passes by insensible degrees. It generally accompanies bed No. 2, but it also extends beyond it and attains to much greater heights, often covering the country like a mantle. This agrees exactly with some of the main physical characters of the rubble-drift in England. The organic remains are too variable a quantity to affect this conclusion. None, in fact, have been recorded in the rubble bed in this district, except when in connection with the caves; but *Helix concinna*, *H. hispida*, *Pupa marginata*, and *Succinea oblonga*, have been found in the loam (1). M. Dupont further describes how intimately much of the Loess of Belgium is connected with these beds. The observations of M. Briart (*ante*, p. 923) confirm this view, which is one I shall have occasion to develop more fully presently.

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* 'Bull. Soc. Géol. France,' 2nd Ser., vol. 24, p. 77, 1866; and 'Bull. de l'Acad. Roy. de Belgique, 2nd Ser., vol. 21, No. 5.

M. Dupont also shows that in the neighbourhood of Dinant, the "*Argile à blocaux*" is frequently present, and forms a well-defined division between the cave beds and the deposits of the Stone Age. One of the most typical of these caves is that of the "Trou du Frontal"* on the banks of the Lesse, and which I had the opportunity of visiting with M. Dupont, not long after its exploration. A copy of this section, and of his description of it, is given below (fig. 12).

M. Dupont divides the cave deposits into two groups, assigning beds 3 and 4 to the Mammoth age, and No. 2 to the Reindeer age in consequence of the scarcity in this latter of the larger Quaternary Mammalia and the abundance of Reindeer remains. But is not this preponderance caused by the circumstance that during the deposition of the beds No. 3, the cave was frequently flooded and only occasionally inhabited by Carnivorous animals or visited by Man; whereas, after the deposition of those beds, the cave, or rather shelter, being then out of the reach of the floods, was often frequented by Palæolithic Man, and became consequently the floor on which were scattered the flint tools he used and the remains of the animals on which he fed, such as the Reindeer, Wild Boar, Ox, Horse, &c., and the other animals named by M. Dupont?

The contents of caves necessarily vary according to their occupants for the time being. The spoil of the Hyæna will differ from that of the Bear, and those, again, from that of Man. For this reason I doubt whether the caves afford a just means of classification. Outside the caves the same animals may have inhabited the woods and plains during the whole of the late Glacial or Post-glacial time, and that seems to me the conclusion to be drawn from the other evidence than that of caves. Certainly the Reindeer existed throughout that period, though its distribution may have varied; while the contents of the rubble-drift, when not connected with the caves, show that the Mammoth and other great extinct Mammalia were common at the time of its formation, and, therefore, at the time of the so-called Reindeer Age.

Though I should agree with M. Dupont in considering the "*Argile à blocaux*" distinct from the cave beds beneath, I look upon the organic remains in that deposit as derived from an old cave floor and hearth (H) spread on the surface of the fluviatile beds No. 3, and caught up by the angular débris as it swept down from the heights above, just as in the same rubble (No. 2) there are fragments of clay and gravel (*a*, *b*) derived from the beds 3*a*, 3*b* beneath it. Not but that this rubble may, as elsewhere, have some remains proper to it, but their origin in this case appears to me to be clearly from the denuded beds. It is important to notice in connection with this section these clay fragments, *a* and *b*, for it shows, as I have had occasion to note everywhere with the Rubble-drift, that the movement has been from above downwards, and that it has been of such short duration that substances so soft as these should have been preserved intact in the body of the rubble. Had the agency been running water, or subaërial, these fragments must have been disintegrated and lost in the mass. Nor would ice have answered the purpose. The case is in some degree analogous to the Gower Cliff sections, where the Rubble-drift has masked the

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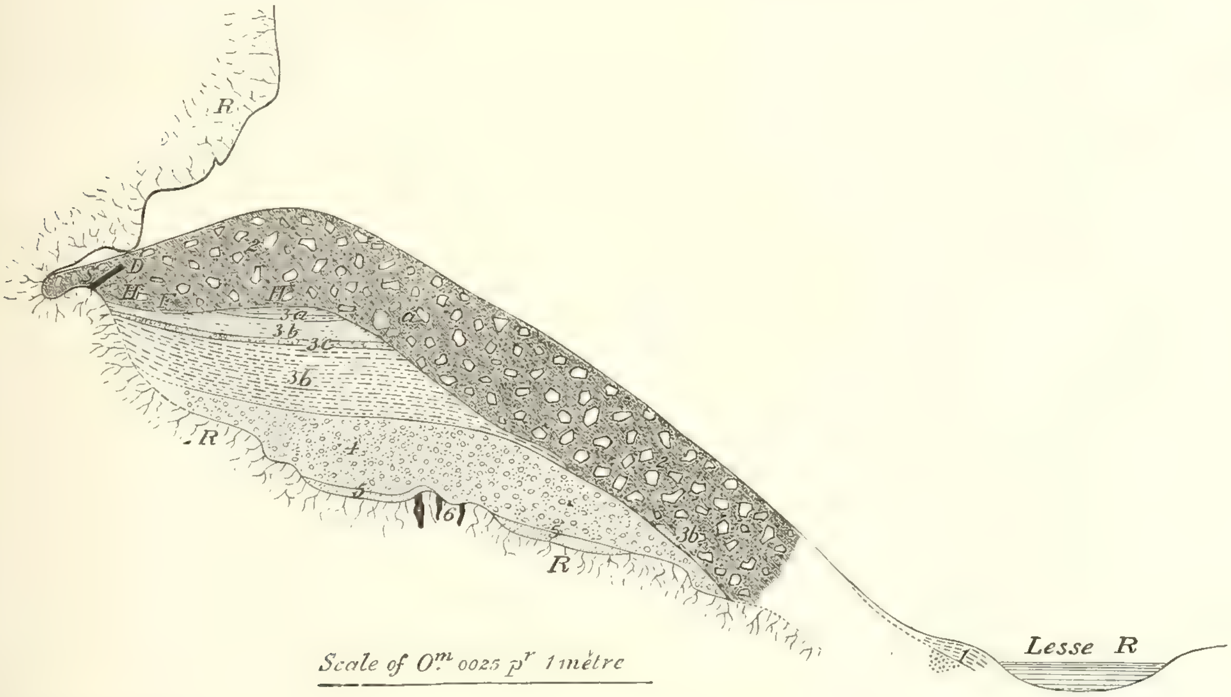
* 'Bull. Acad. Roy. de Belgique,' 2nd Ser., vol. 20, No. 13, Plate 3; and 'L'Homme pendant les Ages de la Pierre dans les Environs de Dinant-sur-Meuse,' 2nd edit., 1872.

Fig. 12.—*Section of the "Trou du Frontal"* (DUPONT).*



1. Recent alluvium.
2. Yellow clay and light grey earth, with angular fragments of limestone, containing at *S*, Human sepulchral bones, and at *H*, débris of repasts and industrial works of (palæolithic) Man; *D*, flagstone shutting the sepulture; *E*, hearth; *a*, *b*, fragments of clays from Bed 3, eroded at the time of deposition of No. 2 (*Argile à blocaux*; Reindeer age).
3. Stratified argilo-arenaceous deposits (fluviatile); 3*a*, reddish yellow clay; 3*b*, seam of gravel; 3*c*, grey clay alternating with yellow sands
4. Rolled pebbles derived from the Ardennes (fluviatile)
5. Greenish quartzose sands, with traces of peat

} Mammoth age.

6. Red clay in veins. *R*, rock.

(Bed No. 2 corresponds with our Rubble-drift, and Nos. 3 and 4 with our fluviatile valley-drifts.)

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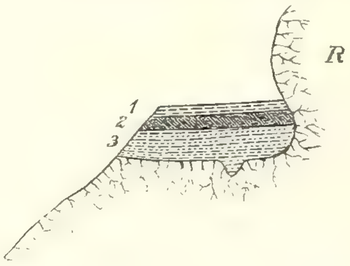
* 'Cong. Int. d'Anthrop. et d'Archéol. Préhis. à Bruxelles,' 1872, Plate 31.

entrance to the caves, and destroyed portions of the "Raised Beach." I should therefore consider the remains of the animals here found in the "*Argile à blocaux*" to be foreign to that deposit, and as belonging to the Mammoth group. It was only near the base of this angular rubble (2) that these derived remains were found.

Another point of analogy rendered evident by the sections of M. DUPONT, is the abrupt transition from the Palæolithic to the Neolithic deposits. This agrees precisely with what we have witnessed on the English coasts, where the stanniferous gravel of Cornwall and the angular rubble of Devonshire sharply separate the Raised Beaches and sands of Quaternary age from the alluvial deposits and forest growth of the recent period. The following (fig. 13) is one of the sections,* where the two deposits are seen in superposition.

Fig. 13.—*Section of the Trou du Sureau, Montaigle* (DUPONT).



1. Débris with Neolithic flints, and animal remains of the same age.
2. Clay with angular fragments (*Argile à blocaux*).
3. Fluviatile beds with animal remains of Quaternary age.

[I am not prepared with evidence relating to the Rubble-drift eastward through Germany, or to the northward of Belgium, but I may mention that Dr. LORIÉ, of Utrecht, has called my attention to a deposit which covers some parts of North Holland, and is described by him under the head of "Phénomènes pseudo-glaciaires."* This drift, which seems to represent some phase of the Rubble-drift, is referred by Dr. LORIÉ to the warp of TRIMMER, and trail of O. FISHER.—J. P., *July*, 1893.]

The Paris Basin.—Several sections of this drift have been noticed in the Paris Basin where deep valleys cut through high plains of Tertiary strata. The steep slopes and escarped edges of these valleys are generally covered with débris or a talus of recent formation, but sometimes of Quaternary age. The latter occasionally contains Mammalian remains, as at Étampes and La Ferté-Aleps (Seine-et-Oise), to south of Paris, and at Auvers and l'Isle-Adam in the valley of the Oise to the north.

At Auvers and l'Isle-Adam the late distinguished geologist M. E. HÉBERT discovered remains of *Hyæna spelæa, Felis, Elephant, Horse, Ox, Irish Elk, Red Deer, Fox? Antelope? Hare* and *Rabbit*:† at Étampes there were *Rhinoceros* and *Bear* also. The bones, which were numerous, were generally much broken. The rubble

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[*] "Contributions à la Géologie des Pays Bas," pp. 70–76. Extrait des 'Archives Teyler,' Sér. ii., Tome iii., Plate 2, 1^{re} partie, 1887.

[†] 'Bull. Soc. Géol. de France,' 2nd ser., vol. 6, p. 604, 1849.

consisted of large blocks and fragments of the Calcaire de St. Ouen which crops out above, with loam and finer débris generally at the base, and in which the bones occurred. The drift lies in crevasses, hollows, or pockets showing, according to M. HÉBERT, strong water action. This agrees very much with what we see in the rubble (or *head*) overlying the Raised Beaches—beds of finer rubble due to currents of small velocity succeeded by a final current of greater force.

The "*Montagne de Genay.*"—An excellent description of this hill, and of the superficial drifts of the district generally, has been given by M. J. J. COLLENOT, with whom I visited the section, which presents some exceptional features.* It is situated a few miles to the north-west of Semur, and consists of horizontal Liassic strata, capped by beds of "*Calcaire à entroques*" (lower Oolite) covered by a drift of red earth, and is nearly isolated from the surrounding ranges. Extending for some distance along the south side of the hill, which rises to the height of 1430 feet above the sea-level, and of 526 feet above the valley, is a mass of breccia lying on the beds of the Upper Lias and descending, apparently, to nearly the base of the slope. It consists mainly of angular detritus of the "*Calcaire à entroques*" in a red earth, and has been in great part converted into a hard breccia by calcareous infiltration. In one place this breccia contains a large quantity of bones. The teeth and more solid bones are generally intact, but the other bones are broken, some transversely, and others in the direction of their length. They show no trace of wear, the angles being perfectly sharp, as are also the innumerable splinters. The bones longitudinally split, M. COLLENOT considered to have been broken by Man to get at the marrow, and others which are black, or partly black, to be carbonized by fire.† He also remarks that none of the bones showed any traces of gnawing, though remains of Carnivora were present. In some parts of the breccia he found land shells —*Pupa*, *Clausilia*, and *Cyclostoma elegans*. The Mammalian remains belong to—

Hyæna spelæa.
Canis lupus.
Elephas primigenius.
Sus scrofa.
Equus.
Bison priscus.
Bos primigenius.
Cervus elaphus.
,, *tarandus.*
,, *megaceros.*

In addition to these, Beaver and Wild Goat (Bouquetin) were mentioned at the Avallon Meeting.

M. COLLENOT also found in this breccia many flint flakes, some of them trimmed on the edges and weathered white, with round and angular fragments of granite and quartzite pebbles, which he believed may have been used as hammers and trimmers

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* 'Description Géologique de l'Auxois,' Semur, pp. 444, 482, 1873. See also 'Bull. Soc. Géol. de France, 2nd ser., vol. 2, "Réunion Extraordinaire à Avallon," p. 721, 1845.

† This appeared to me doubtful.

The rubble is purely local and contains the remains of a land surface only. That its accumulation was an act of short duration, and considerable force is evident from the general fractured condition (admitting some to have been broken by man) of the bones, their innumerable splinters, and the absence of wear. It was suggested by M. NOLD[*] that the bones must have been fresh at the time of entombment, as he found that the thick end of a humerus, which had been shattered and flattened, still held together, in consequence, as he supposed, of the gelatine of the bone preventing the dispersion of the fragments. This, of course, is open to another explanation—that the bone was smashed in the descent of the rubble, and, being held in place by the matrix, was kept there in the general cementation effected by the subsequent calcareous infiltration. There is, I think, reason to believe, as I hope to show when describing the ossiferous fissures, that the bones found in this drift are those of animals which sought refuge on these isolated hills from the rising waters, and were eventually destroyed, their remains being afterwards swept down with the local rubble on the subsequent upheaval of the land.

These detrital taluses are not to be confounded with those later ones produced by the weathering of the rocks and the action of rain, so common on the steep slopes of the Jurassic valleys of Burgundy, and in the Oolite valleys of England. These, in the district we are speaking of, consist of finer débris, generally loose, and locally called *Arène* or *Trasse*.[†] They are inclined at steeper angles, and contain only recent remains—sometimes the skeleton of an animal with the bones entire and not fractured. On the other hand, the older rubble is coarse, often includes large blocks, has prolonged slopes, is frequently cemented so as to form a hard breccia, and contains remains of the extinct mammalia. The action of the one still continues, that of the other ceased with post-glacial times.

Mentone.—One more illustrative section of this breccia on the south coast of France may be mentioned. It was discovered in making the coast railway east of Mentone, at a spot where Jurassic limestone cliffs rise to the height of 260 feet on the sea-board. On the face of these escarped slopes are situated the several well-known ossiferous caves of Mentone[‡] at an average height of 100 feet above the sea-level. According to the late Mr. MOGGRIDGE, "Below these caves a slope of about 180 feet descends to the edge of the sea. Through the upper part of this slope, at distances from the caves of from 0 to 10 feet, is a railway cutting 600 feet long, 54 feet deep, and 60 feet above the sea. The mass removed in making this cutting was composed of angular stones, not waterworn. Loose at the surface, it soon became more or less mature breccia, for the most part so hard that it was blasted with gunpowder. In this breccia and at various depths, some of more than 30 feet, the author has taken out teeth of the Bear (*Ursus spelæus*), and of the Hyæna (*Hyæna spelæa*), while with and below those teeth, he found flints worked by man." Bones and teeth of other various animals also occur, which Mr. G. Busk pronounced to be almost identical with those found in the Gibraltar caves. The analogy between these breccias of the south of France and the "head" and rubble-drift of the coasts of the English Channel is unmistakable.

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[*] 'Bull. Soc. Géol. de France,' 2nd ser., vol. 2, p. 722, 1845.

[†] For further instances and details, see the work of M. COLLENOT before quoted, p. 452 *et seq.*, and the elaborate work of M. BELGRAND, the late eminent engineer, entitled 'La Seine,' vol. 1, Chapters III and XVIII, 1869.

[‡] 'Reports, Brit. Assoc.' Edinburgh, 1871, p. 156. I cannot agree with Mr. MOGGRIDGE in considering the cave of more recent date than the breccia. On the contrary, the breccia partially masked two caves.

[This slope of angular rubble was also noticed by M. E. Rivière,[*] who states that in 1870, when the railway was being carried in front of the caves of Baussi-Raussi, near Mentone, he found in the débris thrown out of the cutting "a number of bones —some entire and others broken—worked flints and flakes, shells, &c., the whole more or less cemented and forming a hard breccia with angular pieces of the adjacent rocks, and here and there carbonaceous matter." The specimens, however, are not specified.—J. P., *April*, 1893.]

The Sub-fossil Wood of Dixmont.—Near Villeneuve-sur-l'Yonne, between Sens and Auxerre, is a very curious deposit of which we have yet but a very insufficient description. I visited it some years ago, but too hurriedly to add much to the account of M. Forestier.[†] That gentleman termed it a subterranean forest. It is a clotted mass of stems and branches of coniferous wood, mostly fir trees, with some chestnuts, piled together in the utmost confusion, but in the part we saw, not compressed much more than an ordinary wood-stack, and the interstices not filled in. The river is at some little distance, but it is not apparently connected with it, and the district is flat. This heap of wood rises some 10 to 20 feet above the surface of the ground, and is said to be nearly 200 feet (60 metres) thick, with a length of about 3 miles (4 to 5 kilometres). It is overlaid by a bed of loam or sand, and under it is a bed of gravel. The lower part is said to pass into the state of lignite, whilst in the upper part, the wood, which has acquired the black colour of ordinary bog-wood, is so well preserved that it can be worked like ordinary wood. No organic remains of any sort have been found either in the overlying drift or amongst the wood. The mass seems to lie in a depression on the surface and not to be connected with a river deposit. The branches are broken into small fragments. Whether it can be connected with the rubble-drift is only a suggestion I venture to make. It resembles no ordinary deposit, and it has occurred to me that it might possibly be, like the land shells and Mammalian remains elsewhere, a portion of the wash of the old land surface.

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[*] 'Paléontologie.' "De l'Antiquité de l'Homme dans les Alpes Maritimes," p. 12, Paris, 1887.

[†] 'Bull. Soc. Géol. de France,' 2nd ser., vol. 7, p. 388, 1850.

OSSIFEROUS FISSURES.

The South Coast of France.—The ossiferous fissures of France present phenomena of great interest. These fissures, which are more common there than in England, show the close connection between their contents and those of the rubble-drift forming the "head" over the Raised Beaches. The local character and condition of the detritus are alike, and the same organic remains are common to both—the only exception being that the land shells found in the "head" have not been recorded in the Devonshire fissures. This deficiency is supplied by the breccia in France, where these fissures are on a much larger scale, and present conditions pointing more definitely to a widespread and deep submergence (see Map).

Such fissures are especially common on the Mediterranean coast. They were described by CUVIER early in the century, but chiefly from the palæontological point of view,[*] and afterwards with a few additions by M. MARCEL DE SERRES.[†] A more complete account of the bone caves and ossiferous fissures was afterwards given by DESNOYERS.[‡] Since then some important additions have been made of similar fissures inland.

The position of the ossiferous fissures of Plymouth offers no salient features. The hills on which they are situated are of small elevation, without any commanding points. At Catsdown and Oreston the limestone rocks rise to the height of 100 to 150 feet, and although they cannot be called isolated, yet they are sufficiently high above the surrounding valleys to render difficult the retreat of animals seeking refuge there from the rising waters. The position of the fissures on the Mediterranean coast is in marked contrast. They there occur on detached and isolated hills rising high above the surrounding plain, and from which no retreat would be possible in case of submergence.

At Nice, the Mont du Château rises 132 feet, and Mont Boron 436 feet, above the sea, and little less above the plains around. At Antibes the rock is 200 to 250 feet, and at Villefranche 450 feet high. But the most remarkable hill is that of Cette, which rises to the height of 355 feet, and bears in its physical features a strong resemblance to the Rock of Gibraltar, projecting like it into the sea, and separated from the mainland by a long and narrow tract of sands.

Some of the fissures are vertical, others are inclined at various angles, and they are very irregular in size. They are filled, as at Plymouth, with angular fragments of the local limestone, in a matrix of red earth or clay forming a breccia, which is in some places loose, whilst in others it forms a hard rock with a calcareous cement. Though there are no such well-marked Raised Beaches as those on the Devonshire coast, they are not altogether wanting. DE LA BECHE mentions that the bottom of one of the fissures at Nice was filled by a conglomerate of rolled (beach) pebbles, while the rock on one side was drilled by Lithodomi. Above this conglomerate, the fissure was filled with a breccia containing bones and land shells.[*] This[†] indicates, therefore, the same relation of "head" or Rubble-drift to an old sea level that exists in the English Channel.

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[*] 'Recherches sur les Ossemens Fossiles,' vol. 4, chap. 4, 1823.

[†] 'Essai sur les Cavernes à Ossements,' 3rd edit., chap. 2, 1838.

[‡] C. D'ORBIGNY'S 'Dictionnaire d'Histoire Naturelle,' vol. 6, p. 343, 1849.

The greater number of the bones, which occur only in places, are broken and in fragments, only a few being entire. They are neither worn nor rolled, nor are they found in any relation to their order in the skeleton.[‡] Nor is it recorded that there are any traces of their having been gnawed.

The bones which have been found at the several places here named belong to :—

Felis leo.
„ *pardus.*
Elephas primigenius ?
Rhinoceros leptorhinus.[§]
Hippopotamus ?
Sus ?
Ursus (*priscus ?*).[§]
Bos.
Equus (*caballus ?*).
Cervus elaphus.
Cervus dama.
Lagomys (*pusillus ?*).
Lepus caniculus.
„ *timidus.*
Antilope ?

Lizard (recent species ?).
Tortoise „
Serpent „

The shells are—

Helix nemoralis.
„ *algira.*
„ *vermiculata.*
„ *lapicida.*
„ *nitida.*
Helix cristallina.
Pupa.
Bulimus decollatus.
Cyclostoma elegans.

These land shells are all of recent species living in the neighbourhood.

Amongst the specimens sent to CUVIER from Nice was a portion of jaw which he recognised as human, though he felt doubt as to whether it was contemporaneous with the other bones. It was found, not in the undisturbed breccia, but in a talus of the breccia which had fallen from the exposed surface of an ossiferous fissure. Some portions of this breccia were hard and compact, and other portions were unconsolidated and earthy, and the bones in those parts were less stained and less well-preserved than in the solid parts. It was amongst this débris that the jaw was discovered. It was coated with the same film of stalagmite as the other bones of which the antiquity could not be doubted. CUVIER, however, thought that the unconsolidated part of the breccia, to which the jaw belonged, might be of more recent date than the consolidated portion, but this is no test, and it must be remembered that, at that time, the belief in the recent creation of Man was universal, and, like the later discovery of Man's works in the cave deposits of Kent's Hole, it led geologists to look with doubt at anything which was not in accordance with that belief. Apart from this cause of doubt, CUVIER's own account affords reasonable evidence of the contemporaneity of the jaw with the other bones.

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[*] 'Trans. Geol. Soc.,' 2nd ser., vol. 3, p. 171, 1835. The height of the perforated rock above the sea is not given.

[†] There is a similar instance of superposition near Mentone, but not so well shown.

[‡] CUVIER, 'Ossemens Fossiles,' vol. 4, 2nd Edit., p. 172.

[§] Mentioned by FALCONER in 'Palæont. Mem.,' vol. 2, pp. 370 and 466.

[Though M. RIVIÈRE's work (*ante* p. 931), is devoted mainly to the description of the Mentone Caves, he touches incidentally upon the breccia of the ossiferous fissures of Nice,* which he considers to be analogous to the breccia on the Mentone slopes. He describes the Nice breccia as consisting of two parts, a lower one, red and compact, with land shells, and an upper one, less compact, of a brown or black colour, with sea-shells and bones sometimes black as if burnt, and some of them split longitudinally as if for the purpose of extracting the marrow. He concluded that the Nice fissures had served, like the Caves of Baussi-Raussi, as habitations for a prehistoric people, and that these people were of late Quaternary age (p. 314). The fissures of Nice may, however, like those of Gibraltar, have been only partly filled by the osseous breccia leaving, owing to the inequalities of their sides, cavities and open spaces, which were subsequently used as habitations by a later race of men and animals. This would be in accordance with the succession of deposits elsewhere—a succession forming three separate stages, namely :—

1st. The older Palæolithic bone-breccia of the caves.

2nd. The angular Rubble-drift and breccia of the slopes and fissures, with similar Quaternary animal remains.

3rd. The more recent cave beds of the Neolithic epoch.

When the Rubble-drift masks the caves, no Neolithic deposits are met with; but when, as at Baussi-Raussi, there is no stalagmite floor or Rubble-drift to separate the first and third of these deposits, they may then appear to pass one into the other. The reason of the small percentage of extinct species from the Mentone caves, noted by M. RIVIÈRE, seems to me to be owing to the circumstance that he takes the whole contents of the cave as belonging to one epoch, and does not give separate lists of the different beds. He thus obtains a total of 281 species (p. 529), consisting of 60 Mammals, 2 Reptiles, 42 Birds, 7 Fishes, 168 Mollusca, 1 Annelid, and 1 Polype, the result being that the average gives an undue proportion of recent species.† There is an absence also, in the upper part of those cave deposits in which M. RIVIÈRE made his interesting discoveries, of flint implements of a special Palæolithic type, whereas those of well-known Neolithic types are common.

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* *Op. cit.*, p. 32.

† It is true that M. RIVIÈRE speaks of Lion and Panther in connection with this upper bed, but neither the depth from the surface nor the other conditions are given, and there are other ways of accounting for their presence.

With regard to the human jaw (now lost) from Nice, M. RIVIÈRE does not agree with CUVIER that it was of a date more recent than the other bones in the breccia, although he considers that it came from his upper and newer bed. CUVIER only mentions that it was an isolated fragment and not attached to the breccia, and it must be remembered that the greater part of the bones examined by him were collected by the eminent naturalist RISSO* from the débris of a fissure that had fallen at the foot of an escarpment.—J. P., *April*, 1893.]

Inland Fissures.—Ossiferous fissures are not, however, confined to near the coast, nor to moderate heights, as on the coasts of France and England. They are found, on the contrary, in the interior of France at very considerable heights. Two of the most remarkable instances are those of Pédémar and Santenay.

The *Montagne de Pédémar* lies a short distance S.S.W. of St. Hippolyte (GARD), and has been described by M. MARCEL DE SERRES.† It is an isolated hill rising to the height of 1128 feet above the sea-level, and of 577 feet above the adjacent valleys It "has the form of a truncated cone, of which the contours represent an elongated oval. The greatest diameter of the oval is 984 feet and the smallest 460 feet." . . . "It is within this limited area that the strange phenomenon has happened of the accumulation of a large quantity of bones of diverse animals" in hollows or fissures on the south slope of the hill near the top. They are found in a red earth, with fragments of the subjacent Neocomian rocks, which cover the summit of the hill, but the bones occur only in the fissures. The rock fragments are perfectly angular, and there is not a trace of rolled pebbles. Almost all the bones are broken into fragments, and M. DE SERRES specially notices that they are scattered without order, without any relation to their position in the skeleton, and that they have *neither been gnawed nor rolled.* There is also an entire absence of coprolites. A few of the bones only were sufficiently perfect to be recognised as belonging to *Rhinoceros lunelensis*, GERVAIS (*R. leptorhinus ?*), *Bos ?*, *Capra ?*, and *Equus.*

MARCEL DE SERRES makes the significant remark that the detritus is such as could only have been derived from rocks on the summit of the hill, showing that the spread of the débris was due to a current originating on its summit, and not from a distance,—such a displacement in fact as would be caused by divergent currents during upheaval.

The "*Montagne de Santenay*" offers another singular instance of a high-level ossiferous fissure. It is situated on an isolated hill near the village of Santenay (Côte-d'Or), and was visited in 1876 by the Geological Society of France, on the occasion of the "Réunion Extraordinaire" at Châlons-sur-Sâone, when the questions to which it gave rise led to an interesting discussion.*

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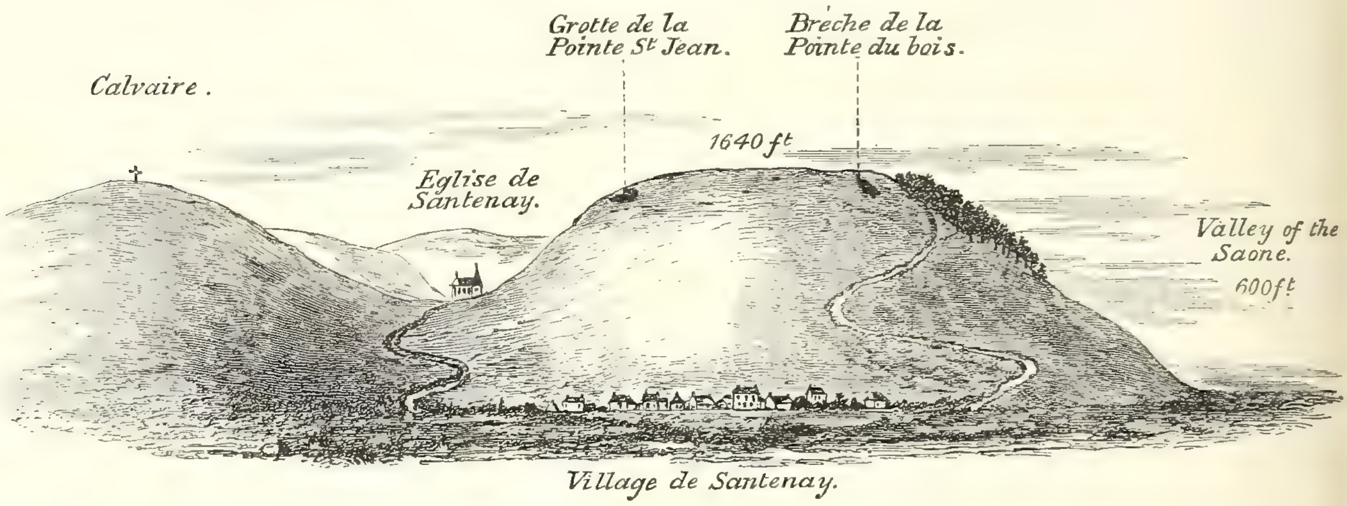
* It is not probable that RISSO, who resided at Nice, would have sent CUVIER any specimens but such as he thought belonged to that deposit. He was of opinion that the fissures had not served as *caves for the habitation* of the animals whose remains are found in the breccia.

† 'Bull. Soc. Géol. de France,' 2nd ser., vol. 15, p. 233, 1858.

Fig. 13*.—*La Montagne de Santenay* (*after a sketch by* M. A. Gaudry).


Grotte de la Pointe St Jean.
Breche de la Pointe du bois.
Calvaire.
1640 ft
Eglise de Santenay.
Valley of the Saone.
600 ft
Village de Santenay.

The hill is a few miles south of Châlons, and rises to the height of 1640 feet, its summit forming a nearly level platform, some 1030 feet above the surrounding plain. On all sides its slopes are steep, except on the one by which it is connected with the range of the Côte-d'Or. On the south side of the hill is an ordinary bone cave (Grotte de Saint Jean), in which M. Hamy found remains of—

Felis leo (*F. spelæa*).
Canis lupus.
,, *vulpes.*
Ursus (probably *U. ferox*).
Equus caballus.
Cervus elaphus (var. *Canadensis*).
Bovidæ of the size of *Bos taurus.*

There is another cave (Grotte de St. Aubin), on the opposite or northern side of the hill, containing the remains of *Bears, Horse, Elephant, Rhinoceros,* and *Ox.* In some sands outside the cave, large horns of *Cervus megaceros* were found.

Close on the summit of this hill, which consists of strata of the middle oolites, at a spot called La Pointe-du-bois, is a fissure† on the surface. It was filled with a breccia composed of the fragments of the adjacent rocks, embedded in a yellow or brownish earth, with bones which were determined by Professor A. Gaudry to be those of—

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* 'Bull. Soc. Géol. de France,' 3rd ser., vol. 4, pp. 681–695.

† It is not said whether it is a fissure of water erosion or a fissure of fracture. That does not, however, affect the argument.

Felis leo (*F. spelæa*).
Felis lynx.
Equus caballus (of large size).
Canis lupus (remains very abundant).
„ *vulpes.*
Meles taxus.
Ursus (intermediate between *U. spelæus* and *U. ferox*).
Lepus timidus.
Rhinoceros merckii.
Sus scrofa.
Bovidæ (size of *Bos taurus*).
Cervus elaphus (and var. *Canadensis*).

The bones were in a very broken state. M. Gaudry observes that their accumulation could neither be attributed *to man nor to animals*, for the fractures in no way resemble those made by man for the purpose of extracting the marrow, and notwithstanding the abundance of Wolves, *none of the bones show traces of having been gnawed by Carnivora.* How then could this collection have been brought together? As M. Gaudry justly remarks, "Why should so many *Wolves*, *Bears*, *Horses*, and *Oxen* have ascended a hill isolated on all sides?" M. Gaudry further remarks that the deposit seems to have been formed by water precipitating the breccia and the bones into a fissure. "But whence," he says, "have come the waters sufficiently abundant to bring together the bones?" The fissure is so near the top of the hill that there is little gathering ground above it, and had the bones and fragments of rock been carried in by a stream or torrential rains, they must have shown more or less wear, and have lost their sharp angles. It was suggested by some members that the event took place during the height of the glacial period, when the Alpine glaciers are known to have advanced as far as Lyons, and might there have formed a barrier which would have greatly modified the conditions of the country through which the Saône flowed, but it is difficult to suppose that the glacier at that distance from its base, retained a magnitude sufficient to dam back the land waters to a height that would cover the hill of Santenay, though it is said that the glaciers of Savoy have left traces on the hills around Lyons, at a height of about 1300 feet; but even this would not have sufficed to cover Santenay, and on the hill itself no traces of glacial action are recorded.

To account for the presence of the bones it was pointed out that in winter snow would cover the ground and hide such fissures, which would thus serve as traps for the animals when pursued, or that on the melting of the snow the remains of those which had died would be washed into them. But as I explained in a former paper, it is impossible to admit that the remains are those of animals that had fallen into the fissures, for in that case all the bones of the skeleton must inevitably have been preserved, although they might have been scattered, whereas the presence of the entire skeleton,[*] or even of the bones of detached limbs in connection, is the rare exception. As a rule, the bones are without order and in no possible relative proportion.[†]

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[*] Not a single such case is recorded of the fissures in France.

[†] That there are instances of open fissures in which animals are lost is not to be doubted. It is said

Nor is it easy to conceive that under extreme glacial conditions, herds of such animals should have resorted to high hills in the midst of ice and snow fields, instead of retreating before the ice and keeping to the open plains and richer pastures. That Wolves and Bears should, before the rising of the waters, have inhabited the caves of St. Aubin and St. Jean lower down the mountain sides, and carried to those caves the prey which they pursued in the plains around, is readily understood ; but it is inconceivable that under any ordinary circumstances the predaceous animals and their victims should have congregated together on the summit of a high, steep, and isolated hill.

Another suggestion was that towards the close of the Quaternary period, the rainfall was so excessive that it flooded the plains and obliged the animals to seek refuge on the higher ground, where they congregated in great numbers, and eventually succumbed to the rain, cold, and hunger, or to the attacks of the predaceous animals of the caves below, their bones being then dispersed and carried by the diluvial rainfall into the open fissures. In support, however, of this suggestion it had to be assumed that the rainfall was one hundred times greater than at present, and that the waters rose at the rate of one metre daily, but although the rainfall during much of the Quaternary period was no doubt very heavy, it is a physical impossibility that without a change in the level of the land, the waters could ever have been piled up in the manner here required. It is evident also from the condition of the bones, that the animals whose remains are entombed in the breccia did not fall a prey to predaceous animals.

The general opinion amongst the members present was that the animals had fallen victims to floods, but whether caused either by dams of ice, the melting of snow-fields, or excessive rainfall, was left indeterminate.

The condition and position of the bones are, on the other hand, at Santenay and Pédémar, as they are at Oreston and Catsdown, such as might result from the effects of a gradual submergence of the land. For a submergence of the character I have described would naturally drive the animals in the plains to seek refuge on the higher hills. Flying in terror and cowed by the common danger, the Carnivora and Herbivora alike sought refuge on the same spot, and alike suffered the same fate wherever the hill was isolated and not of a height sufficient for them to escape the advancing flood. We may suppose the subsidence to have been so slow that there was no sudden rush of water to carry the bodies far away, so that as they decayed, the limbs fell and were scattered and dispersed irregularly on the submarine surface. When that surface was again upheaved, the bones and detached limbs, together with the detritus on that surface, were, as I have before explained,* carried

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that even in extremely hot weather large fissures, in which the young game is lost, are opened in the ground in the south of Spain. It is also well-known that in many countries--Greece, for example—the remains of animals are sometimes carried down by the streams into large swallow holes. But special results attend all these cases, and the swallow holes are situated in low grounds and not on hill-tops.

* 'Quart. Jour. Geol. Soc.,' vol. 48, p. 340.

down by divergent currents to lower levels, or they fell into fissures of the rock over which the detrital matter passed, or else, when facing the coast, over the ledges of the old cliffs rising above the Raised Beaches. Swept down by the intermittent currents produced by the more or less rapid uplifts, and falling with the mass of detritus in a body over the old cliffs or into the open fissures, the bones, in the one case as in the other, were broken and smashed in the extraordinary manner we now find them. Added to this was the fall, caused by the earth tremors inevitable with such movements, of fragments of rock, some of large size, from the sides of the fissures, so that very few of the bones escaped whole. At the same time the action was of too short duration, and the transport was to too short a distance to wear down the sharp angles either of the rock or the bone fragments. Raised again to the surface, the rain waters, percolating through the calcareous rocks traversed by the fissures, and carrying down carbonate of lime, have generally cemented the débris of the fissures, and occasionally of portions of the "head" (Brighton), into a hard brecciated mass from which it is now difficult to extract the bones. Where, on the contrary, the débris remained loose on the surface and formed permeable superficial drift, the effect of water percolation has been to remove the calcareous matter together with the bones, so that where thus exposed, the rubble is more unfossiliferous than when it lies in fissures or hollows where the surface waters could not freely percolate.

Although the localities of Santenay and Pédémar are on levels so very different from those on the shores of the Mediterranean, the conditions which led the animals to congregate there were precisely the same. At Nice, at Antibes, and at Cette the fissured hills form, as before mentioned, high isolated rocks fronting the sea, and connected with the mainland by tracts of lower ground. In each instance these isolated hills would assuredly have been resorted to as places of safety from the rising waters by the animals of the adjacent plains, as well as by those living amongst the rocks and caves of the hills.

Other osseous breccias analogous to those on the Mediterranean coast have been noticed in the departments of the Doubs, Lot, and Haute-Saône, but those I have described are sufficient for our object. M. DESNOYERS[*] also notices an ossiferous deposit on the slopes of the hill of Montmorency, near Paris, which he considered analogous in its character with the Mediterranean breccias. In it he found the remains of *Reindeer*, *Red Deer*, *Horse*, *Wild Boar*, *Lagomys*, *Hare*, *Spermophilus*, *Beaver*, *Pole Cat*, *Marten*, &c., together with land shells of recent species, but the identity with the true ossiferous fissures does not seem to me complete, though it is clear that the deposit was not due to fluviatile agency. The bones are better preserved than in the fissure-breccias, and there were several entire or nearly entire skeletons with the bones nearly in their natural position. M. CONSTANT PRÉVOST spoke of these cavities, which are in gypseous beds of Tertiary age, as pot holes or swallow holes, and as the result of slow but intermittent rain action on lines of drainage, by which surface débris and animal remains were carried underground.

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[*] 'Bull. Soc. Géol. de France,' vol. 13, pp. 290 and 311, 1842.

Spain and Portugal.—The Atlantic waves have left but few traces of Raised Beaches or of the "head" on the western coasts of Spain and Portugal. M. Ch. Barrois figures some broad platforms of marine denudation, apparently of the age of the Raised Beaches, at Cape Vidio, on the north coast of Spain,* but they were without any organic remains by which he could determine their exact age.

The Memoir, however, by Colonel J. F. N. Delgado† on "La Grotte de Furninha, near Cape Carvoeiro, the most westerly point of Portugal, and 49 feet above the sea level, leaves no doubt of the prolongation thus far of a beach in a position analogous to that on the Gower coast. This cave presented some very interesting features. At the bottom of the cave deposits was a bed of rolled pebbles, 3 feet thick, with *Patella vulgata* and *Littorina littorea.* Above this beach were a series of ossiferous deposits, having a total thickness of 31 feet. The lower of these contained the remains of the *striped and spotted Hyæna, Cave Bear, Lynx, Wolf, Felis, Rhinoceros* (*tichorhinus?*), *Deer, Ox, Horse, Hedgehog, Badger, Weasel, Hare, Bird, Fish,* &c., with flint implements of the type of those of St. Acheul. A fragment of a human maxillary bone was also found on the lower beds at a depth of 20 feet. The upper beds contained remains of the Neolithic period. M. Delgado considered that the violence of the waves at this spot had worn back the cliff for a considerable distance, which may be the reason why there is an absence, seemingly, of Rubble-drift.

Besides some interesting glacial phenomena described by M. Fred. de Vasconcellos‡ on the west coast of Portugal, there are some great taluses on slopes, which may possibly represent the Rubble-drift. Mr. D. Sharpe,§ in his paper on the Geology of the neighbourhood of Lisbon, says that the valleys near Lisbon are often lined with detritus washed down from the hills enclosing them, but whether this is of recent date or belongs to the Rubble-drift he gives no information to show.||

The late Señor Carlos Ribeiro¶ has alluded to the occurrence of a Raised Beach near Capes Espiches and Sines, as well as other places, but no particulars are given. He also alludes to a conglomerate (breccia?) which occurs in places on the sides of hills, and is composed of the débris of rocks higher up the slopes; this may possibly represent one form of the Rubble-drift.

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[*] 'Recherches sur les Terrains Anciens des Asturies et de la Gallice,' Lille, 1882, p. 619, and Plate 20, fig. 17.

[†] 'Congrès Intern. d'Anthrop. et d'Archéol. Préh.,' Lisbonne, 1880, p. 207.

[‡] *Ibid.*, p. 155, and 'Trabalhos Geol. de Portugal,' vol. 1, p. 189, 1887.

[§] 'Trans. Geol. Soc.,' 2nd ser., vol. 6, p. 130, 1842.

[||] Mr. G. Maw alludes to the traces of a high-level beach clinging to the Rock of Lisbon at a height of 150 to 180 feet above sea-level, 'Quart. Journ. Geol. Soc.,' vol. 28, p. 87.

[¶] 'Descripcão do Terreno Quaternario,' pp. 2, 15, 17, 22, *et seq.*, 1866.

On the south coast of the peninsula there is more distinct evidence of Raised Beaches and Rubble-drift. General DE LA MARMORA[*] mentions that at Cadiz the remains of a Raised Beach, with foreign pebbles overlaid by a drift of red earth, are to be seen, and that traces of a similar beach are again met with at Cape Trafalgar and Tarifa. Mr. G. MAW gives some further particulars, stating[†] that in "the neighbourhood of Cadiz long ranges of low cliffs occur, closely resembling the Raised Beaches of Devon, and like them composed of a hard concrete of sand and shells" of recent species, implying an elevation of the coast of 40 or 50 feet.

Little is known of these beds on the east coast. Professor ANSTED[‡] speaks of Raised Beaches, at a height of about 40 feet above the sea at various points of the coast (of Malaga), consisting generally of fragments of slate, for the most part angular, and often of a large size. In one of them "fragments of copper ore were found derived from a vein cropping out on the hill above." Similar deposits of angular gravel exist a few miles up on the banks of the Guadalmedina at a rather higher level. He did not observe any shells or fragments of shells in these beds. It would seem from his description that these deposits represent a Rubble-drift or *head* rather than a Raised Beach. In the neighbourhood of Barcelona Professor L. MOLINS FOTI[§] states that there is an extensive development of Loess. A Raised Beach above the sea-level has also been noticed on that coast.

Scanty as is this information, it is in accordance with the general characters of the Rubble-drift, while the absence of intervening links is of less importance, as the phenomena we have noticed at Pédémar and Santenay, and those we are about to notice at Gibraltar, bridge over the intermediate area, and lead us to infer that it must have shared in the submergence which affected high hills on its borders.[||]

Gibraltar.—The well-known Rock[¶] is an isolated hill, separated from the mainland by a few miles of flat ground about 10 feet above the sea-level, and is composed of hard limestones of Jurassic age, forming a high scraggy ridge rather more than 2½ miles long, from 550 to 1550 yards broad, and rising at the north end to the height of 1349 feet, and at the south end of 1370 feet above the sea-level. From the O'Hara Tower, at the south end, the Rock falls rapidly to the Windmill Plain (400 feet), and again to the Europa plain (150 feet). These are plains of marine denudation. The strata dip at a high angle to the west. On the east side the Rock forms an almost sheer precipice: on the west side it slopes rapidly to the sea-level, as shown in the following sections by Messrs. RAMSAY and GEIKIE, which have the advantage of being drawn to a true scale. The interpretation of the drift beds is, however, my own. At the foot of the west slope is the belt of lower ground on which the town is situated.

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[*] DE LA MARMORA'S 'Voyage en Sardaigne,' p. 370, 1857.

[†] 'Geol. Mag.,' vol. 7, p. 553, 1870.

[‡] 'Quart. Journ. Geol. Soc.,' vol. 15, p. 599, 1859.

[§] 'Atti della Società Italiana di Scienze Naturali,' vol. 12, fasc. 3, 1869.

[||] M. MARCEL DE SERRES speaks of an ossiferous breccia of Concud, near Teruel, in Aragon, but gives no particulars and no references.

[¶] CUVIER gives an account of the organic remains of the Gibraltar breccia in his 'Ossements Fossiles.' The general geology, and especially the Raised Beaches, were described by Mr. J. SMITH, of Jordanhill, in 1844 ('Quart. Journ. Geol. Soc.,' vol. 2, p. 41); the caves and their contents, by Dr. FALCONER and Mr. BUSK in 1865 ('Quart. Journ. Geol. Soc.,' vol. 21, p. 364; 'Palæon. Mem.,' vol. 2, p. 556, 1868; and Congr. Preh. Archæol., Norwich Meeting, 1868); the fossil fauna, by G. BUSK in 1877, ('Trans. Zool. Soc.,' vol. 10, pt. 2); and the solid and superficial geology by Sir A. RAMSAY and Professor JAMES GEIKIE in 1878 ('Quart. Journ. Geol. Soc.,' vol. 34, p. 505).

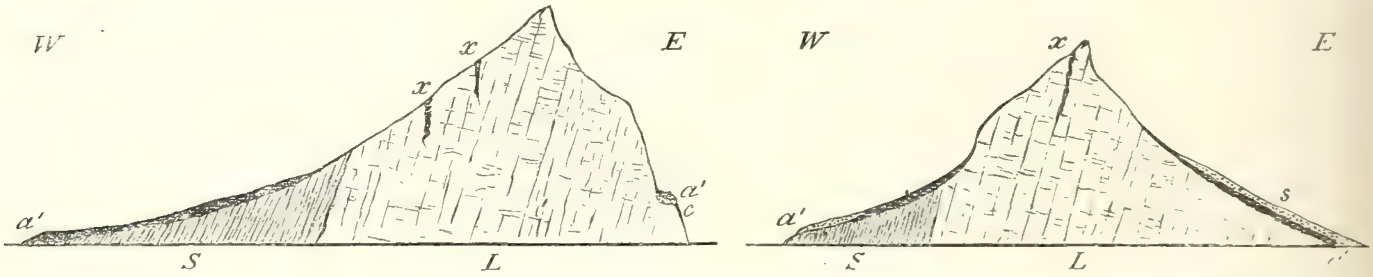

Fig. 14.—*Transverse Sections of the Rock of Gibraltar* (after RAMSAY and GEIKIE).

A. Section across Sugar Loaf Hill, O'Hara Tower, 1370 *feet.*

B. Section across Middle Hill, St. Michael's Gate, 1138 *feet.*

s, sands; *a'*, limestone breccia or rubble-drift (the agglomerates of the authors—their older agglomerate is that on the left and their newer one that on the right side of the sections); *L*, limestone; *S*, shale; *x*, ossiferous fissures (not in original sections*); *c*, Raised Beach.

The Rock has been extensively faulted and fissured. Large rents, some perpendicular, and others inclined at various angles, and extending to great depths, traverse it in different directions. They are met with at all heights up to 1100 feet (St. Michaels).† In the course of constructing fortification works, ossiferous fissures have been discovered from time to time. Amongst the most important is the one known as the Genista Cave on Windmill Hill explored by Captain BROME (1862 and 1868), and investigated palæontologically by Dr. FALCONER and Mr. G. BUSK in 1865. This hill forms a plateau 400 feet above the sea at the south end of the rock.

Many of the fissures have been only partly filled up by the breccia, leaving large open spaces at various depths, some of which have been occupied as caves and places of refuge up to comparatively recent times. Owing to the untimely death of Dr. FALCONER it fell to Mr. BUSK to complete the investigation of the fossil fauna, commenced by the former.‡

The following is Mr. BUSK's revised list of the Mammalian remains:—§

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* These I have merely put in approximately to indicate their general position, but not their depth.

† Sea-caves and beaches also exist at various heights. (See Mr. JAMES SMITH's paper.)

‡ 'Palæontological Memoirs,' vol. 2, p. 554.

§ 'Trans. Zool. Soc.,' vol 10, p. 53, 1876.

Ursus fossilis sive priscus.
Hyæna crocuta.
Felis pardus.
,, *pardina.*
,, *caligata.*
Equus caballus.
Rhinoceros hemitœchus, Falc. (*R. leptorhinus*, Owen, vel *R. Merckii*, Lartet).

Cervus elaphus.
,, *dama.*
Capra ibex.
Bos primigenius.
Sus scrofa.
Lepus caniculus.
,, *timidus.*
Canis vulpes.

Cuvier mentions *Lagomys*, but with doubt.

The bones were much broken and splintered, and Dr. Falconer states that in no instance did they observe "fossil bones *attributable to one complete skeleton* of any one of the larger Mammalia," nor does he or Mr. Busk allude to any of the bones having been gnawed. Had they been so, it would certainly not have escaped the notice of such experienced observers. At the same time they remark that the bones in the newer or prehistoric caves were *gnawed.*

Mr. Busk tells us that "wherever the surface is exposed it is seen intersected by ramifying fissures which occasionally widen into extensive caverns either empty or filled with bone breccia and crystalline spar."* Sometimes the fissures are very narrow, at other times they are many feet wide and extend to great depths; the one on Windmill Hill was followed to a depth of 290 feet from the surface, and another on St. Michael's Hill to a depth of 288 feet. Owing to the size of the blocks in some cases, and in others to the contraction of the sides of the fissures, large open spaces have often been left at various depths.

Human remains, including entire skeletons, together with fragments of pottery, stone implements, worked bones and remains of recent animals, were discovered in many of these spaces or caverns, but they were all of Neolithic and recent date, and do not come within the scope of our inquiry.† Besides these, however, traces of Palæolithic Man were met with. At the depth of 53 feet, in one of the fissures on Windmill Hill, the workmen came upon a quantity of red breccia in which were found two teeth of *Rhinoceros.* In the same mass there was a *human tooth,* which Mr. Busk identified as a molar that had never been cut, together with a *flint knife,* and numerous large pieces of flint.‡

To account for the bones in the breccia, two explanations have been suggested. 1st, that of Buckland and De la Beche, which attributes the ossiferous débris to the remains of animals that had fallen into open fissures. To this I have already stated objections which must, I think, be considered fatal.* 2nd, that suggested by Dr. FALCONER, who supposed that "the wild animals above enumerated, during a long series of ages, lived and died upon the rock. Their bones lay scattered about the surface, and in the vast majority of instances crumbled into dust, and disappeared under the influence of exposure to the sun and other atmospheric agencies, as constantly happens under similar circumstances at the present day. But a certain proportion of them were strewed in hollows along the line of natural drainage when heavy rains fell; the latter, for the time converted into torrents, swept the bones with mud, shells, and other surface-materials, into the fissures that intercepted their course; there the extraneous objects were arrested by the irregularity of the passages, and subsequently solidified into a conglomerate mass by long continued calcareous infiltration."†

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* Many of the later caves were beautifully furnished with stalactites.

† They are described by Mr. Busk in the paper just quoted.

‡ Land shells are also said to have also been found.

Although this last explanation gets rid of some of the difficulties attached to the first, it gives rise to others which seem insuperable. In the first place, it is improbable that all the various wild animals, of which the list is given above, could have at any time, or habitually lived together on the rock. The crags and caves may have been the resort of *Hyænas* and other predaceous animals, but the *Deer*, and other ruminants, the remains of which were numerous, could never have lived in the neighbourhood of these Carnivora. They would naturally have frequented the surrounding plains and forests, where they could have found food, shelter, and water, rather than scrags—dry and in great part barren. It is true that the predaceous animals might have carried there some portions of their prey, but had they done so, either the bones would have been devoured, or such as remained must inevitably have shown marks of the animals' teeth.

In the second place, no animal remains left on the surface could possibly have escaped destruction in the proximity of ground frequented by Hyænas and other Carnivora; or, supposing any bones had escaped, they would have decayed under ordinary atmospheric agencies, and exhibited more or less weathering; had they also been washed down by streams and amongst rocks, they would have been rolled and worn. But there is no evidence of weathering or wear, nor is it shown that the fissures are connected with old watercourses. The bones have clear and sharp fractured edges. Only in two instances it is mentioned that the bones present the appearance of being weathered and sun-cracked, and this seems to refer to those found with human remains and works of art, and not to the older breccia.

For these reasons I think this explanation cannot be accepted, and would again revert to the hypothesis of a submergence of the land. This affords a *vera causa* for the association of animals otherwise so little likely to be found together. It could only have been, as in the cases I have before named, a great and common danger, such as that of the gradual encroachment of the sea on the land, that could have so paralyzed their natural instincts as to have driven those various animals to flock together in search of a common place of refuge from a catastrophe which threatened all alike. Under such circumstances the Ruminants would naturally flee from the plain to the higher hills, and when these were isolated, as in this and the other cases I have named, whenever the waters rose above those hills, they were drowned and their limbs dispersed in the manner I have before described.

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* *Ante*, p. 937, and 'Quart. Journ. Geol. Soc.,' vol. 38, p. 336.

† 'Palæontological Memoirs,' vol. 3, p. 537.

The conclusions I would draw from the important paper of Sir A. RAMSAY and Professor JAMES GEIKIE, on the Geology of the Rock,[*] correspond with those formed on the palæontological evidence, though they differ from those of the authors of that paper; but I speak with reserve from want of personal knowledge of the ground. They describe the superficial deposits as consisting in ascending order of—

1. An older Limestone-agglomerate.
2. Bone breccias in caves and fissures.
3. Raised Beaches and calcareous marine sands.
4. Alameda and Catalan sands.
5. A later Limestone-agglomerate.

Beds 1 and 5 are true *breccias*, but the authors adopt the term *agglomerates* for these beds, to distinguish them from the bone-breccia of the caves and fissures (No. 2). They say: "The oldest (1) of all the superficial accumulations is the remarkable agglomerate or breccia which covers so large an area in the district of Buena Vista and Rosia, and in the neighbourhood of the South Barracks. Similar accumulations are met with in other parts of the Rock; but these, as we shall afterwards point out, belong to a later date (5). In caves and fissures, as is well known, breccias also occur, but these are distinguished from the others by the presence of abundant Mammalian remains." They proceed to say that this older breccia (*a′*, on the left of the sections, fig. 14) is quite unfossiliferous, and rests on the main limestone or on the overlying strata. For these reasons they thought it "better to restrict the term *breccia* to the true cave-and-fissure-accumulations, leaving the term *agglomerate* for the great superficial masses."

In the neighbourhood of Rosia and Buena Vista, this agglomerate or breccia "attains a thickness in some places of not less than 100 feet, and occasionally shows a well-marked dip, . . . as a rule, however, the mass is quite amorphous and devoid of stratification. The matrix is sometimes grey, sometimes reddish, and the included fragments (of the limestone) are almost invariably quite angular, no rounded waterworn stones being visible. The agglomerate is of all degrees of coarseness, the stone varying in size from mere grit up to blocks 12 feet and more in diameter;[†] and larger and smaller fragments are all rudely heaped together without the slightest reference to size or shape, so as to present an appearance as tumultuous and confused as that of a coarse volcanic agglomerate."

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[*] 'Quart. Journ. Geol. Soc.,' vol. 34, p. 505.

[†] They add, "Some of the larger blocks must weigh 20 or 30 tons at least."

This description might be applied almost word for word, to the "head" over the Raised Beaches of Brighton and Devon. There is the same absence of wear and the same local character of the component fragments; the same various degrees of coarseness and occasional inclusion of massive blocks; and the same occasional dip or appearance of bedding; to which may be added the same general rarity of organic remains. Bones, however, have been met with, though very rarely, but when that is the case the authors assign it to a later period. But even at Brighton the greater part of the Elephant-bed is unfossiliferous, and it is only occasionally that bones are found, whilst in Devon,* owing to various causes, they are almost unknown. In the fissure breccia likewise there are large portions (probably the largest) in which no bones are found. The presence or not of bones cannot therefore be considered a character of any fixed value, and alone is insufficient as a test of comparative age.

Messrs. RAMSAY and JAS. GEIKIE state, that the bone breccia of Rosia Bay occupies a "vertical fissure of erosion in the unfossiliferous *limestone-agglomerate,*" and hence that the latter must be of older date. But Major IMRIE,† who was there when the first excavations were going on, says, that it was a cave (with the ordinary cave bone-breccia) that had been "filled up with the concretion" (agglomerate), and Mr. SMITH merely says that it was a fissure in the face of the cliff. Unfortunately no sections are given to assist us in our conclusions. If this breccia (the older agglomerate) were, however, older than the Raised Beaches, we should expect to find portions of it *under* some of those beaches, but *none such are recorded.* On the contrary, a section given by the authors shows an agglomerate or breccia *overlying* a beach (see fig. 17). They explain this section by supposing this agglomerate (their No. 5) to be of later date than the agglomerate or breccia of Rosia and Buena Vista on the west side of the Rock, but the evidence of superposition is wanting, and it is difficult to suppose that the breccias on the east and west side of the Rock, shown in the two sections in fig. 14, are otherwise than contemporaneous.

The authors themselves say "there is no perfectly conclusive evidence to show that any of them (the agglomerates) are older than the marine platforms at Europa and lower levels," and again, "the agglomerates (No. 5) now referred to (those resting on marine deposits, or Raised Beaches, and eroded platforms on the east side of the Rock), are similar in most respects to the older accumulations (No. 1) in Buena Vista. Like them they are made up of angular fragments of limestone of all sizes up to blocks several yards in diameter, set in an earthy matrix which is either red or grey." Another section‡ (fig. 15) is described by them at p. 527, fig. 10. They consider that the agglomerate here overlying the marine sands (a littoral deposit) is newer than that of Buena Vista and Rosia. I do not, however, see sufficient reason for this conclusion, and would suggest another interpretation (see fig. 16).

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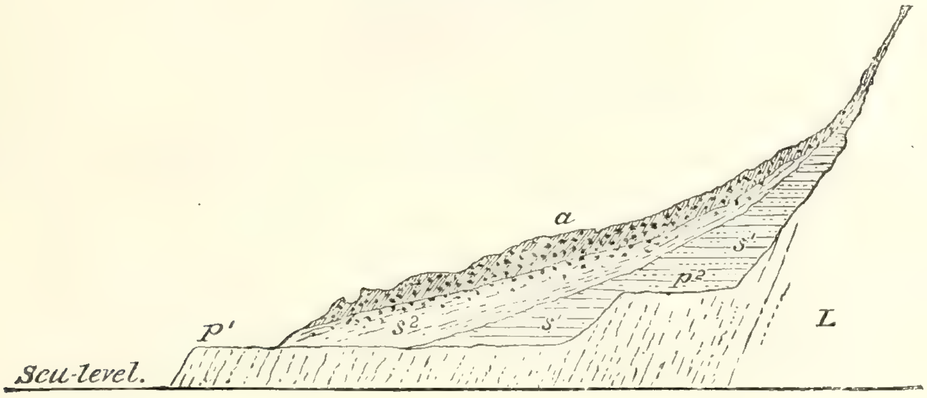
* I am not aware of any except the tooth of Horse, which I found at Hope's Nose, as mentioned in my first paper to the Geological Society.

† 'Trans. Roy. Soc. Edinb.,' vol. 4, p. 191; 1779.

‡ This is described somewhat more fully by Professor J. GEIKIE in 'Prehistoric Europe,' p. 326.

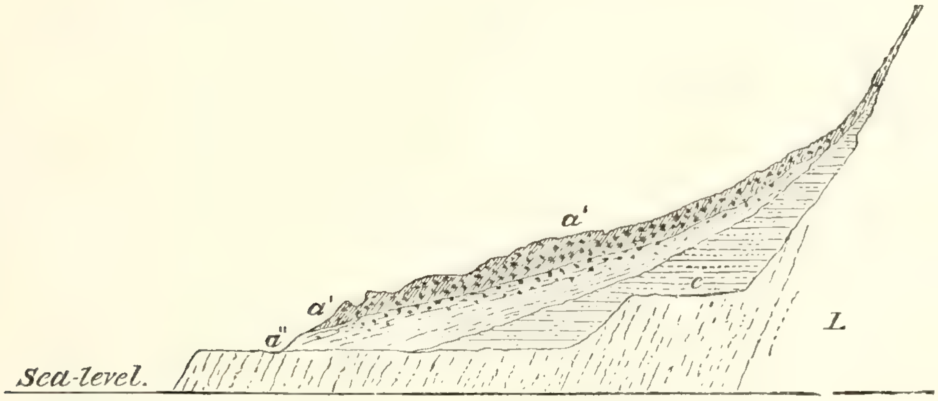
Fig. 15.—*Section from near Windmill Hill Barracks to the sea near Monkey's Cave-road* (Ramsay and Geikie).



L. Limestone.
p^1. Platform of marine erosion 160 feet above the sea.
p^2. Platform of marine erosion 260 feet above the sea.
s^1. Calcareous sandstone, horizontally bedded, marine.
s^2. Calcareous sandstone dipping away from cliffs, the weathered and rearranged upper surface of s'.
a. Unfossiliferous limestone agglomerate (the later agglomerate).

Presuming, therefore, that the surface breccia on both sides of the Rock is all of the same age, my reading of this section would be as under.

Fig. 16.—*Alternative reading of Section No.* 15.



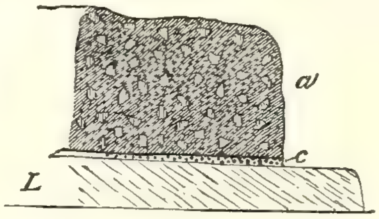
Rubble-drift:
- a'. Coarse limestone detritus.
- a''. Composed mainly of the débris of the marine sands, *c*, with fragments of limestone in its upper part.

c. Marine sands = Raised Beach.

I take it that during upheaval the first effect of the effluent waters has been to denude the marine sands *c*, forming, with the limestone débris from higher up the slopes, a finer mixed detrital bed, which should be connected with the breccia or Rubble-drift a', and not with *c*.

In another section (their fig. 12), the agglomerate is again shown directly overlying the shelly grit, while in fig. 17 (their fig. 11) it rests on a remnant of a Raised Beach, lying on a platform of marine erosion.

Fig. 17.—*Section at Princes Lines* (the sketch only is after Ramsay and Geikie).



a′, Limestone Rubble-drift or *head* (Newer agglomerate, No. 5, of the authors).
c, Raised Beach (Gravelly conglomerate of the authors).

Why this agglomerate is taken to be more recent than that in Rosia Bay, is that it contains fragments of Mammalian bones, which the other is supposed not to do. But it seems to me rather a reason to prove that that breccia (No. 1) is occasionally ossiferous, and also that it is newer than the Raised Beach. The uncertainty of the occurrence of bones is, as before mentioned, one of the features of the Rubble-drift. Mr. Smith speaks only of one breccia with remains of the extinct Mammalia, though he considered that there was a re-constructed breccia of modern date formed by the washings and débris of the older breccia, which it consequently closely resembles, that resemblance being aided by the continued deposition of carbonate of lime and cementation of the mass. This differs, however, from the older cave breccia and from the agglomerates of the authors in containing the remains of later Man and his handiwork, and of recent animals. The distinction has been well shown by Mr. Busk.*

With respect to the origin of the breccia, Messrs. Ramsay and Geikie attribute their older agglomerate on the western slopes to severe cold at the time "when the Rock had a wider area of low ground at its base, and when, so far as we know, it was not tenanted by land animals." They justly, however, observe, "something more than the mere action of strong frost is needed to account for the presence of the widespread and massive agglomerate of Rosia and Buena Vista." How, then, is the transport of large blocks and the accumulation of the débris in so large a volume over the small incline of 8° to 9° (in places not more than 2° to 3°) for a distance of at least 550 yards from the base of the main slope, to be accounted for? Winter torrents will not suffice. A cone of dejection spreads out fan-shaped, and would not extend so far beyond the foot of the main slope, and the débris would show symptoms of wear and rolling, as well as a gradation from coarse to finer materials. "But in the great limestone-agglomerates, not only are the stones all more or less sharply angular and arranged generally without any reference to size or figure, but the agglomerate at the foot of the Rock is not sensibly coarser than that which occurs at Rosia Bay, a distance of 500 or 600 yards away." It is, therefore, impossible to look upon the rubble as a talus, or as thrown out by cones of dejection.

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* 'Trans. Inter. Congr. Prehis. Archæology,' Norwich Meeting, 1869.

From these and other considerations, the authors conclude that there seems to be in this agglomerate "evidence not only of the former action of frosts severe enough to wedge out large blocks of limestone, but also of more or less sudden meltings of thick snow, such as would saturate heavy trains of débris, and cause them to move *en masse* down the steep slopes and over the lower grounds beyond." This might be a true cause if the gathering ground had been larger and higher and the terminal slope less, but with so limited a gathering ground I do not see that it would be possible for snow, even if frozen, to have accumulated in sufficient quantity as to be able to propel so vast a mass of débris, with blocks 12 feet in diameter, over the lower slope, which dips at the small angle of 8° to 9°.* Even were it possible, would not the friction have been such as to cause the blocks to lose their sharp angles? I have shown, also, that this projection of débris with blocks, occurs even where the gathering ground was of much smaller extent, the height less, and a snow-slide improbable. The later agglomerate they suppose to have been formed under similar cold climatic conditions after a long interval, during which the Rock had become inhabited, and the cave and fissure breccias had accumulated under the influence of a genial interglacial climate.

The view I would take respecting the origin of the breccia may be briefly stated. First, however, with regard to the remarkable series of marine terraces, described by Mr. SMITH, of Jordan Hill, at various levels up to 600 feet. Of these we have very insufficient information. We only know from his general statement that in the beach at Europa Point, on the level of 70 feet, he collected nearly 100 species of recent shells said to be such as now inhabit the Mediterranean, but no list is given. At a higher level he makes mention of an oyster bed and of *Pectunculi*, and then of a shelly bed, and of a bed with recent shells at the 600 feet level. But, with the exception of *Pecten maximus* and *Patella ferruginea*, no species are named. Although we cannot assign a date to these several terraces, it is probable that the one at Europa Point is of the same age as the one in the Channel. The others may go back to earlier Quaternary or Tertiary times and correspond with similar traces at high levels in different parts of the Mediterranean coasts. To determine their age we must wait for a better knowledge of their Molluscan fauna.

On the emergence of the Rock, after the formation of these beaches, it became inhabited by wild animals, but there is nothing to show that this took place at an earlier date than the later Glacial or Post-glacial period, for the Mammalia are—with a few exceptions, due to the more southern situation of the Rock—of the same species as are common in the more northern caves. The predaceous animals found ready shelter in the caves and amongst the crags of the Rock, and there they may have carried the remains of the ruminants found in the older bone caves; but it is difficult to suppose that the numerous *Red Deer*, *Fallow Deer*, *Oxen*, and the vast numbers of *Ibex* spoken of by FALCONER—the bones of which show conclusively that they had not been devoured by the Carnivora—could have inhabited the Rock at the same time. When, however, the submergence took place, the animals living on the surrounding plains sought refuge on the Rock in face of this unprecedented danger, and there shared in the common doom. Such of their remains as fell on the submerged surface of the Rock were subsequently carried down with the local detritus and their fragments incorporated, as the Rock emerged from the waters, here and there in the general mass of breccia.

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* A more detailed section is given in 'Prehistoric Europe,' p. 217.

The highest point on the west side of the Rock which the breccia is shown to reach is about 500 feet, and on the east side about 700 feet. But, considering the volume of detritus and the denuded state of the upper portion of the Rock, we may presume that the greater part, if not the whole of the Rock, was submerged. On its upheaval divergent currents swept down, in the manner I have described in speaking of the "head" at Brighton and Sangatte, the great body of limestone débris disintegrated by the cold of the earlier glacial period, and propelled it *en masse* over the flatter ground at the base of the long slopes. With the necessary velocity blocks of 12 or more feet in diameter would be easily moved under such a volume of water, while the friction would be comparatively small. Raised above the sea level the surface waters, after percolation through the calcareous rocks and débris, would rapidly effect the consolidation of the brecciated rubble, as on the slopes of Mont Genay, the coast of Mentone, and in the fissures of Nice and Cette, of which this deposit at Gibraltar is a counterpart on a larger scale. After the final rise of the land the caves and open spaces left in the fissures became the resort of the more recent wild animals and of Neolithic Man.

On this view the superficial deposits would be comprised in three stages or epochs. First and oldest, that of the later Raised Beaches and Quaternary Bone-caves. Second, that of the Breccia on the slopes and in the Ossiferous fissures (or Rubble-drifts). Third, and latest, that of the Neolithic caves.

I have dwelt thus long on Gibraltar not only on account of the extensive scale on which the phenomena are exhibited, but also because the other explanations of the facts being by geologists of high authority and of competent local experience, I did not deem it prudent to advance further on what is still debateable ground, and leave, without an attempt to capture, so important a position in the rear. We can pass more rapidly over Italy and some of the Mediterranean Islands.

Corsica.—It would seem that there are here several Raised Beaches some of which attain a considerable height.* The lowest of them is from 15 to 20 feet above the sea-level, and contains species of shells now living on the coast. To the north of Bastia stretches the promontory of Cape Corso, formed by a mountain range from 2000 to 3000 feet high and intersected by transverse valleys. Fringing this promontory on the eastern side, are numerous quarries of limestone of Cretaceous

* M. D. HOLLANDE, 'Bull. Soc. Géol. de France,' 3me sér., vol. 4, p. 86, 1875.

age, traversed by a large number of ossiferous fissures, 250 to 450 feet above sea-level, that have been described by M. A. Locard.[*] These fissures are generally crooked and narrow, and extend the whole height of the quarries. When the sides of rock come near together at the top, the fissures are sometimes empty, but more generally they are filled with an ochreous and ferruginous earth or loam containing sharp angular fragments of the adjacent rocks, some of large size, with bones and land shells, heaped together without any order. In some places the breccia is concreted and forms a hard rock. M. Locard gives the following list of the fauna that have been found in the breccia :—

Homo.
Lagomys Corsicanus, Cuv.
Myoxus glis, Schreber.
Mus sylvaticus, L.
Canis vulpes, L.
Lacerta.
Ovis musimon, L.
Lepus.
Testudo.
Perdix.

The predominant remains are those of the *Lagomys*, those of the other Rodents are also numerous; of Birds only a few bones have been found. All the bones are much broken and there are no large ones. A very few fragmentary bones of *Man* have been found in association with the bones of *Lagomys*. The condyle of the left upper maxillary, and a fragment of the sphenoid bone were determined by Dr. Lortet. A few species of marine shells were discovered, but M. Locard considers that these were introduced by later Man (?). He further gives a list of thirteen species of *Helix*, three of *Zonites*, two of *Pupa*, and one of *Clausilia*. They are all species now living in Corsica, except one referred to as an Egyptian species.

As the *Lagomys* still survives in Siberia, and its remains are found elsewhere associated with the Mammoth and Woolly Rhinoceros, M. Locard concludes that these ossiferous fissures are of late glacial age, and that Corsica was then separated from the mainland.

Sardinia.—The work of General De la Marmora[†] gives many interesting details of the Quaternary deposits of this island. He shows that at several places on the coast there is a Raised Beach (or littoral deposit) like that of Gibraltar, composed of sand and grit (Grès Quaternaire), with occasional layers of pebbles and shells. The list is useful as affording a term of comparison with the Raised-Beach shells of the English Channel. Like them they are, with few exceptions, all of species now living in the neighbouring seas. There is, therefore, no reason why the Sardinian beach should not also be of late glacial age, although the stratigraphical evidence is not so clear as in the other case.

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[*] 'Arch. Mus. d'Hist. Nat. de Lyon,' vol. 1, p. 37, 1873; and 'Bull. Soc. Géol. de France,' 3me sér., vol. 1, p. 232.

[†] 'Voyage en Sardaigne,' Paris, 1857.

The Raised-Beach Shells of Sardinia.

Cerithium vulgatum, Br.
Conus Mediterraneus, Brüg.
Murex plicatus, Broc.
,, *Sedgwickii*, M.
,, *trunculus*, L.
**Nassa reticulata*, F.
Patella Lamarckii, Lk.
,, *scutellaris*, Lk.
Tritonum nodiferum, Lk.
Turbo rugosus, L.
Vermelia triquetra, L.

Arca Noœ, L.
**Cardium edule*, L.
,, *papillosum*, Poli.
,, *rusticum*, L.
* ,, *tuberculatum*, L.
Donax trunculus, L.
Lucina fragilis, Ph.
,, *lactea*, Lk.
Lutraria rugosa.
**Mactra stultorum*, L.
**Mytilus edulis*, L.
**Ostrea edulis*, L.
,, *lamellosa*, Broc.
,, *plicatula*, L.
**Pectunculus glycimeris*, L.
,, *pilosus*, Lk.
Petricola fragilis, L.
,, *lithophaga*, Retz.
Pinna rudis, L.
,, *tetragona*, Broc.
**Scrobicula piperata.*
Solen vagina, Lk.
Spondylus gardropus, L.
Tellina planata, L.
* *Venus chione*, L.
,, *decussata.*
* ,, *gallina*, L.
,, *multilamella*, Lk.
,, *verrucosa*, L.

Those with a star are found also in the glacial deposits of England.

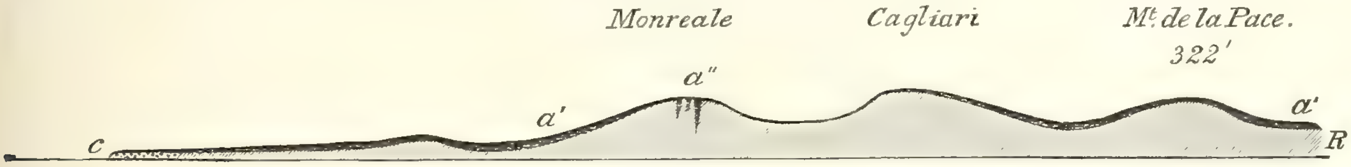
In some cases the shells are whole and with both valves united, but more generally they are broken, and with the colour often preserved as in the Channel beaches. *Ostrea* and *Mytilus* sometimes form entire beds. On the coast the beach is generally about 20 feet above the sea-level, though sometimes less, and is of small thickness, but it is said that as it extends inland it increases in height and thickness, attaining at Fontana, one mile from the sea, a height of 90 feet, and at Marmorata, two miles inland, a thickness of 60 feet, whilst at Alghero it is said to rise to a height of 100 metres.[†] At one place the so-called beach was found three miles inland. This deposit is, as at Gibraltar, generally consolidated by a calcareous cement, and has been quarried as a building stone from time immemorial. The beds are generally nearly horizontal, and sometimes show oblique lamination.

On the summit of the hill of Monreale, near Cagliari, there are fissures (a'', fig. 18), filled with breccia containing an enormous quantity of bones of *Lagomys*, *Myoxus*, *Arvicola*, and other small animals, but without any of the larger extinct Mammalia.

[†] I should imagine these to be not one but separate sea-levels, like those of Gibraltar.

The breccia extends down the slope of the hill on the seaward side, where it seems to impinge on a Raised Beach, as in the following diagram, but the author's description of them is not very clear.

Fig. 18.—*Generalized Section near Cagliari (J.P.).*



a′, Breccia; *a″*, Ossiferous fissures; *c*, Raised Beach.

The breccia on the slopes contains no bones, but land shells have been found in it. In the plain, to the west of Cagliari, the rubble attains a thickness of several metres and contains blocks derived from the neighbouring higher hills inland, in a matrix of cemented red earth.

On the whole, these Quaternary beds accord perfectly with those of Corsica, and also in their essential features with those we have noticed on the Mediterranean coast of France. The projection of the angular local débris to a considerable distance (over the flat land towards the shore) from the base of the hill slopes, is likewise a feature common with the Rubble-drift.

Minorca and Majorca.—General De la Marmora describes similar recent marine beds skirting the coasts of those islands at heights of 30 to 100 feet, and M. Jules Haime* names some of the shells, but further details are wanting of the coast deposits in these islands; they seem identical with those of Sardinia.

Italy.—The phenomena on the coast of Italy are in close agreement with those on other parts of the Mediterranean shores, both with regard to the position of a low level beach and to the position, condition, and contents of the ossiferous fissures. General De la Marmora notes the occurrence of *Lithodomus* perforations on the rocks near Genoa at a height of 82 feet above sea level,† and of a consolidated Raised Beach about 25 feet above sea level in the neighbourhood of Leghorn.‡ At Ischia, also, are well-known marine beds at different, and mostly, high levels. In one of them, the fact was noted, that it abounds with Foraminifera of a species which M. Ernest Van den Broeck found to indicate closer relations with the northern oceanic fauna than with that of the warmer Mediterranean.§

There is also a concreted Raised Beach containing recent shells at Capri, about 30 feet above the sea-level. Other former higher sea levels are indicated by zones of rock with *Lithodomus* perforations.

Osseous breccia is said to occur in several places near Verona, and Vicenza, and in Tuscany. Near Pisa, a red breccia, not distinguishable from that of Gibraltar, lies in vertical fissures of a light coloured limestone. It contains, like the other, remains of Carnivora, Ruminants (especially *Deer*), with species of *Helix* and *Cyclostoma elegans.* In the south of Italy, ossiferous breccias, with remains of various Ruminants, occur in fissures of the limestone at Cape Palinura.

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* 'Bull. Soc. Géol. de France,' 2nd ser., vol. 12, p. 742.

† 'La Sardaigne,' vol. 2, p. 345.

‡ In places it is quarried as a building stone, termed *Panchina*.

§ A. W. Waters, 'Quart. Journ. Geol. Soc.,' vol. 34, p. 196.

Professor CAPELLINI gives a more special account of a bone-breccia discovered a few years since near Spezzia.* On either side of the bay are promontories of Jurassic limestone. The headland of the one on the south side forms, as at Nice, an almost isolated hill. In the course of constructing a fort on this point, a fissure or cavity in the rock was met with at a height of 246 feet (75 metres) above the level of the sea. It was filled with a red ferruginous breccia, cemented by calcite, in which were a great number of bones belonging to *Hippopotamus amphibius* (*H. major*), and *Cervus capreolus,* chiefly the former. A great number of the bones were however dispersed and lost before the attention of Professor CAPELLINI was directed to the spot. The discovery is interesting, as it points to the existence of a fauna similar to that on the Sicilian coast.

Sicily.—On the Atlantic coast, where the successive changes of level have been due to earth movements extending over wide areas, there has been little to interfere with the comparative uniformity in the level of the beaches. But in the Mediterranean area volcanic action has introduced a disturbing element, which no longer allows of uniformity of level being accepted as contributory evidence of contemporaneity of upheaval; and where we have no other evidence, altitude alone cannot be taken as a proof of age. This renders the age of some of the beaches in Sardinia and Corsica uncertain.

In Sicily there is, in fact, proof of considerable uplifts in recent times. According to Signor GEMMELARO,† there are on the coast of Catania several zones of sea-levels on lava-streams of known date. In one case, the lava-stream of the year 1169 has "adhering to it a coarse shelly sand, in which may be distinguished, almost in a normal position, *Cypræa lucida*" and other shells, at a height of 6 to 7 feet above the sea-level. At another spot he found blocks of lava encrusted by *Serpulæ,* at a height of 45 feet.‡ This he considered to be the greatest elevation the coast had undergone during the present period.

Of bone caves there is one with traces of a Raised Beach 70 feet above sea-level, two miles north of Syracuse. The cave contained remains of *Elephant, Hippopotamus,* &c. A bone breccia, which Dr. CHRISTIE classed with that of San Ciro, extended from the entrance to some distance at the base of the cliff.

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* "Breccia Ossifera della Caverna di S. Teresa," 'Mem. dell' Accad. di Scienze dell. Ist. di Bologna,' ser. 3, vol. 10, 1879.

† 'Quart. Journ. Geol. Soc.,' vol. 14, p. 504.

‡ The age of the lava at this spot is not mentioned.

But the caves which more especially bear upon this inquiry are those in the neighbourhood of Palermo that are connected with an ossiferous breccia of a very remarkable and unique character, and which we must therefore describe at greater length. This breccia has been associated with the breccia of the ordinary older bone caves, but, though occurring in front and close to such caves, I think the association accidental, and that the two are independent. Without this disconnection its origin appears unintelligible.

Palermo stands in a plain, encircled, at a distance of about 2 to 4 miles, by an amphitheatre of hills from 2000 to 3000 feet high (fig. 21). Pliocene strata rise very gradually from the shore to the height of about 200 feet, when they abut against highly inclined limestone strata of Cretaceous and Jurassic age, which present a precipitous front towards the plain, while in places a thick accumulation of limestone breccia lies at their base.

There are several caves in the limestone escarpment a little above the line of junction with the Pliocene plain, but the one which most particularly claims attention, from the extraordinary quantity of *Hippopotamus bones* found in connection with it, is that of San Ciro, or Mardolce, situated about 2 miles to the south-east of Palermo, at the foot of Monte Grifone. This cave is 130 feet long, 10 feet wide at the entrance, 30 feet in centre, and 21 feet high, measured from the surface of the cave deposits. It is situated near the base of the escarpment, with ground sloping from its entrance down to the church of San Ciro, a distance of 266 feet (see fig. 19). But the breccia extends apparently beyond this. The cave was described by the Abbate D. SCINÀ* in 1830, and by Dr. TURNBULL CHRISTIE in 1831.† It was visited afterwards by Dr. FALCONER, who included it in his description of the Maccagnone Cave.‡ Owing to the way in which the cave deposits inside and out had been ransacked for bones, first by the peasants and afterwards by order, no systematic exploration was then possible.

When first discovered many hundred quintals of bones were dug out by the peasants. According to SCINÀ the breccia was *crammed with bones, so fresh* that they were cut into ornaments and polished, and that when burnt they gave out ammoniacal vapours. The quantity, however, was so great, that when afterwards exploited for commercial purposes, 20 tons were shipped in the first six months to Marseilles and England, where it is said they were used for the manufacture of lamp black§; if this were the case they could have lost very little of their animal matter. The bones were mostly those of *Hippopotami*, with a few only of *Deer*, *Ox*, and *Elephant*. In a weight of nearly 1½ tons, all except six were bones of two species of

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* 'Rapporto sulle Ossa Fossili di Mardolce,' Palermo, 1831.

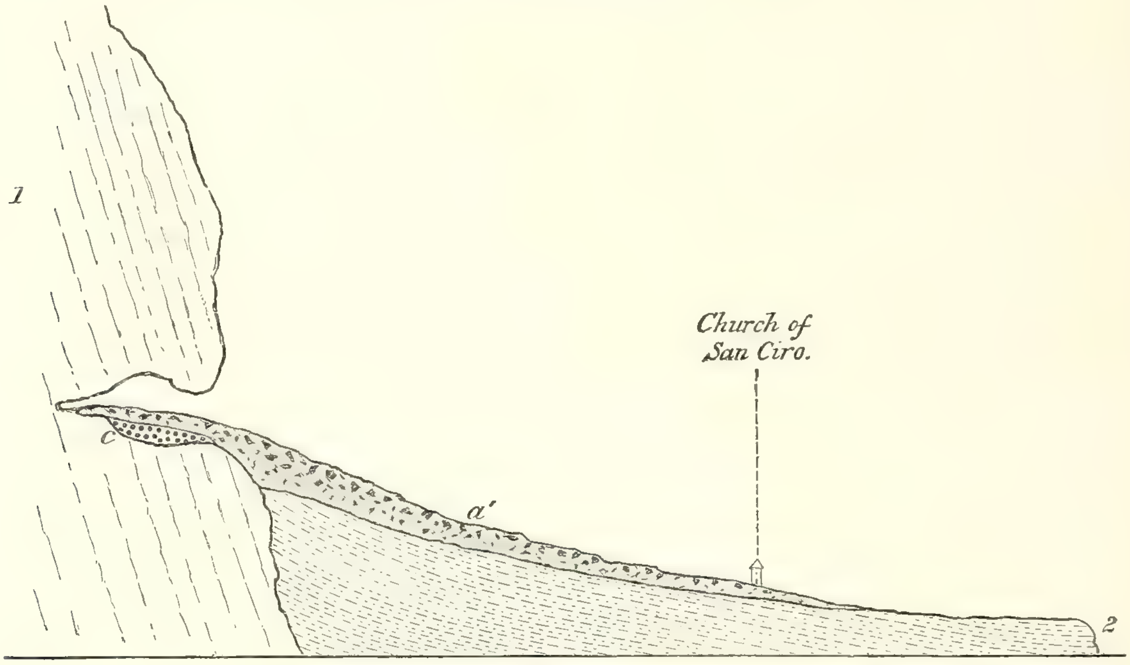
† "On the Newer Deposits of Sicily." 'Phil. Mag.' for October, 1831, p. 1.

‡ 'Quart. Journ. Geol. Soc.,' vol. 16, p. 99, 1860; and 'Palæontological Memoirs,' vol. 2, p. 543, 1868.

§ Or more probably of animal charcoal for the sugar factories. In the rubble-drift at Chilton, near Oxford, the bones likewise retained a large proportion (above 17 per cent.) of animal matter.

Hippopotamus; amongst them were 300 astragali of that animal. Scinà also collected 76 astragali for the Museum of Palermo, together with 14 jaws with teeth, besides numerous single teeth,[*] and specimens of almost all the bones of the body belonging to *animals of all ages down to the fœtus.* The following diagram section (fig. 19) gives the chief features, as described by the above writers. It is not drawn to any scale.

Fig. 19.—*Restored diagram section of the Cave of San Ciro (J.P.).*



1. Hippurite limestone. 2. Pliocene sands and marls.

a′. Bone breccia { Red clay, with angular blocks of limestone, passing down into reddish clay, with smaller fragments and some pebbles of limestone and quartz. (The latter may be derived from the sea-bed, *c*) . . . } 20 feet ?

c. Sand, with sea-shells and corals (?) of recent species. The walls to some height above *c* were perforated by *Lithodomi* † . . . . 1 to 2 „

With respect to the position and condition of the bones, the evidence is conflicting. Scinà says that the bones were without order, and those of the different animals were mixed together, that they were broken, shattered, and dispersed in fragments. Speaking of the Elephant remains, he remarks that no entire tusks were found, and that the fangs of the teeth and epiphyses of the vertebræ were gone, whilst some of the bones were so reduced by wear as to be scarcely recognisable. Two teeth of

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[*] These were still so numerous on the surface of the ground between the cave and the church of San Ciro, at the time of Dr. Falconer's visit, that the women and children of the neighbourhood picked up the teeth and brought them to him in handfuls. Whether these were ploughed up, or had been rejected at the time of the diggings, is, however, uncertain. They were not worn.

† It is possible that there may have been a distinct bone-breccia of the cave age between *c* and *a′*.

Elephant had been worn down to the state of pebbles. It is evident, however, that these are the exceptions; they may have been derived from the old beach- or cave-beds.* CHRISTIE merely says that the breccia contains "a prodigious number of fragments of bones with some rolled pieces and blocks of limestone." Dr. FALCONER makes no allusion to the wear of the bones, and the specimens deposited in the College of Surgeons have neither been worn or gnawed. Speaking also of the analogous breccia at the Grotta di Maccagnone, he says that the bones were all broken and splintered, and that *none of them bore marks of gnawing.*

To obtain the bones, a cutting 6½ feet wide, and apparently about 20 feet deep, was made from the mouth of the cave to the slope outside, but no particulars of the section have been preserved. In the interior of the cave the upper part of the breccia was loose and pulverulent,† and the bones light and adhering to the tongue, but the greater part of it, both in the cave and outside on the slope, was extremely hard.‡ SCINÀ, however, says it was more compact in the cave, and that the bones in it were petrified and solid, carbonate of lime being the cementing and solidifying material. The same writer speaks of the breccia inside and outside the cave as one mass, whereas CHRISTIE thought there was a want of continuity between the two.

Looked upon as a whole, the mass of débris which, near the entrance of the cave had a thickness of from 20 to 30 feet, closely resembles in its general character the Rubble-drift on slopes or over some of our Raised Beaches. CHRISTIE says that it "has some appearance of being divided into strata as if deposited under water," and here, as at Brighton and Sangatte, the coarsest bed lies on the top. According to the same writer, a similar calcareous conglomerate or breccia attains at the foot of Monte Pellegrino, a thickness of 40 to 50 feet, and contains blocks many yards in circumference. In this respect it resembles the breccia on the slopes at Gibraltar. CHRISTIE also remarks that "This bone-deposit has more analogy to the bone-breccias that occur in various parts along the shores of the Mediterranean than to the bone-caves of the more southern parts of Europe."

The traces of old occupation of the cave of San Ciro by predaceous animals is more obscure than in the case of the Gower caves. If an old cave floor existed, it has been swamped under the great mass of breccia with the Hippopotamus bones. That there were, however, dens of Hyæna fringing the declivities of that range of hills is certain. One was discovered by Baron ANCA DI MANGALAVITI § between Palermo and Messina, in which a large number of bones and Coprolites of Carnivora, with Deer horns, and astragali of other animals, bearing the marks of gnawing, have been found. The Baron also points out that, whereas in these caves bones of Carnivora and Deer are common, in the caves of San Ciro and Maccagnone such bones are scarce, whilst those of Hippopotami abounded.

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* Nine Elephants' teeth were found *in the cave.* It looks as though the cave beds and the angular rubble (Rubble-drift) formed two distinct deposits, as in some of the Belgian caves (see fig. 12).

† Apparently there was no stalagmite.

‡ It was quarried outside as a building stone.

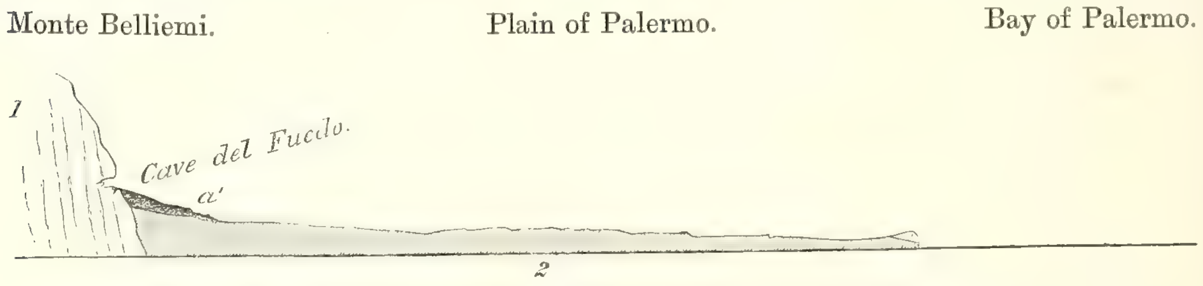
§ 'Bull. Soc. Géol. de France,' 2nd ser., vol. 17, p. 680.

The bones found in the San Ciro breccia belong, according to Falconer, to—

Hippopotamus major.
,, *Pentlandi.*
Elephas antiquus.
Felis (a large sp.).
Canis.
Sus.
Bos.
Cervus.
Ursus.

There are two other caves about four miles westward of Palermo where this breccia has been observed. Both are situated at the foot of Monte Belliemi, and at a height of about 100 feet above that of San Ciro. In neither of them was there any evidence of marine action. In that of Ben Fratelli, which is 320 feet above sea-level, the breccia occurs both inside and outside the cave, whilst in the other, or that of del Fuedo, 332 feet above sea-level and distant one-third of a mile from the former, the breccia is confined to the *exterior* of the cave (fig. 20). Scinà states that both the bones and soil here gave out ammoniacal vapours.

Fig. 20.—*Section from the slope of Monte Belliemi to the Bay of Palermo* (Christie).



1. Limestone rocks.
2. Pliocene strata.

a' Bone breccia (= Rubble-drift ?), outside the cave.
: : : Gravel (of uncertain age).

The upper beds of the cave of Maccagnone, near Carini, which is situated in a similar amphitheatre of hills, have been fully described by Dr. Falconer, but there the same uncertainty exists with respect to the character of the lower beds as at San Ciro. There is also the same profusion of Hippopotamus bones in the breccia on the slope in front of the cave. Dr. Falconer says that he "dug up an enormous quantity of these remains within an area of 12 or 14 feet square." Amongst them were a very large number of astragali. There, likewise, the "bone breccia was strewed over with huge blocks of limestone which had fallen since its deposition."

With regard to the origin of this extraordinary accumulation of Hippopotamus bones, we may at once discard the idea that they were brought to their present position by predaceous animals, as were the bones in the breccia of true bone-caves. In these, the evidence of their having served for food is unmistakable, as traces of the animal's teeth are conspicuously apparent upon the bones, but the bones in this breccia show no traces of gnawing. Besides, in one case, the breccia is confined entirely to the outside of the cave, where it is not probable that the Hippopotamus remains would have escaped destruction by the Hyænas which frequented the same district. Nor could the bones have been washed and transported by river- or sea-action, for they are mostly splintered and in fragments, and are neither worn or rolled.* It is evident also that the animals must have died on or near the spot where their bones are found, and that their accumulation is due to causes other than the foregoing.

It was suggested by Dr. FALCONER, that the bones were those of successive generations of Hippopotami, which went there to die. But this is not the habit of the animal. Sir SAMUEL BAKER, in his many years' experiences in the Soudan, and on the borders of Abyssinia, where Hippopotami live in large herds, makes no mention of such an instinct,† nor does he allude to any circumstances which could possibly, under the *ordinary* conditions of life, have led to such local accumulation of their bones. Under ordinary circumstances, their dead bodies, like those of other wild animals, are devoured by Carnivora, or rapidly disappear from the surface under atmospheric and other agencies. Even supposing the accumulation of the bones to have been effected in some way by successive additions, the usual weathering would have caused great inequality in their state of preservation, whereas, as a rule, all the bones in the breccia seem to have been in the same almost fresh state, with the exception of the specimens near the surface, which have lost more of their animal matter. Besides, as the bones are those of animals *of all ages*—including the unborn—this fact is, I think, conclusive against this explanation.

The circumstances, therefore, which led to these remarkable accumulations of the remains of the Hippopotami must have been *extraordinary*, and I see no hypothesis which meets the case, so well as the one that I have suggested to account for the bones of Mammalia in the Rubble-drift and in the ossiferous fissures, though the local conditions in this case are peculiar.

On the submergence of the Sicilian area, the wild animals of the plains would, as in the case of Santenay, Cette, and Gibraltar, be driven to seek refuge on the nearest adjacent high ground and hills. In the instance before us, the animals must have fled to the amphitheatre of hills which encircle the plain of Palermo on all sides except the sea, and on the slopes of which the Cave of San Ciro and the others are located. As the waters rose, the area of this plain became more and more circumscribed (fig. 21), and retreat more and more impossible, except through a few rare passes in the range of hills, until, at last, the animals were driven together at the base of the hills, where they were stopped by mural precipices impassable to the larger and heavier animals, though some of the more active and agile Ruminants and Carnivores may have, and, judging by the rarity of their remains, probably did escape to the

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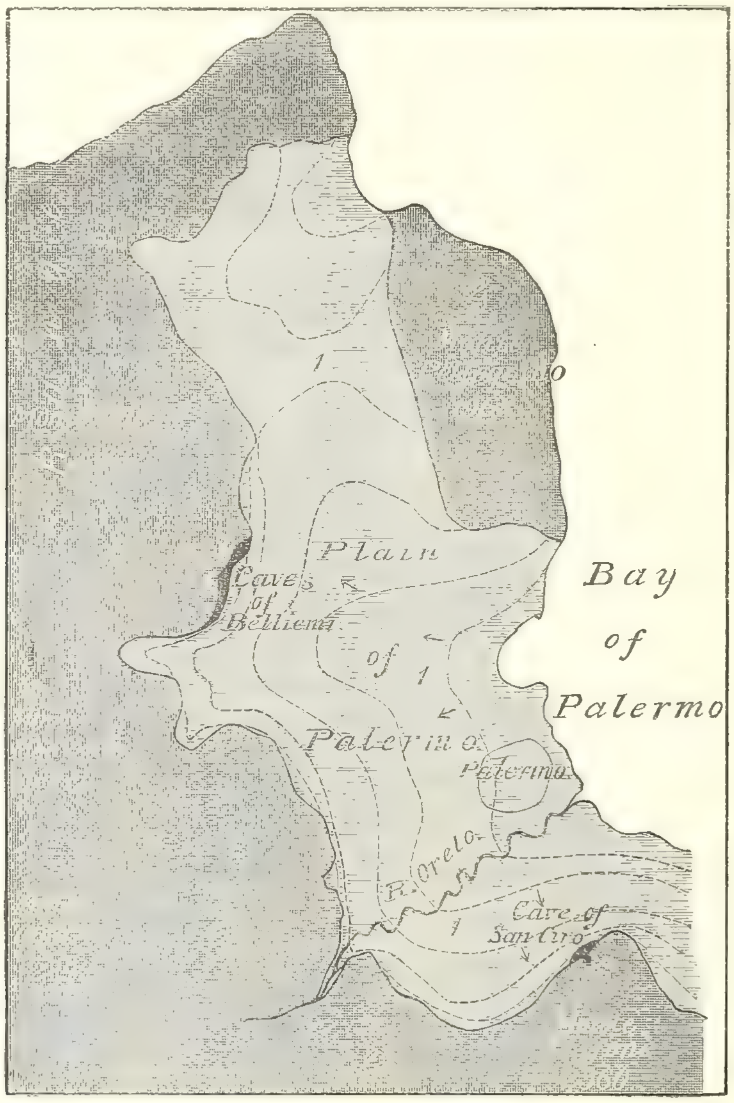
* SCINÀ was of a different opinion. He thought that the bones had been transported from a distance—drifted in by marine currents. But, in that case, all the bones together with the limestone fragments should have suffered equally, whereas, it is evident from his remarks, as well as those of CHRISTIE, that the rolled fragments constitute the exception. These may have been formed when the sea entered the cave (*c*, fig. 19), and caught up afterwards in this Rubble-drift as were the beach pebbles in the *head* at Sangatte and La Motte. Or weathering by the percolation of water may have had something to do with the removal of the angles of the limestone fragments and of some of the bones.

† Mr. HUDSON, however, mentions one instance of an animal resorting to a given spot to die. It is that of the Huanaco: 'The Naturalist in La Plata,' p. 318, 1892. This was also noticed by DARWIN.

Fig. 21.—*The Plain of Palermo with its Amphitheatre of Hills.*



▬ Ossiferous breccia. × Caves.

1. Pliocene strata (with a narrow belt of Eocene beds in places).
2. Cretaceous and Jurassic strata.

The boundary line between 1 and 2 gives approximately a contour line of from 150 to 200 feet above sea-level. The slope thence to the sea is very gradual. Above the Pliocene plain, rocks of Hippurite limestone rise abruptly, forming steep cliffs and mural precipices, with breaks in the direction of Monreale and in the course of the River Oreto. The dotted lines are supposed to represent roughly the portions of the plain gradually occupied by the advance of the sea. These are merely for illustration and are ideal, but the boundary lines and heights, given in feet, are taken from the Geological Survey Map of Sicily.

mountains behind. Retreat entirely cut off by projecting promontories on either side, the only paths yet open to the imprisoned herds were those that led to the caves, which were a little above the general level of the plain. Hither the animals must have thronged in vast multitudes, crushing into the caves and swarming over the ground at their entrance, where they were eventually overtaken by the waters and destroyed, and, as their bodies decayed,* a confused mass of their remains were left and scattered on and near the spot where they had finally congregated.

For reasons before given, the land could not have remained long submerged. As it rose intermittently from beneath the waters, our supposition is that the rocky débris on the sides of the hills was hurled down by the effluent waters on to the pile of bones below, breaking them into fragments, and forming, together with them, the heterogeneous mass of bones and rubble constituting the breccia. The last more rapid uplift, the effects of which are so frequently seen in many sections of the *head*, brought down the larger blocks of rock that now lie on the top of the whole. SCINÀ, an independent witness, inferred from the character of the rock fragments, and from the red clay in which they are imbedded—and which comes from decomposed rock surfaces on the hills above—that, in the case of the Belliemi breccia, both the detritus and the bones had been washed down from Monte Belliemi. All this must have been effected in a space of time comparatively so short, that, though the bodies of the animals decayed, the bones underwent but little change, nor, encased as they became in an almost impermeable breccia, has the change they have since undergone been great.

Thus there is, in all the essential conditions, a close agreement between this Sicilian breccia and the Rubble-drift of the south of England, as likewise with the rubble on the slopes of Mont Genay, of the Rock of Gibraltar, and of other places mentioned in the preceding pages. In all, the débris consists strictly of local materials; the fragments are angular and sharp; the bones are mostly in fragments, and are neither gnawed nor worn; and the faunal remains are those alone of a land surface, and of species such as then were to be found in the district.† This rubble, also, forms in all these cases the last of the drift beds. The only apparent difference arises from the circumstance that, in the Sicilian area, the geographical configuration was that of a land-locked bay with many minor bays or embrasures in the front of the hill-range, so that, as the waters rose, the animals of the plain were driven together, as in a seine, into those bays, where, as a last resource, they sought shelter under the mural precipices and in the more accessible caves. As these precipices were nearly vertical, they formed, as the land rose again, a partial protection from the effluent currents, which otherwise might have carried the débris to a greater distance outwards. Under no other circumstance that I can conceive could the animal remains have been massed as they are at the foot of the escarpments encircling the plain of Palermo.

It may be asked how could large herds of Hippopotami have existed in so limited a plain as that of Palermo. It needed then to have had much greater extent and larger rivers. I have shown that the present height of the Raised Beaches on the English Coast does not give the initial upheaval, but is the sum of the differences of several earth-movements—that the primary upheaval of the beaches was not less than 100 to 150 feet greater than the altitude at which they now stand, and that this led to the conversion of a considerable extent of the area of the Bristol and English Channels into dry land. What little evidence we have on the coast of Malta (*postea*, p. 965), and of Greece,[*] points to similar elevations of the coasts of the Mediterranean, so that large tracts of dry land may then have existed between the Sicilian and Italian shores, and formed suitable pasture grounds for the Hippopotami. With increase of the land area, so would the rivers also have had increased size, and though they may not have been very large, yet as Sir S. BAKER has shown, perennial waters are not indispensable to the Hippopotamus, for in the Settite and other rivers of the Soudan, these huge animals tide over the dry season, by resorting to the few pools left in the dried-up channels of the rivers.

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* Some may have floated and so been carried to a distance and lost.

† SCINÀ seems to imply that on the slope of Belliemi land shells are associated with the breccia.

On this interpretation of the origin of the San Ciro breccia, the reason of the uncertainty as to its position relatively to the cave deposits, felt by FALCONER and CHRISTIE, becomes apparent, for it is obvious that in that case it was formed subsequently to the cave bone-breccia, and fronts the deposits in the same way that the angular rubble ("Argile à blocaux") masked the "Trou du Frontal" (*ante*, fig. 12).

Malta.—There does not appear to be any well-marked Raised Beach on this island, but of bone-caves of the same age as those of Sicily there are several, and their contents are of a very remarkable character. The independence of the caves and the ossiferous breccia of Malta is also more clearly shown than in Sicily. These caves have been described by Dr. LEITH ADAMS,[†] Admiral SPRATT,[‡] Dr. FALCONER,[§] and Mr. G. BUSK.[||]

The island, which is extremely bare, consists entirely of Tertiary strata, covered by a red earth, derived partly from their decomposition. The drift beds lie on the slopes and in the valleys. The island is hilly, but there are no hill ranges like those of Sicily, the greatest heights being under 800 feet.

It is not necessary to give any descriptions of the caves, as they are to be found in the memoirs quoted, but I may give the list of the remarkable cave fauna, which was in occupation of the island at the time of the Rubble-drift. The variation of this fauna from the contemporaneous fauna of Sicily and the mainland was probably due to long isolation of the island during the Quaternary, and possibly the later Tertiary, periods. According to the last summary of Dr. LEITH ADAMS, it consists of :—

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[*] MM. BOBLAYE and VIRLET mention the occurrence of a submarine cliff off the Greek Coast.

[†] 'The Nile Valley and Malta,' pp. 161–238, 1870; and various papers in 'Geol. Mag.,' and in 'Reports, Brit. Assoc.'

[‡] "On the Bone Caves near Crendi Zebbug and Melliha," 'Quart. Journ. Geol. Soc.,' vol. 23, p. 283. It was in these that the dwarf Hippopotamus and pigmy Elephant were first discovered.

[§] 'Palæontological Memoirs,' vol. 2, p. 292.

[||] 'Trans. Zool. Soc.,' vol. 6, p. 119. See also a short notice by Professor F. W. HUTTON, in 'Geol. Mag.,' vol. 3, p. 145; and Dr. JOHN MURRAY'S "The Maltese Islands," in 'Scottish Geographical Mag.' for September, 1890, p. 449.

Elephas Mnaidra, ADAMS . . .	*The large Elephant of Malta.*
,, Melitensis, FALC. . . .	*The dwarf Elephant of Malta.*
,, Falconeri, BUSK . . .	*The pigmy Elephant of Malta.*
Hippopotamus Pentlandi . . .	*The small Hippopotamus.*
Myoxus Melitensis, PARKER . .	*The gigantic Dormouse.*
,, Cartei, ADAMS . . .	*The hollow-jawed Dormouse.*
Arvicola pratensis, BAILLON . .	*The Bank Vole.*
Cygnus Falconeri, PARKER . .	*The great Swan.*
,, Olor, GMELIN	*The mute Swan.*

With these were found bones of a large fresh-water Turtle, and the tooth of a Carnivorous animal.

The *Hippopotamus Pentlandi* occurs in the caves of Sicily ; the *Arvicola pratensis*, a recent species, is found in caves in this country ; the *Cygnus Olor* is a North African bird, and a winter visitor to Malta. The absence, or extreme rarity, of the ordinary cave animals and of Ruminants is another feature of the special insular character of this fauna.

The land shells of the drift beds are not given, but those found in them in the adjacent island of Gozo are all of recent species.*

The breccia is, as it is elsewhere, of a date immediately subsequent to the caves, and contains, in the few instances where it is fossiliferous, the same Mammalian remains. It occurs under three conditions : 1st, in the rock fissures ; 2nd, in cavities or hollows in the surface ; 3rd, on the slopes of the hills and under escarpments. As instances of the first are the Gandia and Shantiin fissures, met with in quarrying the calcareous sandstone near the village of Micabba. Dr. L. ADAMS describes them as rents of small size, crammed to the top with red earth and angular fragments, from a few inches to 2 or 3 feet in circumference, of the adjacent rock, with which numerous bones were interspersed in the greatest confusion. He observes that "the broken and splintered condition of the long bones was remarkable," and that none of them showed traces of having been rolled. He also remarks that "throughout the mass were strewn abundant remains of Elephant bones with the teeth entire or broken, together with fragments of bones of very large aquatic birds and those of the Dormouse, just as if numerous decayed carcasses of Elephants, large Water Birds, and

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* Mr. J. H. COOKE enumerates :—

Helix Pisani, MÜLL.
,, *striata*, DRAP.
,, *virgata ?* MONT.
Helix esoperata ? MONT.
,, *vermiculata*, MÜLL.
Pomatias melitensis, SOW.

Rats, scattered about on the surface, had been suddenly swept pell-mell into the gaping rent" with the detrital matter of the rocks. In the Shantiin fissure upwards of fourteen Elephant molars were discovered, while the Gandia fissure was estimated to contain the remains of no fewer than sixteen individual Elephants. The breccia is coloured red by the red earth derived from the hill tops, and is generally cemented by carbonate of lime, like the Gibraltar and Nice breccias.

Elsewhere Dr. ADAMS says that some of the bones have the appearance of having been partially devoured by Carnivorous animals, but the specimens in question are uncertain as no particulars are given, and they may have been derived from the old bone caves.

The Mnaidra and Berghisa gaps seem to be more in the nature of pot-holes or depressions in the surface, into which the local detritus and carcasses of animals had been swept. There, "among the large blocks of freestone, either impacted or strewn in a heterogeneous manner were lying seemingly entire skeletons of Elephants, some of the skulls and jaws furnishing good evidence of the rough usage they had sustained by being broken and crushed flat by blocks, which, with the force of impact, had cracked the others on which they had impinged." Dr. ADAMS also mentions that at these places seams of rolled pebbles are intermingled with the breccia. These may have been derived from an old watercourse or beach.

Admiral SPRATT and Dr. ADAMS both notice the large accumulation of angular and sub-angular débris in red earth at the mouths of the valleys, and on the slopes of some of the hills. One such bed of special interest is to be seen at Malak, on the south coast of Malta, where for some miles a steep escarpment, 200 to 300 feet high, formed by an old line of fault, fronts the sea. The lower part of this escarpment is covered by a red breccia, hidden in places by a modern talus, sloping down to the water's edge, where it forms steep banks that are being gradually worn away by the waves. In this breccia, which consists of red earth derived from the hills above, and of angular and worn fragments of the overhanging calcareous sandstone and lower limestone, several molars and part of the skull of a *Pigmy Elephant* were found *in situ.*[*] Dr. ADAMS has given a sketch and section of these cliffs, of which the following diagram-section (fig. 22) is a reduction.

The breccia bears a close similarity to the Rubble-drift or "head" over the "Raised Beaches" in the English Channel. There is the same local origin[†] of the detritus with remains of the animals then in occupation of the land. The main difference is that at Brighton and Sangatte the old cliffs are low, so that they are entirely masked by the rubble, whereas the escarpment at Malak is so high, that the mass of débris has only sufficed to cover up the base of the cliff. I also take this breccia to be the equivalent of the breccia on the slopes of Gibraltar (the presence of Mammalian remains being a local condition), and of the breccia on the Mentone Coast. They all testify to the débris of a land surface, carried from higher to lower levels by a force acting vertically.

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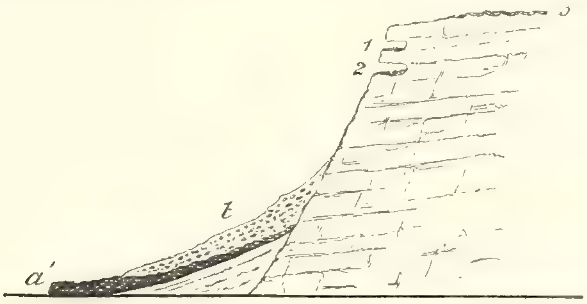
[*] In addition, Professor HUTTON mentions *Sus*, *Arvicola*, and land shells; 'Geol. Mag.,' vol. 3, p. 145, 1866.

[†] Dr. ADAMS supposed that fragments of a black limestone, found in the breccia, were foreign to the island, but Mr. COOKE has recently shown that a bed of this character does exist in the Lower Coralline Limestone; 'Geol. Mag.' for August, 1892, p. 361.

Fig. 22.—*Section showing the position of the Bone Caves and Bone Breccia on the Coast at Malak.*



t, Modern Talus.
a′, Ossiferous Breccia.
1, Malak Cave, with remains of *Hippopotamus Pentlàndi*, and *Pigmy Elephant.*
2, Melleha Cave, with remains of *Hippopotamus.*
3, Red clay on surface.

A fact mentioned by Admiral SPRATT* tends to confirm the opinion I expressed when speaking of the Sicilian coast, that, previously to the submergence, the land stood at a higher level than at present, for in excavating the naval docks at Valetta, the breccia of red earth, with fragments of surface débris and *land shells,* were found in fissures and crevices of the rock at 20 feet and more below the present sea-level.

Judging from the height at which the Rubble-drift is found it is probable that the island was wholly submerged. It is a significant, though not conclusive, fact, also, that not one species or even a single genus of its Quaternary Mammalia are now living on the island, the only indigenous quadrupeds, according to Dr. LEITH ADAMS, being the *Weasel, Hedgehog,* and *Rabbit.*

Carniola and Istria.—Some remarkable facts have been noticed in connection with the mines of Carniola.† The hills around Kropp, which rise to the height of several hundred metres above the sea level, are traversed by large fissures filled with a breccia, with which is associated a hydrated oxide of iron in grains from the size of a pea to that of a nut. This breccia contains, as usual, angular fragments and large blocks of the adjacent limestone rocks. The greater number of these fissures are unfossiliferous, but in a few the teeth and bones of *Ursus spelæus* have been discovered, and in one the tooth of a carnivorous animal was found at a depth of about 250 feet. BRONGNIART classed these fissures, some of which have been worked to the depth of

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* *Op. cit.*, p. 296.

† A. BRONGNIART, 'Annales des Sciences Naturelles,' August, 1828 and 1829.

720 feet, with those of Nice and others on the Mediterranean coast, and considered them all to be due to a common cause. Ossiferous breccias are also found in parts of Istria, and there are large accumulations of red drift and loam filling cavities in the neighbourhood of Trieste ; but without further details it is not possible to judge with certainty of the exact relations of these deposits. So far as known to me, the explanation I have suggested in the other cases might possibly apply to these districts, but the great height of the ground, and the peculiar local conditions, render further investigation necessary.

Dalmatia.—The travels of ALBERTO FORTIS in Dalmatia[*] furnish some incidental notices of the ossiferous breccia on the Adriatic coast, and especially on the islands of Charso and Ossero, which consist of ranges of limestone hills of some height either bare or covered by red earth. The cliffs on the coast are traversed by numerous fissures filled with breccia, and containing, FORTIS says, an extraordinary abundance of bones, but beyond these facts there is no reliable statement as to the character and nature of the bones or breccia.

Dr. F. LANZA,[†] who visited this coast long subsequently, adds little to our information, except that some of the bones belong to a species of Deer, and that the breccia contains land, but no marine, shells. These instances, however, help to show the generality and common characters of the phenomena over all this area.

Ionian Islands.—Except a few incidental notices by ANSTED[‡] and STRICKLAND,[§] such as that the valley of Signies in Cephalonia is covered by an alluvium full of angular flints, I can find no mention of anything referable to a Rubble-drift.

I am unable to extend this inquiry through central and south-eastern Europe, as it would involve many large and separate questions upon which I am not at present sufficiently informed, still, I should wish to call attention to a few points which may be found to bear upon the general subject. The great height of the plains and the vast extent of the area over which Loess deposits are spread in south-eastern Europe, is, as observes[||] Professor JAMES GEIKIE, "fatal, not only to every form of the lacustrine hypothesis, but also to the ingenious view supported by LYELL, as it is to that of BELT."[¶] Marine and fluviatile hypotheses are equally inapplicable, though in some districts, as in Roumania,[**] the Loess, as in Western Europe, belongs both to the valley and plateau types. In other districts it cannot be so referred. Loess is very largely developed in Southern Russia,[††] and with it some geologists associate the Black Earth or Tchernozem of the Steppes. Whatsoever may be their origin, it is evident from the heights at which the Loess and the black earth are found, that the waters in which they were deposited reached to considerable altitudes, and the general character, both of the Loess and of the black earth would accord with a sedimentation from the turbid and barren waters of a temporary submergence. Like the Loess, the black earth occurs at all heights, and covers the valley drifts. From its uniformity and wide extent MURCHISON supposed it to have been deposited under water, and considered that it might have been in great part derived from the black Jurassic shales which are largely developed in Southern Russia, so that its origin, as with other forms of the Rubble-drift, would be local.

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[*] English Translation, London, 1778, p. 440.

[†] 'Bull. Soc. Géol. de France,' 2nd ser., vol. 13, p. 127, 1855.

[‡] 'The Ionian Islands.'

[§] 'Memoirs,' by JARDINE, p. 68.

[||] 'Prehistoric Europe,' p. 160.

[¶] 'Quart. Journ. Geol. Soc.,' vol. 30, p. 490.

[**] STEFANESCU, 'Mémoire relatif à la Géologie du Indet, &c.

[††] MURCHISON'S 'Geology of Russia,' chap. 19.

The area covered by these deposits is so vast—larger than that of some European countries—and so nearly flat and open, that the waters during emergence must have flowed off less rapidly than they would in more restricted channels, and therefore allowed of a greater sedimentation. As in Western Europe, the organic remains of the Loess are all those of a land surface, and though none have been found in the black earth, its highly nitrogenous character is a singular feature, and indicative of the presence of organic if not of animal matter.

Greece.—Since the great work published by the French Government,[*] in which the extent and physical characters of the Quaternary deposits were treated with a fulness unusual for the time at which it was written, I am not aware that anything has been done to throw much additional light upon these drift beds of Greece. MM. BOBLAYE and VIRLET placed these deposits, which they term "Alluvions anciennes," between the Tertiary formations and the last upheaval which marks in Greece the commencement of the present period. The latest of these is composed of a red earth containing angular fragments of the local rocks, some of which are more than *a metre long*, but as the deposit approaches the sea, to the level of which it descends, the fragments became less angular. This drift covers the valleys of the Morea and rises to a considerable height up the sides of the adjacent hills, of which it follows the slopes, and is in places cut off on the coast by cliffs 15 to 20 metres high, while in the valleys, the torrents have cut into it to considerable depths. Though the deposit is unstratified in the upper parts of the valleys, on the coast it becomes interstratified with seams of sand and gravel. Previous to the spread of this drift, there had been an elevation of the coast to the extent of 20 to 25 metres, as shown by lines of *Pholas* perforations. An instance occurs on the cliffs near Nauplia. In other places traces of a Raised Beach exist. Over one of these was a drift bed with *Helix algira.*[†]

The authors say that this ferruginous and argillaceous drift "fills the valleys and forms taluses on the old coast lines . . . . . and seems to be contemporary with the osseous breccia with which it becomes confounded." It is quite distinct from the detritus carried down by the present streams which, though large in quantity, is not spread out in sheets, but forms cones of dejection at the mouths of the valleys.

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[*] 'Expédition Scientifique de Morée; Géologie et Minéralogie,' par MM. DE BOBLAYE et T. VIRLET, vol. 2, 2nd part, pp. 316–375, 1833.

[†] In an overlying bed some rude pottery was found.

This old angular drift, which is often cemented by calcareous infiltration into a hard, rocky mass, forms taluses or banks 40 to 50 metres high against the escarpments of Dolomite at the foot of Mount Taygetus, and is also found at the foot of the escarped shores of the lakes and of the sea-cliffs. The authors state that this drift is prolonged into historic times, with precisely the same characters, but this means I presume, no more than that the more recent deposits formed by the wear and erosion of the older drift, retains very much of the same character and aspect. The limestone hills are covered up to their highest summits with a red earth, due to the decomposition of the rock by atmospheric agencies. It is from this that the matrix of this deposit and of the osseous breccias, and which fills certain caves, is derived.*

Though it is stated in one place that the breccia forms a slope at the foot of the hills inclined at an angle approaching 45°, which might lead to the inference that it was a sub-aërial talus, this statement does not agree with what the authors say about its extending into the plains, nor with the sections they give, which represent the breccia with slight and variable slopes (Plate 2, fig. 2; Plate 3, fig. 4; Plate 6, figs. 4, 5; Plate 7, fig. 2), while it attains in some places in the valleys a thickness of from 50 to 60 feet or more. They also tell us that in some places "the breccia has been nearly denuded away, and all that remains in proof of its existence, is the *osseous breccia* which fills the fissures of the rock, and has acquired great solidity." This drift also blocks the entrance of some bone caves.

No particulars are given of the organic remains. It is merely stated that in this respect they resemble all the others on the coast of the Mediterranean, and that not only is there an absence of marine remains, but that land shells (*Helix algira*) occur in the breccia.

A few miles off the south coast of Greece is the island of Cerigo, which at an early date attracted attention in consequence of the reported occurrence of human remains in the ossiferous breccia filling fissures on the summit of a flat-topped hill near the sea. Amongst these were a jaw and part of a skull. CUVIER discredited the report, but without seeing the specimens, and the discovery was not followed up. Nor were any particulars given of the animal remains. A breccia is said also to lie on the slopes of the hills, while the ossiferous fissures are situated on an isolated hill, where, as at Cette and other places, the animals were likely to have sought refuge from the rising waters.

This *Old alluvium* of the Morea is identical in many of its features with the Rubble-drift of the South of England. There is (1) the same derivation of the detritus from local rocks; (2) the same angularity of the fragments; (3) the same general absence of stratification, though, as in England, with an occasional appearance of rough bedding; (4) the same transport from the higher ground behind to the sea-level; (5) the same fall over the old line of cliffs, and the same cutting back of the rubble slope since the present sea-level has been established; (6) the same blocking of the mouth of caves and filling up of fissures as the rubble swept over the surface; (7) the same powerful propulsion forward of the rubble, and its spread in wide sheets instead of in the cones of dejection formed under ordinary sub-aërial agency, and (8) the same limitation of the organic remains to those of a land surface.

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* This red colour of the ground is common in all the western Mediterranean area.

What differences there are are due merely to the different character and colour of the strata. Soft rocks with light colours prevail on the south coast of England, whereas on the Mediterranean coasts, the red earth, due to the decomposition of the limestones gives, as in the Plymouth district, a general red colour to the ossiferous breccias and Rubble-drift.

With these many points of resemblance there is, I consider, good reason to believe that in both countries these drifts are due to a common cause—one that we have already seen equally affected the intermediate area. These drift beds require, however, further investigation, and may prove as rich in Palæolithic remains as the ground above them has proved to be in prehistoric remains.

Crete.—From M. VICTOR RAULIN's work on Crete* I gather that there is evidence of the elevation of the island within the historical period to the extent of 15 to 25 feet, and further, that at a height of about 65 feet, a Raised Beach of Quaternary age is met with at many points of the coast. In the absence of sections and exact details, we can only surmise that the Rubble-drift is present there from the mention of two facts—(1) that at one place the Raised Beach is overlaid by a calcareous breccia (*brèche calcaire*) or "*Head;*" and (2) that at the foot of the great escarpments of the interior there are immense accumulations of angular detritus. Some of these (*terrains dêtritiques*) are recent, but some seem more widespread and older. They are traceable up to heights of 200 to 300 metres. A red earth, as in Greece, covers much of the surface.

Admiral SPRATT has shown,† that within recent times there has been a subsidence of the east coast of Crete, whilst the west side has been elevated to the extent of 26 feet. Anchor blocks have been found 11 feet above the sea level, and the port of Kissamo has been raised 18 feet out of the sea within Christian times. The two piers of the port of Phalasarna, a city of late Hellenic date, and described by STRABO, are now 22 feet above their original level. SPRATT also found *Pectunculi* of recent species 40 feet above the shore, and indications of another Raised Beach or old sea level at 100 feet. At about the same height he discovered a bone-cave with remains of *Myoxus, Goat, Roebuck,* or *Deer*, in a breccia under stalagmite. Remains of

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* 'Description physique de l'Ile de Crête, partie Géologique,' pp. 616–656, 1861; and 'Bull. Soc. Géol. France,' 2nd ser., vol. 13, p. 439.

† 'Travels and Researches in Crete,' London, 1865.

Hippopotamus have been found on the north-west side of the Lasethe Mountains, but their age is doubtful and more particulars are wanting.

These facts prove how prolonged earth movements have been in this area. They show also that drift deposits of the same character as we have noticed in Greece and Malta are clearly traceable thus far eastward.

Turkey.—What little is known of the superficial deposits of Turkey confirms the impression that the later geological changes we have described, affected that country equally with Greece. On both sides of the Dardanelles there are the remains of a marine deposit with recent species of *Ostrea* and *Cardium*, at a height of about 40 feet above the present sea-level;* and STRICKLAND† says that a thick mass of drift consisting of a ferruginous earth with pebbles and boulders skirts the southern flanks of the lesser Balkans, and extends to the neighbourhood of Constantinople, but he gives no details.‡

M. DE TCHIHATCHEFF§ tells us that in Roumelia there are immense deposits of angular detrital matter sometimes loose, at others agglutinated, covering like a mantle a great part of the country, and especially developed in the neighbourhood of Constantinople. The only fossils he names are land shells—species of *Pupa* and *Clausilia*.

M. VIQUESNEL‖ says that in Macedonia the Tertiary and Cretaceous strata are covered by a drift (*Alluvions anciennes*), over which, in the Plain of Doubnitza, is a loam with fragments of Palæozoic and crystalline rocks, derived from the adjacent hills. This drift forms slopes resting against the hills to the height of 200 to 300 feet. A similar deposit flanks the hills in the Rodonia valley, and again in some of the valleys of Albania.

Although exact particulars and sections are wanting, we recognize in these general descriptions many of the characters of the Rubble-drift.

Asia Minor.—Admiral BEAUFORT,¶ who surveyed the south coast of Asia Minor, speaks of a "petrified beach" as of common occurrence, but from his description it appears to be recent. It contains angular fragments in places. He also incidentally notices that in the Island of Rhodes there is a pudding-stone (breccia ?), considerably elevated above the sea, which was not distinguishable from some he saw on the shores of Greece, except that it was more solid.

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* 'Quart. Journ. Geol. Soc.,' vol. 13, p. 81, 1857.

† 'Memoirs,' Part 2, p. 8, 1836.

‡ Mr. F. CALVERT mentions the finding of two Palæolithic flint implements "like those from Suffolk," in a valley running into the Dardanelles, with boulders, on a ridge 100 to 300 feet high. 'Journ. Anthrop. Inst.,' vol. 10, p. 428, 1881.

§ 'Bull. Soc. Géol. France,' 2nd ser., vol. 8, pp. 307, 311.

‖ "Un Voyage dans la Turquie d'Europe," 'Mém. Soc. Géol. France,' 2 ser., vol. 1, p. 291.

¶ 'Karamania,' 2d edit., pp. 182–5, 1818.

Spratt and Forbes,[*] and Botta also refer to a consolidated beach of recent formation; while Botta[†] notices a bone cave on the slopes of the hills near Beyrout, in which he found a breccia with numerous bones and some shells, but it is apparently of Neolithic age, though he likens the breccia to that of Dalmatia. Mr. J. W. Hamilton[‡] speaks of a breccia covering the sides of the hills near Cnidus, but does not enter into details.

M. de Tchihatcheff in his great work[§] on Asia Minor remarks that Quaternary deposits are much less common than in Europe. He found traces of a Raised Beach in the Troad, and again in the Gulf of Smyrna, but the height above the sea-level is not given. It contains fragments of pottery.[||] Near Mermeridjé the limestone cliffs are drilled by *Pholades* at a height of 10 metres above sea level. In the plain of Tchoukour are some isolated hills capped by beds of gravel, sand, and clay, containing a number of recent marine shells, probably Quaternary, of which a short list is given, but without the height above the sea.

There are detrital deposits of local origin on the slopes of the hills, but M. de Tchihatcheff was uncertain whether they were Quaternary or modern. Elsewhere, in speaking of the detrital deposits in the plain of Smyrna, he says that they may probably be of the same age as the cave deposits. He remarks on the absence of organic remains in the superficial drift of Asia Minor, and on the fact that the detritus is always derived from the neighbouring hills, as a reason for believing that it is of local origin (*formé sur place*) when the country had assumed its present configuration.

Although these observations leave much to be desired, they seem to indicate the presence of a Rubble-drift and its comparatively recent origin.

Cyprus.—Monsieur A. Gaudry has shown in a valuable paper[¶] on the geology of this island, that a Raised Beach or sea bed (*Cordon littoral*) of Quaternary age may be traced nearly all round the island at a height from 3 to 30 feet above the present sea level.[**] In places it forms a belt of some width, occasionally penetrating up the valleys and abutting against hills of Pliocene strata. It consists of shelly sands 3 to 15 feet thick, with seams of pebbles, frequently agglutinated, and is used, like the *Panchina* of Leghorn, as a building stone.

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[*] 'Travels in Lycia, &c.,' vol. 2, pp. 164–209; 1847.

[†] 'Mém. Soc. Géol. France,' vol. 1, pp. 148–159; 1833.

[‡] 'Proc. Geol. Soc.,' vol. 3, p. 293; 1840.

[§] 'Asie Mineure,' part 4, "Géologie," vol. 3, 1869, pp. 382–524.

[||] In western Europe it is not certain that any pottery has been found in beds of Palæolithic age. But I see no reason why this should be the case in southern Europe and the coasts of the Mediterranean. if, as we advance towards the old centres of civilization in the East, Palæolithic man had there been in a more advanced state than in the more western area.

[¶] "Géologie de l'Ile de Chypre," 'Mém. Soc. Géol. de France,' 2nd ser., vol. 7, p. 149, 1859.

[**] In places there is apparently a newer upraised beach of recent date. There are also places where there has been subsidence within historic times.

This deposit is overlaid in places by a fine sandy bed, like "Loess," but in none of the sections given by M. GAUDRY is there the appearance of an overlying Rubble-breccia or "head." Only in three sections are breccias figured, and these are inland and at no great height, lying at the foot of hills, whence the blocks they contain are derived. The matrix is a marl derived also from the adjacent soft Tertiary strata.

Nor is any mention made of osseous breccias or of fissures with angular detritus, which, had they existed, could hardly have escaped M. GAUDRY'S notice. I infer from these facts that the submergence of this island was comparatively slight, and resulted only in a small settlement from the superincumbent turbid waters of the Loess-like deposit which covers much of the lower grounds, where it attains in places a thickness apparently of about 20 feet.

Northern Syria.—Near Lattakia, on the coast opposite Cyprus, Dr. POST[*] discovered at a height of 150 to 250 feet above the sea-level, a bed of marine shells, supposed to be a Raised Beach; but in the absence of a list of the shells, its exact age cannot be settled. It may correspond with the Pliocene beds of Cyprus.

Palestine.—Amongst the incidental geological notices of Canon TRISTRAM[†] are some bearing upon the question before us. In passing over the promontory south of Beyrout, he found a mass of bone breccia with fragments of "flint chips" in crevices in the limestone rock, some feet above the level of the road, but below the level of the old Egyptian track. He traced this bed for a distance of about 120 feet, and it probably extended as far as the face of the cliffs, as blocks of hard breccia with bones were found on the shore. The Canon describes the flints as "elongated chips with sharp edges" (flakes), and says that the teeth were referred by Professor BOYD DAWKINS to *Bison*, probably *Red Deer* or *Reindeer*, and *Elk*. He was of opinion that this breccia formed the flooring of an ancient cavern, but his description agrees as well with the characters of an ossiferous breccia or of breccia on a slope. The height above sea-level is not mentioned.

Allusion is frequently made to the stones of all sizes, from small gravel to large blocks—all angular or but slightly worn—scattered over the surface of the country. Canon TRISTRAM also alludes to the huge hillocks of a soft conglomerate, quite unlike sedimentary beds, containing land shells of existing species, which fringe the plain of the Jordan and stud the sides of the valleys under the hills, but he refers these to freshwater floodings from the Upper Jordan.

The geology of Palestine has been the subject of special investigation by M. LOUIS LARTET[‡] and Professor E. HULL.[§] They inform us that raised beaches or sea-beds are of frequent occurrence on the coast of Syria, but that it is difficult to fix their exact date. M. Lartet says that some of them may go back to the Quaternary period, while some may date from historic times. There is a Raised Beach on the coast near Beyrout, and another at Jaffa, in which were found *Pectunculus violacescens*, Lk., *Purpura hemastoma*, Lk., *Murex brandaris*, L., *Columbella rustica*, Lk., identical with those on the present beach. He also speaks of clastic and erratic deposits, but does not show whether there is any relation between them and the beaches. Some of these deposits may, he thinks, be of Tertiary age, whilst others seem to be subaërial and due to long atmospheric disintegration. They form vast taluses in the interior.

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[*] 'Nature,' for August, 1884.

[†] 'The Land of Israel,' London, 1865.

[‡] 'Essai sur la Géologie de la Palestine,' Paris, 1869, pp. 224–290.

[§] 'The Survey of Western Palestine: Memoir on the Geology and Physical Geography,' 1886, pp. 69–90.

The bone-cave in the valley of the Nahr-el-Kelb near Beyrout, discovered by Botta (*ante* p. 971), was more fully examined by Lartet, who found in it the remains of

Cervus dama.
Antilope (a small species).
Capra Sinaitica.
,, (like that of Crete).

On a platform of rock above the cave there was a calcareous breccia with similar remains (some of the bones calcined) mixed with flint flakes and scrapers.* He concluded that this was a rock shelter of the same age as those of the Dordogne in Central France, in which the flint implements are of the same character. But this correlation is doubtful, as the implements consist only of flakes and scrapers which are equally characteristic of Neolithic as of Palæolithic times. Besides that he mentions, on the authority of M. E. Lartet, that *Cervus dama* is still found in the Lebanon, the *Capra Sinaitica* abounds in Arabia Petræa, and the other species is closely allied to the Wild Goat of Crete, whilst the Antelope is probably the Gazelle so common in some adjacent districts.

Remains of extinct Mammalia are extremely scarce in these regions, though the remains of a fossil Elephant and of a species of Hippopotamus have, it is said, been found on the southern confines of Palestine. M. Lartet states also that stone implements of a true Palæolithic type have been found in some parts of Palestine. One from near Bethlehem has the discoid form resembling some of the Abbeville implements. Others have been found in Arabia Petræa and Babylonia. One from Boucher-Ain resembles in every respect the specimens of the spear-head (*hache*) type found in the Quaternary drifts of the Somme and Thames Valleys. They appear to have been found on the surface, but the exact sites are not well determined.

Professor Hull considers that, throughout the eastern end of the Mediterranean, there is evidence of a submergence after the Miocene epoch to the depth of about 220 feet,† as compared with the present sea-level. He describes a number of Raised Beaches or sea-beds, but in his several sections no overlying angular débris (or *head*) is shown. One noticeable sea-bed is exhibited in the plain of Philistria, where a large tract, little above the level of the Mediterranean, and running some distance up the main valleys, is covered by marine sands and gravels. The raised beach in the neighbourhood of Jaffa was found by the Professor to stretch far inland, and to attain a height of over 200 feet. But this height is open to the objection I have before stated. On the coast these beaches are generally covered by blown sands.

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* See also 'Bull. Soc. Géol. France,' 2nd sér., vol. 22, p. 537.

† The higher sea-levels or beaches are anterior to the changes I am considering, which date from the lower-level beaches.

If we may be allowed to judge from these brief notices, made independently of this special inquiry, it would seem that the Rubble-drift and osseous breccia, so largely spread over Western Europe and Greece, and again in Crete, become less frequent as we proceed eastward, and are but slightly developed in Syria.

Still more significant is the absence of those Fissures filled with local débris and often ossiferous, so common in limestone districts of Southern Europe. They are uncertain in Syria, and seem to be wanting in Cyprus and Palestine,* notwithstanding that Tertiary and Cretaceous limestones of the same characters are largely developed on the coast-lines. I can only infer that the more eastern areas were less affected by the great physical disturbances accompanying the submergence and re-elevation of the land than was the westward area, and that the submergence was of less depth or approaching its outer limits. To this point I shall have occasion to refer again (see Map).

The Coast of North Africa.—Facing Gibraltar there is evidence of upheavals corresponding with those on the Spanish side. At Tangiers, Mr. G. MAW† found horizontal beds of sand and clay resting on nearly vertical strata of the older rocks "at about 40 feet above the sea-level." He did not discover any organic remains, but concluded that this deposit was probably of post-Tertiary age, and synchronous with the Raised Beaches which he had met with near Cadiz. He also noticed a concreted sand, with recent shells, to the south of Cape Spartel, and at Saffe, forming cliffs about 50 feet high.‡

At Tangiers Messrs. RAMSAY and GEIKIE§ observed two lines of marine terraces of elevation. In the lower one, which consists of "semi-consolidated coralline and shell sandy beds, with small pebbles," they discovered at its base, which is on a cliff 12 or 15 feet above high-water mark, a tooth of *Elephas antiquus.* No mention is made of angular rubble at top.

M. COQUAND|| speaks of large ossiferous fissures near Tetuan, in which he found several species of Helix still living in the neighbourhood such as *H. lactea, H. lapicida, H. naticoides, H. erycina,* &c.

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* Unless the breccia near Beyrout should prove to belong to an ossiferous fissure; but, in any case, it is at no great height.

† "On the Evidences of Recent Changes of Level in the Mediterranean Coast-line," 'Geol. Mag.,' vol. 7, p. 548.

‡ 'Quart. Journ. Geol. Soc.,' vol. 28, p. 87.

§ 'Quart. Journ. Geol. Soc.,' vol. 34, p. 514.

|| 'Bull. Soc. Géol. Franc.,' 2nd ser., vol. 4, p. 1246.

Oran.—MM. BAYLE and VILLE[*] give a short list of the shells (all recent) found in the Raised Beach on this part of the coast, and state that Quaternary drifts are largely developed in the province, but the particulars given are insufficient to identify the several stages, though the presence of a detrital drift is indicated. Nor is there any very definite account of the Ossiferous breccia discovered in the fissures in that neighbourhood. It is said to present the same characters as those of Nice and Gibraltar, and contain the remains of *Bear*, *Ox*, *Horse*, and several ruminants.[†]

This is probably the breccia referred to by M. RENOU, who says that the rocks at Santon, near Oran, are traversed in all directions by enormous fissures filled with breccia and a few bones. He also states that the adjacent cliffs of Tertiary black limestone are drilled at a height of about 100 metres by boring shells, whilst, at Arzen, there is a raised beach, 16 to 20 feet above the sea-level, remarkable for the profusion of shells, and in places concreted into a hard mass. *This beach is overlaid by a breccia composed of fragments of slate and limestone.* The thickness of these beds, which, from description of a section near Oran, would seem to correspond with the Raised Beaches and "Head" of our south coast, is not given.

Mr. MAW mentions, on the authority of Canon TRISTRAM, that on the coast, to the west of Oran, there is a series of raised concreted sea-beaches, at heights of 200 to 600 feet. From fragments found in one of these, at a height of 400 feet, Dr. GWYN JEFFREYS determined *Pecten opercularis, Pectunculus glycimeris, Cardium edule, Venus gallica, Turbo rugosus,* and *Fusus corneus,* all common Mediterranean and Quaternary species.[‡] Mr. MAW also remarks upon a fact he had noticed on the coast-lines of Morocco, Corsica, and the Riviera, "that the hill- and valley-system of the land shelves under the adjacent sea without the intervention of distinct escarpments, indicating as he believes that the existing coast-level is so recent that the sea has not yet had time to excavate a cliff-boundary."[§]

Algeria.—In the important governmental work of scientific research in Algeria, M. RENOU[||] gives a few interesting particulars of the Quaternary beds. He mentions that there are several lines of sea-levels—one in particular, general all along the coast, is marked by a horizontal belt 1 foot broad of holes drilled by boring shells, and about 5 feet above the present sea-level. Another bed is composed of a gray clay full of *Cardium edule* and other recent species, with a few fragments of pottery in places, and is about 20 feet above sea-level. This bed is covered by a thickness of 150 to 180 feet of dunes. The higher sea-levels are not described.

At a height of 132 metres, a bone cave was discovered, the entrance to which was blocked by a mass of breccia containing *Helix aspersa*, and a *Bulimus*, while the interior was filled by a bed of clay with Mammalian remains covered by washed-in earth. Three other caves are noticed, one of which seems rather to have been an ossiferous fissure. In this, at a depth of 33 feet and under a mass of stalagmite, was a concreted red breccia containing numerous bones, amongst which M. RENOU discovered a worked flint* flake, very thin, circular in shape, and 2½ inches in diameter. In another cave or fissure, which was also blocked by a mass of breccia with *Helix aspersa*, he found an isolated human molar tooth, with a few Mammalian remains. The bones found in these several caves belonged to *Hyæna*, *Rhinoceros*, *Felis*, *Canis*, *Bos*, *Ovis*, *Capra*, *Sus*, and *Antilope*, but the species are not named. As these discoveries were made in 1840–1842, M. RENOU, though correctly noticing that the species were somewhat dissimilar to those of the present day, came to the conclusion that the caves were synchronous with the Raised Beaches, and that both belonged to the human period as then understood,—that they were pre-Roman, or, as we should now say, pre-historic. Owing to the prepossession then prevailing on this subject, this find failed to attract further attention.†

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[*] 'Bull. Soc. Géol. France,' 2nd ser., vol. 11, p. 505, 1854.

[†] DESNOYERS in C. D'ORBIGNY'S 'Dict. d'Hist. Nat.,' vol. 6, p. 383.

[‡] He states that remains of Pleistocene animals had been found in the neighbourhood of Algiers, but only mentions *Bubalus antiquus*, which was found in rearranged débris of Tertiary beds.

[§] 'Quart. Journ. Geol. Soc.,' vol. 30, p. 106.

[||] 'Exploration Scientifique de l'Algérie,' Paris, 1846; 'Géologie,' par M. E. RENOU.

Non-osseous breccias seem to be widely spread in Algeria, but they have been only incidentally noticed. There is also in places an angular rubble with *Helix* and *Cyclostoma* resting on an uneven surface and overlaid by a bed of red earth. These may be a form of Rubble-drift.‡

Constantine.—In an argillaceous bed, underlying a deposit of Travertine, on the plateau of Mangourah, remains of *Hippopotamus*, *Bos*, *Equus*, and *Antilope*, considered by M. BAYLE§ to be of Quaternary age, have been found. No caves and no ossiferous fissures have, however, been recorded in this province. We are not in possession of sufficient evidence to say whether or not a conglomerate (or breccia) which M. H. COQUAND|| describes as varying from 2 to 50 metres in thickness, should be classed with the Rubble-drift. It resembles it in many respects, overlying all other beds except the travertine, and consisting of débris of the local rocks, of older drift gravels and sands, and of the red surface clays, confusedly mixed together, besides containing occasionally blocks of large size (exceeding 1 metre cube).

Tunis.—A Raised Beach north of Monastir attains a height of 20 metres, decreasing in level southward to 5 or 6 metres. It contains a Strombus (*S. coronatus*) not found

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* This material is foreign to the district.

† [Of the occurrence of Palæolithic flint implements there can be no doubt, Sir JOHN LUBBOCK having picked up one on the surface, near Algiers. He mentions also the discovery of one by Dr. BLEICHER in a rock shelter, near Oran. 'Journ. Anthrop. Inst.,' vol. 10, p. 316, 1881.—J. P., *July*, 1893.]

‡ 'Bull. Soc. Géol. de France,' 2nd ser., vol. 11, p. 343: *ibid.*, 3rd ser., vol. 16, p. 877, 1888.

§ M. BAYLE remarks on the extent inland and on the coast of Quaternary drifts, and states that some beds contain, in places, numerous *Helices*.

|| "Description Géologique de la Province de Constantine," 'Mém. Soc. Géol. de France,' 3rd ser., vol. 6, p. 219, 1878.

in the adjacent sea, but occurring in the Raised Beach at Arzen, near Oran, and living on the Atlantic coast.*

Further to the south M. POMEL† describes the country as consisting of rolling hills of Cretaceous rocks in a sea of Quaternary drift composed of a loam more or less sandy, often of a red colour, and with but slight traces of bedding, though in places showing intercalated beds of sand. Fossils are of very rare occurrence, and consist of fragments of land shells, including specimens of *Zonites candidissimus,* a shell still common in the district. This deposit forms in places low cliffs on the coast. Elsewhere it plunges under the sea. The Island of Karkenna, opposite Sfax, is entirely composed of it. Inland it rises, as a Rubble-drift might do, on the slopes of the hills to heights of 60 metres. Its thickness is not given. Near Gabès a conglomerate, of which the materials are derived from the adjacent hills, caps the gypseous loam, but there is no mention of any breccias or of ossiferous fissures in that district.

Tripoli.—Judging from a remark of M. POMEL, I should imagine that the Quaternary loam of South Tunis ranged into Tripoli, and eastward toward the Lybian desert, but details are wanting.

The evidence thus far on the North African coast, imperfect as it is, shows the persistence of the Raised Beaches; while, as we proceed from west to east, we have to note the increasing rarity of Ossiferous Fissures so characteristic of the submerged areas. Though of frequent occurrence in the neighbourhood of Oran, and not uncommon in Algeria, there is no record of them in the provinces of Constantine and Tunis. Ordinary surface breccias continue, however, into the former province. These facts would seem to imply that, as on the coast of Asia Minor, the depth of the submergence decreased eastward, but closer investigation is needed (see Map).

Egypt.—It may be a question whether the submergence extended beyond the Lybian desert. There is no well-defined evidence in proof of it in Egypt, for the breccia described by General PITT-RIVERS near Thebes,‡ and in which he found chert or flint implements, does not appear to be a Rubble-drift, though it possesses some of its characters. It is composed of débris of chert and limestone derived from the adjacent hills, cemented together by carbonate of lime. In this, worked chert flakes occurred at a depth of from 6 to 10 feet from the surface. That these implements are of great antiquity is evident from the fact that there are tombs excavated in the breccia which are supposed to be not less than 3500 years old. There is, however, nothing in the shape of the implements to indicate their age, and no organic remains of any sort were found. The hardness of the breccia is no criterion, for, as we have before mentioned, the recent beach on the coast of Syria also forms a perfectly hard conglomerate, as do some of the recent breccias in Greece. Besides, the deposit is said to have been washed down a valley and "spread out on the plain in a fan-shape between the gorge of the valley and the Nile," which is not the usual form assumed by the Rubble-drift. It seems more probably to be a detrital bed carried down from the adjacent hills at a time when a heavy seasonal rainfall, like that which now takes place in Abyssinia, prevailed in this part of Egypt, and is not necessarily of geological antiquity.

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* DE LA MARMORA states that near Tunis and Carthage a "Panchina," like that at Leghorn, and about 8 metres above the sea-level, is worked as a building-stone.

† 'Bull. Soc. Géol. de France,' 3rd ser., vol. 6, p. 219, 1878.

‡ 'Journ. Anthrop. Inst.,' vol. 11, p. 382, 1881.

In the many accounts we have of the geology of Egypt no mention is made of any deposit that corresponds with the Ossiferous Breccias and Fissures of the Mediterranean coasts, though limestone rocks have been largely exploited and exhibit great lengths of mural escarpments. It would seem that these proofs of submergence are wanting in Egypt. The earlier stages, however, or those corresponding with the period of some of the high-level beaches of the Mediterranean are represented in Egypt by the terraces of the Lower Nile valley, described by Sir J. W. DAWSON.* One of these terraces on the Mokattam Hill, near Cairo, is about 200 feet above the sea level, and associated with it have been found recent species of *Ostrea*, *Pecten*, *Terebratula*, *Lithodomus*, and *Balanus*. Other terraces are mentioned higher up the valley, and likewise in the neighbourhood of Alexandria.

The Mokattam beach possibly corresponds also with the river terrace above Assouan, about 120 feet above the level of the Nile, described by Dr. LEITH ADAMS.† In the river drift on this terrace he found several freshwater shells, and amongst others the *Corbicula fluminalis* so common in pre-glacial and post-glacial times in England. If, as in Western Europe, the valley gravels and Raised Beaches had been covered by a Rubble-drift, it should have shown in some of these sections, but none have been recorded.

Again, in the European area, the Quaternary deposits are separated from the alluvium of modern or Neolithic date by beds of coarse gravel, as in the valleys of the Thames and Somme, or by the stanniferous gravel in the valleys of Cornwall. In the valley of the Nile no such drift beds have been found at the base of the alluvium of that river, though the numerous deep borings‡ that have been made, both below and above Cairo, offered unusual facilities for their detection. In all cases, where the boring was of sufficient depth, the alluvium was found "to consist of variable proportions of blown sand and alluvial mud," and nothing to represent a Rubble-drift was met with. Nor were any marine beds encountered.

We may, therefore, assume that before the deposit of the alluvium over the Nile valley, a land surface existed which would seem to have subsided gradually to an extent corresponding with the thickness of the Nile sediments. Although, therefore,

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* 'Geol. Mag.,' Dec. iii., vol. 1, p. 289, 1884; and "Egypt and Syria," chapters 2 and 4, 1887.

† 'Quart. Journ. Geol. Soc.,' vol. 20, p. 6, 1864.

‡ LEONARD HORNER in 'Phil. Trans.' for 1855, Part I., p. 105, and for 1858, Part II., p. 53, and Prof. JUDD, in 'Roy. Soc. Proc.,' vol. 39, p. 213, 1885.

Egypt may not have felt the full effects of the great earth movements which told upon the lands to the westward, it would appear not to have altogether escaped.

As the rate at which the Nile sediment accumulated admits of a rough approximation, we may in this way possibly get a clue to the date when this change of level took place, and though Mr. HORNER's estimate of 13,500 years[*] cannot claim entire reliance, it may serve as a starting point subject to correction.[†] A secular addition of sediment, uniform throughout, cannot be admitted, for as the sediment results from the persistent seasonal rains of the Abyssinian uplands, the quantity of sediment carried down annually must be a diminishing quantity in proportion as the rocks become gradually denuded and barer by the loss of surface and disintegrated soil.

Judging by analogy, it is most probable that Palæolithic Man did exist in Egypt, for flint or chert implements of the precise type of those found in the river drifts of the Thames and Somme Valleys have been found there, but apparently *only on the surface.* Sir JOHN LUBBOCK[‡] has discovered them on the hills and lower plateaux of the Nile Valley. Three pointed flint implements which he found at Abydos, and are stained chocolate colour, might be matched with specimens from the high-level gravels of St. Acheul. General PITT-RIVERS figures one of an elongated pointed shape, 4 inches in length, which was found on the surface near Koorneh. In the collection made by Professor H. W. HAYNES, there are specimens of the long pointed form so common at Amiens, and in the British Museum there is another specimen of pointed type also common in the valley of the Somme. It was found by Mr. W. M. FLINDERS PETRIE on the surface of the spur of a hill about 200 feet above the Nile level, near Esneh, 18 miles south of Thebes.[§] He describes it as river-worn and rolled. To me it, however, seems to have suffered more from sand- than river-wear, for although on one side the sharp angles are rounded off, and the surface of the flint has received a bright polish, on the other side the chipped surface retains its sharp angles and the flint its dull aspect.

Another circumstance, which may be merely a coincidence, or may have a collateral bearing on the question, is that several of the animals which lived in the South of Europe prior to the time of the Rubble-drift and Ossiferous fissures, disappeared in that area after that event, whereas they survived in the Nile Valley to Historic times; such, for example, are—

Lion.	Spotted Hyæna.	Hippopotamus.
Panther.	Caffir Cat.	African Elephant.

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[*] *Op. cit.*, p. 74; and LUBBOCK's 'Pre-historic Times,' p. 320.

[†] M. MORLOT's estimate, based on the growth of the delta of the Tinière, gives 8000 to 11,000 years as the lapse of time since the commencement of the Neolithic epoch in Switzerland, thus agreeing approximately with the age assigned by Mr. HORNER to the alluvium of the Nile.

[‡] 'Journ. Anthrop. Inst.,' vol. 4, p. 215, 1874.

[§] 'Ten Years' Diggings in Egypt,' p. 79, 1892.

These reasons, although not conclusive, and requiring corroboration, afford some grounds for supposing that the submergence did not extend to Egypt.

Conclusion.

The seemingly confused accumulations of superficial débris, lying on the surface of the land without apparent order or stratification, led the early geologists to conclude that they were all due to the transient action of water and had a common origin, but the explanation wanted the necessary geological data and definition.* Subsequent research has introduced order, and discovered agencies adequate to the explanation of most of the phenomena. Glacial, fluviatile, and meteoric action have claimed a large share of the work, and accounted for much which was then obscure. Nevertheless a residue, which supports to a certain extent the contention of the early geologists, remains, and which, as I have already explained, cannot be placed to the account of any of these agencies. It is to the various forms that this residue assumes, that I have applied the general term of "Rubble-drift." The reason why I have not retained the original term of "Diluvium," is that this has been, and still is on the Continent, so widely applied to fluviatile, subaërial, and other drift beds, that it is desirable to avoid a term so variously used, whilst, at the same time, it does not embrace some of the more important phases of the Rubble-drift.

I am well aware that several objections, more or less formidable, may be raised to the hypothesis which I have suggested to account for the origin of this drift. A few of these I may allude to here, though it would not be possible to discuss in these pages the wide and important general questions involved. Those who hold uniformitarian views will object to the want of known precedents and to the exceptional character of the agency proposed. In this difficulty I cannot share. I must repeat what I have long contended for,† that it is impossible to suppose that our very limited experience—say of 2000 years—could furnish us with standards applicable to the comparatively illimitable past. In fact, those that are relied on depend upon unstable conditions and are liable to vary with every passing century. While admitting the *permanence of the laws of Nature*, it is impossible, under the conditions through which this globe has passed, to suppose that at all former periods the effects, which have resulted from the operation of those laws, though equal *in kind*, were *equal in degree*. As in other similar questions, we must judge of the hypothesis not by an *à priori* assumption, but by the agreement of the consequences which it involves with the facts, and by the extent to which it satisfies the various conditions of the problem.

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* A very complete and valuable body of references to, and abstracts of, these early opinions will be found in Sir Henry Howorth's 'The Mammoth and the Flood,' and in his various papers in the 'Geol. Mag.' for 1882. It must, however, be borne in mind that the larger proportion of the phenomena upon which those opinions were based have since been shown to be referable to glacial and fluviatile agencies, so that some of them are no longer applicable to the conditions as they were then interpreted. Sir Henry also adopts an hypothesis somewhat analogous to my own, but based on different grounds, and treated from a different point of view. It relates also to a different area.

† 'Inaugural Lecture,' Oxford, p. 33, 1875.

Another objection may be raised on the grounds that the thickness of the Earth's crust is such as to render the movements spoken of improbable; but there again the physical data upon which that thickness has been estimated depend upon observations of such extreme delicacy that they scarcely yet afford a sure basis for calculation,[*] whilst the geological facts are antagonistic to the great thickness and rigidity contended for. No conclusion can, at all events, be correct which does not satisfy the *geological* as well as the *physical* conditions.

[Whatever may be the present rigidity of the crust of the Earth, the facts I have briefly alluded to in the preceding pages respecting the presence of Raised Beaches at many different levels and at great heights on the shores of the Mediterranean, show how great its mobility must have been throughout Quaternary times, and up to the date of the Rubble-drift. The latter merely marks the last stage of a long series of earth movements of variable intensity and duration.—J. P., *June*, 1893.]

Though some geologists may not admit that the presence of marine shells at the height of 1300 feet, on Moel Tryfaen, is a proof of upheaval of the sea-bed, none deny that the terraces and shell beds in Norway and Sweden, at heights of from 200 to 600 feet, have been caused by elevation of the land during the same Quaternary times with which we are now dealing. The main question for consideration in connection with these great earth movements is the rate at which the upheavals have been effected. Here again, for reasons before stated, I cannot hold that the present affords a just criterion for the past; but as these are questions which I have discussed elsewhere,[†] I merely refer to them here that the geological argument may not be overruled by postulates that are not founded on a more certain basis.

Other, and to my mind more serious, objections may, however, be raised, which will require careful investigation, such, for instance, as that there is nothing to show, in the absence of marine sediments, the occupation of the land by the sea waters. But, if that occupation were of short duration, it would not have been possible for ordinary sedimentation to have taken place; it could only have been such sedimentation as might have fallen in a limited time from the turbid superincumbent waters —like that which now forms the beds of brick-earth or Loess in the Channel Islands and in certain continental areas. Nor would there, for the same reason, have been time for the migration and establishment of a marine fauna on the submerged area. I can only conclude from these and other circumstances that the submergence of the land was of short duration, in which case this objection would no longer hold.

The questions, also, which arise in connection with the effect that such a widespread submergence would have upon both the land and the marine life of the period, present many difficulties. How far both the land and sea faunas would suffer, and in what way could their re-occupation of the emerged lands and uplifted sea-bed have been established, are matters for much consideration. In certain areas, where the waters rose above the highest summits of the land, entire faunas must have been destroyed, but where the higher summits were not submerged, room for the escape of large sections of both the fauna and flora existed, and local centres were left for their subsequent redistribution. Then, again, how far would a marine fauna be able to withstand the changes of level and pressure; or would its survival depend upon immigration from adjacent unsubmerged areas? But these are questions for naturalists. No answer can be given at present, as the occasion for their discussion has not hitherto arisen. I have necessarily confined myself to the geological phenomena.

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[*] It is also a question on which physicists have arrived at very different results.

[†] 'Roy. Soc. Proc.,' vol. 41, p. 158, 1885; and 'Geology,' vol. 1, p. 2, 1886.

Whatever phase of the Rubble-drift we may examine, we recognize in all of them physical and faunal conditions referable to the agency of one and the same cause. Whether we look at (1) the débris in one section of the Loess, (2) the Breccia on slopes, (3) the "*Head*" over the Raised Beaches, (4) the Basement gravels of many valleys, or (5) the Ossiferous fissures, we discern a complete absence of that wear which results from maintained river, sea, or ice action. Nor is there any indication of that transport of débris from a distance which attends river or tidal action. On the contrary, all the component materials are of *local origin*, derived from the adjacent slopes or hills, and they are *all unworn.* The evidence of the organic remains is to the same effect, in that they are those of *a land fauna alone*, with an entire absence of marine and fluviatile remains. The bones found in the Rubble-drift are not only in the same unworn condition as the rock fragments, but they are *free from all marks of gnawing.* This is a proof that the animals had not, as in the caves, fallen a prey to Carnivora, but must have met their death in a way which was unusual—such as from drowning,—for had their bodies remained on a land surface after death, they would have been subject to being devoured by predaceous animals, or else the bones would have shown traces of weathering and wear. At the same time the *sharply* fractured state and dispersion of the bones show that they must have been subjected to considerable violence and displacement. These conditions, as well as the mode of distribution of the rubble from many independent centres, accord in all points with the results that would ensue from the submergence and re-elevation of a land surface from beneath deep waters after a temporary submergence.

These conclusions, startling though they may appear,* have been forced upon me, not only by my own observations in the South of England, and parts of the Continent, but also by the independent evidence of other geologists, though their interpretation of the facts may be different. Looked at in all its aspects, I see no alternative that equally well answers to all the conditions of the problem. Other explanations[*] may satisfy some of the conditions in particular cases, but none of them satisfy all, whereas I think it will be found that the submergence hypothesis not only meets the requirements of each particular case, but that it also shows them all to be concordant, and such as would pertain to one common and general cause.

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* Viewed by our own standard the depth of submergence appears excessive, but in dealing with a body of the volume of the globe and with a surface length of 3000 miles or more, the deflection of the crust would appear really comparatively slight.

Another important conclusion hinges upon this question. I have before[†] pointed out the bearing that the position of the Rubble-drift should have in limiting our estimate of the time elapsed since the close of the Glacial period. In a paper already referred to I had shown cause why that time was not to be measured by Dr. CROLL'S reckoning of 80,000 years, as not being supported by the facts of geology. The position and character of the Rubble-drift show that the transition from the so-called Post-glacial beds to the recent Alluvial deposits is very abrupt, and that there is an absence of sedimentation or of anything indicative of lapse of time between those two series. This conclusion is confirmed by the sections of the Belgian caves. There, as we have seen (*ante*, pp. 927, 928), the Quaternary cave deposits are separated by only a few feet of Rubble-drift from deposits of the stone or Neolithic age. Nowhere are there any intervening sedimentary beds, or any deposits requiring length of time for their accumulation—the only subsequent work requiring time being comprised in the alluvial accumulation of our great rivers.

Besides, on CROLL'S hypothesis, Man must have remained comparatively stationary during a vastly long period. But how does this accord with the facts. Take the earliest works of Man with which we are acquainted—the rude implements of the Chalk plateau—and note the difference between them and the implements of the later Valley gravels. The former consist of rude flints picked up on the surface, and given only such an amount of trimming as to bring an angle to a point, or to form a cutting edge out of a blunt natural fracture, or else the stones, just as they were found, were used as hammers and trimmers. The valley implements, on the other hand, comprise flint tools and implements carefully worked all over and trimmed to certain definite patterns, the workmanship, apart from the want of grinding, being in some cases so fine as almost to equal that of the implements of the Stone age.

The caves of Central France and of Belgium afford still clearer evidence of the progress made by early Man in the interval between these two stages. His work in the last Quaternary stage exhibits an intelligence higher than that of many modern savages. His harpoons and bone implements were skilfully made, and that he possessed some artistic taste is shown by the sculptured bones and horns, and by the rude, but sufficiently accurate representations, of the contemporary fauna. How can we, then, believe that Man, who had showed himself thus progressive early in the Quaternary period, could towards its close have remained for say 70,000 years without further progress than that shown by Man of the early Stone period.

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[*] These have been considered *seriatim* in a former paper, 'Quart. Journ. Geol. Soc.,' vol. 48, pp. 325–328.

[†] 'Quart. Journ. Geol. Soc.,' vol. 48, p. 342.

There is certainly nothing to represent geologically that long period of time, nor have biologists been able to detect any essential structural differences between Palæolithic Man and Neolithic Man in support of such a conclusion. All the evidence tends, on the contrary, to prove that late glacial (or post-glacial) Man, together with the great extinct Mammalia, came down approximately to within some 10,000 to 12,000 years of our own times,* and that the Rubble-drift marks the stroke of the pendulum when the Glacial period came to a close, and the Neolithic age commenced.

EXPLANATION OF MAP (PLATE 33).

This map merely gives the position of the typical sections of the Rubble-drift and Raised Beaches mentioned in the text. They indicate the extent of the submerged area, but not its limits, and may serve as points of departure for more extended and detailed work. Pending the delimitation of the two divisions of the Loess, I have not attempted to apportion the areas they respectively occupy. A line, however, drawn from Brest to Odessa may be taken as a median line of a broad and irregular belt traversing the countries in which the high-level Loess as a whole is most largely developed—not continuously, but in certain areas and at certain heights.

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* This point is discussed at length in my paper on "The Glacial Period and Antiquity of Man," in 'Quart. Journ. Geol. Soc.,' vol. 43, p. 393, 1887.

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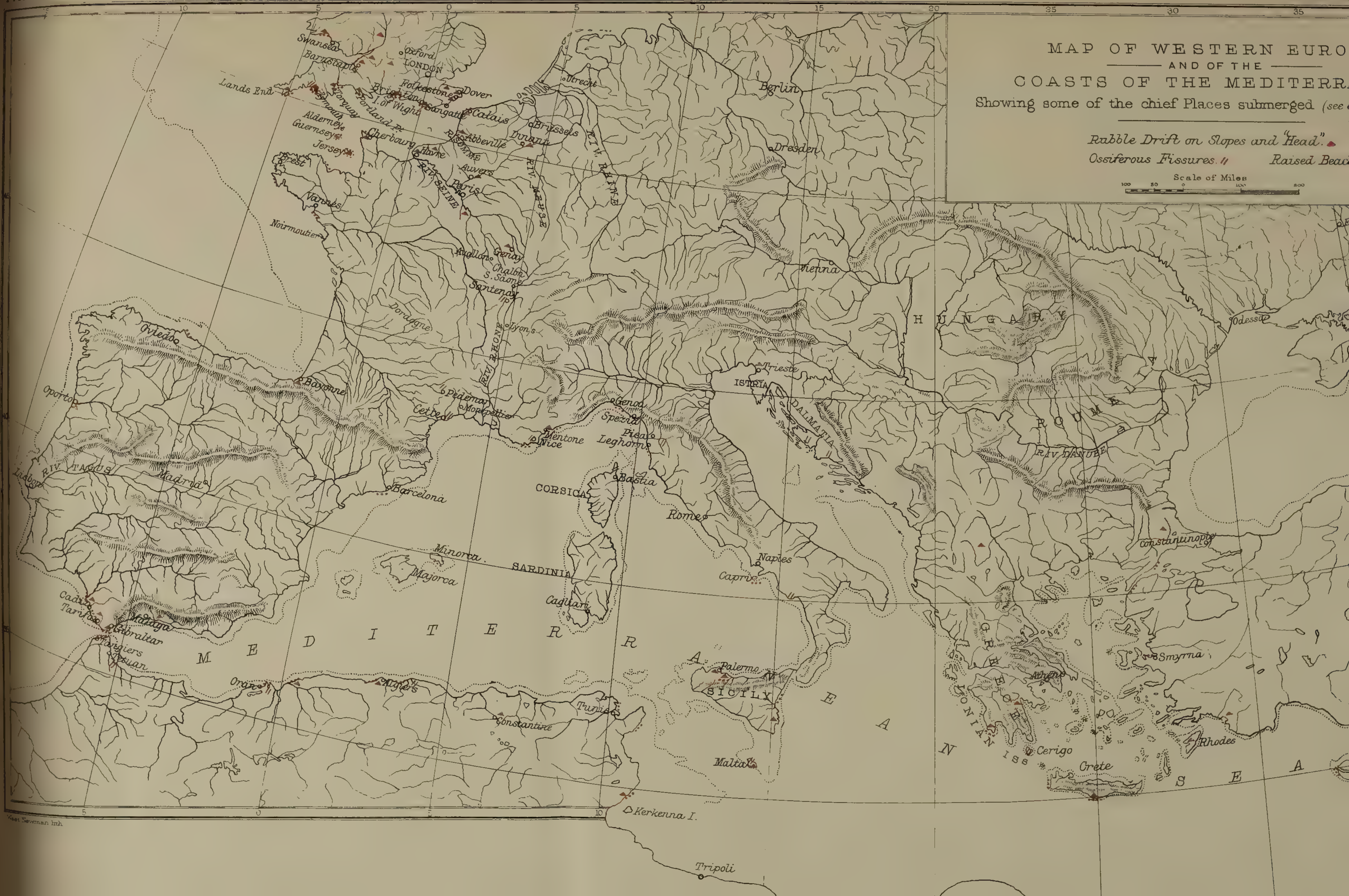
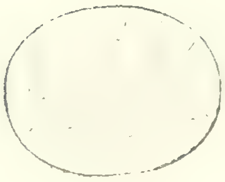


MAP OF WESTERN EURO
AND OF THE
COASTS OF THE MEDITERRA
Showing some of the chief Places submerged (see

Rabble Drift on Slopes and "Head".
Ossiferous Fissures.
Raised Beac



XIX. *Magnetic Qualities of Iron.*

By J. A. Ewing, *M.A.*, *F.R.S.*, *Professor of Mechanism and Applied Mechanics in the University of Cambridge, and Miss* Helen G. Klaassen, *Lecturer in Physics, Newnham College.*

Received June 7,—Read June 15, 1893.

[Plates 34–41.]

Recent applications of electricity, and especially the extended use of transformers, have added particular interest to the study of cyclic magnetizing processes in iron. It has become a matter of consequence to investigate, in various specimens of metal, not only the amount of the energy dissipated by hysteresis in a magnetic cycle, but the relative amounts under various degrees of magnetization and various intensities of magnetizing force. Other questions arise with regard to the dependence of this loss on the frequency of the cyclic process and on the manner in which it is performed. The experiments to be described in this paper deal mainly with the effects of cyclic variations of magnetizing force. They are intended to contribute some additions to existing data, to answer one or two specific questions, and to exemplify certain more or less novel methods of experimental inquiry. A section at the end of the paper relates to the molecular theory of magnetization, and its adequacy to explain some characteristic manifestations of magnetic hysteresis.

Experiments on Rings, by the Ballistic Method.

In a paper published eight years ago by one of us,* experiments were described in which a piece of soft iron was carried through a numerous series of cycles of magnetization, of graded amplitude, with the object of determining the form taken by the curve of magnetization and magnetizing force during the process of reversal between any assigned limits, and of comparing the work spent in the process with the amplitude of the magnetization. A similar experiment was described for steel. Since then the importance of such information has been recognized by electrical engineers, and some experiments with a similar object have been made by Messrs. Evershed and

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* "Experimental Researches in Magnetism." 'Phil. Trans.,' Part II., 1885, p. 523.

VIGNOLES,* and by Mr. C. P. STEINMETZ.† Notwithstanding, however, the increased interest which now attaches to the matter in consequence of its practical bearing, the available data are still meagre. By way of adding to them, we have made a detailed examination of some ten samples of wire and sheet iron, arranged in the form of rings to be operated on by the ballistic method.

In the former tests, referred to above, the method used was to make direct observations with a magnetometer, the samples being then long, straight pieces of wire. When the metal to be examined is in the form of wire, which may be taken long enough to secure practical endlessness, or of rods or bars big enough to be turned down into ellipsoids, the direct magnetometric method is entirely suitable. Several of our samples, however, were to be cut out of plate or sheet metal, such as is now largely used in the manufacture of transformers; and for this reason, as well as because direct measurements were to be made of the heating effect of magnetic reversals (which made ring samples most eligible), the ballistic method was selected, and a somewhat novel form was given to it, to allow points on the cyclic curve to be accurately and conveniently determined by its means.

In the paper already cited, some ballistic observations of magnetic cycles were described, in which the successive points on the cycle were determined by summing the ballistic effect of successive steps, in each of which the magnetic force underwent a small sudden change. This method is, at the best, laborious, and requires great care to avoid the carrying forward and accumulation of errors. The accuracy of each point depends on that of the points before it, though at the end of the process a check may be applied by comparing the ballistic effect of a sudden reversal with the sum of the effects of successive steps. There is an obvious advantage in making the determination of each point independent of the previous readings of the ballistic galvanometer, and, recognizing this, Messrs. EVERSHED and VIGNOLES introduced, in their experiments, the plan of reading each point by a single step from the terminal condition of magnetization represented by one or other extremity of the cycle. They wound their ring with two coils, through one of which a constant current was maintained; through the other coil a current, opposite in magnetizing quality, could be passed, and its strength could be varied up to a maximum which made its ampère-turns just twice the ampère-turns of the other coil. Then, when current flowed in the first coil only, the magnetization had its extreme positive value. By suddenly applying a current in the second coil, it was altered to any value between the positive and negative extremes. The use of two currents is an undesirable complication, especially as the symmetry of the resulting curves depends on the exactness with which the magnetizing effect of the one is adjusted to be, at its maximum, just twice that of the other. We have devised, and used throughout all these experiments, a method which, while requiring only one current and one magnetizing coil, retains the advantage that each point in the cycle is reached by a single step from the end of the cycle.

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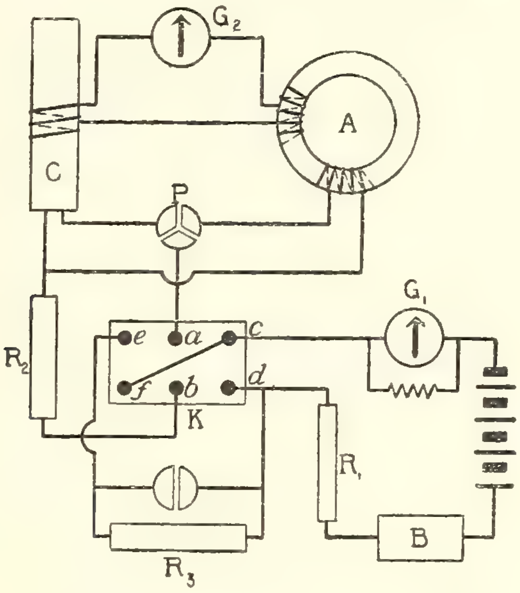
* EVERSHED and VIGNOLES, 'The Electrician,' May 15, 1891.

† STEINMETZ, 'Transactions of the American Institute of Electrical Engineers,' vol. 9, No. 1.

The diagram (fig. 1) shows the arrangement used.

Fig. 1.



The magnetizing current passes from the battery through the adjustable resistance R_1 and the galvanometer G_1 to a key, K, and thence, through the resistance R_2, to a two-way plug, P, which allows it to be sent either into the magnetizing coil of the ring A, which is the sample under test, or into the primary of a special induction coil, C, which is used for standardizing the ballistic galvanometer. A and C each have a secondary coil; the two secondaries are kept in series with one another, and their circuit is completed through the ballistic galvanometer G_2. A Kelvin balance, B, which is short-circuited when it is not required, serves to determine the constant of the current galvanometer G_1. The key K is a rocker, working in mercury cups, which, when rocked to the left, connects *a* with *e* and *b* with *f*, and, when rocked to the right, connects *a* with *c* and *b* with *d*. In the latter position of the key, the full strength of the magnetizing current is in action, and hence, if the plug P is set so as to send the current into A, the sample is magnetized to a degree which represents one extremity of a magnetic cycle. Between the cups *e* and *d* of the rocking key are a short-circuit plug and also an adjustable resistance, R_3. When this short-circuit plug is inserted, the key K is simply a reversing key, and then, when rocked to the left, it reverses the magnetization of the ring, carrying it at once to the opposite end of the cycle. But when, instead of a short circuit between *c* and *d*, there is a resistance, the rocking of the key to the left applies to the sample a magnetizing current opposite in sign to the primitive current, but less than it in amount, and, consequently, carries its magnetism, not to the end of the cycle, but to a point which falls short of the end by an amount depending on the magnitude of the resistance R_3. The peculiarity of the method is that, by a single movement of the key, the magnetizing current is at once reduced in amount and reversed in sign.* By giving R_3 a series of values, each movement of the key to the left determines a new point on the curve, while each movement to the right brings back the magnetization to its extreme value. In this way as many points are found as may be required to define that part of the curve which lies between the zero of magnetizing force and its extreme negative value. To find points lying between the positive extreme and the zero of magnetizing force, a different process is adopted. The key is kept to the left, and a sudden reduction of the current (without change of sign) is effected by withdrawing a plug from the resistance box R_2. This gives a point in the first quadrant of the descending curve, and a series of such points are found by selecting successive values of R_2, the magnetization of the sample being, however, restored to its initial full value before each such step.

To make the matter plainer, we may describe the operation of taking a cycle in detail. First, the strength of the magnetizing current was adjusted by selecting a proper number of cells (a storage battery was used throughout) and by means of the resistance R_1, so that the limits of magnetization should have the desired value. At this stage there was no resistance in R_2, and the short-circuit plug between *e* and *d* was in its place. The key K was now rocked back and forth many times to bring the magnetic changes into a thoroughly cyclic state—a precaution especially necessary when only a weak degree of magnetization was in question. The ballistic effect of this reversal was observed, several times over. Then, with the key K leaning to the right, a small sudden reduction of current was made by removing a plug from R_2, and the ballistic effect of this was observed. The plug in R_2 was replaced, the key K rocked back and forth to put the metal through a cycle, and then a second resistance, larger than the first, was suddenly opened in R_2. Operating in this way, a series of points were found for values of the magnetizing force H, lying between the positive maximum and zero. To continue the curve beyond the axis into the region of negative H, the resistance R_3 was adjusted to have, first, a rather high value. The key was then rocked from right to left, the ballistic effect being read; then this resistance was short-circuited; the specimen was taken through its full cycle by rocking the key; R_3 was set to be less than before, and its short-circuit plug was removed, the key was again rocked to the left, and so on. The curve determined by these successive operations was the curve ABCD (fig. 2), in which (ordinates being magnetization and abscissæ magnetizing force) A is the extreme state, always reverted to between each operation and the next; AB is the position determined by plugging out resistances in R_2, and BCD by rocking the key K with successive (lessening) values of R_3. This curve completely defines the magnetic cycle, inasmuch as the return limb

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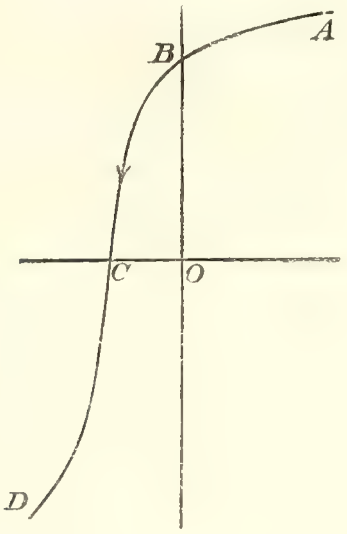
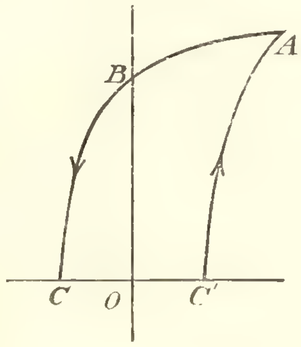
* Since this description was written, a paper has been communicated to the Royal Society by Dr. John Hopkinson and Messrs. Lydall and Wilson, in which ballistic experiments are described where the method was essentially the same as that used by us. A single movement of a key reversed the magnetizing current and introduced resistance. ('Roy. Soc. Pro.,' vol. 53, p. 352.)

Fig. 2



is symmetrical with the one sketched. As a matter of convenience in exhibiting the results, the portion CD may be sketched as extending from a point C′ back to the starting point A, as in fig. 3, OC′ being taken equal to OC, and in this way the relations of the rising and falling curve to one another are more clearly shown. There is, of course, no need to draw the other half of the cycle. The ordinate of A is taken as equal to half the magnetic change which occurs on reversal of the full strength of the magnetizing current.

Fig. 3.



G_1, used in measuring the magnetizing current, was a mirror galvanometer, strongly shunted and provided with a strong controlling field. Its constant in the shunted state was determined daily by means of the Kelvin centi-ampère balance. The ballistic galvanometer was provided with a usual arrangement for applying electromagnetic impulses to stop the swing of the needle. Its readings were reduced to absolute measure by means of the coil C, an air induction coil, whose dimensions were accurately measured, and the coefficient of mutual induction of its coils calculated from them. Its induction was sufficient to give a ballistic deflection of the same order of magnitude as that given by the sample under test. The resistance of the ballistic circuit was not altered throughout an experiment, and the two secondaries were permanently connected in series

The use of the coil C made the magnetic induction, as well as the magnetizing force, depend for its interpretation in absolute measure on the standard furnished by the Kelvin balance. The resistance R_1 was a frame of platinoid wire, wound for these experiments; R_3 was, in the first instance, a liquid slide consisting of amalgamated zinc plates, immersed in a jar containing solution of zinc-sulphate, but in the later experiments we found that a Varley carbon resistance, consisting of a pile of discs adjustable by compression, formed a very convenient substitute, especially when supplemented by short shunts of platinoid for the lower values of R_3.

All the samples tested in this set of experiments were arranged in the form of rings, with a mean diameter, generally, of about 10 centims., and an area of cross-section of from one to two square centims. The magnetizing coil was wound uniformly round the whole ring, in a single layer or in two layers. A few turns of wire, wound outside of that, formed the secondary or induction coil, the number of these turns being chosen so that reversal of the strongest magnetization produced as large a throw as could conveniently be read on the ballistic galvanometer. In several instances pairs of rings were prepared, exactly alike in all respects, for the purpose of making certain special tests which will be described later.

With each sample a series of cycles were determined, generally ten or twelve, beginning with one in which the limiting magnetization was strong, and working down to cycles on which the range was so small that the effects of hysteresis nearly disappeared. In the case of one or two samples a supplementary examination of the smaller cycles was made on a finer scale, by means of another secondary coil with a considerably larger number of turns, while the scale of the current galvanometer was at the same time altered by using a weaker shunt. We shall now give the results of the tests in detail.

Ring I.—In this ring the core was of fine iron wire, insulated by a winding of cotton. The diameter of the wire was 0·02475 centim. It was supplied (by Messrs. Glover) as a specimen of soft iron, but the tests showed that it had either not been softened, or had been insufficiently softened after its last passage through the draw-plate, the curve of magnetization presenting, to some extent, the characteristics of those for unannealed metal. Eleven cycles were examined; the results are stated below and are shown by the curves of fig. 4 (Plate 34). The figures give the magnetic force H and the magnetic induction B, both in C.G.S. units.

Ring I.—Fine Iron Wire.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).	
H.	B.	H.	B.	H.	B.	H.	B.
18·23	10870	13·01	8840	8·80	6780	6·59	5260
12·12	10220	9·52	8510	7·06	6610	3·76	4880
5·14	8840	7·50	8190	5·87	6450	2·29	4630
2·98	8290	4·58	7690	3·91	6130	1·39	4440
1·61	7620	2·80	7370	2·52	5850	0·78	4200
0	6910	1·55	6990	1·47	5640	0	3900
− 0·67	6890	0	6450	0·65	5370	−0·39	3740
− 1·11	6310	− 0·79	5750	0	5040	−1·11	3310
− 1·48	6040	− 1·30	5420	−0·81	4610	−1·85	2660
− 1·85	5750	− 1·85	4930	−1·48	4170	−2·31	1790
− 2·41	4800	− 2·78	2930	−2·22	3310	−2·78	270
− 2·78	3820	− 3·71	− 430	−2·78	1630	−3·24	−1250
− 3·33	1490	− 4·17	−1630	−3·22	− 110	−3·71	−2280
− 3·89	− 300	− 4·64	−2490	−3·71	−1460	−4·47	−3580
− 4·45	− 1540	− 5·56	−3960	−4·63	−3310	−6·59	−5260
− 5·37	− 3230	− 6·49	−5200	−5·56	−4500		
− 6·48	− 4830	− 8·38	−6670	−6·48	−5370		
− 8·34	− 6580	−13·01	−8840	−8·80	−6780		
−10·18	− 7780						
−18·23	−10870						

Cycle (5).		Cycle (6).		Cycle (7).		Cycle (8).	
H.	B.	H.	B.	H.	B.	H.	B.
4·40	3280	2·91	1410	1·94	560	1·23	270
2·93	3060	2·18	1330	1·24	450	0·73	220
1·94	2900	1·10	1110	0·59	340	0	80
1·24	2760	0·31	870	0	200	0·39	0
0·51	2550	0	760	−0·48	30	0·65	80
0	2360	−0·46	600	−0·74	− 40	0·93	160
−0·43	2110	−0·93	320	−1·02	−150	1·23	270
−1·11	1730	−1·39	50	−1·39	−310		
−1·85	590	−1·85	− 380	−1·67	−420		
−2·31	0	−2·31	− 920	−1·94	−560		
−2·78	−1110	−2·91	−1410				
−3·25	−1980						
−4·40	−3280						

In addition to these, three higher cycles were taken, with limiting values of H and B, as under :—

Limit of H.	Limit of B.
55·8	16640
34·4	14810
25·3	12940

These, however, are omitted from the figure, as they would have required it to be drawn to an inconveniently small scale.

Inspection of fig. 4 brings out one curious feature in the relation of the successive cycles to one another, which, however (as other figures will presently show), is of quite general occurrence. The extremity of each cycle lies outside of the rising curve of the immediately higher cycle, provided there is no very great difference in their ranges of magnetization. It is only when we come down to the lowest cycles, where the steps by which B is reduced from cycle to cycle are relatively great, that this does not happen. It should be remembered that each of these cycles is taken only after a large number of reversals of that particular magnetizing force which determines its range. During these reversals the range became somewhat reduced, especially when the magnetism does not approach saturation.

From these curves, by measuring the enclosed areas, values have been found of the energy dissipated in performing the magnetic cycle, namely, $\int H\,dI$ or, $\frac{1}{4\pi}\int H\,dB$.

RING I.—Fine Iron Wire. Energy dissipated in Cyclic Process of Double Reversal of Magnetism.

Limit of H.	Limit of B.	$\int H\,dI$, ergs.
55·8	16640	24930
34·4	14810	20320
25·3	12940	17130
18·23	10870	13410
13·01	8840	9900
8·80	6780	6420
6·59	5260	4180
4·40	3280	1940
2·91	1410	450
1·94	560	62
1·23	270	18

These results are given graphically in figs. 7 and 8, along with corresponding ones for the next ring.

Ring II. was of steel wire, of nearly the same fineness as the iron wire of Ring I. The diameter was 0·0257 centim., and, like the last, it was insulated throughout with cotton. Eleven cycles were taken in the first instance, as follows :—

Ring II.—Fine Steel Wire.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).	
H.	B.	H.	B.	H.	B.	H.	B.
43·91	17680	32·42	16720	21·89	15040	13·66	12070
23·25	16900	19·50	16160	15·02	14680	7·27	11570
15·63	16470	13·93	15870	9·25	14260	3·80	11180
9·69	15900	8·91	15440	5·24	13830	1·83	10960
6·05	15440	5·78	14940	2·79	13430	0	10500
2·31	14830	2·99	14510	0	12820	− 1·36	10040
0	14260	0	13760	− 0·61	12650	− 2·73	9430
− 0·95	13980	− 1·43	13330	− 2·04	12110	− 3·47	9010
− 2·04	13590	− 2·38	13010	− 3·40	11500	− 3·74	8830
− 3·40	12870	− 3·40	12480	− 4·76	10130	− 4·07	8580
− 5·09	11050	− 4·76	11300	− 5·44	7770	− 4·76	7360
− 5·78	8560	− 5·78	8050	− 6·18	4570	− 5·44	4800
− 6·80	4350	− 6·80	3350	− 6·87	920	− 6·12	1950
− 8·16	− 2210	− 8·16	− 3310	− 9·14	− 6270	− 6·80	− 1910
−11·49	− 9270	−10·53	− 8160	−10·19	− 8410	− 8·57	− 6970
−13·59	−11520	−12·27	−10590	−11·56	−10340	− 8·84	− 7820
−16·51	−13400	−17·06	−13830	−13·53	−12190	−10·40	− 9960
−19·44	−14860	−19·91	−14730	−18·01	−14120	−12·10	−11280
−43·91	−17680	−32·42	−16720	−21·89	−15040	−13·66	−12070

Cycle (5).		Cycle (6).		Cycle (7).		Cycle (8).	
H.	B.	H.	B.	H.	B.	H.	B.
10·98	10190	8·91	8190	7·48	6170	6·12	3960
6·51	9840	5·71	7770	5·03	5960	3·40	3680
4·55	9630	3·29	7530	2·58	5600	2·04	3460
2·45	9350	1·77	7320	1·09	5350	0	3060
1·43	9170	0	6890	0	5060	−1·09	2680
0	8810	−1·36	6490	−0·88	4850	−2·04	2350
− 0·34	8770	−2·72	5910	−1·70	4570	−3·40	1470
− 0·68	8700	−3·74	5170	−2·72	4030	−4·08	610
− 1·70	8270	−4·49	4280	−3·40	3600	−4·76	−1390
− 2·72	7840	−5·09	2250	−4·08	2890	−5·44	−3030
− 3·40	7530	−5·44	710	−4·76	990	−6·12	−3960
− 4·08	6920	−6·12	−2680	−5·50	−1900		
− 4·76	5630	−6·80	−5240	−6·12	−4240		
− 5·44	2890	−7·89	−7090	−7·48	−6170		
− 6·12	− 70	−8·91	−8090				
− 6·80	− 3530						
− 7·40	− 5700						
− 8·16	− 7340						
− 9·66	− 9130						
−10·98	−10190						

Cycle (9).		Cycle (10).		Cycle (11).	
H.	B.	H.	B.	H.	B.
4·76	1820	4·08	1180	2·24	360
2·92	1610	1·70	900	0·88	210
1·50	1390	0	610	0	70
0	1110	1·70	110	−1·57	−210
−1·70	610	2·99	− 530	−2·24	−360
−3·40	− 430	4·08	−1180		
−4·08	−1250				
−4·76	−1820				

These results are also shown in fig. 5 (Plate 34), where, however, Cycle 2 and the extremity of Cycle 1 are omitted to avoid overloading the diagram. Cycles 10 and 11 are also omitted. In drawing the curves the points corresponding to the readings actually taken are marked by dots enclosed within circles. They give satisfactory evidence that the method of observation we have used will yield a smooth curve. The curves here have the square-shouldered form, which is often found in steel as well as almost always in *annealed* soft iron. (The absence of this characteristic in Ring I., as well as its comparatively low permeability, was an indication of its being in a somewhat hard state.)

To test more particularly the action of weak magnetic forces, Ring II. was further taken through a series of low cycles, the sensibility of the apparatus in regard both to ballistic and current measurements having been first increased in the manner already indicated. That is to say, an induction coil of a larger number of turns was now used, and the current galvanometer was less strongly shunted. Five small cycles were observed as follows :—

Ring II. (Small cycles.)

Cycle (12).		Cycle (13).		Cycle (14).		Cycle (15).		Cycle (16).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
3·930	925	3·105	590	2·145	320	1·340	170	0·787	89
2·315	750	1·995	480	1·215	220	0·905	135	0·422	52
1·260	605	1·155	380	0·840	185	0·680	110	0·203	34
0·695	505	0·630	295	0·535	135	0·385	75	0	12
0	395	0·325	245	0	75	0	30	−0·267	−21
−0·265	320	0	195	−0·265	25	−0·160	10	−0·534	−58
−0·535	265	−0·425	125	−0·535	− 10	−0·535	− 35	−0·787	−89
−1·070	130	1·070	− 25	−1·070	−110	−0·800	− 85		
—	− 50	1·605	−160	−1·550	−210	- 1·070	−120		
−2·140	−210	3·100	−590	−2·140	−320	−1·340	−170		
−2·670	−405	2·185	−320						
−2·885	−520								
−3·930	−925								

These, with the exception of No. 15, are drawn out in fig. 6 (Plate 35).

Measuring areas as before we find the energy dissipated in the cycles :—

Ring II.—Fine Steel Wire. Energy dissipated in cyclic process of double reversal of magnetism.

No. of Cycle.	Limits of H.	Limits of B.	$\int H\,dI$, ergs.
1	43·91	17680	45770
2	32·42	16720	41320
3	21·89	15040	34330
4	13·66	12070	23460
5	10·98	10190	18100
6	8·91	8090	12820
7	7·48	6170	8300
8	6·12	3960	4310
9	4·76	1820	1130
10	4·08	1180	540
12	3·93	925	307
13	3·105	590	125
14	2·145	320	30
16	0·787	89	1·4

Fig. 7 gives the value of $\int H\,dI$ for this ring and also for Ring I., in relation to H, and fig. 8 (Plate 36) gives them in relation to B.

Fig. 8*a* further shows to a larger scale the relation of $\int H\,dI$ to B in the low-cycle tests of Ring II. It is interesting, as an example of the growth of $\int H\,dI$ in the region of low magnetic force, when the effects of hysteresis are only beginning to be felt. The figures show in a striking way how small the loss of energy through hysteresis becomes when the limits of magnetization or of magnetic force are restricted within certain values.

A characteristic of the curves of fig. 5 is the remarkably uniform rate at which B changes with respect to H during a great part of the process of magnetic reversal. After the shoulder of the descending curve has been turned, by applying a sufficiently strong demagnetizing force, the quantity dB/dH takes a large and nearly constant value which it retains until a tolerably strong reversed magnetization has been produced. The steep and nearly straight portion of the curve which corresponds to this part of the process has, moreover, nearly the same gradient in all except the smallest cycles. The same characteristic will be found in examples of annealed iron, to be given later, and the gradient in them is of course even steeper than it is here. In the present sample of steel wire (Ring II.), the maximum value of dB/dH is about 4950; nearly half the whole magnetic change in the highest cycle occurs at approximately this rate. In some of our soft iron samples dB/dH reaches a value nearly three times as great as this.

The amount of demagnetizing force just sufficient to bring the magnetism to zero has been called by HOPKINSON[*] the *coercive force*. It will be seen from these tests that the coercive force rises progressively as the magnetization is increased. We have measured its values for the cycles of Rings I. and II., and these may conveniently be expressed in relation to the limits of H and of the intensity of magnetization I.

RING I.

Limits of H.	Limits of I.	Coercive force.
58·36	1320	4·75
35·93	1176	4·55
26·48	1027	4·20
18·23	863	3·84
13·01	703	3·59
8·80	539	3·20
6·59	418	2·88
4·40	261	2·25
2·91	112	1·46
1·23	22	0·36

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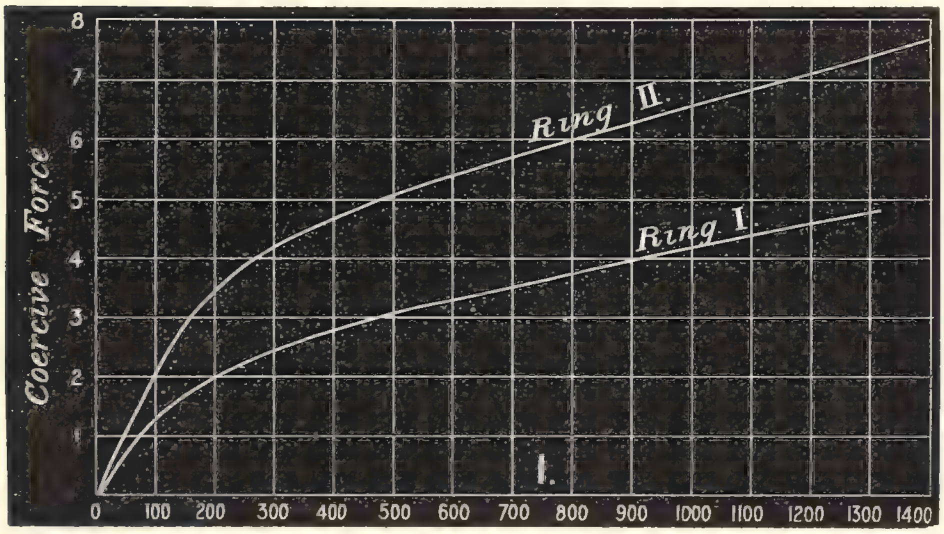
[*] J. HOPKINSON, "Magnetization of Iron," 'Phil. Trans.,' 1885, p. 460.

Ring II.

Limits of H.	Limits of I.	Coercive force.
43·21	1404	7·71
32·42	1328	7·43
21·89	1195	7·06
13·66	960	6·48
10·98	810	6·00
8·91	643	5·61
7·48	490	5·01
6·12	317	4·28
4·76	145	2·83
4·08	94	1·94
2·14	25	0·45

Fig. 9.



Coercive Force and I. Rings I. and II.

In fig. 9 the coercive force is plotted in relation to I. From this figure it is possible, by extrapolation of the curve, to arrive at a probable estimate of a definite physical constant of the material which forms a good criterion of magnetic hardness, namely, the coercive force which would correspond to the state of magnetic saturation. This, in other words, is the demagnetizing force which would be just sufficient to remove magnetism from a piece which had been magnetized to literal saturation. We know that the saturation value of I. in wrought iron and steel is about 1700.* A conjectural extension of the curves along the straight lines which they follow in the region of strong magnetization shows that the coercive force of saturation is probably about 5·5 in the (rather hard) iron of Ring I., and 8·5 in the steel of Ring II.

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* 'Phil Trans,' 1889, A, p. 221.

The next three rings had cores of sheet iron. Ring III. was made up of comparatively thick annular discs which were turned out of discs that had been supplied for the core of a dynamo armature. The mean thickness of the discs was 0·195 centim.; the internal diameter of the annulus was 13·55 centims. and its external diameter 1545 centims. Rings of paper were interposed to prevent the discs from touching one another. The tests showed that this iron compared well with most other specimens in the matter of magnetic softness.

The following cycles were measured :—

Ring III.—Thick Sheet Iron.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).	
H.	B.	H.	B.	H.	B.	H.	B.
32·34	15920	21·60	14920	14·62	13730	10·42	12460
22·51	15520	16·66	14660	12·19	13560	9·11	12330
17·23	15190	13·59	14390	10·44	13400	8·12	12130
10·10	14290	8·73	13730	7·29	12900	6·08	11940
5·97	13330	5·47	12860	4·86	12260	4·33	11470
3·30	12220	3·08	11870	2·09	10940	2·70	10940
0	9350	1·37	10670	1·37	10380	1·29	9950
− 1·33	5700	0	9060	0	8950	0	8560
− 1·54	4180	− 0·72	7820	− 0·95	7230	− 0·83	7100
− 1·90	1390	− 1·33	5800	− 1·33	5630	− 1·33	5100
− 2·28	− 1060	− 1·52	4300	− 1·52	4640	− 1·52	3720
− 2·85	− 3720	− 1·90	1390	− 1·90	1390	− 1·90	660
− 3·80	6370	− 2·28	− 1190	− 2·28	− 1330	− 2·28	− 1730
− 4·75	7820	− 2·66	− 2980	− 2·66	− 3120	− 2·66	− 3580
− 5·69	9220	− 3·04	− 4440	− 3·23	− 4970	− 3·23	− 5500
− 6·65	10220	− 3·80	− 6500	− 3·80	− 6570	− 3·80	− 6830
− 7·59	10870	− 4·75	− 8220	− 4·75	− 8230	− 5·69	− 9550
− 9·53	12000	− 5·69	− 9290	− 5·69	− 9350	− 7·59	−11010
−11·30	12740	− 7·59	−10840	− 7·59	−11010	− 8·73	−11670
−18·22	14460	− 9·49	−12000	−10·82	−12660	−10·42	−12460
−24·15	15320	−11·39	−12770	−12·15	−13000		
−25·44	15460	−15·18	−13790	−14·62	−13730		
−32·34	15920	−18·22	−14460				
		−21·60	−14920				

Cycle (5).		Cycle (6).		Cycle (7).		Cycle (8).	
H.	B.	H.	B.	H.	B.	H.	B.
8·05	11340	5·59	9520	4·01	7790	2·89	5830
7·26	11270	4·83	9460	3·42	7700	2·39	5770
6·61	11170	4·03	9260	2·89	7590	1·60	5500
5·17	10940	3·15	9060	2·17	7390	0·95	5230
3·84	10610	2·20	8720	1·48	7100	0	4280
2·51	10150	1·14	8290	0·80	6800	−0·76	3380
1·22	9490	0·61	7990	0·30	6530	−1·14	1930
0·65	8830	0	7130	0	6060	−1·52	− 530
0	8160	−0·76	5940	−0·95	4270	−1·90	−2920
−0·76	6900	−1·14	4740	−1·14	3490	−2·28	−4380
−1·33	4790	−1·52	2090	−1·52	700	−2·66	−5300
−1·52	3260	−1·90	− 900	−1·96	−2020	−2·89	−5830
−1·90	200	−2·28	−3090	−2·28	−3750		
−2·28	− 2260	−2·66	−4540	−2·85	−5600		
−2·66	− 3780	−3·23	−6200	−3·42	−6730		
−3·23	− 5830	−3·80	−7320	−4·01	−7790		
−3·80	− 7100	−4·75	−8660				
−4·75	− 8490	−5·59	−9520				
−5·69	− 9620						
−6·84	−10550						
−8·05	−11340						

Cycle (9).		Cycle (10).		Cycle (11).		Cycle (12).	
H.	B.	H.	B.	H.	B.	H.	B.
2·05	3720	1·32	1360	0·76	400	0·45	170
1·63	3620	0·95	1300	0	140	0	30
1·10	3460	0·68	1160	−0·49	−140	0·42	−100
0·68	3160	0·45	1060	−0·57	−230	0·45	−170
0	2750	0	830	−0·76	−400		
−0·76	1590	−0·57	200				
−1·14	260	−0·76	− 100				
−1·52	−1890	−0·95	− 560				
−1·90	−3380	−1·14	− 960				
−2·05	−3720	−1·32	−1360				

These cycles, with the exception of No. 1, are shown in figs. 10 and 10*a* (Plate 37), where, to avoid confusion, the odd and even cycles are drawn separately. Measurement from these curves of the area, and of the coercive force, gives the following results :—

Ring III.—Thick Sheet Iron: Values of $\int H\,dI$ and Coercive Force.

Number of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H\,dI$.	Coercive force.
1	32·34	15920	1265	13960	2·10
2	21·60	14920	1187	12050	2·10
3	14·62	13730	1092	10240	2·10
4	10·42	12460	991	8550	2·01
5	8·05	11340	902	7270	1·96
6	5·59	9520	757	5240	1·79
7	4·01	7790	620	3760	1·61
8	2·89	5830	464	2270	1·44
9	2·05	3720	296	1130	1·20
10	1·32	1360	108	230	0·69
11	0·76	400	32	..	0·28
12	0·45	170	14	..	0·12

The values of $\int H\,dI$, in relation to B, will be found plotted in fig. 21, along with corresponding results for other samples of sheet and wire iron.

The next sample to be tested (Ring IV.) was of thin sheet iron, which, like the last, had been supplied to form the core of a dynamo armature. This iron, though supplied for a purpose in which magnetic softness is important, did not compare favourably with the thick sheet of Ring III. The ring was built up of 30 flat annuli, which were turned out of larger punched discs. The thickness of the sheet varied in different discs from 0·044 to 0·050 centim. The external diameter was 10·9 centims. and the internal diameter 8·45 centims., and the mean thickness of the sheets was 0·47 millim. The discs were insulated from one another with paper. Nine cycles were measured in the first instance, and then six small cycles, with a larger induction coil and a more sensitive adjustment of the current galvanometer. The results are given numerically below, and graphically in figs. 11 and 12. The larger cycles, with the exception of the second, are drawn on fig. 11. Fig. 12 exhibits the small cycles. The induction coil used in measuring the large cycles had 12 turns, but in the small cycles a coil with 195 turns was used to give a large magnification of the ballistic readings, the sensibility of the ballistic galvanometer itself being unchanged. The curves of the small cycles have been determined in this instance with particular minuteness, as a study of the behaviour of the iron under reversals of weak magnetizing forces. In the lowest cycle the limits of the magnetizing force are so narrow that the condition is approaching that of quasi-elastic magnetization: there is, however, still a distinct trace of hysteresis.

Ring IV.—Thin Sheet Iron.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).		Cycle (5).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
23·61	14720	17·20	13760	12·99	12640	8·89	10880	6·45	9030
15·25	14220	11·33	13420	9·38	12410	7·02	10740	4·63	8830
10·13	13790	8·35	13090	7·26	12170	5·80	10610	3·24	8630
4·97	12960	4·63	12420	4·36	11710	3·78	10280	2·19	8370
2·93	12260	2·76	11890	2·66	11170	2·39	9940	1·33	8100
0	10590	0	10290	1·46	10700	0	8730	0	7430
−0·68	9990	−0·68	9660	0	9770	−0·85	7930	−0·68	6830
−1·19	9320	−1·36	8750	−0·73	9100	−1·71	6520	−1·36	5890
−1·71	8420	−2·05	6880	−1·36	8200	−2·39	2650	−2·05	3420
−2·39	4980	−2·73	1870	−2·05	6220	−2·90	−1020	−2·73	−1250
−2·90	1370	−3·41	−2070	−2·73	1350	−3·41	−3750	−3·41	−4460
−3·41	−1570	−4·09	−4740	−3·41	−2520	−4·26	−6220	−4·09	−6190
−4·26	−4690	−4·78	−6480	−4·09	−5120	−5·15	−7430	−4·78	−7200
−5·12	−6780	−5·46	−7880	−4·78	−6800	−6·48	−9300	−5·63	−8300
−5·97	−8250	−6·14	−8750	−5·80	−8470	−7·47	−9980	−6·45	−9030
−6·82	−9250	−6·82	−9420	−6·82	−9540	−8·89	−10880		
−8·53	−10720	−8·53	−10820	−8·53	−10810				
−10·92	−11840	−10·24	−11760	−10·24	−11670				
−13·65	−12920	−11·94	−12420	−12·04	−12340				
−20·57	−14520	−13·65	−12960	−12·99	−12640				
−23·61	−14720	−17·20	−13760						

Cycle (6).		Cycle (7).		Cycle (8).		Cycle (9).	
H.	B.	H.	B.	H.	B.	H.	B.
4·93	7250	4·01	5710	3·24	4160	2·32	1800
3·78	7120	2·49	5510	2·19	4020	1·36	1670
2·80	6980	1·81	5380	1·63	3920	0·75	1530
1·95	6780	0·85	5080	0·82	3690	0	1270
1·22	6580	0	4640	0	3360	−0·78	800
0	6010	−0·85	4000	−0·73	2790	−1·19	400
−0·82	5310	−1·71	2430	−1·36	1980	−1·36	130
−1·71	3710	−2·05	500	−1·71	1080	−1·71	−600
−2·39	−430	−2·39	−1500	−2·05	−650	−2·05	−1330
−3·07	−3840	−2·73	−3040	−2·39	−2150	−2·32	−1800
−3·75	−5580	−3·07	−4040	−2·73	−3190		
−4·43	−6580	−3·41	−4780	−3·24	−4160		
−4·93	−7250	−4·01	−5710				

Ring IV.—continued. Small Cycles.

Cycle (10).		Cycle (11).		Cycle (12).		Cycle (13).		Cycle (14).		Cycle (15).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
1·884	927	1·687	677	1·400	420	1·105	263	0·825	157	0·320	43
1·696	906	1·403	644	1·193	401	0·974	251	0·659	140	0·227	34
1·537	886	1·118	607	0·981	372	0·827	232	0·547	126	0·158	27
1·204	840	0·834	564	0·756	341	0·659	209	0·408	108	0	6
0·882	787	0·551	512	0·518	304	0·470	186	0·232	85	−0·172	−18
0·573	722	0·276	447	0·265	255	0·254	152	0	46	−0·243	−31
0·278	647	0·150	418	0	195	0	98	−0·221	5	−0·320	−43
0·150	610	0	373	−0·221	131	−0·221	49	−0·331	−23		
0	561	−0·221	299	−0·441	59	−0·441	−12	−0·441	−50		
−0·236	470	−0·441	208	−0·662	−29	−0·662	−86	−0·662	−105		
−0·331	429	−0·551	167	−0·772	−80	−0·882	−172	−0·825	−157		
−0·441	380	−0·662	106	−0·882	−129	−0·992	−217				
−0·551	327	−0·772	44	−0·992	−186	−1·105	−263				
−0·662	261	−0·882	−17	−1·105	−249						
−0·772	195	−0·992	−91	−1·213	−319						
−0·882	121	−1·103	−177	−1·400	−420						
−0·992	43	−1·213	−264								
−1·103	−51	−1·324	−363								
−1·213	−154	−1·456	−486								
−1·324	−272	−1·544	−568								
−1·434	−401	−1·687	−677								
−1·544	−552										
−1·654	−693										
−1·783	−824										
−1·884	−927										

From measurement of the curves of figs. 11 and 12, the following values of $\int H\,dI$ and of the coercive force have been determined.

Ring IV.—Thin Sheet Iron. Values of $\int H\,dI$ and Coercive Force.

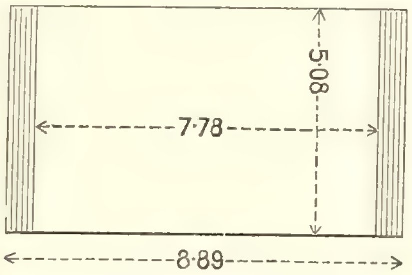
Number of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H\,dI$.	Coercive force.
1	23·61	14720	1170	16670	2·66
2	17·20	13760	1093	14830	2·56
3	12·99	12640	1005	12670	2·46
4	8·89	10880	865	9750	2·31
5	6·45	9030	718	6940	2·12
6	4·93	7250	577	4980	1·94
7	4·01	5710	454	3440	1·79
8	3·24	4160	331	2220	1·62
9	2·32	1800	143	660	1·20
10	1·884	927	73·6	225	
11	1·687	677	53·7	134	
12	1·400	420	33·3	59	
13	1·105	263	20·9	24	
14	0·825	157	12·4	8	
15	0·320	43	3·4	0·3	

These results are also exhibited, with others, in fig. 21. A reference to fig. 12 will show that the overlapping of one cyclic curve by the next lower curve (corresponding to a slightly narrower magnetic range) is characteristic of low cycles as well as of high ones. The loss of energy in reversal (measured by the area enclosed within the curve) becomes very small, both absolutely and relatively to the magnetization, in the lowest cycles; the molecular movements which constitute magnetization are there approaching the almost perfect reversibility which, as Lord RAYLEIGH has shown, is attained when the range of force is still further reduced.

By way of further illustrating the behaviour of this sample under the weak magnetizing forces of the small cycles, fig. 13 has been drawn to show the total induced magnetism, and also the residual magnetism, in each small cycle, both in relation to H. The figure forms an interesting example of the early stages of the magnetizing process.

The next sample (Ring V.) was also of thin sheet iron. It was sent by Mr. PARKER, of the Electric Construction Corporation, as a sample of the iron used in building up the cores of his transformers. This iron is supplied in sheets about 6 feet long and 3 feet wide, and its thickness is 0·367 millim. To make the ring, a strip 2 inches wide and 6 feet long was cut from the sheet, and this was coiled (along with a strip of paper for insulation) into a close spiral of the dimensions shown in the sketch (fig. 14), the ends of the spiral overlapping one another for a centimetre or so. This is a convenient way of giving sheet metal the ring-form suitable for ballistic tests; it is open, however, to the criticism that the bending of the metal is liable to harden it somewhat, especially near both surfaces, and it is possible that this consideration partly explains the comparatively sloping character of the curves (figs. 15 and 15a) obtained with this ring. A similar remark, of course, applies in the case of rings formed by winding soft iron wire.

Fig. 14.



Section through Ring V.

Ring V.—Soft Thin Sheet Iron (used in transformer cores).

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).		Cycle (5).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
15·72	12360	12·41	11390	9·15	10120	6·95	8940	5·03	7530
8·43	11420	7·36	10720	6·02	9730	4·99	8690	3·92	7420
5·74	10640	5·23	10170	3·54	9070	3·20	8280	2·75	7120
3·51	9760	3·31	9460	1·96	8360	1·41	7450	1·30	6600
1·62	8720	1·55	8520	0	6760	0	6300	0	5550
0	7230	0	7090	−0·69	5830	−0·69	5420	−0·69	4620
−0·86	5970	−0·69	6160	−1·38	3680	−1·38	3050	−1·38	2140
−1·72	1460	−1·38	4010	−1·72	600	−1·72	−80	−1·72	−880
−2·41	−2560	−1·72	1040	−2·06	−1700	−2·06	−2230	−2·06	−2640
−3·10	−4590	−2·06	−1260	−2·75	−4180	−2·75	−4540	−2·75	−4780
−4·30	−6680	−2·75	−3850	−3·44	−5660	−3·44	−5800	−3·27	−5610
−5·16	−7620	−3·44	−5500	−4·30	−6870	−4·30	−6960	−3·92	−6430
−6·88	−8880	−4·30	−6760	−5·18	−7640	−5·03	−7620	−5·03	−7530
−8·50	−9870	−5·16	−7640	−6·09	−8470	−6·29	−8550		
−11·97	−11370	−6·88	−9020	−7·74	−9400	−6·95	−8940		
−12·76	−11640	−9·90	−10560	−9·15	−10120				
−15·72	−12360	−12·41	−11390						

Cycle (6).		Cycle (7).		Cycle (8).		Cycle (9).		Cycle (10).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
3·68	6100	2·75	4730	2·04	3240	1·27	1210	0·61	220
2·58	5940	2·09	4670	1·30	3080	0·76	1100	0·34	200
1·82	5690	1·55	4480	0·61	2800	0·41	930	0	80
1·20	5440	1·06	4290	0	2360	0	710	−0·34	−150
0	4620	0·67	4070	−0·69	1460	0·34	410	−0·61	−220
−0·69	3680	0	3550	−1·38	−1320	0·69	−50		
−1·38	710	−0·69	2720	−1·72	−2530	1·03	−710		
−2·06	−3350	−1·39	−550	−2·04	−3240	1·37	−1210		
−2·75	−4950	−2·06	−3460						
−3·03	−5280	−2·32	−3960						
−3·68	−6100	−2·75	−4730						

These cycles are shown in figs. 15 and 15*a*. Measuring from the curves we have these results, which will also be found in fig. 21.

Ring V.—Soft Thin Sheet Iron. Values of $\int H dI$ and Coercive Force.

Number of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H\, dI$.	Coercive force.
1	15·72	12366	982	8310	1·92
2	12·41	11390	905	7250	1·87
3	9·15	10120	882	6020	1·80
4	6·95	8940	711	4900	1·71
5	5·03	7530	599	3670	1·62
6	3·68	6100	485	2520	1·47
7	2·75	4730	376	1760	1·42
8	2·04	3240	258	900	1·02
9	1·27	1210	96	180	0·68
10	0·61	220	17	6	0·10

The remaining specimens dealt with in this section of our experiments were of iron wire. In Ring No. VI. the core was of fine cotton-covered wire 0·34 millim. in diameter, which was kindly supplied to us for the purpose of these experiments by a leading firm of manufacturers at the instance of Mr. W. H. Preece. It was described as Swedish charcoal iron, but the tests show that it was decidedly hard in the magnetic sense, having a rather high coercive force and other characteristics resembling those of mild steel. Six cycles were measured, with results which are given below and graphically in fig. 16.

Ring VI.—Fine Iron Wire.

Cycle (1).		Cycle (2).		Cycle (3).	
H.	B.	H.	B.	H.	B.
12·34	11330	9·14	9430	7·30	7730
6·54	10760	5·45	9070	4·71	7410
3·29	10280	2·44	8540	2·27	7000
1·64	9960	0·61	8140	0·77	6680
0·64	9720	0	7980	0	6480
0	9470	−0·64	7730	−1·28	6030
−1·31	8910	−1·54	7330	−2·57	5230
−2·57	8020	−2·57	6360	−3·85	1840
−3·93	5110	−3·85	3610	−4·50	−1640
−4·37	2520	−4·37	870	−4·86	−3410
−5·17	−1760	−4·88	−2040	−6·12	−6560
−5·68	−4180	−5·40	−4460	−6·38	−6880
−6·41	−6360	−7·35	−8100	−7·30	−7730
−9·28	−9680	−7·71	−8420		
−12·34	−11330	−9·14	−9430		

Cycle (4).		Cycle (5).		Cycle (6).	
H.	B.	H.	B.	H.	B.
6·14	6230	5·03	4220	3·55	1290
3·18	5830	2·85	3940	2·31	1130
1·85	5600	1·72	3690	0·95	850
0	5150	0	3250	0	650
−1·28	4620	−1·28	2760	−1·03	240
−2·57	3810	−2·57	1880	−2·06	− 200
−3·21	2800	−3·21	750	−2·79	− 690
−3·85	100	−3·68	− 950	−3 55	−1293
−4·21	−1520	−4·47	−3290		
−5·27	−5070	−5·03	−4220		
−5 48	−5390				
−6·14	−6230				

Ring VI.—Fine Iron Wire. Values of $\int H\,dI$ and Coercive Force.

Number of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H\,dI$.	Coercive force.
1	12·34	11330	901	16490	4·83
2	9·14	9430	750	12250	4·52
3	7·30	7730	614	9050	4·20
4	6·14	6230	495	6580	3·86
5	5·03	4220	335	3670	3·43
6	3·55	1290	105	490	1·65

The next specimen (Ring VII.) was also of cotton-covered iron wire, of larger diameter than the last, namely 0·975 millim. It turned out to be only moderately soft in the magnetic sense. In addition to the measurements of magnetic cycles, the results of which are given below and in figs. 17 and 17*a* (Plate 39), several other experiments were made with this ring with the view of determining directly the heating effect of magnetic reversals. These will be described in a later part of this paper. The ring had a mean diameter of 9·6 centims., and the net area of section of the iron wire forming its core was 1·00 sq. centim.

Ring VII.—Iron Wire.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).	
H.	B.	H.	B.	H.	B.	H.	B.
37·40	15980	21·73	14340	14·59	12860	10·28	11420
25·65	15490	10·48	13300	8·47	12230	8·21	11130
11·42	13880	5·86	12250	5·18	11580	4·51	10570
3·04	11330	3·38	11370	1·75	10310	1·69	9520
1·58	10610	0·79	9970	0·77	9780	0·39	8670
0·00	9310	0·00	9240	0·00	9130	0·00	8550
− 2·78	− 410	− 0·75	7840	− 0·73	8430	− 0·73	7570
− 3·85	− 5130	− 1·28	7480	− 0·90	7910	− 1·28	6720
− 5·15	− 7740	− 2·14	3820	− 1·71	6050	− 2·14	2860
− 6·97	− 9600	− 3·42	− 3650	− 2·57	520	− 3·00	− 2860
− 9·41	−11070	− 4·71	− 7080	− 3·85	− 5520	− 4·02	− 6290
−13·86	−12710	− 6·84	− 9520	− 5·56	− 8460	− 5·15	− 8190
−22·67	−14600	−10·82	−11670	− 7·72	−10350	− 6·33	− 9430
−24·81	−14860	−21·73	−14340	− 8·68	−10820	−10·28	−11420
−37·40	−14980			−14·59	−12860		

Cycle (5).		Cycle (6).		Cycle (7).		Cycle (8).		Cycle (9).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
7·66	10090	5·61	8520	3·60	5780	2·20	2260	1·05	390
5·56	9840	3·34	8170	1·92	5430	1·07	1960	0·45	260
2·63	9180	1·50	7680	1·28	5270	0·58	1730	0·28	190
0·73	8240	0·71	7350	0·64	4970	0·00	1440	0·00	130
0·00	7770	0·00	6730	0·00	4550	−0·56	880	−0·58	−130
−0·73	6900	−0·73	6020	−0·64	4080	−1·07	230	−1·05	−390
−1·28	6050	−0·98	5620	−1·07	3200	−1·50	− 820		
−2·14	1860	−1·43	4640	−1·71	950	−1·93	−1800		
−3·00	− 3630	−2·14	690	−2·35	−2740	−2·20	−2260		
−4·28	− 7130	−3·00	−4180	−3·00	−4680				
−5·26	− 8170	−4·28	−7130	−3·60	−5780				
−7·66	−10090	−4·53	−7390						
		−5·61	−8520						

Ring VII.—Iron Wire. Values of $\int H\,dI$ and Coercive Force.

Number of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H\,dI$.	Coercive force.
1	37·40	15980	1268	16580	2·81
2	21·73	14340	1140	13390	2·74
3	14·59	12860	1023	11270	2·61
4	10·28	11420	908	8790	2·52
5	7·66	10090	802	7390	2·41
6	5·61	8520	677	5600	2·22
7	3·60	5780	460	3020	1·84
8	2·20	2260	182	650	1·17
9	1·05	390	31	26	0·32

For the last sample on our list (Ring VIII.) we are indebted to Mr. James Swinburne, who sent a hank of the soft iron wire used in the cores of his "hedgehog" transformers. The diameter of the wire was 0·602 millim., and its surface was bright. On winding it into a ring with a mean diameter of 2·75 centims., without, in the first instance, taking any steps to secure insulation between neighbouring turns of the bright iron wire, we found that the Foucault currents in the ring were so influential as to make the ballistic tests of it in this condition worthless. In all the previous examples the wire or sheet metal forming the core had been insulated by cotton or paper. The ring was then soaked in a hot bath of boiled linseed oil, which was allowed to penetrate freely to the core—a treatment which, we believe, is similar to that actually followed in the building of the transformers. It appears that this was effective in preventing the Foucault currents from passing from wire to wire. Tests made with the ring in this state, which we shall distinguish as the first state, are shown in figs. 18 and 18*a*. The following are the observed values of B and H in nine graded cycles :—

Ring VIII.—Soft Iron Wire of "Hedgehog" Transformer. First State.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).	
H.	B.	H.	B.	H.	B.	H.	B.
22·23	15600	15·04	14200	11·33	13020	8·62	11690
9·58	14400	7·89	13410	6·72	12550	4·60	11030
4·45	13140	4·05	12530	3·69	11820	1·84	10290
2·16	12130	2·00	11840	1·92	11160	0·75	9830
1·06	11540	0	10140	0·98	10630	0	9300
0	10750	— 1·21	9080	0	9960	— 1·53	7500
— 1·38	9150	— 1·96	7550	— 1·45	8300	— 1·96	6380
— 2·16	7020	— 2·77	1380	— 1·96	7110	— 2·37	3590
— 2·44	4830	— 3·14	— 2080	— 2·40	4190	— 2·75	— 3301
— 2·79	1450	— 3·53	— 4280	— 2·75	800	— 3·14	— 3130
— 3·02	— 550	— 4·34	— 6790	— 3·14	— 2520	— 3·73	— 5980
— 3·34	— 2950	— 4·71	— 8120	— 3·54	— 4780	— 4·32	— 7430
— 3·93	— 5530	— 5·89	— 9520	— 4·15	— 6770	— 5·07	— 8630
— 4·71	— 7520	— 6·88	—10580	— 4·71	— 8040	— 5·91	— 9630
— 5·89	— 9430	— 8·24	—11580	— 5·70	— 9430	— 7·66	—11030
— 7·85	—11310	—12·29	—13310	— 6·68	—10430	— 8·62	—11690
— 9·81	—12500	—15·04	—14200	— 9·69	—12350		
—16·72	—14710			—11·33	—13020		
—22·23	—15600						

Cycle (5).		Cycle (6).		Cycle (7).		Cycle (8).		Cycle (9).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
6·56	10160	5·03	8490	3·65	5950	2·51	2490	1·73	900
3·65	9630	3·14	8170	2·56	5740	1·42	2150	0·82	630
1·57	9030	1·57	7700	1·69	5470	0	1490	0	300
0·90	8760	0	6910	0	4680	1·10	430	0·66	— 30
0	8230	—1·10	5780	—1·10	3480	1·77	— 690	1·18	—360
—0·98	7240	—1·57	4980	—1·77	1960	2·12	— 1020	1·73	—900
—1·38	6710	—2·16	2320	—2·36	— 1630	2·51	— 2490		
—1·96	5120	—2·75	— 2660	—2·87	— 4150				
—2·36	2320	—3·10	— 4650	—3·65	— 5950				
—2·76	— 1530	—3·95	— 6910						
—3·14	— 4190	—5·03	— 8490						
—3·53	— 5780								
—4·13	— 7300								
—4·83	— 8360								
—6·01	— 9630								
—6·56	—10160								

These cycles give the following values of ∫HdI and the coercive force :—

Ring VIII.—Soft Iron Wire of "Hedgehog" Transformer. First State. Values of $\int H dI$ and Coercive Force.

No. of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H dI$.	Coercive force.
1	22·23	15600	1240	15980	2·95
2	15·04	14200	1128	13620	2·89
3	11·33	13020	1035	11890	2·85
4	8·62	11690	930	9780	2·71
5	6·56	10160	808	7820	2·61
6	5·03	8490	673	5850	2·40
7	3·65	5950	474	3500	2·17
8	2·51	2490	198	870	1·47
9	1·73	900	72	130	0·60

The fact that the cyclic curves obtained from the metal in this state had somewhat rounded outlines, and gave values of $\int H dI$, which, though by no means exceptionally great, are greater than those given by the softest iron, led us to suspect that the metal had been slightly hardened, either in the manufacturer's hands, or in our coiling it into a ring for testing. Accordingly, we resolved to make another trial after annealing the wire in the coiled up state. The ring (No. VIII.) was stripped of its magnetising coil, and was heated to redness in a forge-fire, after which it was again thoroughly soaked in a hot bath of boiled linseed oil. This treatment, besides removing any hardness that might have been produced by mechanical operations, no doubt had the effect of improving the insulation of the coils from one another, by coating the wire with a film of oxide; and this may have contributed to produce an apparent change of magnetic quality. On rewinding the magnetising coil, the following tests were made with results which differ in a noteworthy way from those of the preceding tests, although the expectation that $\int H dI$ would be reduced was not fulfilled.

RING VIII.—Second State. Soft Iron Wire of "Hedgehog" Transformer after special annealing.

Cycle (1).		Cycle (2).		Cycle (3).		Cycle (4).		Cycle (5).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
23·60	16170	15·25	15490	11·24	14870	8·05	13840	6·15	12640
9·63	15700	7·89	15220	6·74	14660	4·08	13510	3·50	12450
5·77	15450	4·01	14830	3·70	14460	1·81	13330	1·27	12120
3·46	15180	2·04	14570	1·92	14070	0	12720	0	11650
2·12	14970	0	13970	0	13460	−1·50	11470	−1·54	10540
0	14310	− 1·23	13040	− 1·54	12150	−2·16	8480	−2·35	4890
− 1·54	13380	− 1·92	11720	− 2·31	7650	−2·73	1780	−2·89	− 2110
− 2·58	5560	− 2·69	3570	− 2·69	2800	−3·27	− 5160	−3·46	− 6950
− 3·08	− 1060	− 3·27	− 3700	− 3·08	− 2350	−3·85	− 8360	−4·58	−10410
− 3·46	− 4960	− 3·85	− 7880	− 3·47	− 5990	−4·62	−10330	−6·15	−12640
− 3·91	− 8020	− 4·62	−10340	− 4·23	− 9500	−7·05	−13190		
− 4·70	−10410	− 7·05	−13170	− 5·27	−11490	−8·05	−13840		
− 5·77	−12180	−10·82	−14690	− 7·05	−13280				
− 7·74	−13730	−15·25	−15490	− 9·59	−14400				
−12·24	−15180			−11·24	−14870				
−16·17	−15640								
−23·60	−16170								

Cycle (6).		Cycle (7).		Cycle (8).		Cycle (9).		Cycle (10).	
H.	B.	H.	B.	H.	B.	H.	B.	H.	B.
4·66	10940	3·63	8640	2·50	4000	1·92	1390	1·41	660
2·19	10610	1·42	8250	0	3140	0	730	0	260
1·19	10480	0	7780	−1·39	1480	−1·06	− 120	−0·81	− 70
0	10080	−1·54	6310	−1·73	30	−1·35	− 450	−1·41	−660
−1·42	9010	−2·27	1890	−2·07	−2101	−1·92	−1390		
−1·92	7550	−2·87	−4660	−2·50	−4000				
−2·50	1060	−3·63	−8640						
−2·89	− 3570								
−3·66	− 8750								
−4·27	−10140								
−4·66	−10940								

Inspection of these curves (figs. 19 and 19*a*, Plate 40) will show that a result of the special annealing, which the ring received after it was coiled, was to make the stages of the cyclic process be much more sharply differentiated from one another than they were before. In other words, the curves are now exceedingly square-shouldered. The residual magnetism is high, and the rate of descent when reversal of magnetism is taking place is extremely rapid and uniform. dB/dH in the higher cycles is about 13,000 at its maximum, and maintains a value little short of this, while B changes by as much as 20,000.

Fig. 20 has been drawn in further illustration of the magnetic qualities of this ring (Ring VIII. after special annealing). The curves in it show the whole induced magnetism and the residual magnetism (for each cycle) in relation to H. Taking the inclination of a tangent to the former curve drawn through the origin to measure the maximum permeability we find its value to be 2420.

In the same diagram (fig. 20) the curve drawn in broken line shows, in relation to H, the ratios of the residual to the induced value of B in each cycle. This curve rises with a rapid, nearly uniform, gradient (from a point to the right of the origin, since a finite amount of magnetising force may be applied without causing the iron to acquire any residual magnetism). It bends over fast when H is about 2·5, passes a maximum which is not sharply defined, and slowly falls as the magnetism is pushed into the region of saturation. Throughout the steep part of the magnetization curve the difference between the residual and induced magnetisms scarcely increases at all, while B is increasing tenfold. The relation of the residual to the induced magnetism presents, in this and other respects, just those characteristics which the molecular theory (referred to below) would anticipate. In the present instance when the ratio is a maximum the residual induction is 93 per cent. of the induced.

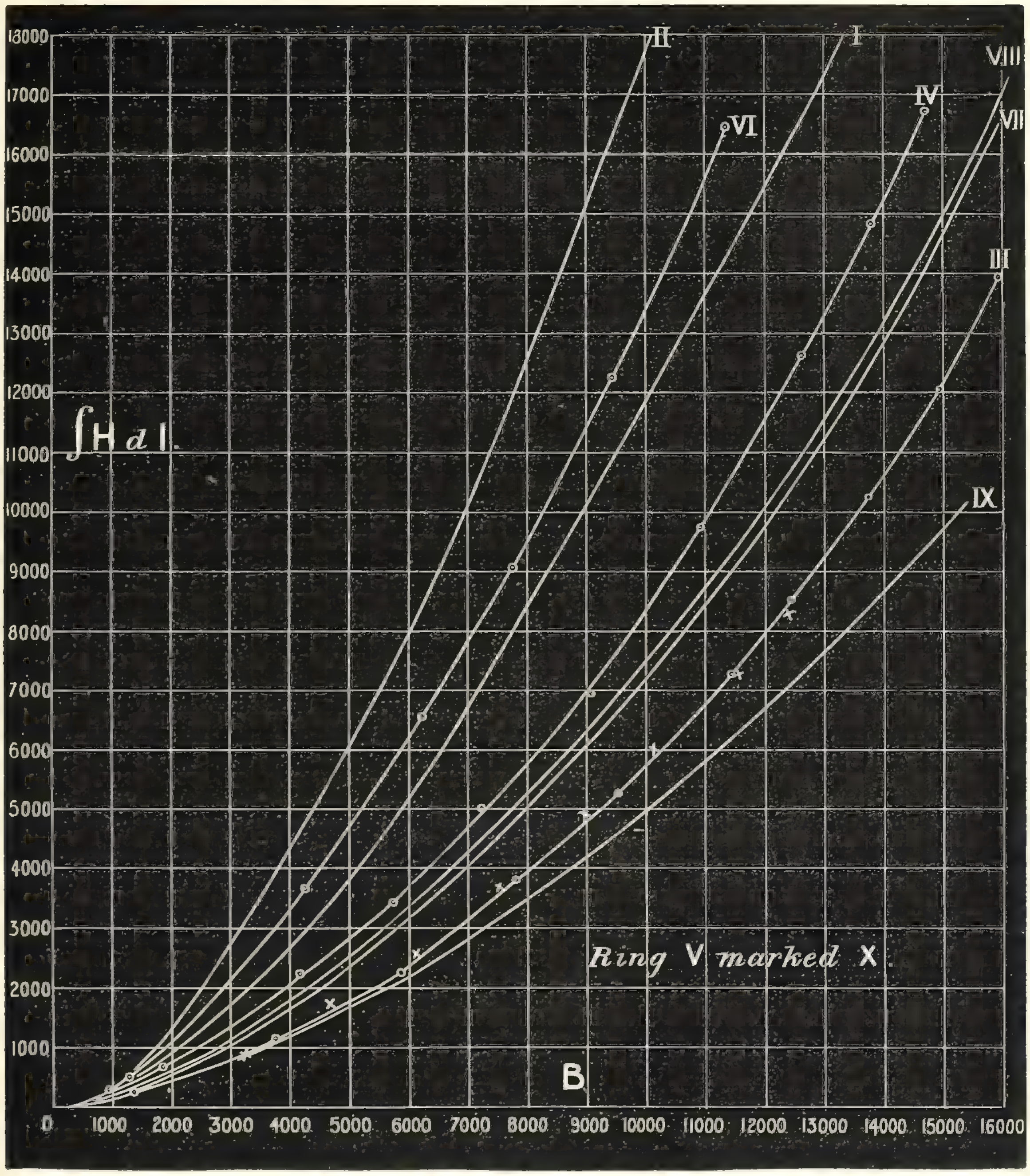
The values of $\int H dI$ and the coercive force for Ring VIII. in its second (annealed) state are as follows :—

RING VIII.—Second state. Soft iron wire of "Hedgehog" Transformer after special annealing. Values of $\int H\,dI$ and coercive force.

No. of cycle.	Limits of H.	Limits of B.	Limits of I.	$\int H dI$.	Coercive force.
1	23·60	16170	1285	17380	2·98
2	15·25	15490	1231	15790	2·95
3	11·24	14870	1183	14800	2·89
4	8·05	13840	1101	12990	2·85
5	6·15	12640	1005	10770	2·73
6	4·66	10940	871	8780	2·56
7	3·63	8640	688	6120	2·44
8	2·50	4000	318	1770	1·71
9	1·92	1390	111	310	1·00
10	1·41	660	53	70	0·70

It will be seen from these figures that the special annealing had hardly any influence on the value of $\int H dI$ and of the coercive force, notwithstanding its very considerable effect in changing the form of the cyclic curves and in improving the permeability. The curve from Ring VIII. in fig. 21 gives the relation of $\int H dI$ to B in this final state. The corresponding curve for the ring in its previous state would, if drawn, be nearly coincident with this one.

Fig. 21.



Curves of ∫HdI and B. Collected results.

For the sake of comparison with these various specimens of iron we add here the values of ∫HdI (copied from ‘Phil. Trans.,’ 1885, p. 556) obtained in tests made in Japan of a piece of soft iron wire. The method used in the Japanese tests differed from our present method; the specimen was a single straight piece, long enough to be practically endless, and its magnetism was changed slowly and was measured by a magnetometer. We have numbered these old tests IX. for convenience of reference.

IX.—Old tests made in Japan of a Soft Iron Wire.

Limits of H.	Limits of B.	Limits of I.	$\int H\,dI$.
75·2	15560	1230	10040
26·5	13700	1090	8690
7·04	11960	951	6590
6·62	11480	913	6160
4·96	10590	842	5560
3·76	8790	699	3990
3·01	7180	571	2940
2·56	5950	473	2190
1·95	3830	304	1160
1·50	1970	157	410

None of the specimens tested at this time have given so low values of $\int H\,dI$ for the higher cycles, though the values in Rings III. and V. are practically identical with those in the Japanese iron for cycles in which B is less than 4000. The superior magnetic softness of this wire (No. IX.) was further shown by the fact that its coercive force after strong magnetization was 1·75, whereas in all the rings now tested the coercive force exceeded 2. At least one other specimen of wire used in the Japanese experiments was as good.

The comparison suggests a doubt whether all that is possible has yet been done by manufacturers of iron to produce a quality specially suitable for such uses as the construction of transformer cores. The conditions of manufacture which secure low hysteresis appear to be imperfectly known, or, at least, to be less generally recognized than electricians might wish. It must, however, be admitted that with respect to hysteresis losses under the comparatively low intensities of magnetization usual in transformers, the sheet iron of Rings III. and V. is very nearly as good as the wire tested in Japan.

This will be apparent from fig. 21, where the values of $\int H\,dI$ are plotted in relation to B for all the rings, and also for the wire in question (No. IX.). The figure shows in a striking way the extent to which the dissipation of energy in a cyclic magnetizing process may vary in different specimens even of soft iron.

To facilitate comparison, we also give a Table of Numerical Values of $\int H\,dI$, measured from these curves, for the various specimens of iron.

VALUES of $\int H \, dI$.

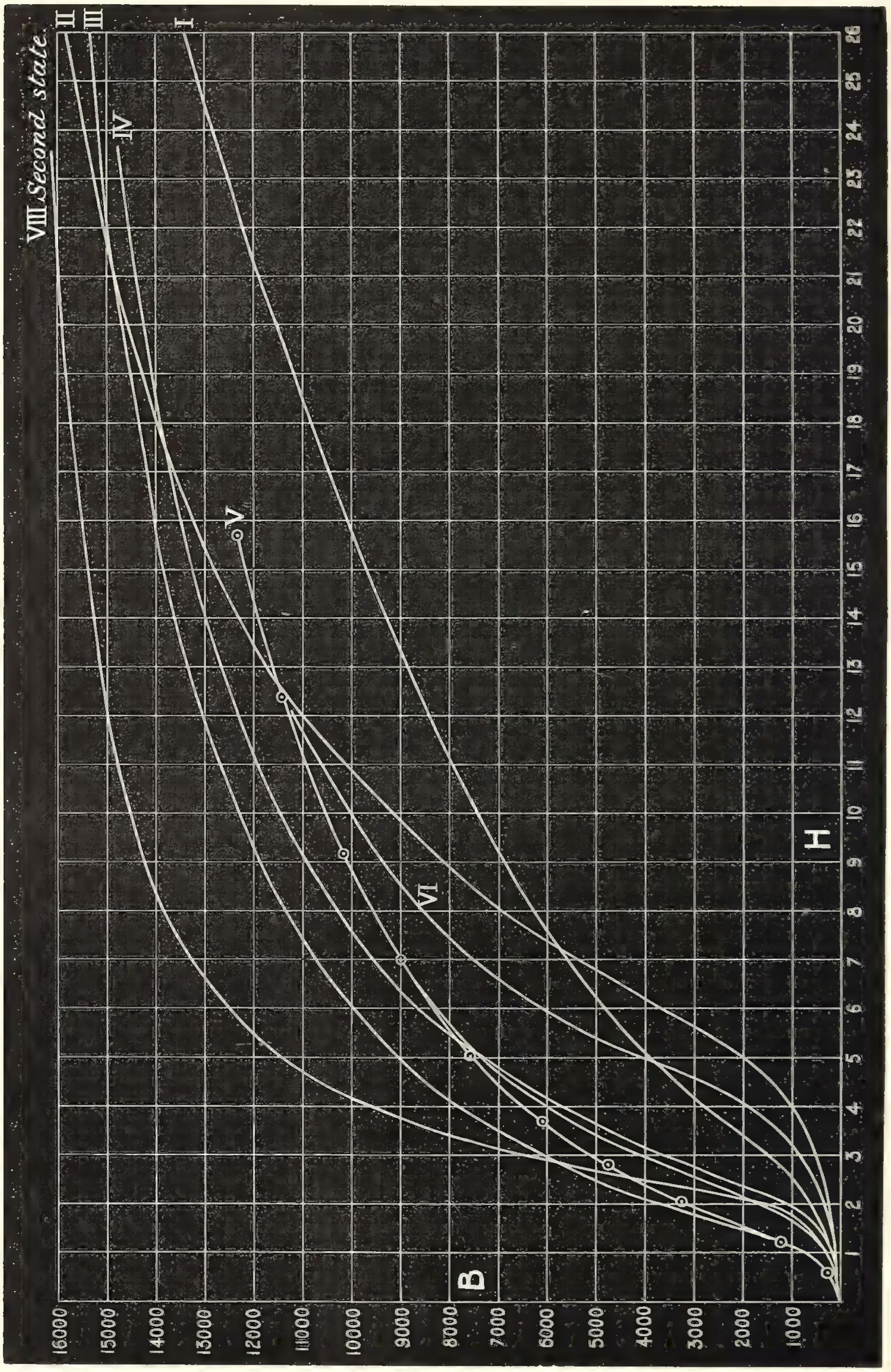
B.	IX.	III.	VII.	VIII.	IV.	I.	VI.
2000	400	420	530	600	750	930	1100
3000	780	800	1050	1150	1350	1700	2150
4000	1200	1260	1670	1780	2030	2600	3300
5000	1680	1770	2440	2640	2810	3800	4700
6000	2200	2370	3170	3360	3700	5200	6200
7000	2800	3150	4020	4300	4650	6600	7800
8000	3430	3940	5020	5300	5770	8400	9500
9000	4160	4800	6100	6380	6970	10100	11400
10000	4920	5730	7200	7520	8340	11800	13400
11000	5800	6800	8410	8750	9880	13600	15600
12000	6700	8000	9750	10070	11550	15400	
13000	7620	9200	11200	11460	13260	17300	
14000	8620	10500	12780	13100	15180		
15000	9730	12150	14600	14900	17300		

The values for Ring V. are nearly the same as for Ring III.

IX. Very soft iron wire (Japanese tests).
III. Sheet iron, 1·95 millim. thick.
V. Thin sheet iron, 0·367 millim. thick.
VII. Iron wire, 0·975 millim. diameter.
VIII. Iron wire (of "Hedgehog" transformers), 0·602 millim. diameter.
IV. Thin sheet iron, 0·47 millim. thick.
I. Fine iron wire, 0·2475 millim. diameter.
VI. Fine wire, 0·34 millim. diameter. Supplied as Swedish charcoal iron.

Again, fig. 22 shows the relation of B to H for the various rings. The values of B are the limiting values in the successive cycles, and are measured, as we have already pointed out, after the corresponding magnetic force has been many times reversed. They are consequently rather lower, especially in the early parts of the curve, than those values of B which a first application of H would produce. Comparing the curves of fig. 22 we see that high maximum permeability does not necessarily imply small hysteresis losses. Ring VIII. with its "square-shouldered" curve has a higher maximum of permeability than any of the other rings, though its hysteresis losses are much greater than those of Rings III. and V. And Rings III. and V. differ decidedly in permeability though their hysteresis losses are as nearly as possible equal. A reference back to fig. 15 will show that the curves exhibiting magnetic cycles may assume a sloping form, implying comparatively little permeability, and yet enclose but small areas. When iron is spoken of as magnetically soft the implication sometimes is that its permeability is high, sometimes that its hysteresis losses are small. The two characteristics are however, in great measure,

Fig. 22.



Curves of B and H. Collected Results.

independent. It is interesting to notice also that the easy slope of some of the soft iron curves in fig. 22 allows them to be crossed by curves for steel, even under moderate magnetizing forces. For instance, a force of 13 *c.g.s.* induces more magnetism in the rather hard steel of Ring II. than in the very soft iron of Ring V. A comparison of permeabilities in respect to any such force as this would furnish no criterion of magnetic softness. It may, however, be added that the specimens which are soft in the sense of having small hysteresis losses excel in permeability in the initial stage of the magnetizing process. Compare, for instance, the relative permeabilities of the rings when B is say 1000: it will be seen that the order in which they stand in regard to smallness of hysteresis loss, and in regard to permeability, is at this stage the same. If softness is to be judged by the B-H curve it is clear that the early portion of the curve is the best guide.

In a paper read before the American Institute of Electrical Engineers* on the "Law of Hysteresis," Mr. C. P. STEINMETZ has discussed certain experiments on the relation of $\int H\,dI$ to B, and contends that the empirical formula

$$\int H\,dI = \eta B^{1\cdot6},$$

where η is a constant factor, is in good agreement with the results of experiment. From the figures Mr. STEINMETZ gives, and from those which our own experiments supply, it appears that within a certain range of values of B this formula may be taken as giving a fairly close approximation to the real value of $\int H\,dI$. As an empirical formula of the kind is of use to designers of transformers, we have been at some pains to examine how nearly and within what range a formula of this type may be taken to represent the facts. It will suffice to refer to one example in some detail.

Taking Ring IV. (thin sheet iron) for which we have a numerous and very consistent series of determination of $\int H\,dI$, extending from $B = 43$ to $B = 14{,}720$, we have tested the constancy of the index ϵ in the formula

$$\int H\,dI = \eta B^{\epsilon}$$

by plotting log B in relation to log $\int H\,dI$.

From $B = 2000$ to $B = 8000$, the curve obtained in this way is a good straight line giving the value 1·475 for the index ϵ, and 0·01 for η. When B is about 8000 the inclination of the line changes, and from $B = 8000$ to $B = 14{,}000$ we again obtain very nearly a straight line giving $\epsilon = 1\cdot70$ and $\eta = 0\cdot00134$.

Again, the measurements obtained from the small-cycle tests of Ring IV., show that from $B = 200$ to $B = 500$ the curve of log B and log $\int H\,dI$ is very closely a straight line, giving $\epsilon = 1\cdot9$.

Where B is above 500 the gradient of the line gradually changes. Taking the region from $B = 500$ to $B = 1000$ the value of ϵ is 1·68, while from $B = 1000$ to $B = 2000$ it is 1·55.

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* 'Transactions of the Institute,' Jan. 19, 1892, vol. 9, p. 3.

From these results it is clear that no formula of the type under consideration, with a constant index ϵ, will serve to represent the results within anything like the limits of experimental accuracy. The index begins by being 2 or nearly 2 (a result which follows also from Lord RAYLEIGH's experiments*): in the ring referred to, this decreases to about 1·47 in the region of high permeability, and then increases again to 1·7 when the "wendepunct" is passed. The changes in the index, indeed, correspond to the passage from one to another of the familiar successive stages in the process of magnetization: comparatively high values of the index are found first in the initial stage of low permeability, and again in the stage of strong magnetization when the permeability is reduced by the approach towards saturation, while in the intermediate stage where the curve of B and H is steep the index is decidedly low. The well-marked changes of gradient curve which characterize the magnetizing are accompanied by scarcely less well-defined changes on the part of the index ϵ and the factor η, in the empirical formula devised by Mr. STEINMETZ.

While, therefore, a formula of this type cannot be admitted to have any physical significance, it may still be serviceable in giving rough approximations for the purposes of the electrical engineer. Though the formula is by no means to be accepted as an equation to the actual curve of $\int H\,dI$ and B, the curve which it gives by a suitable choice of index ϵ and factor η lies fairly close to the actual curve, intersecting it at an intermediate point as well as at two extremes. And it is the case that an index of 1·6, or a number approximating to that, gives a curve lying generally in the neighbourhood of the true curve throughout the range of B which is of most practical importance. In the case of our Ring IV., for instance, the formula

$$\int H\,dI = 0{\cdot}0034\, B^{1{\cdot}6}$$

give values of $\int H\,dI$ which are nowhere (within the range of these experiments) so grossly divergent from the truth as to unfit them for use in calculations connected with transformer design. Thus we have:—

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* 'Phil. Mag.,' March, 1887.

B.	Actual value of $\int H\,dI$.	Value calculated from the above formula.
150	7	10·3
300	31	31
500	80	71
1,000	256	215
2,000	750	650
4,000	2,030	1,970
6,000	3,700	3,780
8,000	5,820	5,980
10,000	8,340	8,540
12,000	11,550	11,430
14,000	15,200	14,630

The divergence is becoming considerable at the top of the range, but elsewhere the agreement is fair. It should, however, be observed that if we wish to make the curve given by the empirical formula coincide with the real curve throughout any short part of its range, other, and often very different values must be given to ϵ and η. Thus, if the part from $B = 2000$ to $B = 8000$ only is considered—a part which includes those values of B which are usual in transformer cores—the formula

$$\int H\,dI = 0{\cdot}01\,B^{1{\cdot}475}$$

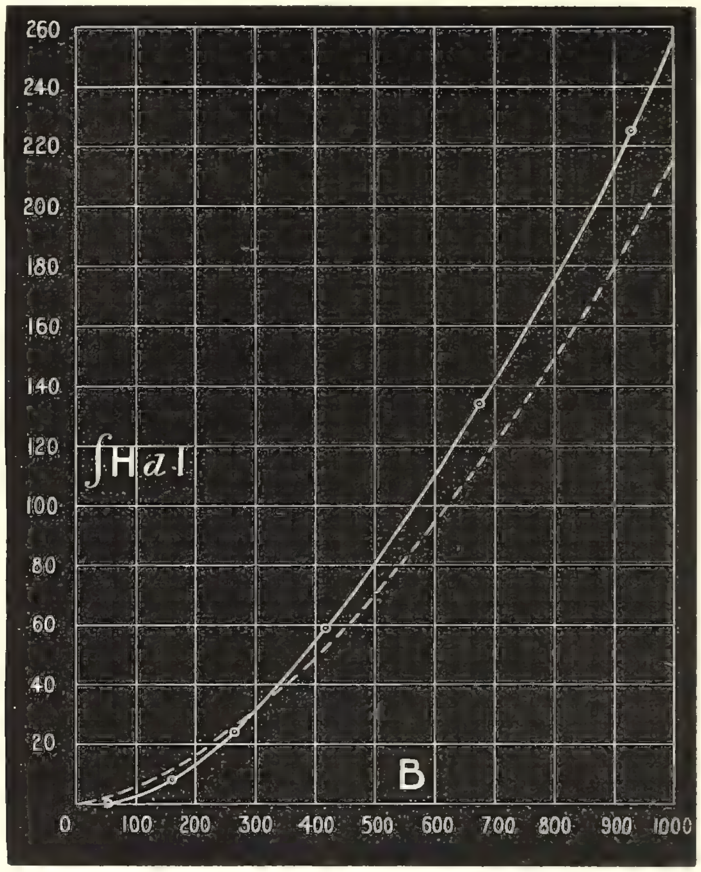
will express the results very much more closely. For example:—

B.	Actual value of $\int H\,dI$.	Value calculated from the expression $0{\cdot}01\,B^{1{\cdot}475}$.
2,000	750	740
3,000	1,350	1,345
4,000	2,030	2,060
5,000	2,820	2,860
6,000	3,700	3,740
7,000	4,650	4,690
8,000	5,800	5,720

Throughout this not inconsiderable range of magnetization (the region, namely, of high permeability) the agreement is good, but other empirical constants are required to fit the earlier, and again the later parts of the curve.

In further illustration of the point, we add a curve (fig. 22), showing the actual values of $\int H\,dI$ in relation to B for the low cycles in the tests of Ring IV., and on the same diagram a dotted line is drawn, to show the value found for Mr. STEINMETZ' formula, namely, $\eta B^{1{\cdot}6}$, the constant η being taken as 0·0034, since that value best accords with the tests of Ring IV. throughout the whole range of magnetization. It will be evident from the diagram that this early portion of the curve is not even approximately expressed by the formula in question. The character and extent of the divergence is, perhaps, more obvious from inspection of the curve, than from comparison of the numbers which have been given above.

Fig. 23.



∫H dI and B for the low cycles of Ring IV. The dotted line represents values of 0·0034 $B^{1·6}$.

Experiments on the Heating Effect of Reversals of Magnetism.

In another section of our experiments direct measurements were made of the amount of heat generated by reversals of magnetism.

Some of the rings employed in the tests which have just been described were again used in these experiments, and under precisely the same conditions.

The method employed (a short account of which was published in 'The Electrician' of October 9, 1891) depended on the use of two exactly similar rings, one of which had its iron core subjected to magnetic reversals, while heat was developed in the other by the passage of a steady electric current. This steady current was adjusted until the rate at which heat was generated by the current in one ring became the same as the rate at which heat was generated by reversals of magnetism in the other ring, equality of temperature as between the two rings being tested by means of a thermoelectric circuit with a pair of junctions, one embedded within the core of each ring. The current which was used to magnetize the one ring passed also round the other, in a coil which had the same dimensions as the magnetizing coil of the first ring, but which was wound non-inductively. Hence, whatever heating effect the magnetizing current had on one ring, it also had on the other, though one ring only became magnetized. Both rings were subjected to precisely the same thermal influence, except that in the active ring the magnetism of the core was periodically reversed, and in the inactive ring a special heating current was used to balance the heat generated in the other by these magnetic reversals. The arrangements were in all cases such that the inactive ring could be made magnetically active and *vice versâ*, without any change in the winding or structure of the rings, by merely changing the electrical connections of the divided coils which were wound in identical fashion on each. In those of the experiments that dealt with rings which had a core of insulated iron wire, the heating current was caused to pass through the wire of the core itself in the inactive ring—a particularly satisfactory arrangement, inasmuch as the heat generated by this current in the inactive ring was then distributed throughout the core in just the same way as the heat generated by magnetic reversals in the active ring. In experiments dealing with sheet metal cores a special coil was wound as close as possible to the core of the inactive ring to take the heating current.

This method of measuring the heat due to magnetic hysteresis by securing a thermal balance between two rings, alike in all respects, was originally devised to give an answer to the question whether, when a transformer is in action, the heating of the core through magnetic hysteresis is the same when the transformer is "loaded," or when it is not; in other words, whether the taking of current from the secondary coil has any influence on the amount of heat generated in the core, the limits of magnetization and the frequency of the reversals being unaltered. Independent experiments by Professor RYAN, Mr. MORLEY, and Professor AYRTON, on the efficiency of transformers, had apparently agreed on showing a marked reduction in the loss of energy due to magnetic reversals when the "load" was put on. Adding a secondary coil to each of our rings to make it into a transformer, we applied the above balance method to compare the rate of magnetic heating when the secondary was open with the rate when the secondary was closed through a low resistance, so as to apply a heavy "load," arrangements being at the same time used to keep the magnetic range and frequency unchanged. The result was that no change could be detected in the rate at which heat was produced by the reversal of magnetism when the "load" on the secondary coil was varied from nothing to an amount greatly in excess of any "load" used in practical transformers.* The conclusion being of this negative kind, a very short account of the experiments will suffice.

The electrical connections are shown in fig. 24. On each ring of the pair a primary coil was wound in two layers, and a secondary coil, also in two layers. In each coil the two layers had the same number of turns, and the number was the same on both rings. The two layers, both of primary and of secondary, could be connected either inductively or non-inductively. They were each wound uniformly round the whole of the ring. On one ring they were connected inductively, and were respectively in series with the non-inductive primary and secondary of the other ring. Hence all direct thermal action of primary and secondary currents was the same for both rings. The magnetic reversals were caused by passing through the primary circuit the current from an alternating dynamo, which could be run at various speeds, and the strength of this current was adjusted, by means of a liquid rheostat, R_1, in its circuit, to secure that the magnetization should have the same limiting value when the secondary circuit was closed (through the resistance R_3) as when it was open. To test the constancy in the limit of magnetization, a voltmeter, V_1, of special construction was used, which was connected to a special induction coil of a few turns wound on each ring. The voltmeter was of the hot wire type; it consisted of a piece of very fine platinum wire, stretched between supports, but deflected sideways, after the manner of the wire in Professors Perry and Ayrton's voltmeter, by a light spring pulling transversely in the middle of the wire's length. The middle portion of the wire stood in the field of a microscope, and its transverse movements were read on a micrometer scale in the eye-piece. This was found to form a very convenient means of reading alternating electromotive forces of one or two volts and under. Owing to the small section of the wire and the freedom from friction which was secured by having no mechanical indicating gear, the instrument responded quickly and exactly to change of electromotive force, and allowed the necessary adjustment of the primary alternating current to be made with great nicety. In most of the experiments a Cardew voltmeter, V_2, was also connected across the terminals of the primary circuit. Its response to small changes was much less rapid

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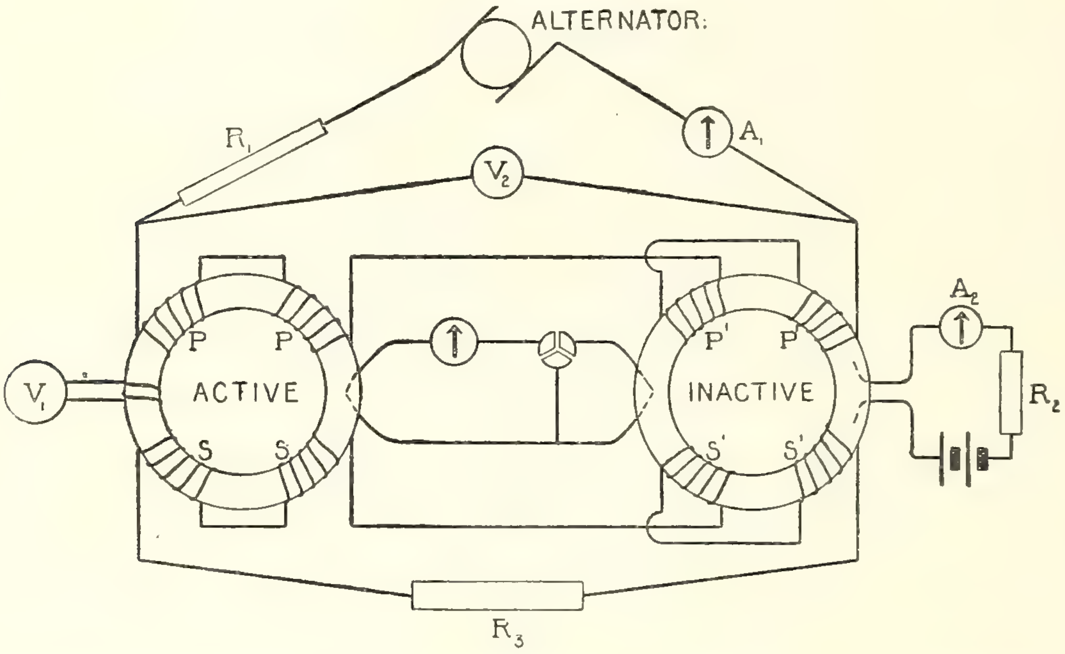
* Since these experiments were made (and briefly published in 'The Electrician' of Dec. 4, 1891), later tests of transformers, by Professor FLEMING and others, have borne out our conclusion that the amount of "load" has no influence on the waste of energy through magnetic hysteresis

Fig. 24.



and certain, but its absolute graduations furnished a convenient means of estimating the intensity of magnetization from the amount of back electromotive force.

The heating due to magnetic reversals in the active ring was determined by seeing what strength of steady current in the core of the inactive ring would maintain a balance in the thermo-electric circuit. The heating current was adjusted, by means of R_2, so that, after both rings had been brought to the same temperature, the two temperatures remained equal while both rose. The thermo-electric junctions were connected to oppose each other through a sensitive galvanometer, but an alternative connection could be made in which one junction only acted on the galvanometer, so that the temperature of the ring might be estimated at any stage in the experiment. It was necessary, in fact, to record the temperature of the ring from time to time, to allow for change in the resistance of the iron wire forming its core.

The following will serve as an example of the readings taken. It refers to an experiment upon a pair of precisely similar rings, one of which was the Ring VII. of the tests already described. The core consisted of iron wire, insulated with cotton. The frequency in this experiment was 133 periods per second, and the greatest value of B was about 7500.

(*a.*) With secondary open :—

Heating current, in ampères.	Reading of Cardew voltmeter.	Resistance of iron core.
0·75 (too strong)	22	5·85
0·65 (too weak)	22	to
0·7 (balance)	22	5·90

(*b.*) With secondary closed through a low resistance :—

Heating current, in ampères.	Reading of Cardew voltmeter.	Resistance of iron core.
0·75 (too strong)	24½	5·80
0·65 (too weak)	to	to
0·7 (balance)	24	5·90

The reading of the microscope hot-wire voltmeter was the same in both parts of the experiment, the difference in the Cardew readings being due to the greater volume of magnetizing current that was required when the secondary circuit was closed. It will be seen that no difference could be detected in the amount of balancing current necessary to maintain equality of temperature. Other experiments gave confirmatory and quite conclusive results. One more may be quoted :—

Frequency 69 periods per second. Greatest value of B about 13500.

(*a.*) Secondary open. Balancing current 236 on arbitrary scale.

(*b.*) Secondary *short-circuited.* Balance with current between 231 and 244.

The heating effect of reversals is, of course, due to Foucault currents, as well as to magnetic hysteresis. An attempt was made to compare the total heating effect, measured directly by reference to the balancing current in the inactive core with the value of $\int H\,dI$ as found in the preceding experiments, with the view of estimating the influence of Foucault currents. But the value of such a comparison depends on our being able to estimate the maximum of magnetization in the cycle, and this is by no means a simple matter. The observed electromotive force gives the mean value of $(dB/dt)^2$, and to calculate B from this it is often assumed that the mode of variation of B may be treated as simply harmonic, in which case we should have

$$\text{Maximum } B = \frac{10^8 . E}{\pi n \, NA \sqrt{2}},$$

where E is the observed electromotive force in a coil of N turns, the area of curve-section of the iron being A and the frequency n. But even if the magnetizing force were simply harmonic it is otherwise with B, whose increment and decrement is largely affected by magnetic hysteresis. The relation of maximum B to mean $(dB/dt)^2$ could be determined if the form of the curve of B and t were known. Recent experiments—amongst others, those of Dr. JOHN HOPKINSON and Messrs. WILSON and LYDALL,[*] in which B as well as dB/dt has been determined as a function of t in a ring submitted to alternating magnetic forces—exemplify how wide the variation from a sine-function is likely to be. In view of this consideration, and of the fact that no such data were at hand in the case of our own tests, we have not followed up this method of directly measuring the heat produced by reversals.

EXPERIMENTS MADE WITH THE MAGNETIC CURVE TRACER.

When the magnetism of a piece of iron is reversed rapidly more work is spent than when the reversal takes place slowly. Whether all the extra work done in the former case is to be set down to Foucault currents is, perhaps, an open question; there may be a genuine time-lag in the process of magnetization, independent of the effect of Foucault currents, though not readily distinguishable experimentally from their effect. This quasi-viscosity, distinct from Foucault-current effects, will, if it exists, require higher values of H to be applied than would otherwise be necessary when any assigned limits of B are to be quickly reached. At high speeds of reversal the curve which shows the relation of the externally applied magnetizing force to B becomes swelled out and encloses a larger area, corresponding to the greater amount of work then done, whether in generating Foucault currents or in overcoming this other possible source of quasi-viscous resistance. That an increased amount of energy is expended is clear from experiments of the kind which have just been referred to, even if the exact interpretation of these results is somewhat doubtful through uncertainty as to the value of B. Direct evidence of the change in form of the B—H, or rather B-current, curve is furnished by the magnetic curve-tracer.

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[*] 'Roy. Soc. Proc.,' 1893.

This instrument, which was exhibited by one of us at the Edinburgh meeting of the British Association (1892), draws the magnetic curve by giving to a small mirror two component motions, one (horizontal) proportional to the magnetizing current and the other (vertical) proportional to the magnetism of the metal under examination. The metal is arranged to form a magnetic circuit which is complete except for a narrow gap between a pair of opposed poles. In this gap a wire is tightly stretched, through which a constant current passes. The wire consequently sags up or down in the gap as the magnetism of the circuit varies, and this gives movement in altitude to the mirror. At the same time, the varying magnetizing current passes through another similar wire tightly stretched between the poles of another magnet. The magnetism of this second magnet is kept constant, and hence the sagging of its wire depends on the varying strength of the current which the wire carries, namely, the magnetizing current of the other magnet. This gives azimuthal movement to the same mirror. A spot of light reflected from the mirror to the screen consequently traces out the curve of magnetism and magnetizing current. The movements are so deadbeat that with a light mirror and tightly stretched wire a magnetic cycle of double reversal may be traversed ten times or even twenty times per second without material distortion from the inertia of the working parts. The variation of magnetizing current must not, however, be sudden, and to secure this a commutator is used in which a revolving boss carrying two zinc plates turns within a cylinder containing a weak solution of zinc sulphate, in which there are two other (fixed) plates of zinc. As the revolving pair of plates reverse their position between the fixed pair, the potential of a battery connected to the fixed plate becomes reversed on the revolving pair, and from them the current is taken to the magnetizing coil.*

By the slow manipulation of a rheostat of any form the magnetizing force may be carried throughout any desired series of changes and the character of the corresponding process of magnetization be studied. The instrument affords a ready means of testing the comparative magnetic quality of given samples of metal. It also allows features in the magnetizing process to be examined which it would be difficult to observe in other ways. We have made a number of experiments with it, some of which may be cited as showing the application of a novel method in magnetic research, and also as bearing incidentally on points that have been already spoken of in connection with the ballistic tests.

Fig. 25 gives the curves drawn by the magnetic curve tracer in a series of tests of a pair of bars built up of soft thin sheet iron. Four cycles are shown, the magnetizing current having had as extreme values three, two, one and one-half ampères respectively. The zero was shifted a little before each cycle to separate the curves. Here the changes of current took place slowly, a pause being made at each of the marked points to allow the position of the reflected spot of light to be noted. The curves are sheared over to the right in consequence of the gap in the magnetic circuit under

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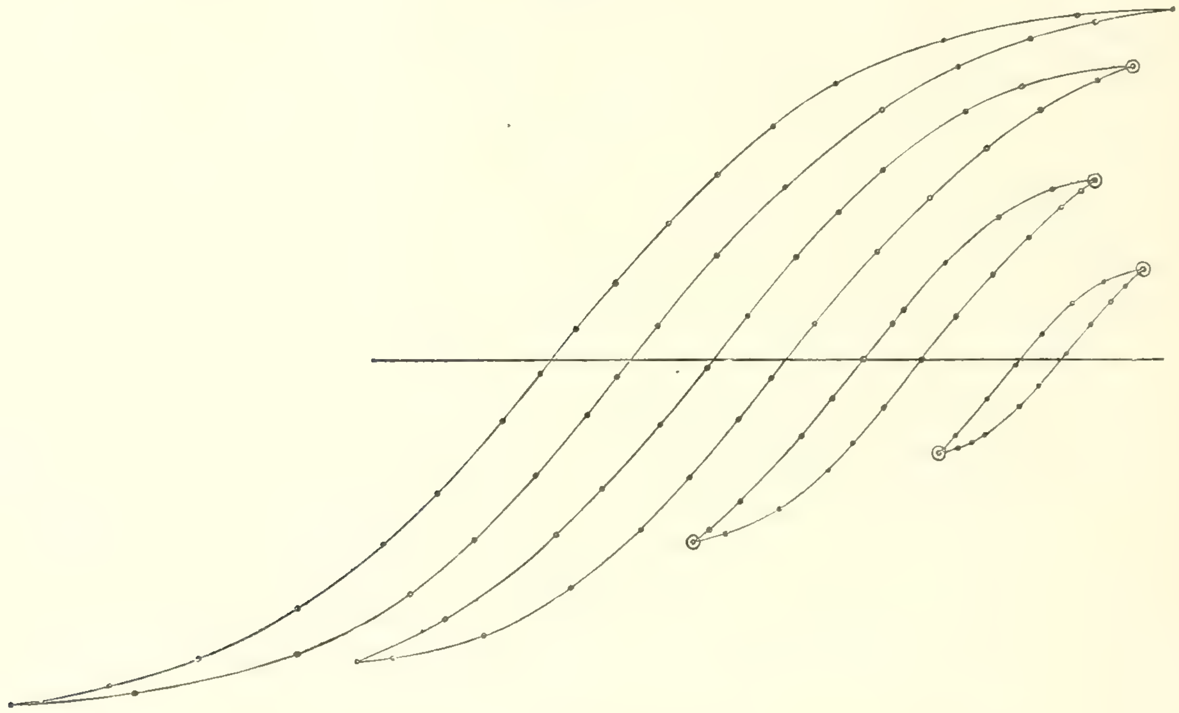
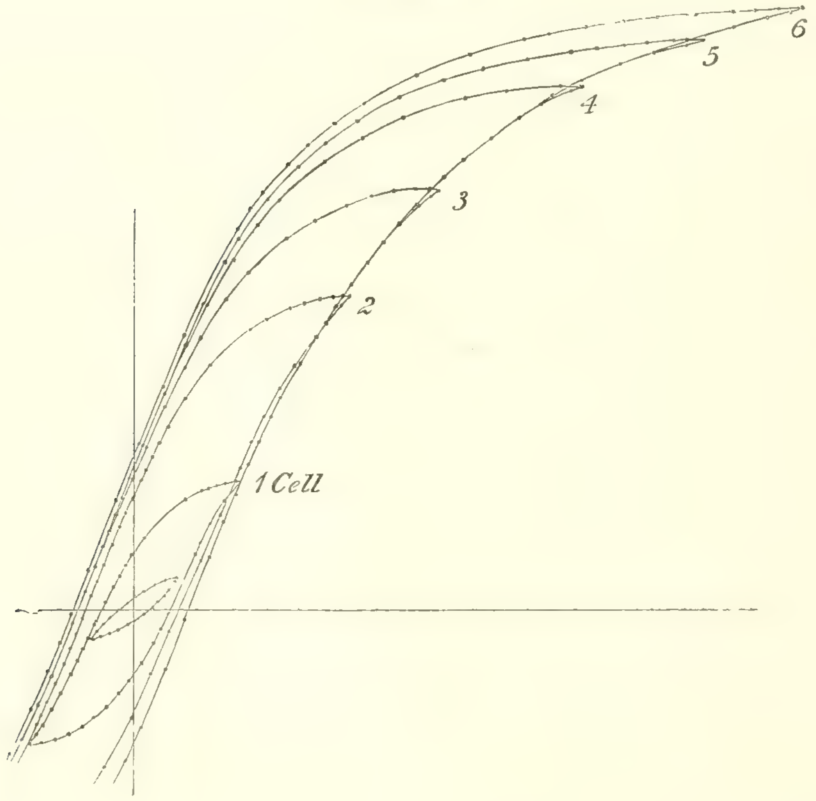
* For diagrams and description of the magnetic curve tracer, see the 'Electrician' of May 26, 1893.

Fig. 25.



Cyclic magnetization of soft iron shown by the magnetic curve tracer.

Fig. 26.

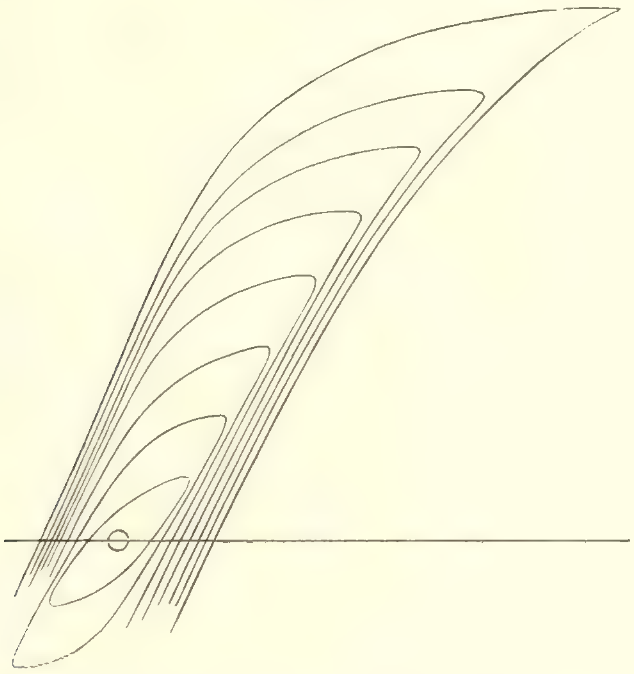


test. The areas of these curves, which are not affected by this shearing, when plotted in relation to B are found to lie on a smooth curve of the type drawn in fig. 21.

Fig. 26 is another test of the same class, made with a different curve tracer and with other iron. Here the successive cycles are shown in their natural relation to one another, without change of zero, and it will be seen that they show well a feature which was referred to in the ballistic tests, namely the projection of the extremities of each cyclic curve over the rising limb of the cycle next above it. Many other examples of curve-tracer tests might be cited as confirming this observation.

Fig. 27 is closely related to fig. 26. It shows a group of curves taken in immediate succession to the group of fig. 26, the iron, the instrument, and the condition of the experiment being in every way the same, except that in fig. 27 the cycles were gone through comparatively fast—at the rate of $3\frac{1}{2}$ cycles per second—whereas in fig. 26 the process was quite slow. The rounded corners and increased wideness of these curves should be noticed. Their increased area shows that more work is here being done on the iron, the effect probably—at least in great part—of Foucault currents. The main portion of the magnetic circuit consisted, in this instance, of well laminated metal, but thick solid pole pieces were used.

Fig. 27.



The curves of fig. 27 were recorded by marking with a pencil the luminous line described upon the screen in the periodic repetition of each cycle. In fig. 28, where the results of a group of tests with a different instrument are shown, the curves were automatically recorded by allowing the spot of light to fall on a sensitive plate.* In all except the last figure in this group each cycle was gone through in about half a second—the commutator being driven uniformly by means of a small electric motor—and the exposure lasted during some twenty to thirty cycles. The curves, therefore, give good evidence of the constancy with which the movement repeats itself. In the last figure in the group of fig. 28 the commutator was slowly turned by hand, and two minor loops were superposed on the main cycle.

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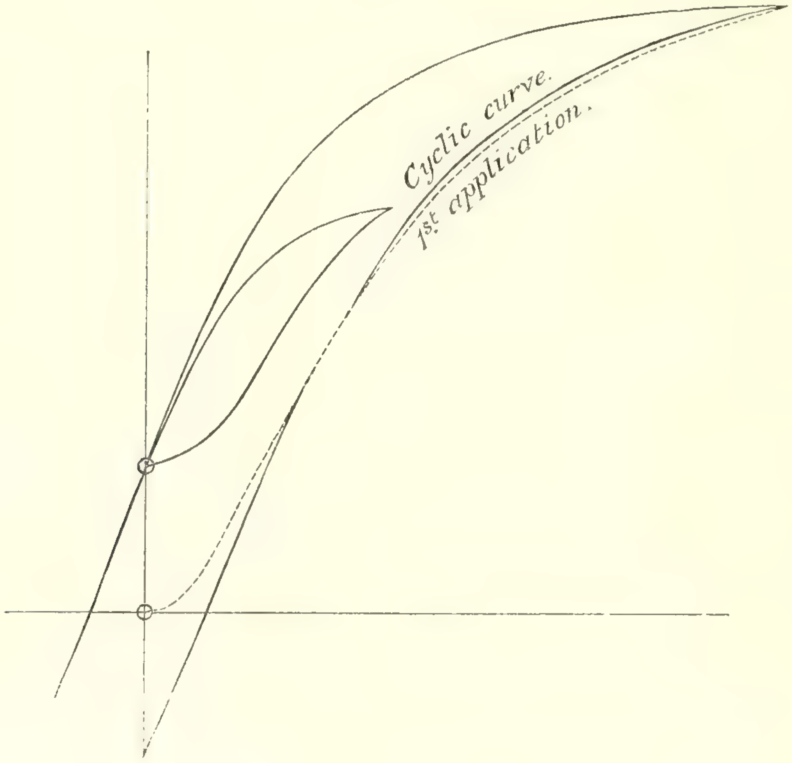
* We are indebted to Mr. A. G. Dew Smith for these photographic records.

Fig. 29 is another test made at the same time, and on the same metal as the tests of figs. 26 and 27. It shows a small loop superposed on a main cycle, and also a curve (drawn in broken line), which is the magnetization curve starting from zero, taken after the metal had been reduced to neutrality by the process commonly known as "demagnetizing by reversals"—namely by numerous reversals of a gradually diminishing magnetic force. Here is found a feature which was first remarked, we believe, by Dr. John Hopkinson, namely, that in magnetizing from the neutral state the curve crosses the rising limb of the cyclic curve. The same feature has been looked for, and always found, in many other curve-tracer tests.

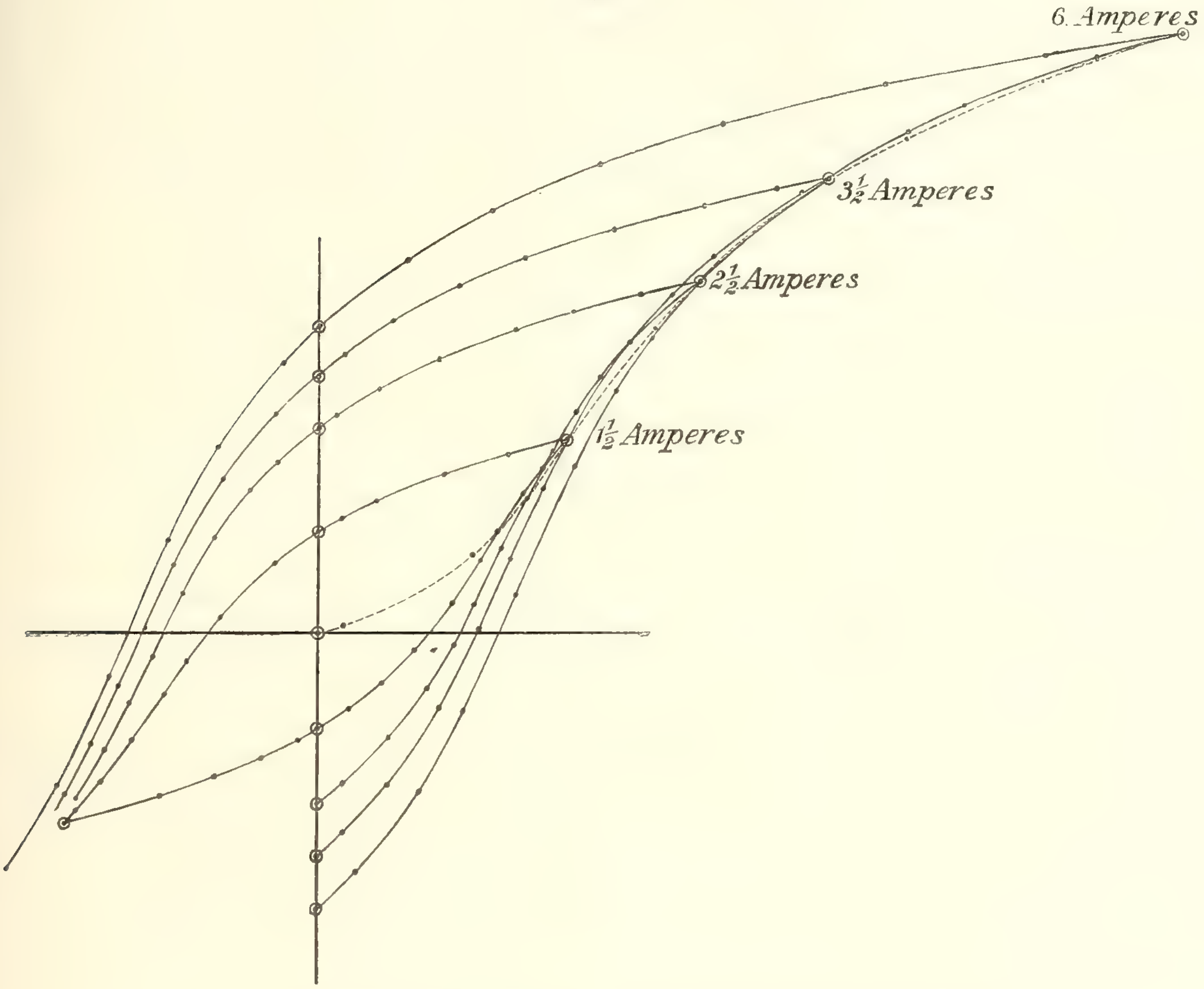
Fig. 29.



This crossing is in agreement with the fact already noticed, that the extremity of one cyclic curve projects beyond the rising limb of a higher cycle. For, after a piece has been put through the process of "demagnetizing by reversals," the magnetizing curve starting from zero passes, as nearly as we can judge, through the projecting extremities of the cycles. Hence it lies outside of the upper part of the rising limb of any cyclic curve.

Fig. 30 gives a further illustration of these remarks. It shows a magnetic curve-tracer test of moderately hard steel, the dotted curve being obtained by magnetizing from zero, after the process of demagnetizing by reversals had been performed. This line will be seen to pass through the extreme points of the several cyclic curves, which thus overlap one another in the manner already described.

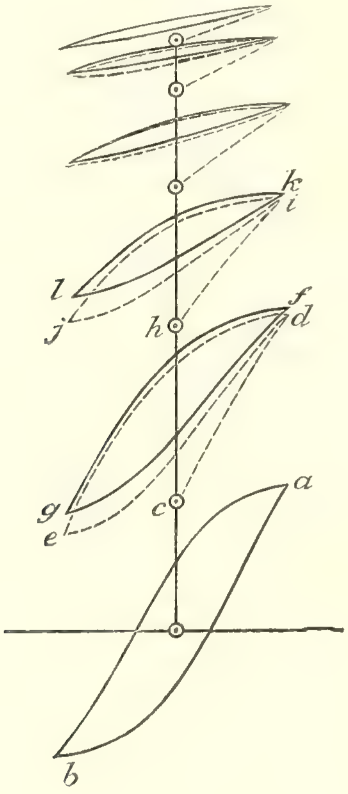
Fig. 30.



Cyclic magnetization of steel, shown by the magnetic curve tracer.

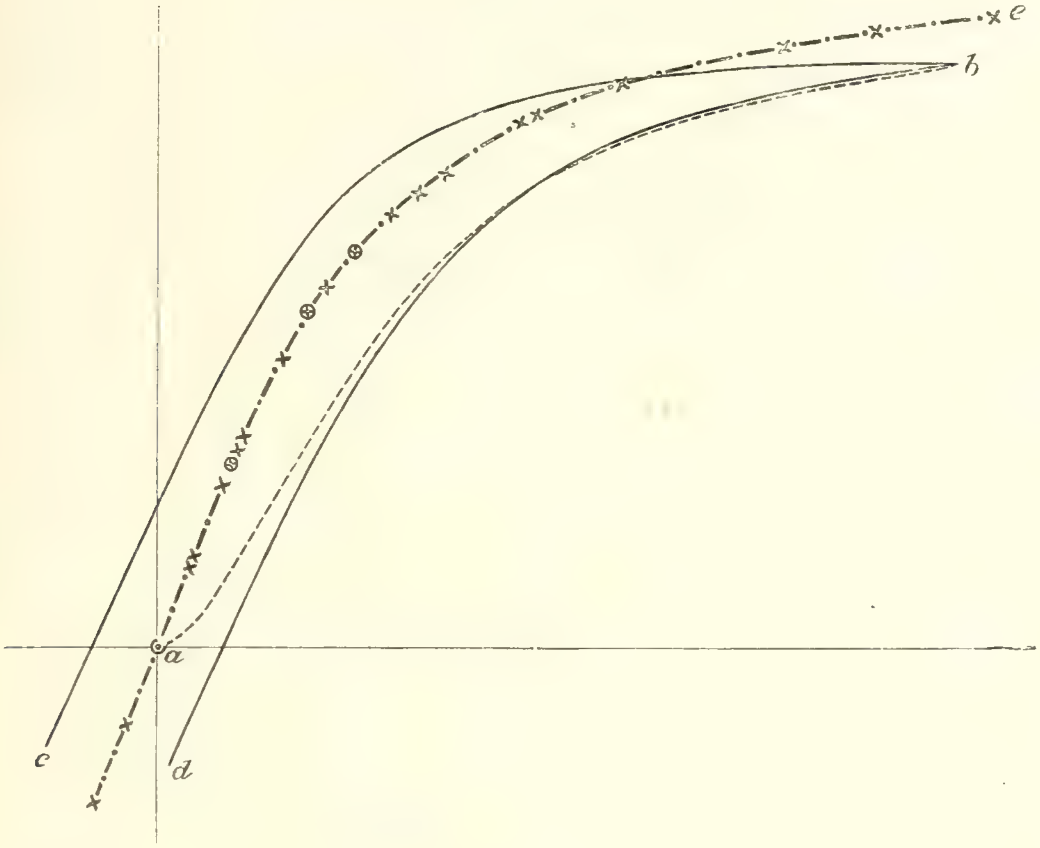
Fig. 31 is a study in superposed magnetizations in soft iron. Here the magnetic curve tracer was arranged with two independent magnetizing coils upon the magnetic circuit under examination. One of these was used to apply a cyclic variation of magnetic force, while the other was used to apply a steady force of greater or less intensity, on which the cyclic force could be superposed. For brevity, we may distinguish the two circuits as P and Q. In the first place, after the iron had been demagnetized by reversals, the current in Q was applied, and was reversed repeatedly between equal positive and negative values, there being, at the time, no current in P. This gave the usual cyclic curve *ab* (fig. 31), from which it will be seen that the magnetizing force then applied by Q was comparatively small. Then (the piece having been again demagnetized) a current was set up in P, which brought the magnetism up to a value represented by the point *c* in the figure. This current in P being now kept constant, the current in Q was applied. This gave the line *cd*; then the current in Q was reversed, giving the dotted line *de*, and re-reversed. Finally, after several reversals of the current in Q, the process of reversal gave the closed figure shown by the full line *fgf*. Next (the current in Q having been stopped), the steady current in P was increased to a value which brought the magnetism up to *h*. Then the application and reversal of Q's current gave the lines, *hi*, *ij*, *jk*, the cycle finally settling down after several reversals into the form shown by the full lines, *klk*. The same description applies to each of the loops which are seen higher in the diagram. The highest ones lie in what is often called the region of saturation. Throughout all these successive operations the current in Q, whose reversal formed the loops, had the same extreme value. The experiment shows how, as might be expected, the reversals of a given magnetizing current produce less and less magnetic change as the total magnetism is increased. It also illustrates one of the minor consequences of magnetic hysteresis, namely, that a periodic force, applied in addition to a constant force, does not immediately produce cyclic effects; it is only after several repetitions of the reversal that the curves become closed. During the interval of "accommodation" (to use the term which has been applied in other instances where the same feature occurs in the phenomena of magnetization) the mean value of the magnetism is rising, as a consequence, one may say, of the molecular shaking which is involved in the process of reversal.

Fig. 31.



A more striking example of the influence of the molecular shaking which is brought about by reversals of a magnetizing force will be found in the experiment of fig. 32.

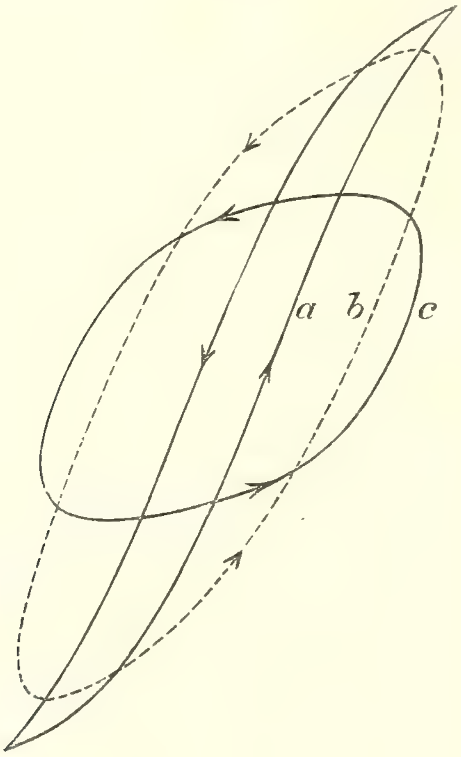
Fig. 32.



This experiment, like the last, was made with a magnetic curve-tracer in which the metal under examination had two independent magnetizing coils wound on it. Through one of these, Q, a current was passed which, when acting alone and carried through a cycle of values, gave the usual curve *abcd*, the dotted line *ab* being obtained after the metal had been neutralized by reversals in the usual way. Next the current in Q was brought to some steady value and a violent molecular shaking was produced by rapid reversals of a current in P which, starting with a rather high value, was gradually reduced to zero while a rapid reversing key in its circuit was kept in motion. It would be inappropriate to call this process by the accustomed name (of "demagnetising by reversals") inasmuch as its effect was to leave the iron more or less strongly magnetized in consequence of the existence of the constant magnetizing force due to the current in Q. The maximum magnetizing force produced in this process of reversals by the current in P was about equal to the maximum force which was excited by the current Q at the extreme point *b* of the curve. Reversals of a force beginning with this high value and decreasing to zero gave the metal so thorough a molecular shaking up that the magnetizing effect of the steady current became, under their influence, almost wholly independent of the previous magnetic history of the piece. In other words the effects of hysteresis nearly disappeared. The points marked thus + in fig. 32 were obtained in this way. They represent the relation of

B to the magnetizing force after reversals of an auxiliary force had been used to produce violent molecular agitation. Some of the points are marked thus ⊕, and in these cases the process of magnetic shaking was performed after the steady current had been so manipulated as to lead to a point on the descending limb of the ordinary cyclic curve : the application of reversals then brought the magnetism down to the marked value. On the other hand the points marked + were reached by applying reversals to raise the magnetism from a point on the ascending limb of the main cycle. It will be seen that the points marked ⊕ lie as nearly as possible on the same smooth curve with those marked +. That is to say, the effects of hysteresis are obliterated by this method of agitating the metal even more completely than they are obliterated by mechanical vibration in soft iron.* Ascending and descending limbs alike become merged on the single curve *ae*.

Fig. 33.



Magnetic curve-tracer curves for soft iron bars.
(*a*) Cycle performed slowly.
(*b*) Period of cycle 3 seconds.
(*c*) Period of cycle 0·43 second.

Reference has been made above to the evidence of time-lag in magnetization which the magnetic curve tracer furnishes. It is in the softest iron that these are most apparent, and especially when the pieces under test are solid or imperfectly laminated. When solid rods of soft iron are used, three quarters of an inch or so in diameter, the movements of the mirror of the curve traces take place as if it were controlled by a dash-pot, or were immersed in a viscous fluid. Even a comparatively slow gradual change in the magnetizing current is followed, after it ceases, by a continued creeping of the reflected spot of light lasting for some seconds,* and, when a cycle of reversal is performed at even a very moderate pace, the curve assumes a wholly different form from that which would correspond to a quite slow process. This is illustrated by fig. 33, where (*a*) is the curve taken quite slowly for a pair of soft iron rods 1·9 centim. in diameter, and (*b*) and (*c*) show the results when the period of a cycle was 3 seconds and 0·43 second respectively.

---

* Compare a corresponding experiment in which tapping was found to make the ascending and descending limbs of the magnetic cycle approach coincidence, 'Phil. Trans.,' 1885, Part II. p. 564–566.

The influence which even the lower of these speeds has on the form of the curve is extraordinary. The maximum of the magnetizing current is nearly the same in all three curves (owing to the self-induction of the magnetizing coil it is a little less in the faster cycles, but enough non-inductive resistance was put in the circuit to make this difference small). It will be seen that the effect of time-lag in lowering the maximum of magnetization (namely, the maximum of the mean magnetization taken over the cross-section of the rods) is enormous; the effect of change in frequency being most noticeable when the period is comparatively long. Again, the influence of time-lag in augmenting the work spent in magnetic reversals is conspicuous, and here, again, it is remarkable how great this influence is at very moderate speeds of reversal. The exact form which the curve takes at any assigned speed will, of course, depend on what particular function of the time the applied electromotive force is caused to be, as a consequence of the exact form of the commutator plates and other conditions of the experiment. In the present case it appeared that the applied electromotive force was made to vary rather quicker near its mean values and rather less quick near its extreme values than would have been the case had it been a simple harmonic function of the time.

It is noticeable that the area of the curve increases at first and finally decreases as the frequency is increased. In other words, with a given maximum of magnetizing current the work expended in reversals is a maximum, per cycle, for some one frequency. This, of course, results from the fact that at higher frequencies the magnetization fails to penetrate sufficiently into the interior of the bar.

In steel these phenomena are much less apparent than in soft iron, but they take place to some extent. Fig. 34 shows the results of a similar experiment made with a pair of bars of tool steel of the same dimensions as the iron rods of fig. 33. Here again (*a*) shows the effects of perfectly slow reversal, while (*b*) and (*c*) are cycles performed in periods of 3 seconds and 0·6 second respectively. In both of these experiments (figs. 33 and 34) the cycles marked (*b*) and (*c*) were produced by using an electric motor to drive the commutator, regularity of speed being essential to steadiness of the figure, especially at the slower speeds.

Whether these manifestations of time-lag in magnetization are entirely explicable by reference to Foucault currents is not easy to answer. It appears possible that another cause may contribute to produce time-lag. In the upsetting of a group of molecular magnets time is needed for the disturbance to gradually spread from the point, or points, where the mutual controlling forces which produce equilibrium are first overcome. Let a group of pivotted compass needles, for example, be subjected to a gradually augmented deflecting magnetic force; the first to yield will in general be some outlying member; its yielding will reduce the stability of its neighbours, and the disturbance spreads to them and from them to others. In a bar of iron we may conceive the molecules next the surface to be less strongly controlled than those in the interior, and to be, therefore, more likely to act as origins of disturbance when the equilibrium is upset by application of magnetizing force. From this point of view, the greater amount of surface a mass of iron presents the more quickly it is likely to take up the full amount of magnetic induction proper to any applied magnetic force, and hence in laminated iron the influence of time-lag should be less conspicuous than in a solid piece. To distinguish an effect of the kind here suggested from the effect of Foucault currents is evidently difficult, since both effects depend on the metal being tested in relatively large solid masses. A possible experiment might be arranged in which a rod built up of a number of concentric cylinders should be compared with an equal rod of the same iron built up of a number of sectors parted by radial planes. Both modes of lamination would reduce whatever part of the time-lag is due to the cause just suggested, but the second mode only would be effective in preventing Foucault currents.

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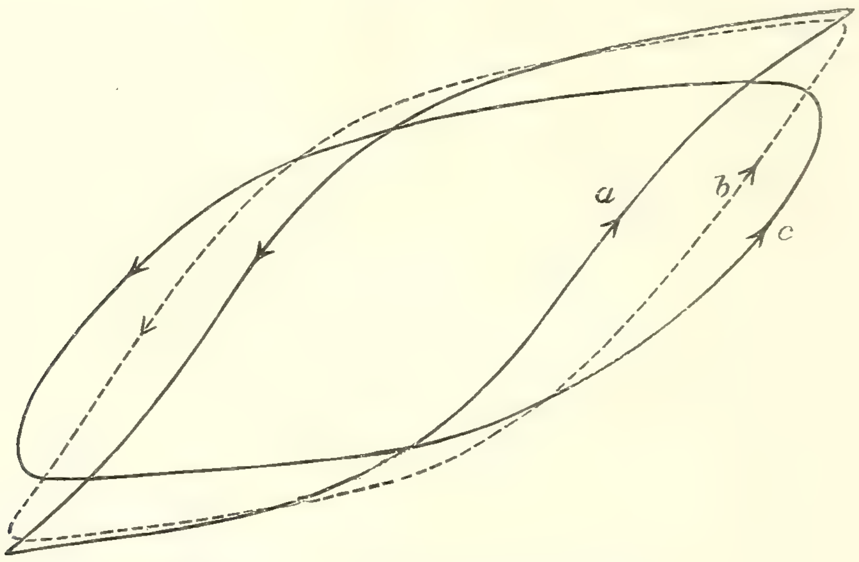
* Compare the direct magnetometric observations described in 'Roy. Soc. Proc.' June 20, 1889.

Fig. 34.



Magnetic curve-tracer curves of steel.
(*a*) Cycle performed slowly.
(*b*) Period of cycle 3 seconds.
(*c*) Period of cycle 0·6 second.

It is clear from the experiments of fig. 33, and even those of fig. 34, that ordinary ballistic tests are liable to serious error when the specimen tested is a solid bar of any considerable section, especially when the material is soft iron, unless the ballistic galvanometer has a much longer period than is usual. The need of insulation, as well as lamination, is well exemplified by the case of Ring VIII., described above at p. 1008.

Experiments with Groups of Pivotted Magnets.

The remainder of our experiments relate to the molecular theory of magnetic induction discussed in a former paper by one of us.* It was there shown that, if we adopt Weber's view that the molecules of a magnetizable metal possess magnetic polarity, the phenomena of hysteresis and other prominent characteristics of the magnetizing process follow as consequences of the control which the molecular magnets exert on one another by their polar forces. Thus, in the application of magnetic force to an unmagnetized piece, the molecular magnets are at first deflected stably, and in this stage there is no hysteresis. But when the force is sufficiently increased certain molecules are deflected so far as to become unstable, and a breaking up and re-arrangement of molecular groups supervenes. Finally, after this process has occurred throughout the piece a third stage is entered on, when further stable deflections through a small range bring about the completely "saturated" state. Hysteresis is found whenever any of the molecules pass from one stable configuration to another through a phase of instability, and this transition, being mechanically irreversible, involves a dissipation of energy. It was shown that this conception of molecular structure gave an explanation of very many known facts regarding the magnetic quality of iron and steel, and that some of these may be effectively illustrated by means of a model consisting of a group of small magnets pivotted like compass needles on fixed centres, and placed near enough to one another to allow their mutual polar forces to be strongly felt.

In further elucidation of this theory, we have made a number of experiments with such a model. In some of them as many as 130 little magnets have been used. These were cut out of sheet steel. They were mostly lozenge-shaped, and were about an inch long and half an inch wide, but some were smaller; each had a central recess punched in it to form a socket for the supporting pivot, which was an inverted drawing-pin with a sharp point. The pins were stuck through a sheet of paper to hold them at the desired distance apart, and the paper was laid on a horizontal sheet of glass which stood between a pair of coils, through which current was passed to produce a deflecting magnetic field. Two layers of such magnets were generally used, one above the other. The aggregate magnetic moment of the group was measured by means of a mirror magnetometer, which was furnished with an adjustable compensating coil to neutralize the direct effect of the current in the two field-coils of the model.

The group having been well stirred up and allowed to settle in the absence of directing force, the current in the coils was slowly increased by means of a liquid rheostat, and galvanometer readings of it were taken, along with magnetometer readings of the aggregate moment. The numbers (with arbitrary scales) are given below, and are drawn as a curve in fig. 35. The highest value of the current was strong enough to bring the group of magnets to a configuration approaching "saturation." It will be seen that the three stages of the magnetizing process are well reproduced in the behaviour of the model, and, notwithstanding the limited number of "molecules" which took part in the operation, the readings lie fairly well on a smooth curve, which has all the general character of the curve of magnetization in actual iron.

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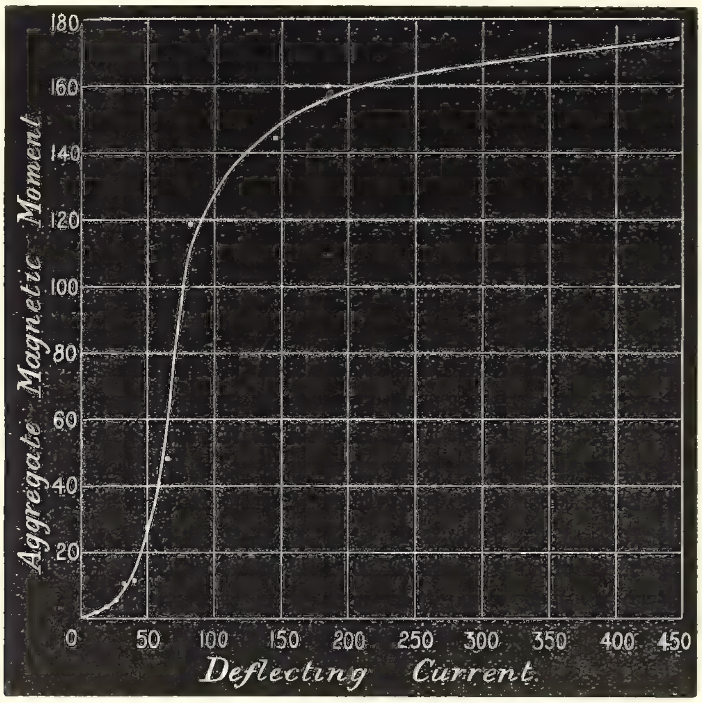
* "Contributions to the Molecular Theory of Induced Magnetism," 'Roy. Soc. Proc.,' vol. 48, p. 342 (June, 1890).

Fig. 35.

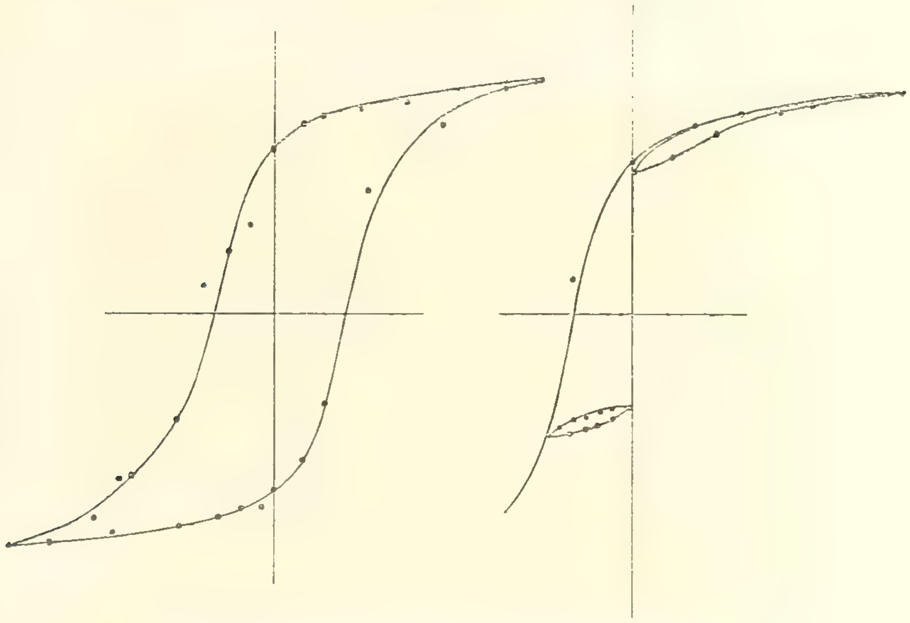


Curve showing the behaviour of a group of 130 pivotted magnets subjected to increasing directive force.

Current.	Moment.	Current.	Moment.
0	0		
4	0	58	41
10	1	63	48
17	3	83	119
26	5	145	145
33	11	184	160
41	11	447	174
50	28		

In the same way fig. 36 shows the results obtained in carrying a group of pivotted magnets through a complete cycle, by double reversal of the directing field, after the cycle had already been gone through once or twice. Fig. 37 shows an immediately succeeding experiment on the same group, designed to illustrate the formation of small loops by the removal and reapplication of magnetic force. In these operations the behaviour of the model agrees in all respects with that of a magnetic metal.

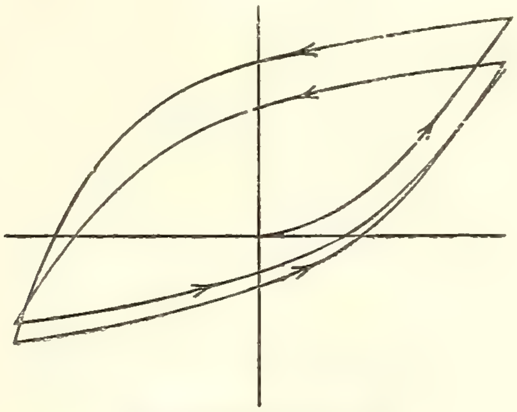
Fig. 36. Fig. 37.



Cyclic processes in a group of pivotted magnets.

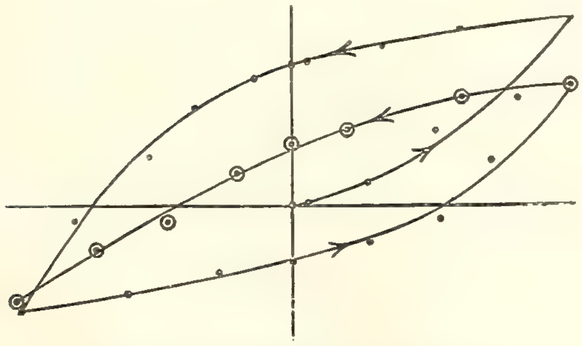
The same close correspondence is found even in less obvious features of the magnetizing process. This is illustrated in figs. 38, 39, and 40.

Fig. 38.



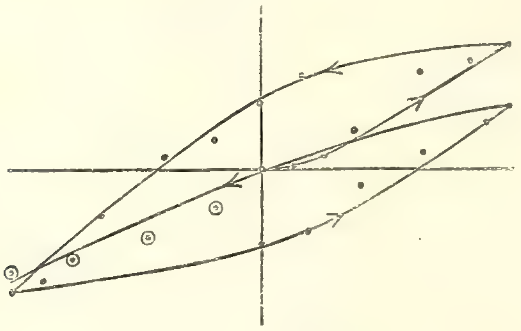
Magnetisation of steel.

Fig. 39.



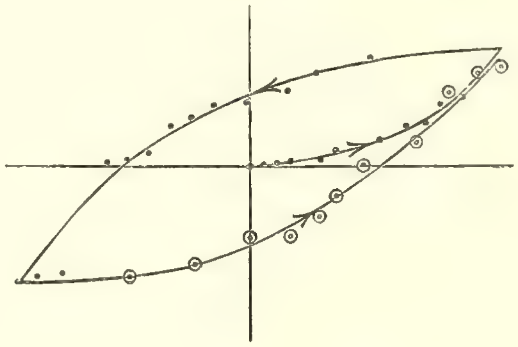
Corresponding process in model.

Fig. 40.



When a piece of previously unmagnetized iron or steel is acted on by a magnetic force sufficiently strong to carry the metal to the stage of high permeability, but not strong enough to make it approach saturation, and this force is then caused to vary between equal positive and negative values, it is found that the first application induces more magnetism than is induced by subsequent applications, and that the changes of magnetism do not become cyclic until the force has been several times reversed. And during this gradual approach towards a cyclic *régime* the range through which the magnetism varies between its positive and negative extremes becomes reduced. These effects of hysteresis are well shown by fig. 38, which is copied from a former paper,* and represents a test of a piece of annealed steel.

Fig. 41.



The same phenomena occur in the model, provided the little magnets have been previously shaken up and allowed to settle in the absence of any directing field : a condition which may be supposed to correspond with the annealing of the solid metal. Fig. 39 is a characteristic instance. It represents a test of the same group of pivotted magnets as was used in figs. 36 and 37. After all effects of earlier deflecting fields had been eradicated by shaking up the group, a rather weak directing force was applied, first to the positive side, and then this was reversed three times. At the third reversal the changes of aggregate moment on the part of the group had become nearly cyclic, but with a range of variation which was decidedly less than that indicated by the first application of the field. Corresponding results were found in another experiment where the first application of the field was towards the side called negative.

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* 'Phil Trans.,' 1885, Plate 61, fig. 28. See also fig. 26 on the same plate.

Again, using a model in which there were only 36 magnets, notes were made of the position assumed by each member of the group when the directing field was (1) applied, (2) reversed, (3) re-reversed, and so on. It was seen that the configuration assumed in the first application was not reproduced in the third, nor that of the second in the fourth. But the fifth application made all the members of the group take up the same position as they took in the third, and again the sixth agreed with the fourth.

Further, tests of iron and steel show that if we have to deal, not with a previously unmagnetized piece but with a piece on which the process of "demagnetizing by reversals" has been performed, the phenomena which have just been described are not found, or, at least, are much less conspicuous than in virgin metal.

They reappear after the primitive state is reproduced by annealing. In this respect the molecular state of iron which has been demagnetized by reversals of a strong magnetic force (diminishing to zero) differs from that of a piece which has been demagnetized by annealing—or even, it may be added, by mechanical vibration.*

The same difference is found in the behaviour of the model after the two corresponding kinds of preliminary treatment. If a neutral initial state has been arrived at by a casual stirring up of the little magnets, the model behaves in the way shown in fig. 38. On the other hand, if the neutral state has been produced by repeated reversals of a strong deflecting field, very slowly performed, so as not to throw the group into general agitation, and gradually reduced in range to zero while the reversals go on, the displacements of the magnets under a subsequently applied weaker field are found to fall at once into a cyclic *régime* when the field is reversed. An example of this is given in figs. 40 and 41, which show experiments made with a group of 120 magnets. In fig. 40 the magnets had been shaken up to begin with, and, in consequence, when the magnetic field was applied the displacements were not at once cyclic. Fig. 41 shows the behaviour of the same group under precisely the same changes of deflecting field after the process of "demagnetizing by reversals" had been used to bring the group into a neutral state. Here we find a cyclic *régime* established at once, and as near an approach to symmetry in the curve as a model with a limited number of magnets can well be expected to show. Observations made on such a model are necessarily somewhat rough, and too much stress must not be laid on particular features of the curves. But in several repetitions of these experiments the same general distinction has always been apparent in the behaviour of the model after one and the other mode of preliminary treatment, a distinction which, as we have seen, forms a striking analogue to what is observed in actual iron or steel.

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* In illustration of this difference, reference may be made to figs. 26 and 27 in Plate 61, 'Phil. Trans.,' 1885.

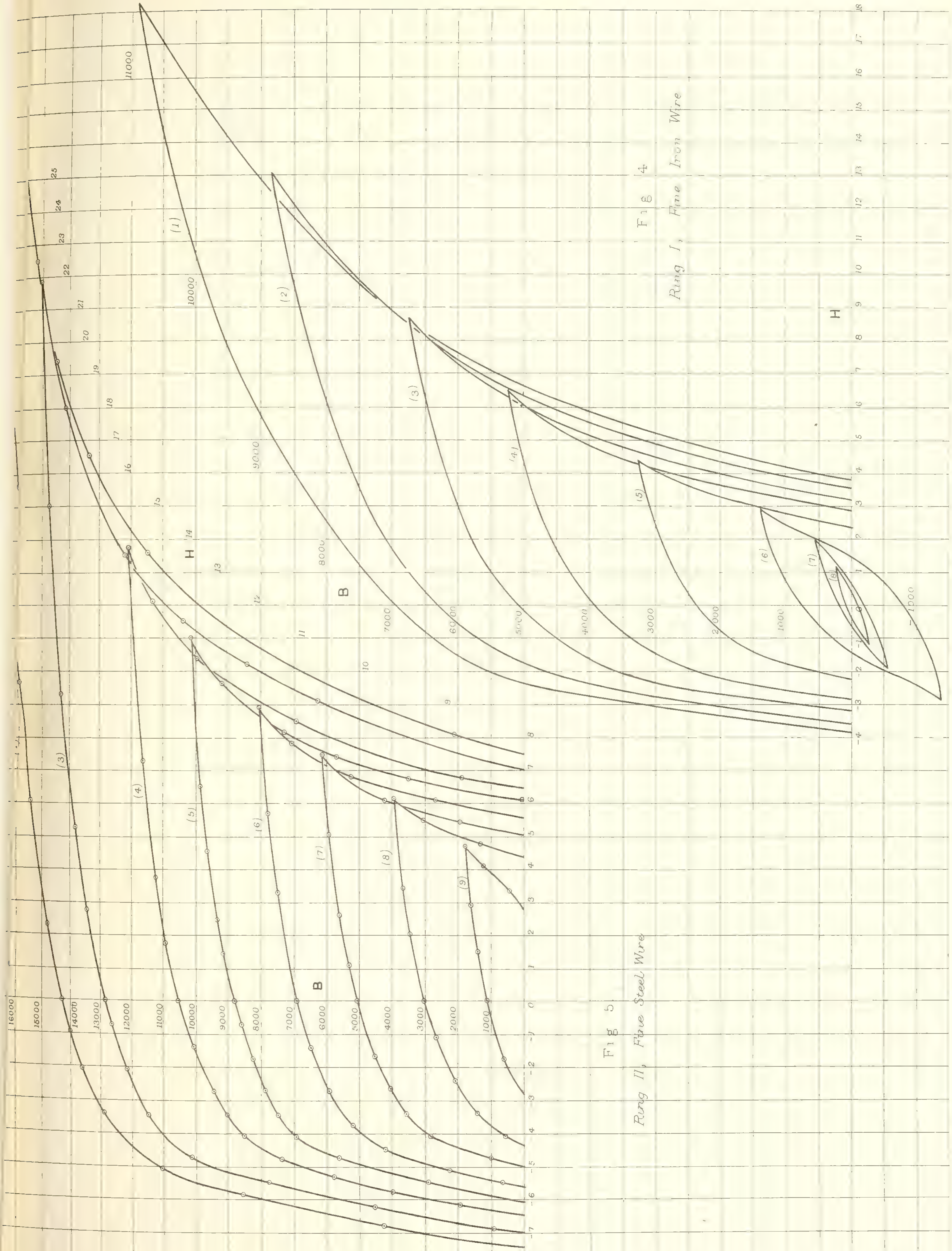
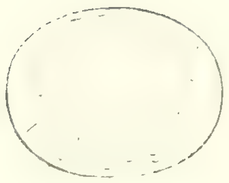


Fig 4. Ring I, Fine Iron Wire

Fig 5. Ring II, Fine Steel Wire

West, Newman, lith.



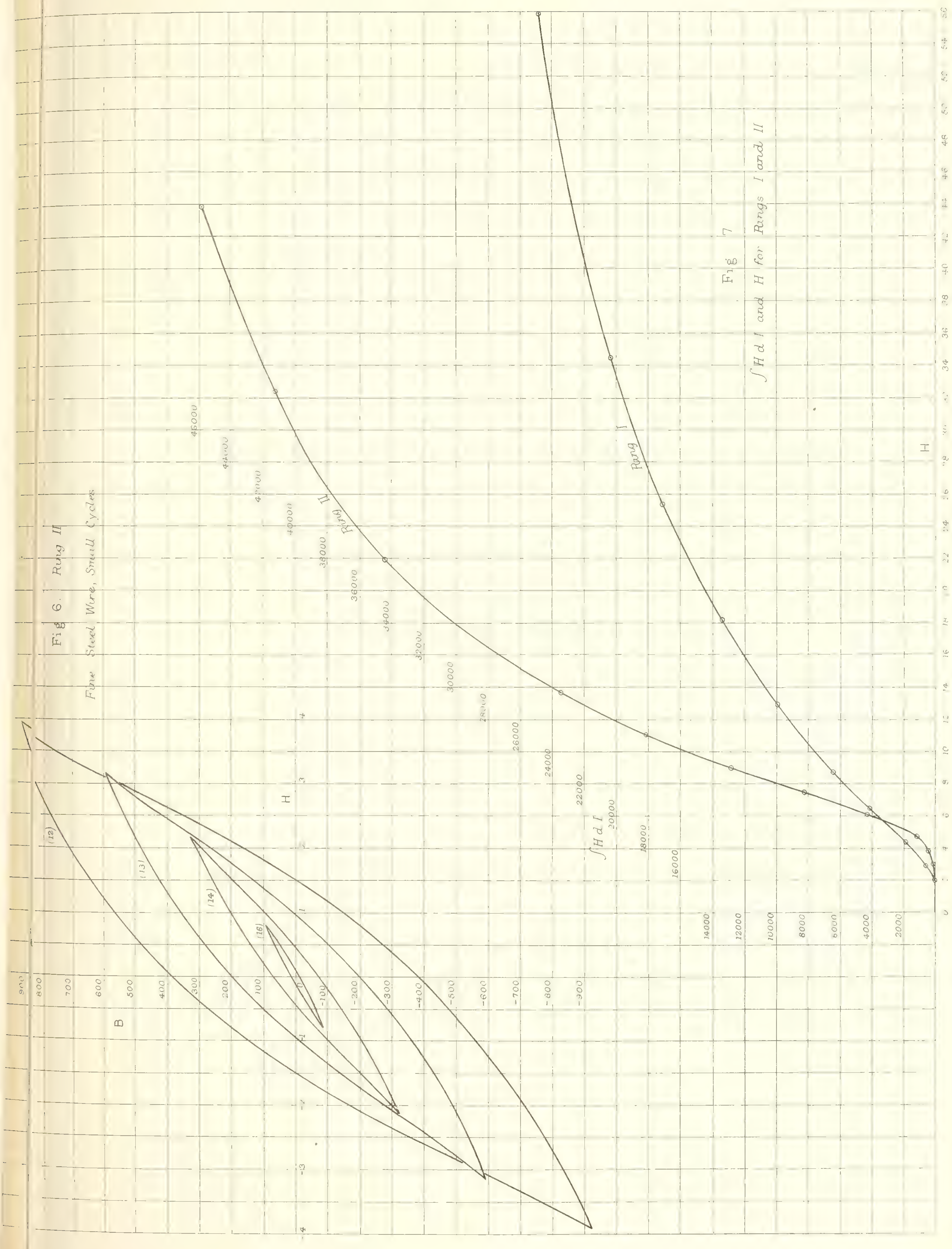


Fig 6. Ring II
Fine Steel Wire, Small Cycles

Fig 7
∫H dI and H for Rings I and II

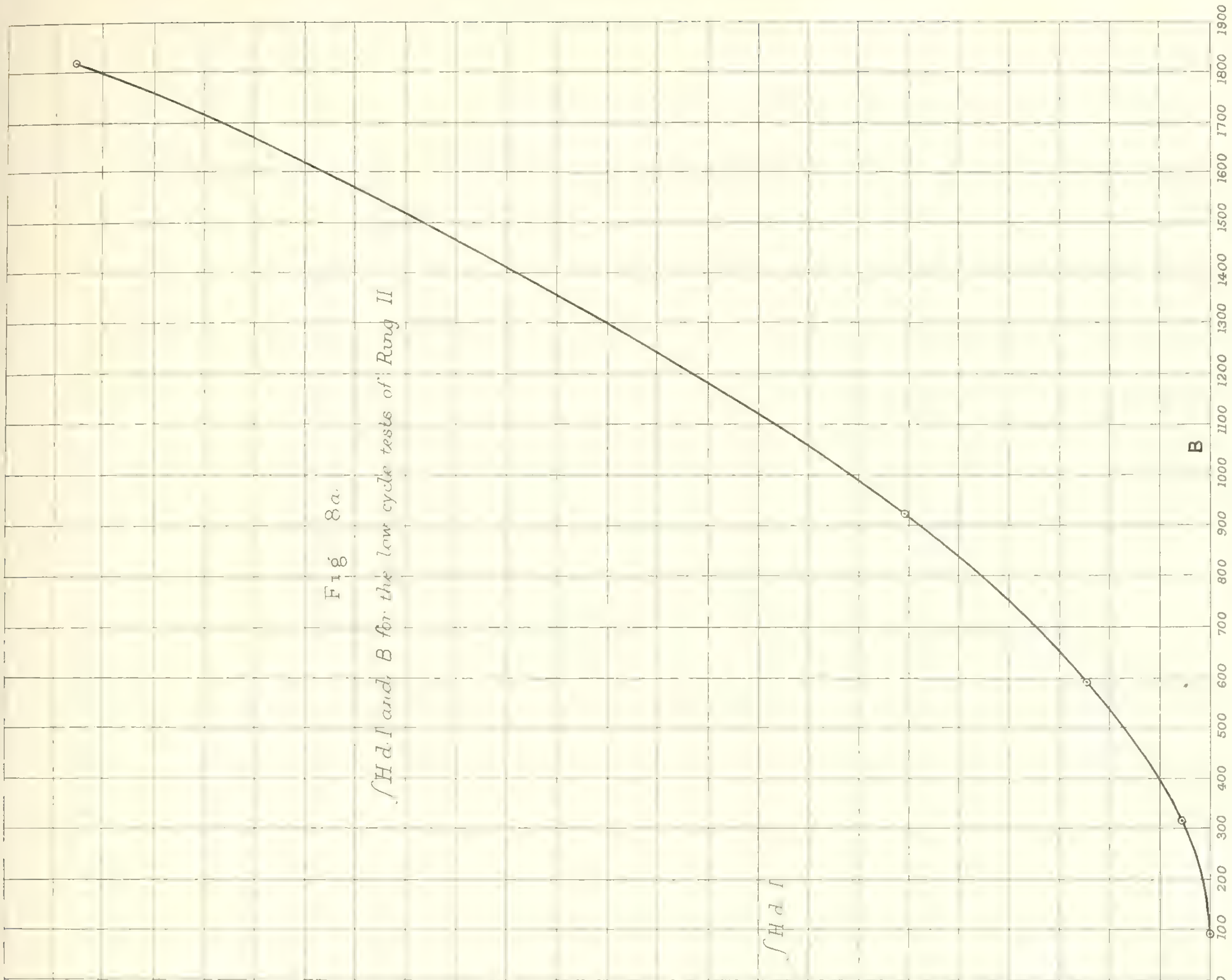


Fig. 8a.

∫H d I and B for the low cycle tests of Ring II

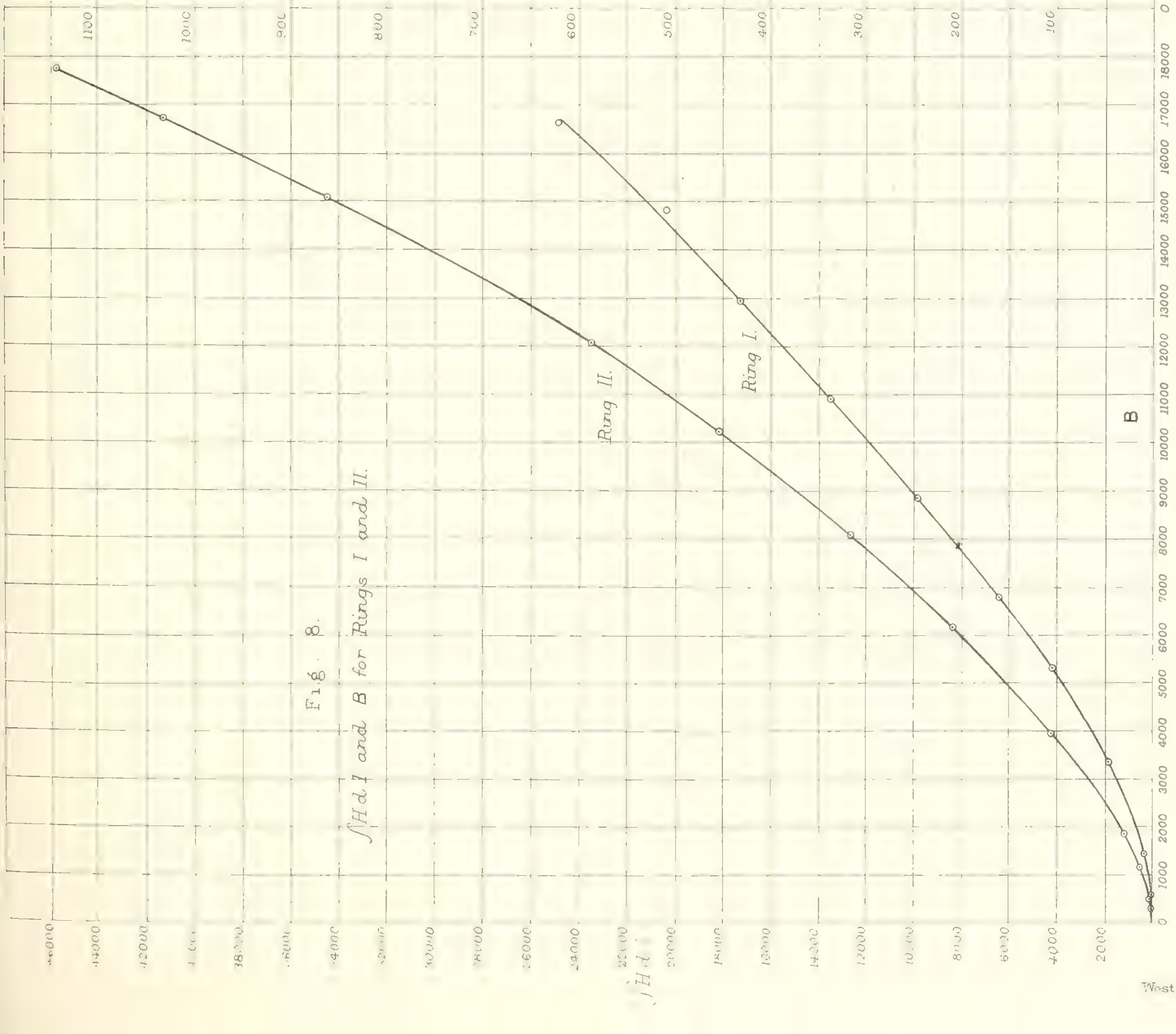
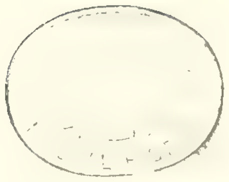


Fig. 8.

∫H d I and B for Rings I and II.

West, Newman lith.



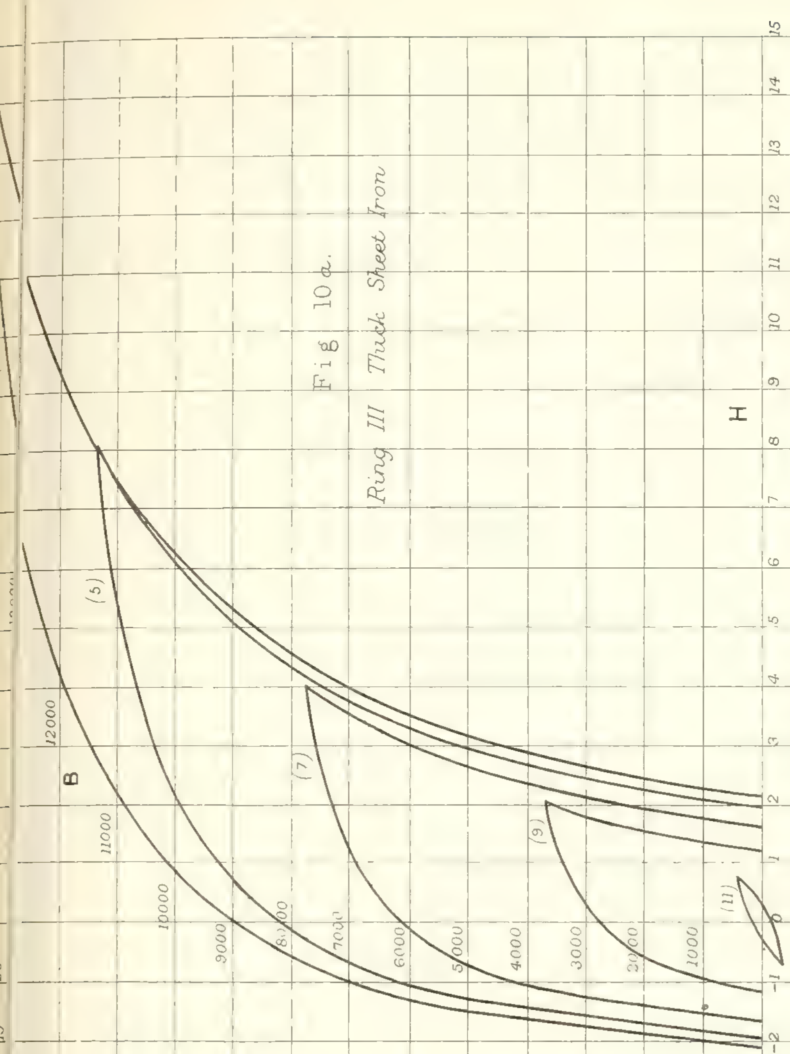


Fig. 10 a.
Ring III Thick Sheet Iron

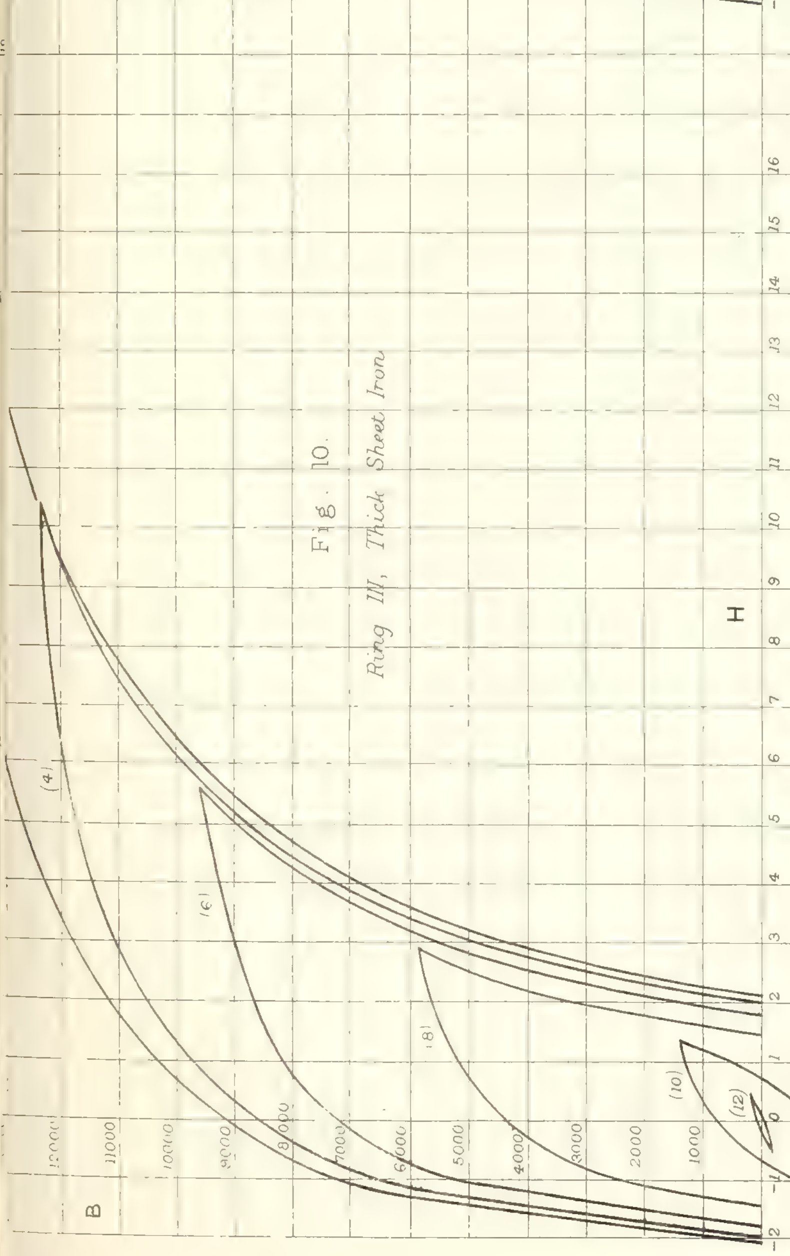


Fig. 10.
Ring III, Thick Sheet Iron

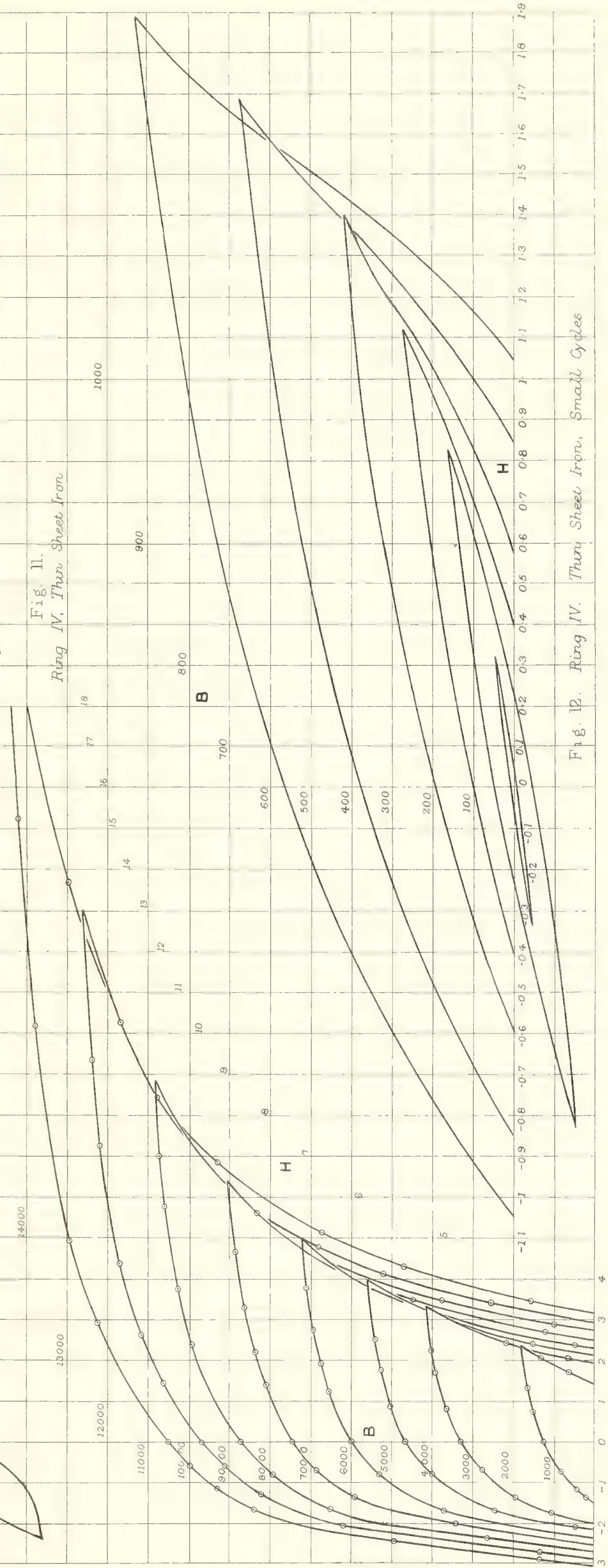
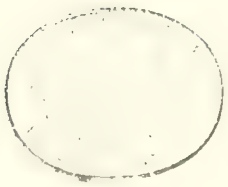


Fig. 11.
Ring IV, Thin Sheet Iron

Fig. 12. Ring IV. Thin Sheet Iron, Small Cycles



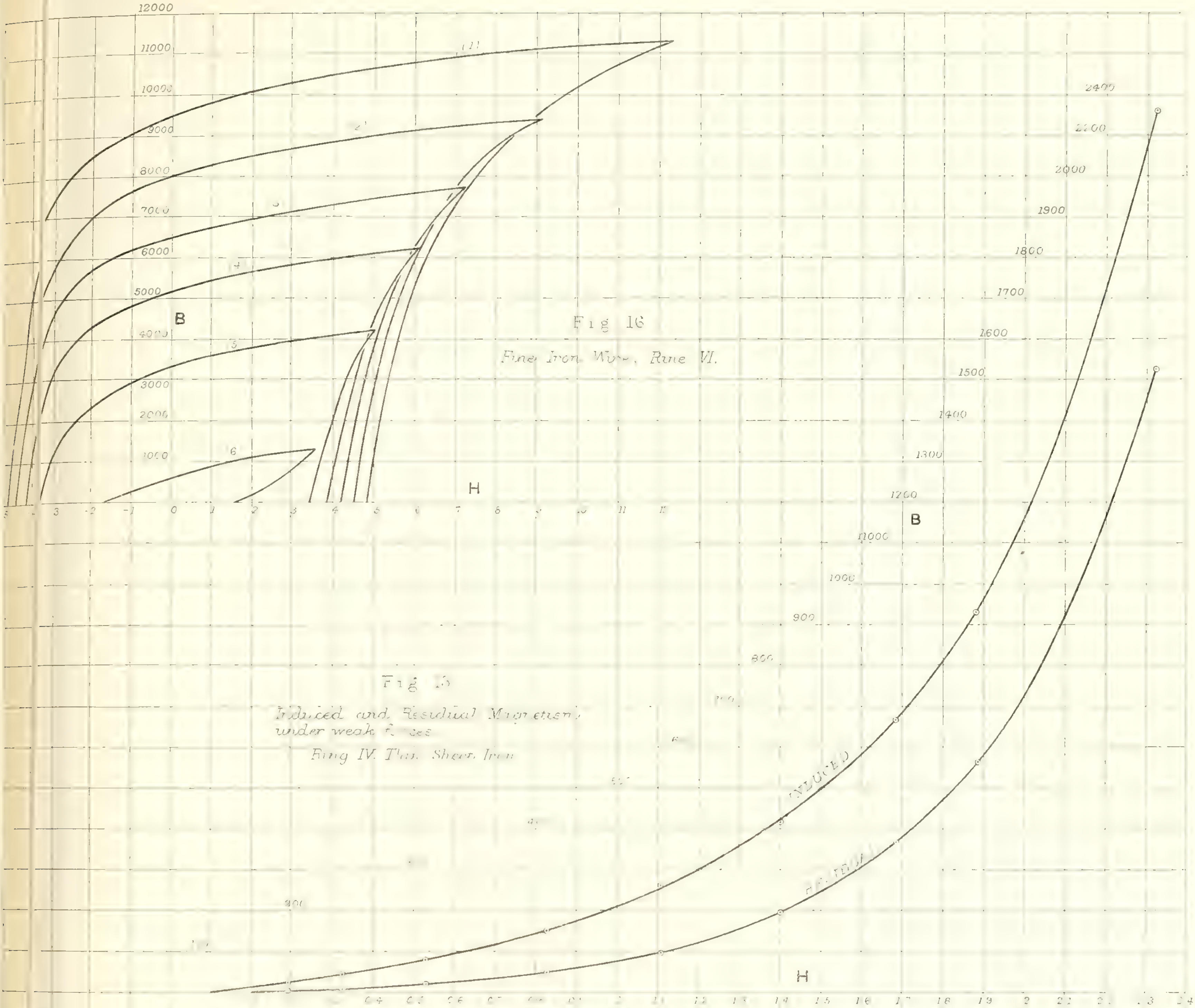


Fig 16

Fine Iron Wire, Ring VI.

Fig 13

Induced and Residual Magnetism under weak forces

Ring IV. Thin Sheet Iron

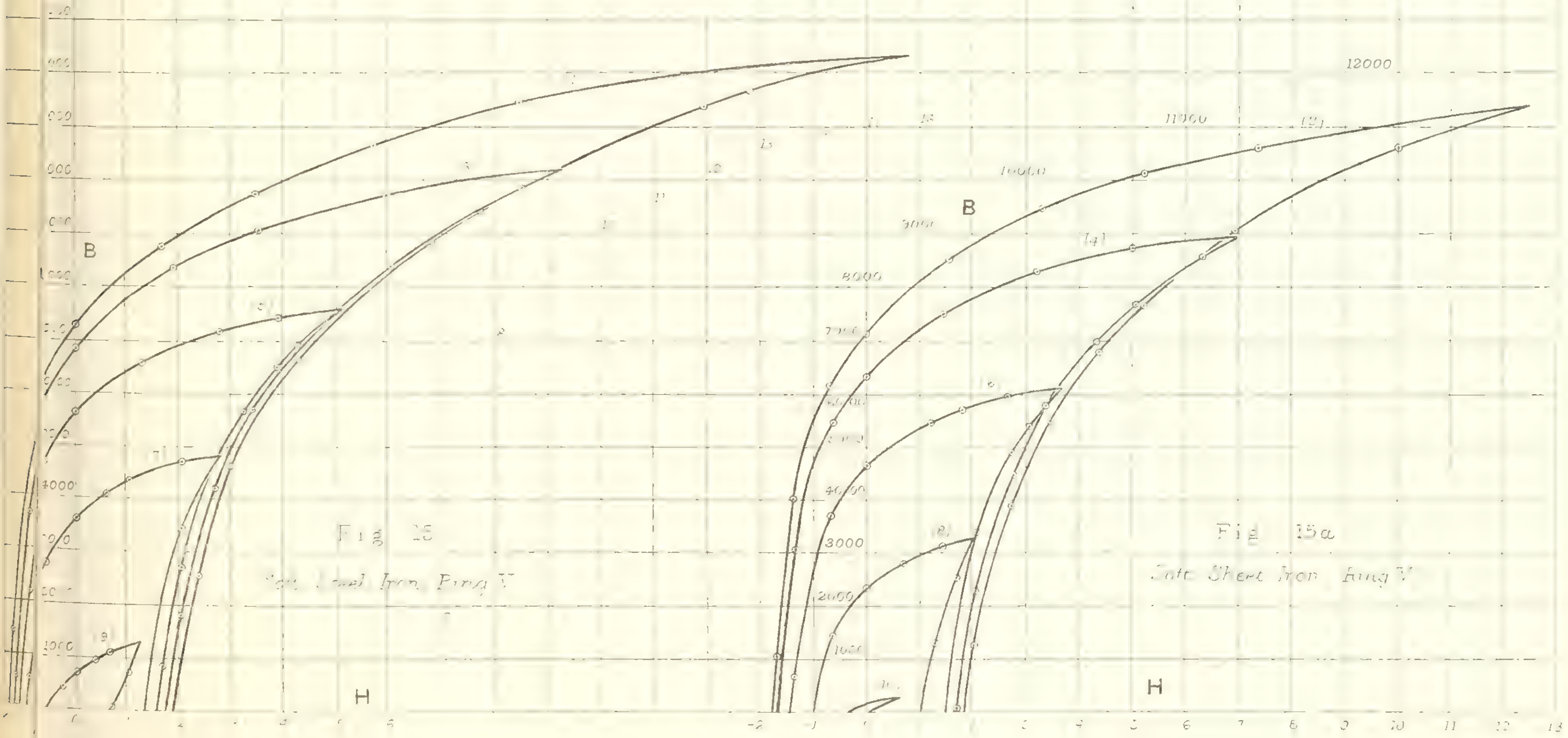


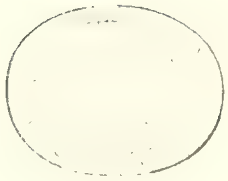
Fig 15

Soft Steel Iron, Ring V

Fig 15a

Soft Sheet Iron, Ring V

West Newman, lith



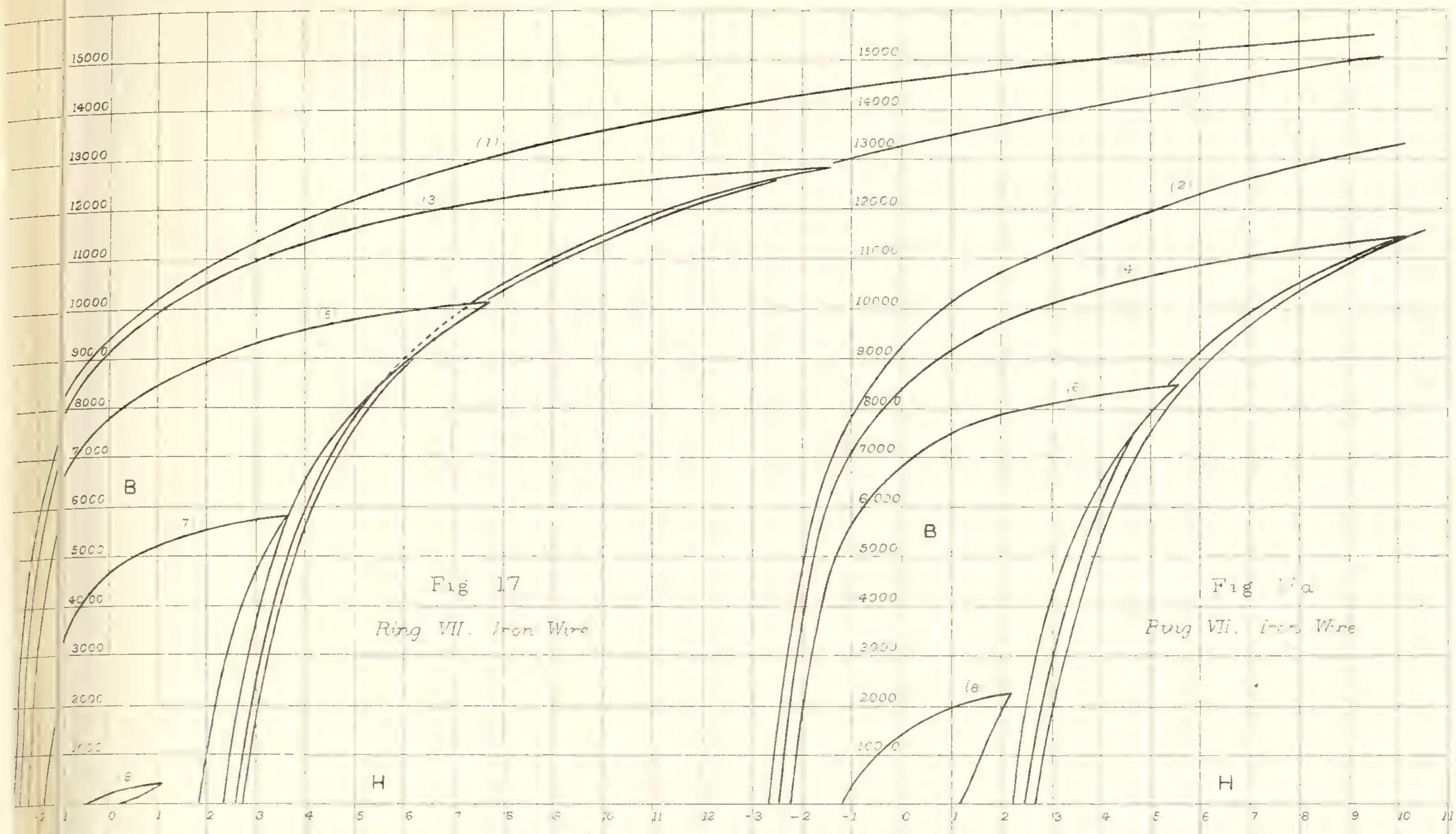


Fig 17.

Ring VII. Iron Wire

Fig 17a.

Ring VII. Iron Wire

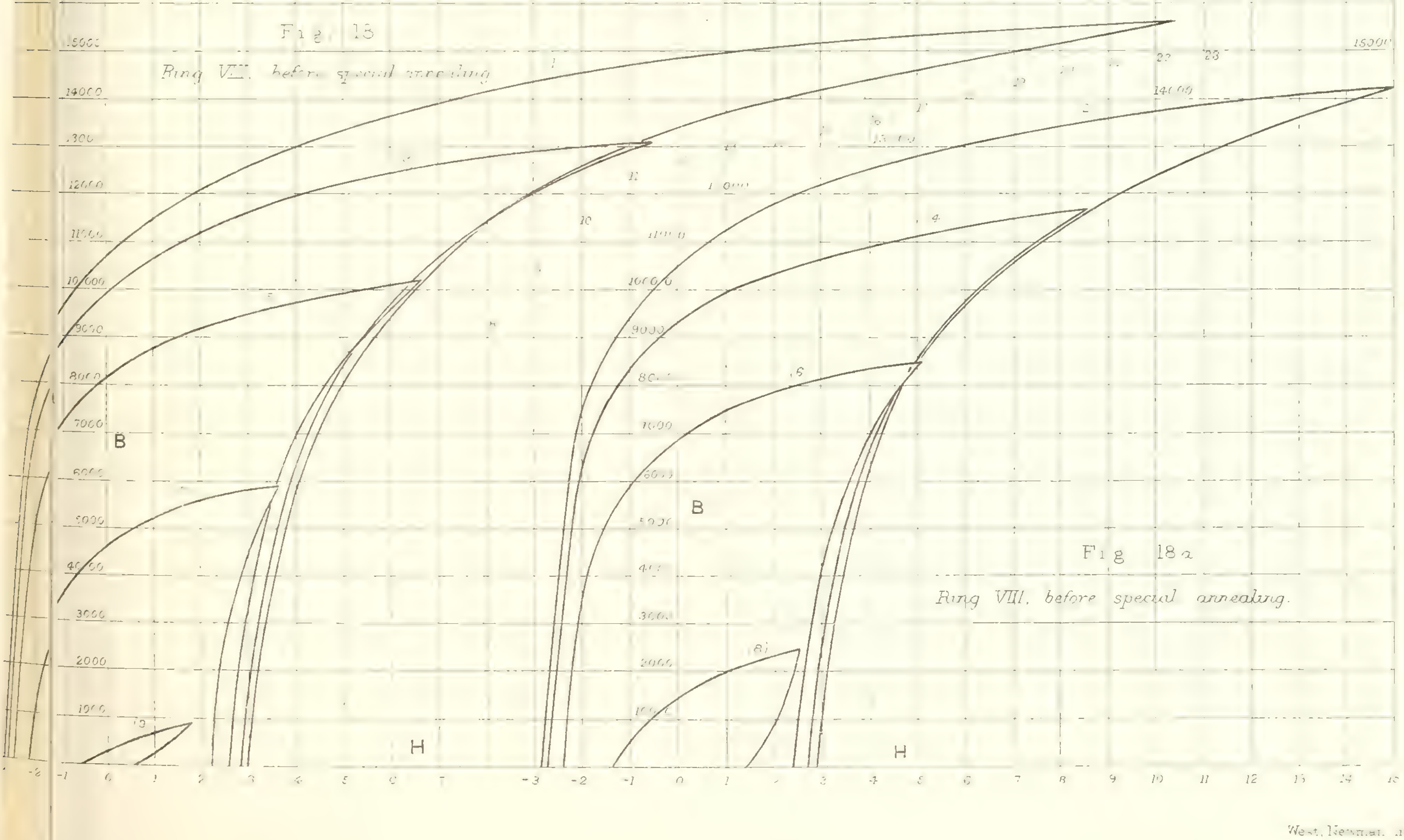


Fig 18.

Ring VII. before special annealing

Fig 18a.

Ring VIII. before special annealing.

West, Newman. lith.

H.S

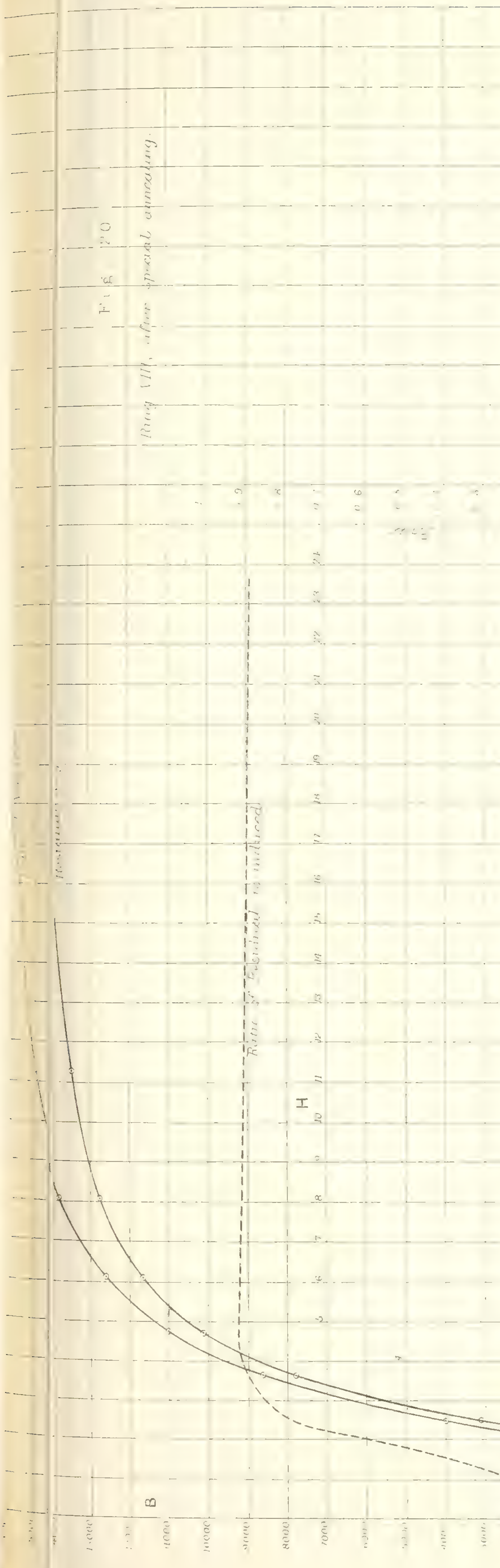


Fig. 20.

Ring VIII, after special annealing.

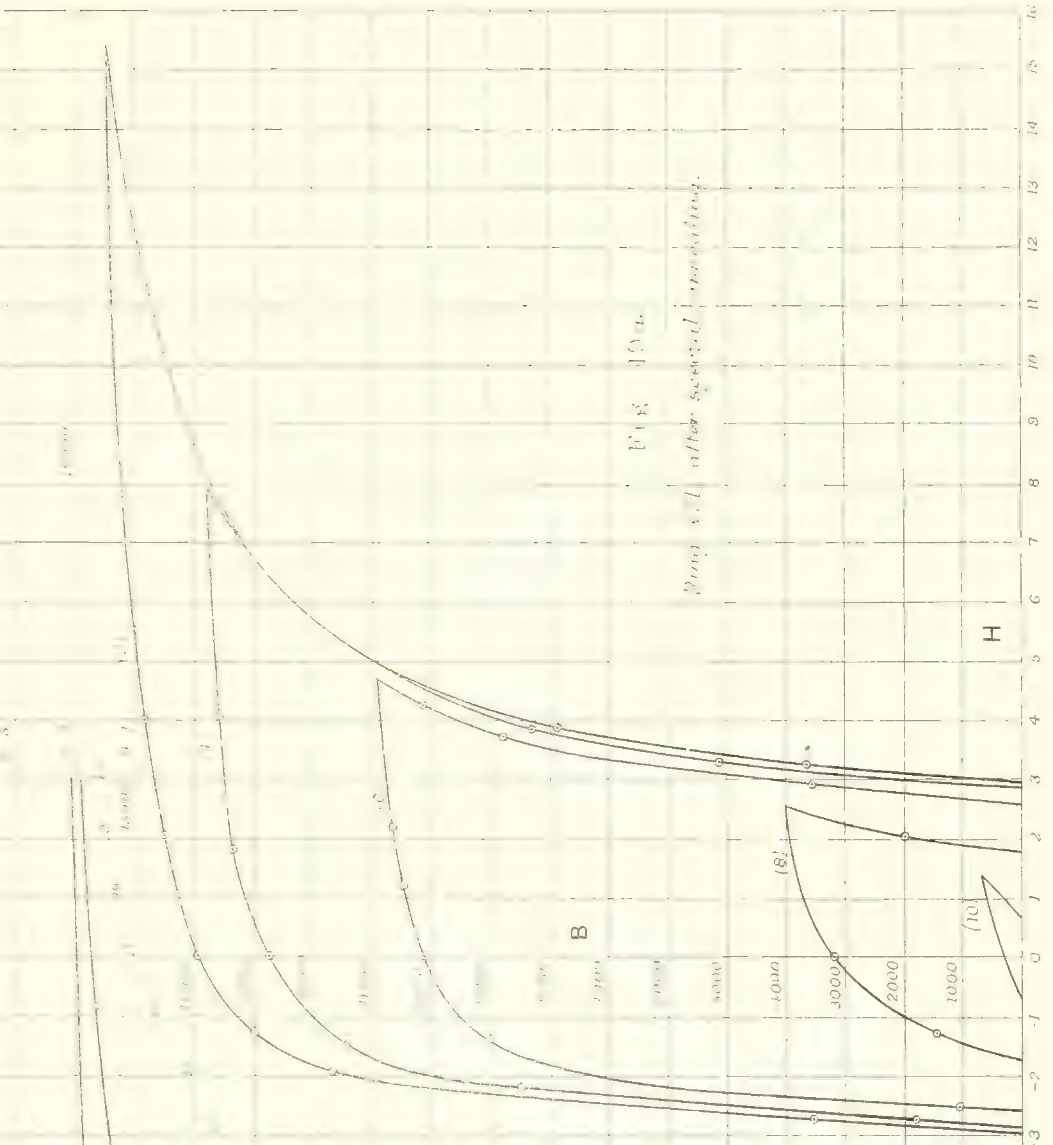


Fig. 19a.

Ring VII, after special annealing.

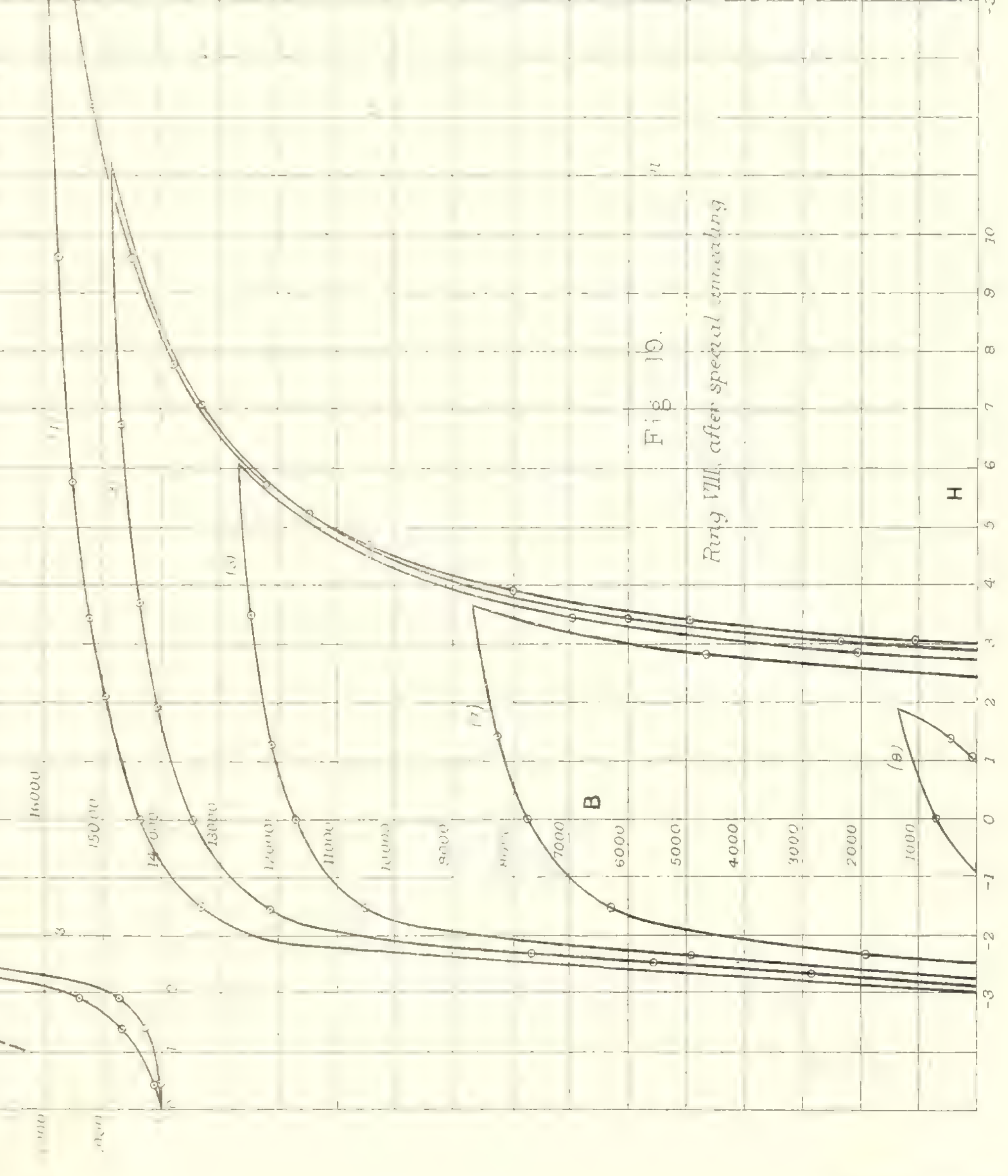


Fig. 19.

Ring VIII, after special annealing

West Newman, lith.

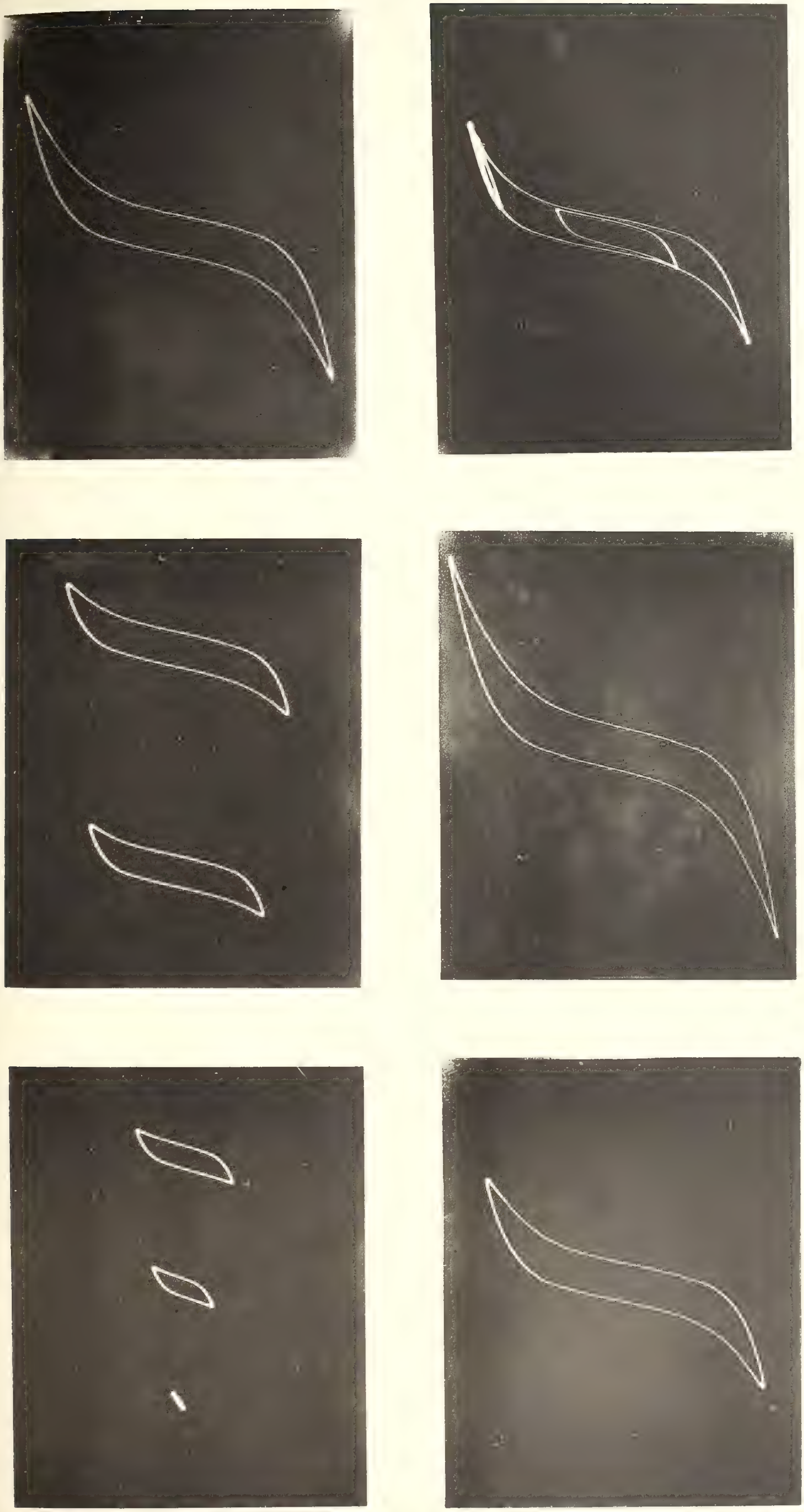


Fig. 28.—Photographic Records by the Magnetic Curve Tracer.

West, Newman imp.

XX. *The Potential of an Anchor Ring.*—Part II.

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Communicated by Professor J. J. Thomson, *F.R.S.*

Received March 16,—Read April 20, 1893.

Introduction.

This paper is a continuation of that at pp. 43–95 *suprà*, on "The Potential of an Anchor Ring." In that paper the potential of an anchor ring was found at all external points; in this, its value is determined at internal points. The annular form of rotating gravitating fluid was also discussed in that paper; here the stability of such a ring is considered. In addition, the potential of a ring whose cross-section is elliptic, being of interest in connection with Saturn, is obtained. The similarity of the methods employed, as well as of the analysis, has led me to give in this paper also a determination of the steady motion of a single vortex-ring in an infinite fluid, and of several fine vortex rings on the same axis.

In *Section I.* solutions of Laplace's equation applicable to space inside an anchor ring are obtained. These results are applied to obtain the potential of a solid ring at internal points, and also of a distribution of matter on the surface of the ring. The work done in collecting the ring from infinity is obtained.

In *Section II.* the stability of an annulus of rotating gravitating fluid is considered for three kinds of disturbances.

(1) *Fluted: i.e.,* those in which the ring remains symmetrical about its axis,[*] but the cross-section is deformed.

(2) *Twisted: i.e.,* those in which the cross-section remains circular, but the circular axis of the ring is deformed.

(3) *Beaded: i.e.,* those in which the circular axis of the ring is undisturbed, but the cross-section is a circle of variable radius.

The ring is found to be stable for fluted and twisted waves, but is broken up by long beaded waves.

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[*] Axis of the ring throughout the paper means the axis of revolution; the central circle of the ring is called the *circular* axis.

Section III. is devoted to Saturn's rings. In LAPLACE's proof (' Méc. Cél.,' Book 3, c. 6) that the rings are not continuous fluid, he assumes the attraction of a ring of elliptic cross-section on a point at the surface to be the same as that of an elliptic cylinder. Mme. KOWALEWSKI, in her memoir on the ring of Saturn (' Astronomische Nachrichten,' No. 2643, vol. 111, 1885) uses a method which applies only to rings of nearly circular section. Here I have attempted to find the potential of a ring of elliptic cross-section. Applied to Saturn, the results obtained agree fairly with LAPLACE'S.

In *Section IV.* the steady motion of a single vortex-ring of finite cross-section is discussed. If m be its strength, c its mean radius, and its cross-section be given by

$$\mathrm{R} = a\,\{1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi + \ldots\},$$

it is shown that β_2, β_3, $\beta_4 \ldots$ are of the 2nd, 3rd, &c., orders in a/c, and their values are found as far as $(a/c)^4$.

The velocity of the ring

$$= \frac{m}{2\pi c}\left\{\log\frac{8c}{a} - \tfrac{1}{4} - \frac{12\log\frac{8c}{a} - 15}{32}\left(\frac{a}{c}\right)^2 \ldots\right\}$$

$$\beta_2 = -\frac{12\mathrm{L} - 17}{32}\left(\frac{a}{c}\right)^2.$$

The results agree with those given by Mr. HICKS, obtained by means of Toroidal Functions. (' Phil. Trans.,' 1884–1885.)

In *Section V.* the motion of a number of fine vortex rings on the same axis is discussed. Equations are obtained giving the forward velocity and the rate of increase of the radius of each ring. Let m_1 be the strength, c_1 the mean radius, a_1 the radius of the cross-section, and z_1 the distance of the centre along the axis of z for one of the rings.

It is shown that the kinetic energy of the system is given by

$$\mathrm{T} = 8\Sigma\left\{\frac{m_1^2 c_1}{2}\left(\log\frac{8c_1}{a_1} - \tfrac{7}{4}\right) + m_1 m_2 \int_0^\pi \frac{c_1 c_2 \cos\phi\, d\phi}{\sqrt{\{(z_2 - z_1)^2 + c_2^2 - 2c_1c_2\cos\phi + c_1^2\}}}\right\}.$$

The equations of motion are

$$\left.\begin{aligned} m_1 c_1 \dot{z}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial c_1} \\ -\, m_1 c_1 \dot{c}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial z_1} \end{aligned}\right\}.$$

The momentum integral takes the simple form

$$\Sigma\,(m_1 c_1^2) = \text{const.}$$

The special cases of one ring pursuing another of equal strength, of the direct approach of a ring towards a fixed plane, and of the motion of a ring over a spherical obstacle are considered in detail.

Section I.

§§ 1–4. Solution of LAPLACE'S equation inside an anchor ring.
§§ 5–7. Potential of a solid anchor ring at internal points.
Exhaustion of potential energy.
§§ 8–9. Potential of a distribution of matter on the surface of a ring, at all internal points, and at external points near the ring.

Section II.

§§ 10–13. Potential and exhaustion of potential energy of a ring whose cross-section is $R = a\,(1 + \Sigma\,\beta_n \cos n\chi)$.
§§ 14–15. Small fluted oscillations of a gravitating fluid ring.
§§ 16–19. Exhaustion of potential energy of a ring which has beaded waves on it.
§ 20. Such waves render the ring unstable.
§§ 21–22. Exhaustion of potential energy of a ring whose central circle is deformed.
§ 23. Such deformation does not produce instability.

Section III.

§§ 24–26. Potential of a ring, whose cross-section is elliptic, at external points.
§ 27. When the elliptic section is very flat.
§ 28. Application to Saturn.

Section IV.

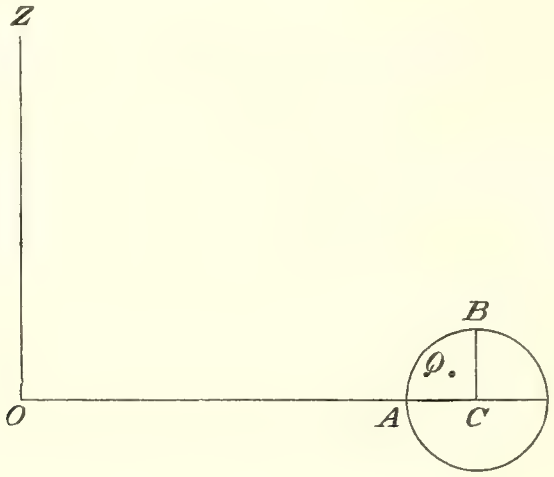
§ 29. Steady motion of a single thick vortex ring.
§§ 30–34. The stream-line function.
§§ 35–36. Determination of the velocity and the form of the cross-section.
§ 37. Small fluted oscillations.

Section V.

§ 38. Stream-line function for any number of fine vortex rings on the same axis.
§ 39. The equations of motion of the rings.
§§ 40–42. The integrals of momentum and energy.
§§ 43–47. Special cases of two rings.

SECTION I.

§ 1. *To find solutions of* LAPLACE'S *Equation applicable to space inside an anchor ring.*



Let O be the centre of the ring, OZ its axis.

Let any section through OZ cut the ring in the circle whose centre is C.

Let Q be any point in that section.

Let OC $= c$, CA $= a$, and let the coordinates of Q, referred to CA and CB as axes, be x and z; also let the polar coordinate of Q, referred to C as origin, CA as initial line, be R and χ, so that CQ = R and $\angle$ ACQ $= \chi$.

In cylindrical coordinates, LAPLACE'S equation is

$$\frac{d^2V}{d\varpi^2} + \frac{1}{\varpi} \cdot \frac{dV}{d\varpi} + \frac{d^2V}{dz^2} + \frac{1}{\varpi^2} \cdot \frac{d^2V}{d\phi^2} = 0.$$

Writing $\varpi = c - x$, this becomes

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} = \frac{1}{c-x} \cdot \frac{dV}{dx} - \frac{1}{(c-x)^2} \cdot \frac{d^2V}{d\phi^2}.$$

We shall find solutions of this equation in descending powers of c.

First, consider the case where V is independent of ϕ.

Then

$$c\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2}\right) = x\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2}\right) + \frac{dV}{dx}.$$

Let

$$V = U_n + \frac{1}{c} U_{n+1} + \frac{1}{c^2} U_{n+2} + \&c.,$$

then

$$\left.\begin{array}{l}
\dfrac{d^2U_n}{dx^2} + \dfrac{d^2U_n}{dz^2} = 0 \\[2ex]
\dfrac{d^2U_{n+1}}{dx^2} + \dfrac{d^2U_{n+1}}{dz^2} = x\left(\dfrac{d^2U_n}{dx^2} + \dfrac{d^2U_n}{dz^2}\right) + \dfrac{dU_n}{dx} \\[2ex]
\dfrac{d^2U_{n+2}}{dx^2} + \dfrac{d^2U_{n+2}}{dz^2} = x\left(\dfrac{d^2U_{n+1}}{dx^2} + \dfrac{d^2U_{n+1}}{dz^2}\right) + \dfrac{dU_{n+1}}{dx} \\[2ex]
\qquad\qquad\qquad \&c., \qquad \&c.
\end{array}\right\}$$

Let

$$U_n = (x + iz)^n,$$

then

$$\frac{d^2U_{n+1}}{dx^2} + \frac{d^2U_{n+1}}{dz^2} = n\,(x + iz)^{n-1},$$

therefore

$$U_{n+1} = \tfrac{1}{4}(x - iz)\,(x + iz)^n + C\,(x + iz)^{n+1}.$$

Take

$$C = 0,$$

then

$$U_{n+1} = \tfrac{1}{4}(x - iz)\,(x + iz)^n.$$

From this U_{n+2} might be found, and so on.

§ 2. It is, however, more convenient to transform the equations to polar coordinates. Let

$$x = ae^\rho \cos\chi, \qquad z = ae^\rho \sin\chi.$$

For points inside the ring ρ varies from $-\infty$ to 0, and χ from 0 to 2π.

Now,

$$\frac{dU}{dx} = \frac{1}{a}\,e^{-\rho}\left(\cos\chi\,\frac{dU}{d\rho} - \sin\chi\,\frac{dU}{d\chi}\right),$$

$$\frac{dU}{dz} = \frac{1}{a}\,e^{-\rho}\left(\sin\chi\,\frac{dU}{d\rho} + \cos\chi\,\frac{dU}{d\chi}\right),$$

and

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dz^2} = \frac{1}{a^2}\,e^{-2\rho}\left(\frac{d^2U}{d\rho^2} + \frac{d^2U}{d\chi^2}\right).$$

Substituting in the above equations, we find

$$\left.\begin{array}{l}
\dfrac{d^2U_n}{d\rho^2} + \dfrac{d^2U_n}{d\chi^2} = 0 \\[2ex]
\dfrac{d^2U_{n+1}}{d\rho^2} + \dfrac{d^2U_{n+1}}{d\chi^2} = ae^\rho\left\{\cos\chi\left(\dfrac{d^2U_n}{d\rho^2} + \dfrac{d^2U_n}{d\chi^2} + \dfrac{dU_n}{d\rho}\right) - \sin\chi\,\dfrac{dU_n}{d\chi}\right\} \\[2ex]
\dfrac{d^2U_{n+2}}{d\rho^2} + \dfrac{d^2U_{n+2}}{d\chi^2} = ae^\rho\left\{\cos\chi\left(\dfrac{d^2U_{n+1}}{d\rho^2} + \dfrac{d^2U_{n+1}}{d\chi^2} + \dfrac{dU_{n+1}}{d\rho}\right) - \sin\chi\,\dfrac{dU_{n+1}}{d\chi}\right\} \\[2ex]
\qquad\qquad\qquad \&c. \qquad\qquad\qquad \&c.
\end{array}\right\} \quad . \quad . \quad (1).$$

Let

$$\mathrm{U}_n = e^{n\rho} \cos n\chi.$$

Then

$$\frac{d^2\mathrm{U}_{n+1}}{d\rho^2} + \frac{d^2\mathrm{U}_{n+1}}{d\chi^2} = ae^{\overline{n+1}\rho}\, n \cos \overline{n-1}\chi.$$

Therefore

$$\mathrm{U}_{n+1} = an\, \frac{e^{\overline{n+1}\rho} \cos \overline{n-1}\chi}{(n+1)^2 - (n-1)^2},$$

$$= a\, \tfrac{1}{4}\, e^{(n+1)\rho} \cos (n-1)\chi.$$

This gives

$$\frac{d^2\mathrm{U}_{n+2}}{d\rho^2} + \frac{d^2\mathrm{U}_{n+2}}{d\chi^2} = a^2 e^{(n+2)\rho} \left\{\frac{3n}{4} \cos (n-2)\chi + \frac{2n+1}{4} \cos n\chi\right\}.$$

Therefore

$$\mathrm{U}_{n+2} = a^2 e^{(n+2)\rho} \left\{\frac{(2n+1)}{16\,(n+1)} \cos n\chi + \tfrac{3}{32} \cos (n-2)\,\chi\right\}.$$

In the same way we find

$$\mathrm{U}_{n+3} = \left(\frac{a}{2}\right)^3 e^{(n+3)\rho} \left\{\tfrac{1}{2}\, \frac{(2n+1)\,(2n+3)}{(2n+2)\,(2n+4)} \cos (n+1)\,\chi + 2\, \frac{1\,.\,3}{2\,.\,4}\, \frac{2n+1}{2n+2} \cos (n-1)\,\chi \right.$$
$$\left. + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6} \cos (n-3)\,\chi\right\}.$$

This suggests the form of U_{n+p}, which is easily verified.

$$\mathrm{U}_{n+p} = \left(\frac{a}{2}\right)^p e^{(n+p)\rho} \left\{\tfrac{1}{2}\, \frac{(2n+1)\ldots(2n+2p-3)}{(2n+2)\ldots(2n+2p-2)} \cos (n+p-2)\,\chi \right.$$
$$+ \frac{p-1}{1}\, \frac{1\,.\,3}{2\,.\,4}\, \frac{(2n+1)\ldots(2n+2p-5)}{(2n+2)\ldots(2n+2p-4)} \cos (n+p-4)\,\chi$$
$$\left. + \frac{(p-1)(p-2)}{2\,!}\, \frac{1.3.5}{2.4.6}\, \frac{(2n+1)\ldots(2n+2p-7)}{(2n+2)\ldots(2n+2p-6)} \cos (n+p-6)\chi + \ldots\right\} \quad (2).$$

This formula for U_{n+p} holds whether p be $>$ or $<$ n. The number of terms in U_{n+p} is p.

§ 3. The series $\mathrm{U}_n + \frac{1}{c}\,\mathrm{U}_{n+1} + \frac{1}{c^2}\,\mathrm{U}_{n+2} + \ldots$ is a solution of LAPLACE'S equation. It is convergent at all internal points.

For U_{n+p} is $< \left(\frac{a}{2}\right)^p e^{(n+p)\rho}(1+1)^{p-1}$.

Thus the series $\mathrm{U}_n + \frac{1}{c}\,\mathrm{U}_{n+1} \ldots$ converges more rapidly than the geometrical progression

$$1 + \frac{a}{c} + \frac{a^2}{c^2} + \text{\&c.},$$

Writing sines instead of cosines, we obtain another set of solutions of LAPLACE's equation applicable to space inside a ring.

Writing $-n$ for n solutions of LAPLACE's equation applicable to space just outside an anchor ring are obtained. They can, however, only be obtained as far as the nth term. At this point $\log \mathrm{R}$ arises in the series. In obtaining each term from the preceding, as an integration is performed, a constant should be added; and for the series to be convergent these constants must have definite values. To find the series further, the method of my former paper is necessary.

The following are the solutions of LAPLACE's equation inside a ring for $n = 0$, 1, 2, 3, or 4.

Constant.

$$\left.\begin{aligned}
&\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^2}{a^2} + \left(\frac{a}{2c}\right)^2\tfrac{3}{4}\cdot\frac{\mathrm{R}^3}{a^3}\cos\chi + \left(\frac{a}{2c}\right)^3\frac{\mathrm{R}^4}{a^4}\left(\tfrac{5}{8}\cos 2\chi + \tfrac{9}{16}\right) \\
&\qquad + \left(\frac{a}{2c}\right)^4\frac{\mathrm{R}^5}{a^5}\left(\tfrac{35}{128}\cos 3\chi + \tfrac{45}{64}\cos\chi\right) + \ldots, \\
&\frac{\mathrm{R}^2}{a^2}\cos 2\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^3}{a^3}\cos\chi + \left(\frac{a}{2c}\right)^2\frac{\mathrm{R}^4}{a^4}\left(\tfrac{5}{12}\cos 2\chi + \tfrac{3}{8}\right) \\
&\qquad + \left(\frac{a}{2c}\right)^3\frac{\mathrm{R}^5}{a^5}\left(\tfrac{65}{96}\cos 3\chi + \tfrac{5}{8}\cos\chi\right) + \ldots, \\
&\frac{\mathrm{R}^3}{a^3}\cos 3\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^4}{a^4}\cos 2\chi + \left(\frac{a}{2c}\right)^2\frac{\mathrm{R}^5}{a^5}\left(\tfrac{65}{96}\cos 3\chi + \tfrac{5}{8}\cos\chi\right) + \ldots, \\
&\frac{\mathrm{R}^4}{a^4}\cos 4\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^5}{a^5}\cos 3\chi + \ldots
\end{aligned}\right\} \quad . \quad (3).$$

§ 4. In discussing the stability of a fluid ring, approximate solutions of the equation

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dx^2} - \frac{1}{c-x}\frac{d\mathrm{V}}{dx} + \frac{1}{(c-x)^2}\frac{d^2\mathrm{V}}{d\phi^2} = 0$$

are required.

Writing $\mathrm{V}\cos p\phi$ for V, it becomes

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2} - \frac{1}{c-x}\frac{d\mathrm{V}}{dx} - \frac{p^2}{(c-x)^2}\mathrm{V} = 0.$$

If p is of the order c/a the last term becomes of the same order as the terms $\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2}$, and the equation may be written

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2} - \frac{p^2}{c^2}\mathrm{V} - \frac{1}{c}\left\{\frac{d\mathrm{V}}{dx} + \frac{2p^2}{c^2}x\mathrm{V}\right\} - \frac{1}{c^2}\left\{x\frac{d\mathrm{V}}{dx} + \frac{3p^2}{c^2}x^2\mathrm{V}^2\right\} - \&\text{c.} = 0$$

and the equation must be regarded as an approximation to

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} - \frac{p^2}{c^2}V = 0$$

or

$$\frac{d^2V}{dR^2} + \frac{1}{R}\cdot\frac{dV}{dR} + \frac{1}{R^2}\frac{d^2V}{d\chi^2} - \frac{p^2}{c^2}V = 0,$$

of which $\cos n\chi \,.\, J_n\left(\frac{ipR}{c}\right)$ is a solution; and starting from this, an approximate solution may be obtained.

We shall only need the cases in which p is small.

The equation may then be written

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} = \frac{1}{c}\frac{dV}{dx} + \frac{1}{c^2}\left(x\frac{dV}{dx} + p^2V\right) + \frac{1}{c^3}\left(x^2\frac{dV}{dx} + 2p^2xV\right) + \dots$$

An approximate solution of this is

$$\cos p\phi \left\{\frac{R^n}{a^n}\cos n\chi + \frac{a}{4c}\frac{R^{n+1}}{a^{n+1}}\cos(n-1)\chi + \frac{a^2}{8c^2}\frac{R^{n+2}}{a^{n+2}}\left(\frac{4p^2+2n+1}{2n+2}\cos n\chi + 3\cos(n-2)\chi\right) + \&c.\right\} \quad . \quad . \quad . \quad (4).$$

When $n = 0$ the solution is

$$\cos p\phi\left\{1 + \frac{a^2}{4c^2}p^2\frac{R^2}{a^2} + \&c.\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

§ 5. *To find the Potential of a solid anchor ring at an internal point.*

At any external point whose polar coordinates are r, θ, ϕ,

$$V = \frac{M}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} + \frac{a^2}{8}\cdot\frac{d}{cdc}\int_0^\pi\frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} - \frac{a^4}{192}\left(\frac{d}{cdc}\right)^2\int_0^\pi\frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} - \&c.\right\}.^*$$

Using the expansions of the integrals given in that paper, we find that at the surface of the ring

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* *Suprà*, p. 59.

$$\begin{aligned}V = 2\pi a^2 \Big\{\lambda + 2 &+ \frac{\lambda + \frac{3}{4}}{2}\,\sigma\cos\chi + \left[\frac{\lambda - \frac{1}{2}}{16} + \frac{3\,(\lambda + \frac{17}{36})}{16}\cos 2\chi\right]\sigma^2 \\ &+ \left[\frac{3\,(\lambda - \frac{1}{12})}{32}\cos\chi + \frac{5\,(\lambda + \frac{7}{24})}{64}\cos 3\chi\right]\sigma^3 \\ &+ \left[\frac{9\lambda}{256} + \frac{7\,(\lambda - \frac{19}{168})}{128}\cos 2\chi + \frac{35\,(\lambda + \frac{19}{120})}{1024}\cos 4\chi\right]\sigma^4 + \&c.\Big\}. \quad (6),\end{aligned}$$

where $\sigma = a/c$, and $\lambda = \log\,(8c/a) - 2$.

Now, inside the ring,

$$\frac{d^2V}{dz^2} + \frac{d^2V}{d\varpi^2} + \frac{1}{\varpi}\frac{dV}{d\varpi} + 4\pi = 0.$$

Let

$$V = -\,2\pi z^2 + V_1,$$

then

$$\frac{d^2V_1}{dz^2} + \frac{d^2V_1}{d\varpi^2} + \frac{1}{\varpi}\frac{dV_1}{d\varpi} = 0.$$

We may, therefore, assume that

$$\begin{aligned}\frac{V}{2\pi a^2} = &-\frac{z^2}{a^2} + A_1 \\ &+ A_2\left\{\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2} + \frac{3\sigma^2}{16}\frac{R^3}{a^3}\cos\chi + \sigma^3\left(\frac{R}{a}\right)^4(\tfrac{5}{64}\cos 2\chi + \tfrac{9}{128}) + \ldots\right\}, \\ &+ A_3\Big\{\frac{R^2}{a^2}\cos 2\chi + \frac{\sigma}{4}\frac{R^3}{a^3}\cos\chi + \sigma^2\frac{R^4}{a^4}(\tfrac{5}{48}\cos 2\chi + \tfrac{3}{32}) \\ &\qquad + \sigma^3\frac{R^5}{a^5}(\tfrac{35}{768}\cos 3\chi + \tfrac{15}{128}\cos\chi) \\ &\qquad + \sigma^4\frac{R^6}{a^6}(\tfrac{21}{1024}\cos 4\chi + \tfrac{35}{512}\cos 2\chi + \tfrac{25}{512}) + \ldots\Big\}, \\ &+ A_4\left\{\frac{R^3}{a^3}\cos 3\chi + \frac{\sigma}{4}\frac{R^4}{a^4}\cos 2\chi + \ldots\right\}, \\ &+ A_5\left\{\frac{R^4}{a^4}\cos 2\chi + \ldots\right\}.\end{aligned}$$

Therefore, at the surface of the ring,

$$\begin{aligned}\frac{V}{2\pi a^2} = &-\tfrac{1}{2} + A_1 + A_2\left(\frac{\sigma}{4} + \frac{9\sigma^3}{128}\right) + A_3\left(\frac{3\sigma^2}{32} + \frac{25\sigma^4}{512}\right), \\ &+ \cos\chi\left\{A_2\left(1 + \frac{3\sigma^2}{16}\right) + A_3\left(\frac{\sigma}{4} + \frac{15\sigma^3}{128}\right)\right\}, \\ &+ \cos 2\chi\left\{\tfrac{1}{2} + A_2\frac{5\sigma^3}{64} + A_3\left(1 + \frac{5\sigma^2}{48} + \frac{35\sigma^4}{512}\right) + A_4\frac{\sigma}{4}\right\}, \\ &+ \cos 3\chi\left\{A_3\frac{35\sigma^3}{768} + A_4\right\}, \\ &+ \cos 4\chi\left\{A_3\frac{21\sigma^4}{1024} + A_5\right\} \quad \ldots\ldots\ldots\ldots \quad (7).\end{aligned}$$

Comparing this with the value of the potential at the surface of the ring, given above, we have equations to determine the constants A_1, A_2, &c.

These give

$$\left.\begin{aligned} A_1 &= +\lambda\tfrac{5}{2} - \frac{\lambda + \frac{7}{4}}{16}\sigma^2 - \frac{3(\lambda + \frac{3}{4})}{512}\sigma^4 \\ A_2 &= \frac{\lambda + 1}{2}\sigma - \frac{3\lambda + 5}{64}\sigma^3 \\ A_3 &= -\tfrac{1}{2} + \frac{3(\lambda + \frac{3}{4})}{16}\sigma^2 - \frac{12\lambda + 19}{512}\sigma^4 \\ A_4 &= \frac{5(\lambda + \frac{7}{12})}{64}\sigma^3 \\ A_5 &= \frac{35(\lambda + \frac{11}{24})}{1024}\sigma^4 \end{aligned}\right\} \quad \ldots\ldots\ldots (8).$$

Thus the potential is found at all internal points as far as terms in σ^4.

Writing L (log $8c/a$) instead of $\lambda + 2$, as far as terms of the second order

$$\begin{aligned} V = 2\pi a^2 \Bigg\{ & L + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left[\frac{L-1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right]\cos\chi \\ & + \sigma^2\left[-\frac{L - \frac{1}{4}}{16} + \frac{L-1}{8}\frac{R^2}{a^2} - \tfrac{3}{64}\frac{R^4}{a^4} + \frac{3(L - \frac{5}{4})}{16}\frac{R^2}{a^2}\cos 2\chi - \tfrac{5}{96}\frac{R^4}{a^4}\cos 2\chi\right] + \ldots \Bigg\} \quad (9). \end{aligned}$$

§ 6. The value of V is a maximum, when

$$\chi = 0,$$

and

$$-2R + \sigma\left[(L - 1)a - \tfrac{3}{4}\frac{R^2}{a}\right] = 0,$$

that is, when $R = \sigma \,.\, \tfrac{1}{2}(L - 1)a$ + terms in σ^3.

This maximum value

$$= 2\pi a^2\left\{L + \tfrac{1}{2} - \frac{\sigma^2}{8}(L-1)^2 + \frac{\sigma^2}{4}(L-1)^2 - \ldots\right\}$$

$$= 2\pi a^2\left\{L + \tfrac{1}{2} + \frac{\sigma^2}{8}(L-1)^2 + \ldots\right\}.$$

§ 7. The exhaustion of potential energy

$$= \tfrac{1}{2} \iiint \mathrm{V}\, dm$$

$$= \tfrac{1}{2} \,.\, 2\pi \int_0^a \int_0^{2\pi} \mathrm{V}\,(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi$$

$$= 4\pi^3 a^2 c \int_0^a \left\{ \mathrm{A}_1 - \frac{\mathrm{R}^2}{2a^2} + \mathrm{A}_2\left(\frac{\sigma}{4}\,\frac{\mathrm{R}^2}{a^2} + \frac{9\sigma^3}{128}\,\frac{\mathrm{R}^4}{a^4}\right) + \mathrm{A}_3\left(\frac{3\sigma^2}{32}\,\frac{\mathrm{R}^4}{a^4} + \frac{25\sigma^4}{512}\,\frac{\mathrm{R}^6}{a^6}\right)\right\}\mathrm{R}\,d\mathrm{R}$$

$$- 2\pi^3 a^2 \int_0^a \left\{ \mathrm{A}_2\left(\frac{\mathrm{R}}{a} + \frac{3\sigma^2}{16}\,\frac{\mathrm{R}^3}{a^3}\right) + \mathrm{A}_3\left(\frac{\sigma}{4}\,\frac{\mathrm{R}^3}{a^3} + \tfrac{15}{128}\,\sigma^3\,\frac{\mathrm{R}^5}{a^5}\right)\right\}\mathrm{R}^2\,d\mathrm{R}$$

$$= 2\pi^3 a^4 c \left\{ -\tfrac{1}{4} + \mathrm{A}_1 - \mathrm{A}_2\left(\frac{\sigma}{8} + \frac{\sigma^3}{128}\right) - \mathrm{A}_3\left(\frac{\sigma^2}{96} - \frac{5\sigma^4}{1024}\right)\right\}$$

$$= \frac{\mathrm{M}^2}{2\pi c}\left\{ \mathrm{L} + \tfrac{1}{4} - \frac{\mathrm{L} - \frac{2}{3}}{8}\,\sigma^2 - \frac{3\,(\mathrm{L} - \frac{19}{12})}{512}\,\sigma^4 - \ldots \right\} \quad \ldots\ldots\ldots\ldots (10).$$

If we may judge from these few terms this series is very convergent. When $\sigma = 1$, *i.e.*, in the case of a ring so thick that it touches itself at the origin, we have $\mathrm{L} = \log_e 8 = 2{\cdot}080$.

The exhaustion of energy

$$= \frac{\mathrm{M}^2}{2\pi c} \,.\, \{2{\cdot}330 - {\cdot}177 - {\cdot}003\}$$

$$= 2{\cdot}150 \,.\, \frac{\mathrm{M}^2}{2\pi c}.$$

The first three figures would seem to be correct.

[Let the same mass be collected into rings of different mean radii, and consequently different thicknesses; let the final mean radius of the ring, that is, the mean radius when the ring is so thick that it touches itself at its centre, be taken as unity; let the exhaustion of energy in this position be also taken as unity. Then the exhaustion of energy for different mean radii of the ring is given by the following Table :—

Mean radius	1·25	1·5	1·75	2	3	4	5
Exhaustion of potential energy	·9488	·8874	·8284	·7749	·6144	·5162	·3506

July 22, 1892.]*

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* This Table was given at the suggestion of one of the Referees, replacing a Table given in the paper as read.

§ 8. *To find the potential of a distribution of matter of density* $\beta_n \cos n\chi$ *on the surface of an anchor ring, at all internal points and at external points near the surface of the ring.*

We know that for a cylinder

$$\left.\begin{aligned} \frac{n\mathrm{V}_i}{2\pi\beta_n} &= \left(\frac{\mathrm{R}}{a}\right)^n \cos n\chi \\ \text{and}\qquad \frac{n\mathrm{V}_0}{2\pi\beta_n} &= \left(\frac{a}{\mathrm{R}}\right)^n \cos n\chi \end{aligned}\right\}.$$

Let us assume for a ring

$$\begin{aligned} \frac{n\mathrm{V}_i}{2\pi\beta_n} = {} & \left(\frac{\mathrm{R}}{a}\right)^n \cos n\chi \\ & + \frac{\sigma}{4}\left(\frac{\mathrm{R}}{a}\right)^{n+1} \cos (n-1)\chi + \frac{\sigma^2}{8}\left(\frac{\mathrm{R}}{a}\right)^{n+2} \left\{\frac{2n+1}{2n+2}\cos n\chi + \tfrac{3}{4}\cos (n-2)\chi\right\} + \dots \\ & + \frac{\sigma}{4}\mathrm{A}_1\left\{\left(\frac{\mathrm{R}}{a}\right)^{n-1}\cos (n-1)\chi + \frac{\sigma}{4}\left(\frac{\mathrm{R}}{a}\right)^{n}\cos (n-2)\chi + \dots\right\} \\ & + \frac{\sigma}{4}\mathrm{A}_2\left\{\left(\frac{\mathrm{R}}{a}\right)^{n+1}\cos (n+1)\chi + \frac{\sigma}{4}\left(\frac{\mathrm{R}}{a}\right)^{n}\cos n\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{A}_3\left\{\left(\frac{\mathrm{R}}{a}\right)^{n-2}\cos (n-2)\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{A}_4\left\{\left(\frac{\mathrm{R}}{a}\right)^{n}\cos n\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{A}_5\left\{\left(\frac{\mathrm{R}}{a}\right)^{n-2}\cos (n+2)\chi + \dots\right\} \end{aligned}$$

and

$$\begin{aligned} \frac{n\mathrm{V}_0}{2\pi\beta_n} = {} & \left(\frac{a}{\mathrm{R}}\right)^n \cos n\chi \\ & + \frac{\sigma}{4}\left(\frac{a}{\mathrm{R}}\right)^{n-1} \cos (n+1)\chi + \frac{\sigma^2}{8}\left(\frac{a}{\mathrm{R}}\right)^{n-2} \left\{\frac{2n-1}{2n-2}\cos (n-2)\chi + \tfrac{3}{4}\cos (n-2)\chi\right\} + \dots \\ & + \frac{\sigma}{4}\mathrm{B}_1\left\{\left(\frac{a}{\mathrm{R}}\right)^{n-1}\cos (n-1)\chi + \frac{\sigma}{4}\left(\frac{a}{\mathrm{R}}\right)^{n-2}\cos n\chi + \dots\right\} \\ & + \frac{\sigma}{4}\mathrm{B}_2\left\{\left(\frac{a}{\mathrm{R}}\right)^{n+1}\cos (n+1)\chi + \frac{\sigma}{4}\left(\frac{a}{\mathrm{R}}\right)^{n}\cos (n+2)\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{B}_3\left\{\left(\frac{a}{\mathrm{R}}\right)^{n-2}\cos (n-2)\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{B}_4\left\{\left(\frac{a}{\mathrm{R}}\right)^{n}\cos n\chi + \dots\right\} \\ & + \frac{\sigma^2}{16}\mathrm{B}_5\left\{\left(\frac{a}{\mathrm{R}}\right)^{n+2}\cos (n+2)\chi + \dots\right\}. \end{aligned}$$

The constants are to be chosen so that

and

$$\left.\begin{aligned} \mathrm{V}_i - \mathrm{V}_0 &= 0 \\ \frac{d\mathrm{V}_0}{d\mathrm{R}} - \frac{d\mathrm{V}_i}{d\mathrm{R}} + 4\pi\beta_n \cos n\chi &= 0 \end{aligned}\right\}$$

at the surface of the ring.

Thus

$$\left.\begin{aligned} 1 + \mathrm{A}_1 &= \mathrm{B}_1 \\ (n+1) + (n-1)\,\mathrm{A}_1 &= -(n-1)\,\mathrm{B}_1 \\ \mathrm{A}_2 &= 1 + \mathrm{B}_2 \\ (n+1)\,\mathrm{A}_2 &= -(n-1) - (n+1)\,\mathrm{B}_2 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \tfrac{3}{2} + \mathrm{A}_1 + \mathrm{A}_3 &= \mathrm{B}_3 \\ \tfrac{3}{2}(n+2) + n\mathrm{A}_1 + (n-2)\,\mathrm{A}_3 &= -(n-2)\,\mathrm{B}_3 \\ \frac{2n+1}{n+1} + \mathrm{A}_2 + \mathrm{A}_4 &= \frac{2n-1}{n-1} + \mathrm{B}_1 + \mathrm{B}_4 \\ \frac{2n+1}{n+1}(n+2) + (n+2)\,\mathrm{A}_2 + n\mathrm{A}_4 &= -\frac{2n-1}{n-1}(n-2) - (n-2)\,\mathrm{B}_1 - n\mathrm{B}_4 \\ \mathrm{A}_5 &= \tfrac{3}{2} + \mathrm{B}_2 + \mathrm{B}_5 \\ (n+2)\,\mathrm{A}_5 &= -\tfrac{3}{2}(n-2) - n\mathrm{B}_2 - (n+2)\,\mathrm{B}_5 \end{aligned}\right\}.$$

These equations give

$$\begin{array}{ll} \mathrm{A}_1 = -\dfrac{n}{n-1}. & \mathrm{B}_1 = -\dfrac{1}{n-1}. \\[2ex] \mathrm{A}_2 = \dfrac{1}{n+1}. & \mathrm{B}_2 = -\dfrac{n}{n+1}. \\[2ex] \mathrm{A}_3 = -\dfrac{n}{2(n-2)}. & \mathrm{B}_3 = -\dfrac{(2n-3)}{(n-1)(n-2)}. \\[2ex] \mathrm{A}_4 = -2. & \mathrm{B}_4 = -2. \\[2ex] \mathrm{A}_5 = \dfrac{2n+3}{(n+1)(n+2)}. & \mathrm{B}_5 = -\dfrac{n}{2(n+2)}. \end{array}$$

Thus we find that to the second power of σ,

$$\begin{aligned} \frac{\mathrm{V}i}{2\pi a\beta_n} = {} & \left(\frac{\mathrm{R}}{a}\right)^n \frac{\cos n\chi}{n} \\ & + \frac{\sigma}{4}\left[\left\{\frac{1}{n}\left(\frac{\mathrm{R}}{a}\right)^{n+1} - \frac{1}{n-1}\left(\frac{\mathrm{R}}{a}\right)^{n-1}\right\}\cos(n-1)\chi + \frac{1}{n(n+1)}\left(\frac{\mathrm{R}}{a}\right)^{n+1}\cos(n+1)\chi\right] \\ & + \frac{\sigma^2}{32}\left[\frac{2(2n+3)}{n(n+1)(n+2)}\left(\frac{\mathrm{R}}{a}\right)^{n+2}\cos(n+2)\chi + \frac{4}{n}\left\{\left(\frac{\mathrm{R}}{a}\right)^{n+2} - \left(\frac{\mathrm{R}}{a}\right)^{n}\right\}\cos n\chi\right. \\ & \left. + \left\{\frac{3}{n}\left(\frac{\mathrm{R}}{a}\right)^{n+2} - \frac{2}{n-1}\left(\frac{\mathrm{R}}{a}\right)^{n} - \frac{1}{n-2}\left(\frac{\mathrm{R}}{a}\right)^{n-2}\right\}\cos(n-2)\chi\right]. \quad (11), \end{aligned}$$

and

$$\frac{\mathrm{V}_0}{2\pi a\beta_n} = \left(\frac{a}{\mathrm{R}}\right)^n \frac{\cos n\chi}{n}$$
$$+ \frac{\sigma}{4}\left[\left\{\frac{1}{n}\left(\frac{a}{\mathrm{R}}\right)^{n-1} - \frac{1}{n+1}\left(\frac{a}{\mathrm{R}}\right)^{n+1}\right\}\cos(n+1)\chi - \frac{1}{n(n-1)}\left(\frac{a}{\mathrm{R}}\right)^{n-1}\cos(n-1)\chi\right]$$
$$+ \frac{\sigma^2}{32}\left[-\frac{2(2n-3)}{n(n-1)(n-2)}\left(\frac{a}{\mathrm{R}}\right)^{n-2}\cos(n-2)\chi + \frac{4}{n}\left\{\left(\frac{a}{\mathrm{R}}\right)^{n-2} - \left(\frac{a}{\mathrm{R}}\right)^{n}\right\}\cos n\chi\right.$$
$$\left. + \left\{\frac{3}{n}\left(\frac{a}{\mathrm{R}}\right)^{n-2} - \frac{2}{n+1}\left(\frac{a}{\mathrm{R}}\right)^{n} - \frac{1}{n+2}\left(\frac{a}{\mathrm{R}}\right)^{n+2}\right\}\cos(n+2)\chi\right] \quad (11a).$$

These agree in giving at the surface of the ring

$$\frac{\mathrm{V}}{2\pi a\beta_n} = \frac{\cos n\chi}{n} + \frac{\sigma}{4n}\left\{\frac{\cos(n+1)\chi}{n+1} - \frac{\cos(n-1)\chi}{n-1}\right\}$$
$$+ \frac{\sigma^2}{16n}\left\{\frac{2n+3}{(n+1)(n+2)}\cos(n+2)\chi - \frac{2n-3}{(n-1)(n-2)}\cos(n-2)\chi\right\} + \&c. \quad (12).$$

I have verified the above value of the potential at external points near the ring by evaluating the integral. The method here given is suggested by LAPLACE ('Méc. Cél.,' Book 3, c. 6). It is not satisfactory, as it will not give the value of V further than the terms in σ^n. The form of the potential at points near the ring changes after these terms. Log R is introduced, and it is found that some of the equations coincide so that there are not enough to determine the necessary constants.

§ 9. [As illustrating this, and furnishing a result which will be used later, consider a distribution of matter of density $a\beta_2 \cos 2\chi$ on the surface of the ring.

By § 22 of my previous paper, it is seen that

$$\frac{\mathrm{V}_0}{2\pi a^2\beta_2} = \tfrac{1}{2}\frac{a^2}{\mathrm{R}^2}\cos 2\chi + \frac{\sigma}{4}\left\{\left(\tfrac{1}{2}\frac{a}{\mathrm{R}} - \tfrac{1}{3}\frac{a^3}{\mathrm{R}^3}\right)\cos 3\chi - \tfrac{1}{2}\frac{a}{\mathrm{R}}\cos\chi\right\}$$
$$+ \frac{\sigma^2}{32}\left\{-\frac{4\log\frac{8c}{\mathrm{R}} + 1}{2} + 2\left(1 - \frac{a^2}{\mathrm{R}^2}\right)\cos 2\chi + \left(\tfrac{3}{2} - \tfrac{2}{3}\frac{a^2}{\mathrm{R}^2} - \tfrac{1}{4}\frac{a^4}{\mathrm{R}^4}\right)\cos 4\chi\right\} + \&c.$$

This agrees with (11a), except for the term

$$-\frac{\sigma^2}{32}\,\frac{4\log\frac{8c}{\mathrm{R}} + 1}{2}.$$

The value of the potential, inside, is consequently given by

$$\frac{\mathrm{V}_i}{2\pi a^2\beta_2} = \tfrac{1}{2}\frac{\mathrm{R}^2}{a^2}\cos 2\chi + \frac{\sigma}{4}\left\{\left(\tfrac{1}{2}\frac{\mathrm{R}^3}{a^3} - \frac{\mathrm{R}}{a}\right)\cos\chi + \tfrac{1}{6}\frac{\mathrm{R}^3}{a^3}\cos 3\chi\right\}$$
$$+ \frac{\sigma^2}{32}\left\{\tfrac{7}{12}\frac{\mathrm{R}^4}{a^4}\cos 4\chi + 2\left(\frac{\mathrm{R}^4}{a^4} - \frac{\mathrm{R}^2}{a^2}\right)\cos 2\chi + \tfrac{3}{2}\frac{\mathrm{R}^4}{a^4} - 2\frac{\mathrm{R}^2}{a^2} - 2\log\frac{8c}{a}\right\} + \&c.$$

Adding to this the value given in (9) for the potential inside the ring $\mathrm{R} = a$, we have correct to the first power of β_2, the potential inside $\mathrm{R} = a\,(1 + \beta_2 \cos 2\chi)$.

The terms in β_2 of the exhaustion of potential energy of this ring are given by

$$2\pi^2 a^2 \int_0^{2\pi}\!\!\int_0^{a(1+\beta_2\cos 2\chi)} \left\{ \sigma\left(\frac{\mathrm{L}-1}{2}\,\frac{\mathrm{R}}{a} - \frac{\mathrm{R}^3}{8a^3}\right)\cos\chi \right.$$

$$\left. + \sigma^2\left[\tfrac{3}{16}\left(\mathrm{L}-\tfrac{5}{4}\right)\frac{\mathrm{R}^2}{a^2} - \tfrac{5}{96}\frac{\mathrm{R}^4}{a^4}\right]\cos 2\chi \right\}(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi$$

$$+ 2\pi^2 a^2 \beta_2 \int_0^{2\pi}\!\!\int_0^{a} \left\{ \frac{\sigma}{4}\left(\frac{\mathrm{R}^3}{2a^3} - \frac{\mathrm{R}}{a}\right)\cos\chi + \frac{\sigma^2}{32}\left(\tfrac{3}{2}\frac{\mathrm{R}^4}{a^4} - 2\frac{\mathrm{R}^2}{a^2} - 2\mathrm{L}\right)\right\}(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi,$$

all the other terms vanishing on integration.

Therefore this part of the exhaustion of energy

$$= 2\pi^2 a^4 c^2\ \sigma^2 \beta_2 \int_0^{2\pi} \left\{\left[-\frac{4\mathrm{L}-5}{16} + \tfrac{3}{16}\left(\mathrm{L}-\tfrac{5}{4}\right) - \tfrac{5}{96}\right]\cos^2 2\chi + \tfrac{1}{24}\cos^2\chi - \tfrac{1}{32}\left(\mathrm{L}+\tfrac{1}{4}\right)\right\} d\chi$$

$$= -\,2\pi^3 a^4 c\ \beta_2 \frac{\sigma^2}{8}\left(\mathrm{L} - \tfrac{5}{12}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

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SECTION II.

§ 10. An annular form is possible for gravitating fluid rotating round an axis in relative equilibrium. When the cross-section of the annulus is small compared with its radius, the cross-section is nearly circular.

Let the ring be disturbed so that the cross-section takes the form given by the equation

$$\rho = a\,\{1 + \beta_2 \cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots\}.$$

We shall prove that the ring is stable for disturbances of this kind, and find the periods of the various oscillations.

It is necessary to find the exhaustion of potential energy. Let this be called U. Then

$$\mathrm{U} = \tfrac{1}{2}\int \mathrm{V}_i\, dm$$

$$= \pi \int_0^{\pi}\!\!\int_0^{\rho} \mathrm{V}_i\,(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi$$

where V_i is the potential at an internal point.

It is necessary to know U as far as the second order of the small quantities β_2, &c., β_n. Suppose V_i is obtained in the form $v_0 + \Sigma v_n \cos n\chi$, where v_0, v_n are functions of R. Then it is necessary to know v_0 and v_1 to the second order, but $v_2 \ldots v_n \ldots$ only to the first order.

Now

$$\frac{V_i}{2\pi a^2} = \log\frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left(\frac{\log 8c/a - 1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right)\cos\chi + \&c.$$

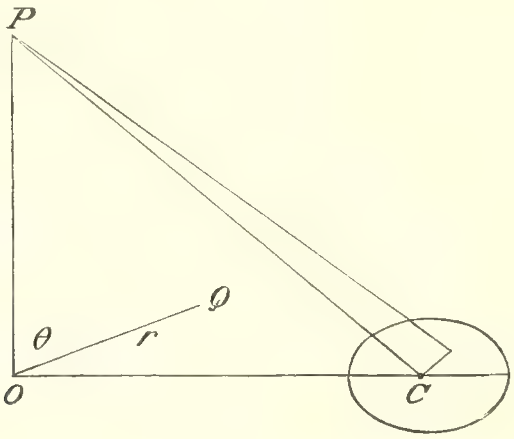
$$+ \Sigma\,\beta_n\left\{\left(\frac{R}{a}\right)^n\frac{\cos n\chi}{n} + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{R^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{R^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi\right.\right.$$

$$\left.\left. + \frac{1}{n(n+1)}\frac{R^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\}$$

$$+ A + B\left(\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2} + \&c.\right),$$

where A and B are small quantities of the second order in $\beta_2 \ldots \beta_n \ldots$, which must be found by comparing the value V_i at the surface of the ring with the value of V_0 there.

§ 11. We shall now find V_0 the potential of the ring at an external point.



Let Q be an external point whose coordinates are r and θ.

Let P be a point on the axis, C be the centre of gravity of the cross-section. Let PCO $= \alpha$ and let CP $= r_1$.

The potential at P is

$$2\pi\int_0^{2\pi}\int_0^{\rho}\frac{(c - R\cos\chi)\,R\,dR\,d\chi}{\sqrt{\{r_1^2 + R^2 - 2Rr_1\cos(\chi-\alpha)\}}}$$

$$= 2\pi\int_0^{2\pi}\left\{\frac{c}{r_1}\frac{\rho^2}{2} + \frac{c}{r_1^2}\frac{\rho^3}{3}\cos(\chi-\alpha) + \frac{c}{r_1^3}\frac{\rho^4}{4}P_2[\cos(\chi-\alpha)] - \frac{1}{r_1}\frac{\rho^3}{3}\cos\chi\right.$$

$$\left. - \frac{c\rho^4}{4\rho_1^2}\cos\chi\cos(\chi-\alpha)\ \ldots\right\}d\chi.$$

Substituting for $\cos\alpha$, c/r_1, and for ρ^p,

$$a^p\left\{1 + p\,\Sigma\,\beta_n \cos n\chi + \frac{p(p-1)}{2}\,\Sigma\left(\frac{\beta_n^2}{2} + \beta_n\,\beta_{n+1}\cos\chi\right)\right\},$$

and retaining only for the present the terms of the second degree in $\beta_2 \ldots \beta_n \ldots$, we find that at P

$$\mathrm{V}_0 = 4\pi^2a^2c\left\{\frac{1}{2r_1}\,\Sigma\left(\frac{\beta_n^2}{2} - \sigma\,\beta_n\,\beta_{n+1}\right) + \frac{ac}{r_1^3}\,\Sigma\,\frac{\beta_n\,\beta_{n+2}}{2}\right\}.$$

Therefore at Q

$$\mathrm{V}_0 = 2\pi a^2\,\Sigma\left(\frac{\beta_n^2}{2} - \sigma\,\beta_n\,\beta_{n+1}\right)\int_0^\pi \frac{c\,d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$
$$- 2\pi a^3c\,\Sigma\,(\beta_n\,\beta_{n+1})\,\frac{d}{dc}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}.$$

[*Supra*, p. 59.]

Expanding these integrals, we find that at a point R, χ just outside the ring,

$$\frac{\mathrm{V}_0}{2\pi a^2} = \Sigma\left(\frac{\beta_n^2}{2} - \sigma\,\beta_n\,\beta_{n+1}\right)\left(\log\frac{8c}{\mathrm{R}} + \frac{\log\frac{8c}{\mathrm{R}} - 1}{2}\,\frac{\mathrm{R}}{c}\cos\chi\right)$$
$$+ \sigma\,\Sigma\,\beta_n\,\beta_{n+1}\left(\frac{c}{\mathrm{R}}\cos\chi + \frac{2\log\frac{8c}{\mathrm{R}} - 1}{4} + \ldots\right).$$

Adding this to the values already obtained for the potential of a ring whose cross-section is an exact circle of radius a, and to that for a distribution of surface density $a\,\Sigma\,\beta_n\cos n\chi$ on such a ring, we find that

$$\frac{\mathrm{V}_0}{2\pi a^2} = \log\frac{8c}{\mathrm{R}} + \frac{\log\frac{8c}{\mathrm{R}} - 1}{2}\,\frac{\mathrm{R}\cos\chi}{c} + \ldots - \frac{\sigma^2}{8}\left(\frac{c}{\mathrm{R}}\cos\chi + \frac{2\log\frac{8c}{\mathrm{R}} - 1}{4} + \ldots\right)$$
$$+ \Sigma\,\beta_n\left\{\left(\frac{a}{\mathrm{R}}\right)^n\frac{\cos n\chi}{n} + \frac{\sigma}{4}\left[\left(\frac{a}{\mathrm{R}}\right)^{n-1}\frac{\cos(n+1)\chi}{n}\right.\right.$$
$$\left.\left. - \left(\frac{a}{\mathrm{R}}\right)^{n+1}\frac{\cos(n+1)\chi}{n+1} - \left(\frac{a}{\mathrm{R}}\right)^{n-1}\frac{\cos(n-1)\chi}{n(n-1)}\right] + \ldots\right\}$$
$$+ \Sigma\left(\frac{\beta_n^2}{2} - \sigma\,\beta_n\,\beta_{n+1}\right)\left(\log\frac{8c}{\mathrm{R}} + \frac{\log\frac{8c}{\mathrm{R}} - 1}{2}\,\frac{\mathrm{R}}{c}\cos\chi\right)$$
$$+ \sigma\,\Sigma\,\beta_n\,\beta_{n+1}\left(\frac{c}{\mathrm{R}}\cos\chi + \frac{2\log\frac{8c}{\mathrm{R}} - 1}{4} + \ldots\right) \quad \ldots\ldots\ldots \quad (14).$$

§ 12. Now when $R = \rho = a(1 + \Sigma\beta_n \cos n\chi)$, V_i and V_0 must have the same value. This will enable us to find the terms of the second order in V_i.

Remembering that for a solid ring of circular section, V and dV/dR are continuous at the surface, and $d^2V_0/dR^2 = (d^2V_i/dR^2) + 4\pi$, and that for a distribution of surface density $a\beta_n \cos n\chi$

$$\frac{dV_0}{dR} - \frac{dV_i}{dR} + 4\pi a\beta_n \cos n\chi = 0 ,$$

we see that when $R = \rho = a(1 + \Sigma\beta_n \cos n\chi)$

$$\begin{aligned}\frac{V_0 - V_i}{2\pi a^2} = {} & \frac{4\pi}{2\pi a^2}\frac{a^2}{2}(\beta_2 \cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots)^2 \\ & - \frac{4\pi a\,\Sigma\beta_n \cos n\chi}{2\pi a^2}\, a(\beta_2 \cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots) \\ & + \Sigma\left(\frac{\beta_n^2}{2} - \sigma\beta_n\beta_{n+1}\right)\left(\log\frac{8c}{a} + \frac{\log\frac{8c}{a} - 1}{2}\sigma\cos\chi\right) \\ & + \Sigma\beta_n\beta_{n+1}\left(\cos\chi + \frac{2\log\frac{8c}{a} - 1}{4}\sigma\right) \\ & - A - B\left(\cos\chi + \frac{\sigma}{4}\right).\end{aligned}$$

Since $V_0 - V_i = 0$, we find

$$\left.\begin{aligned} A + B\frac{\sigma}{4} &= (L - 1)\Sigma\frac{\beta_n^2}{2} - \frac{2L + 1}{4}\sigma\Sigma\beta_n\beta_{n+1} \\ B &= -\frac{L - 1}{4}\sigma\Sigma\beta_n^2 \end{aligned}\right\}$$

Therefore

$$\begin{aligned}\frac{V_i}{2\pi a^2} = {} & L + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left(\frac{L - 1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right)\cos\chi + \&c. \\ & + \Sigma\beta_n\left\{\frac{1}{n}\frac{R^n}{a^n}\cos n\chi + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{R^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{R^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi \right.\right. \\ & \qquad \left.\left. + \frac{1}{n(n+1)}\frac{R^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\} \\ & + (L - 1)\Sigma\frac{\beta_n^2}{2} - \frac{2L + 1}{4}\sigma\Sigma\beta_n\beta_{n+1} - \frac{L - 1}{16}\sigma^2\Sigma\beta_n^2 \\ & + \frac{L - 1}{4}\sigma\Sigma\beta_n^2\left(\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)\end{aligned}$$

§ 13. The exhaustion of potential energy is given by

$$\mathrm{U} = \tfrac{1}{2}\int \mathrm{V}_i\, dm$$

$$= \pi a^2 \iiint \left\{\mathrm{L} + \tfrac{1}{2}\left(1 - \frac{\mathrm{R}^2}{a^2}\right) + \sigma\cos\chi\left(\frac{\mathrm{L}-1}{2}\frac{\mathrm{R}}{a} - \frac{\mathrm{R}}{8a^3}\right)\right\}(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi\,d\phi$$

$$+ \pi a^2 \iiint \Sigma\,\beta_n \left\{\frac{1}{n}\frac{\mathrm{R}^n}{a^n}\cos n\chi + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{\mathrm{R}^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{\mathrm{R}^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi \right.\right.$$

$$\left.\left. + \frac{1}{n(n+1)}\frac{\mathrm{R}^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\}(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi\,d\phi$$

$$+ \pi a^2 \iiint \left\{(\mathrm{L}-1)\,\Sigma\frac{\beta_n^2}{2} - \frac{2\mathrm{L}+1}{4}\sigma\,\Sigma\,\beta_n\,\beta_{n+1} - \frac{\mathrm{L}-1}{16}\sigma^2\,\Sigma\,\beta_n^2 \right.$$

$$\left. + \frac{\mathrm{L}-1}{4}\sigma\,\Sigma\,\beta_n^2\left(\frac{\mathrm{R}}{a}\cos\chi + \frac{\sigma}{4}\frac{\mathrm{R}^2}{a^2}\right)\right\}(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi\,d\phi$$

$$= 2\pi^2 a^2 \int_0^{2\pi}\left[c\left\{(\mathrm{L}+\tfrac{1}{2})\frac{\rho^2}{2} - \frac{\rho^4}{8a^2}\right\} - \cos\chi\left\{(\mathrm{L}+\tfrac{1}{2})\frac{\rho^3}{3} - \frac{\rho^5}{10a^2}\right\}\right.$$

$$\left. + c\sigma\cos\chi\left\{\frac{\mathrm{L}-1}{2}\frac{\rho^3}{3a} - \frac{\rho^5}{40a^3}\right\}\right]d\chi$$

$$+ 2\pi^2 a^4 c\int_0^{2\pi}\Sigma\,\beta_n\cos n\chi\;\Sigma\left\{\frac{\beta_n}{n}\cos n\chi - \frac{\sigma\beta_n}{4n(n-1)}\cos(n-1)\chi\right.$$

$$\left. + \frac{\sigma\beta_n}{4n(n+1)}\cos(n+1)\chi\right\}(1 - \sigma\cos\chi)\,d\chi$$

$$+ 2\pi^3 a^4 c\left\{(\mathrm{L}-1)\,\Sigma\frac{\beta_n^2}{2} - \frac{2\mathrm{L}+1}{4}\sigma\,\Sigma\,\beta_n\,\beta_{n+1}\right\}.$$

In the first integral, substitute

$$\rho^p = a^p\left\{1 + p\,\Sigma\,\beta_n\cos n\chi + \frac{p(p-1)}{2}\Sigma\left(\frac{\beta_n^2}{2} + \cos\chi\,\Sigma\,\beta_n\beta_{n+1} + \ldots\right)\right\}.$$

On integration this gives

$$2\pi^3 a^4 c\left\{\mathrm{L} + \tfrac{1}{4} + (\mathrm{L}-1)\,\Sigma\frac{\beta_n^2}{2} - \frac{2\mathrm{L}+1}{4}\sigma\,\Sigma\,\beta_n\beta_{n+1}\right\}.$$

The second integral may be written in the form

$$2\pi^2 a^4 c\int_0^{2\pi}\Sigma\,\beta_n\cos n\chi\;\Sigma\left(\frac{\beta_n}{n} - \frac{\sigma\beta_{n+1}}{4n(n+1)} + \frac{\sigma\beta_{n-1}}{4n(n-1)}\right)\cos n\chi\,(1 - \sigma\cos\chi)\,d\chi,$$

giving on integration

$$2\pi^3 a^4 c \left\{ \Sigma \frac{\beta_n^2}{n} - \sigma \Sigma \frac{(2n+1)\beta_n\beta_{n+1}}{2n(n+1)} \right\}.$$

Therefore, adding the three integrals together,

$$\mathrm{U} = 2\pi^3 a^4 c \left\{ \mathrm{L} + \tfrac{1}{4} + \Sigma \left(\mathrm{L} - 1 + \frac{1}{n} \right) \beta_n^2 - \sigma \Sigma \left[\frac{2\mathrm{L}+1}{2} + \frac{2n+1}{2n(n+1)} \right] \beta_n \beta_{n+1} \right\} \quad (16).$$

Let a_0 be the mean radius of the cross-section.
Then

$$\pi a_0^2 = \pi a^2 \left(1 + \Sigma \frac{\beta_n^2}{2} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Let $2\pi^2 a_0^2 c_0$ be the volume of the ring.
Then

$$2\pi^2 a_0^2 c_0 = 2\pi^2 a^2 c \left(1 + \Sigma \frac{\beta_n^2}{2} - \sigma \Sigma \beta_n \beta_{n+1} \right) \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Substituting, we obtain

$$\mathrm{U} = 2\pi^3 a_0^4 c_0 \left\{ \log \frac{8c_0}{a_0} + \tfrac{1}{4} - \Sigma \left[\beta_n^2 \frac{n-1}{n} - \sigma \beta_n \beta_{n+1} \frac{(n-1)(3n+1)}{4n(n+1)} \right] \right\}.$$

M. POINCARÉ gives

$$\mathrm{U} = 2\pi^3 a_0^4 c_0 \left\{ \log \frac{8c_0}{a_0} + \tfrac{1}{4} - \Sigma \frac{n-1}{n} \beta_n^2 \right\}.$$

(TISSERAND, 'Méc. Cél.,' vol. 2, p. 166.)

[This is correct to the first power of σ. The term in $\sigma^2\beta_2$ is important, for in the equilibrium position of the ring β_2 is of order σ^2, and therefore this term is of the same order as the term in β_2^2. By (10) and (13), the more correct value of U is

$$2\pi^3 a_0^4 c_0 \left\{ \mathrm{L} + \tfrac{1}{4} - \frac{\mathrm{L} - \frac{2}{3}}{8} \sigma^2 - \frac{3\left(\mathrm{L} - \frac{19}{12}\right)}{512} \sigma^4 - \beta_2 \frac{\sigma^2}{8} \left(\mathrm{L} - \tfrac{5}{12} \right) \right.$$
$$\left. - \Sigma \left[\beta_n^2 \frac{n-1}{n} - \sigma \beta_n \beta_{n+1} \frac{(n-1)(3n+1)}{4n(n+1)} \right] \right\} \quad . \quad . \quad (19).$$

July 22, 1893.]*

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* This was inserted in consequence of an observation of one of the Referees, who noticed, what I had overlooked, that the less exact value of U failed in § 14 to give the correct equilibrium value of β_2.

§ 14. To find the fluted oscillations, it is necessary to determine the kinetic energy in the disturbed motion.

Let Φ be the velocity potential of this motion.

Since the bounding surface is given by

$$\mathrm{R} = a\{1 + \Sigma\beta_n \cos n\chi\},$$

therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} + \frac{\partial\mathrm{R}}{\partial c}\dot{c} = \dot{a}(1 + \Sigma\,\beta_n \cos n\chi) + a\,\Sigma\,\dot{\beta}_n \cos n\chi - a\,\Sigma\,(n\beta_n \sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} + \frac{\partial\chi}{\partial c}\dot{c}\right)$$

at the surface of the ring.

Now

$$\frac{\partial\mathrm{R}}{\partial c} = \cos\chi \quad \text{and} \quad \frac{\partial\chi}{\partial c} = -\frac{\sin\chi}{\mathrm{R}}.$$

Therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} = \dot{a} - \dot{c}\cos\chi + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi - a\,\Sigma\,(n\beta_n \sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} - \frac{\sin\chi}{\mathrm{R}}\dot{c}\right).$$

Since

$$a^2c\left(1 + \Sigma\frac{\beta_n^{\,2}}{2}\right) = \text{const.},$$

$$\frac{\dot{c}}{c} + \frac{2\dot{a}}{a} + \Sigma\beta_n\dot{\beta}_n = 0.$$

Therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right) - \frac{a}{2}\Sigma\beta_n\dot{\beta}_n + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi$$
$$- a\,\Sigma\,(n\beta_n \sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} - \frac{\sin\chi}{\mathrm{R}}\dot{c}\right).$$

Approximately

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right).$$

Therefore

$$\Phi = -a\dot{c}\left\{\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{4c}\frac{\mathrm{R}^2}{a^2}\right\}.$$

This gives

$$\frac{\partial\Phi}{\partial\chi} = a\dot{c}\frac{\mathrm{R}}{a}\sin\chi.$$

Substituting in the last term, we see that it vanishes: the second term may also be omitted, giving

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right) + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi$$

at the surface.

Therefore

$$\Phi = - \dot{c}a\left(\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{4c}\frac{\mathrm{R}^2}{a^2}\right) + \frac{1}{n}\Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\,a\,\frac{\mathrm{R}^n}{a^n}\cos n\chi.$$

The term in $\dot{a}\beta_n$ may be omitted.

It is easily seen, by § 4, that a more approximate solution is obtained by assuming

$$\Phi = - a\dot{c}\mathrm{A}_1\left\{\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{4c}\frac{\mathrm{R}^2}{a^2} + \frac{a}{32c}\frac{\mathrm{R}^3}{a^3}\cos\chi\right\}$$

$$+ \Sigma\,\mathrm{A}_n\left\{\frac{\mathrm{R}^n}{a^n}\cos n\chi + \frac{a}{4c}\frac{\mathrm{R}^{n+1}}{a^{n+1}}\cos(n-1)\chi\right\} \quad . \quad . \quad . \quad . \quad (20).$$

The constants are easily found to be

$$\mathrm{A}_1 = 1 - \frac{3\sigma^2}{32},$$

$$\mathrm{A}_n = a^2\left(\frac{\dot{\beta}_n}{n} - \frac{(n+2)}{4n\,(n+1)}\sigma\dot{\beta}_{n+1}\right).$$

At the surface of the ring, therefore,

$$\Phi = - a\dot{c}\left\{\cos\chi\left(1 - \frac{\sigma^2}{16}\right) + \frac{\sigma}{4}\right\} + a^2\Sigma\left(\frac{\dot{\beta}_n}{n} - \frac{\sigma\,(n+2)}{4n\,(n+1)}\dot{\beta}_{n+1}\right)\left(\cos n\chi + \frac{\sigma}{4}\cos(n-1)\chi\right)$$

$$= - a\dot{c}\left\{\cos\chi\left(1 - \frac{\sigma^2}{16}\right) + \frac{\sigma}{4}\right\} + a^2\Sigma\left(\frac{\dot{\beta}_n}{n} - \frac{\sigma\dot{\beta}_{n+1}}{2n\,(n+1)}\right)\cos n\chi$$

and

$$\frac{d\Phi}{dn} = - \dot{c}\left(\cos\chi + \frac{\sigma}{2}\right) + a\,\Sigma\,\dot{\beta}_n\cos n\chi.$$

The kinetic energy is given by

$$2\mathrm{T} = \iint \Phi\frac{d\Phi}{dn}\,d\mathrm{S}$$

$$= 2\pi ac\int \Phi\frac{d\Phi}{d\mathrm{R}}(1 - \sigma\cos\chi)\,d\chi$$

$$= 2\pi^2 a^2 c\left\{\dot{c}^2\left(1 - \frac{9\sigma^2}{16}\right) + a^2\Sigma\left(\frac{\dot{\beta}_n{}^2}{n} - \frac{\sigma\dot{\beta}_n\dot{\beta}_{n+1}}{n}\right)\right\}. \quad . \quad . \quad . \quad (21).$$

To this must be added the kinetic energy of the undisturbed motion, which is given by

$$2\mathrm{T} = \pi\dot{\phi}^2 \int_0^{2\pi}\int_0^{\rho} (c - \mathrm{R}\cos\chi)\, 3\mathrm{R}\, d\mathrm{R}\, d\chi$$

$$= \mathrm{M}c^2\dot{\phi}^2 \left(1 + \tfrac{3}{4}\sigma^2 + \tfrac{3}{2}\sigma^2\beta_2\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

where $\dot{\phi}$ is the undisturbed angular velocity of the fluid.

Therefore

$$2\mathrm{T} = \mathrm{M}\left\{\dot{c}^2\left(1 - \frac{9\sigma^2}{16}\right) + a^2\Sigma\frac{\dot{\beta}_n^{\,2} - \sigma\dot{\beta}_n\dot{\beta}_{n+1}}{n} + c^2\dot{\phi}^2\left(1 + \tfrac{3}{4}\sigma^2 + \tfrac{3}{2}\sigma^2\beta_2\right)\right\} . \quad (23).$$

§ 15. Also

$$\mathrm{U} = \gamma\frac{\mathrm{M}^2}{2\pi c}\left\{\log\frac{8c}{a} + \tfrac{1}{4} - \&\mathrm{c.} - \frac{\beta_2\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\right.$$

$$\left. - \Sigma\left(\frac{n-1}{n}\beta_n^{\,2} - \frac{(3n+1)(n-1)}{2n(n+1)}\sigma\beta_n\beta_{n+1}\right)\right\} \quad . \quad . \quad (24),$$

where a and c now denote the mean radii of the cross-section and of the ring respectively; $\sigma = a/c$, and is small; γ is the constant of gravitation.

Retain only the terms of the highest order; LAGRANGE'S equations give

$$\mathrm{M}a^2\frac{\ddot{\beta}_n}{n} + \frac{\mathrm{M}^2}{2\pi c}\gamma\frac{n-1}{2n}\beta_n = 0.$$

Thus the period of the oscillation of the type $\beta_n \cos n\chi$ is given approximately by

$$\ddot{\beta}_n + \pi\frac{(n-1)}{2}\gamma\beta_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

The time of a complete oscillation is $\sqrt{\dfrac{8\pi}{(n-1)\gamma}}$, and the time a fluted wave of this type would take to travel round the ring is

$$n\sqrt{\frac{8\pi}{(n-1)\gamma}}.$$

[The equation giving the period of the disturbance of type $\beta_2 \cos 2\chi$ is

$$\mathrm{M}\left\{a^2\frac{\ddot{\beta}_2}{2} - \tfrac{3}{4}c^2\sigma^2\dot{\phi}^2\right\} + \gamma\frac{\mathrm{M}^2}{2\pi c}\left\{\beta_2 + \frac{\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\right\} = 0.$$

Therefore the value of β_2 in the undisturbed motion is given by

$$\begin{aligned}\beta_2 &= \tfrac{3}{4}\,\pi\gamma\dot{\phi}^2 - \frac{\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\\ &= \frac{3a^2}{4c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) - \frac{a^2}{8c^2}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right), \text{ by (26)},\\ &= \tfrac{5}{8}\,\frac{a^2}{c^2}\left(\log\frac{8c}{a} - \tfrac{17}{12}\right),\end{aligned}$$

agreeing with the result given *supra*, p. 94.—July 22, 1893.]*

An interesting oscillation of the ring is that in which the mean radius increases and diminishes, necessitating a decrease and increase of the cross-section.

We have

$$\left.\begin{aligned} \mathrm{M}c^2\dot{\phi} &= \text{const.}\\ \mathrm{M}\ddot{c} - \mathrm{M}c\dot{\phi}^2 - \gamma\,\frac{\mathrm{M}^2}{2\pi}\,\frac{d}{dc}\left(\frac{\log\frac{8c}{a} + \frac{1}{4}}{c}\right) &= 0 \end{aligned}\right\}$$

Remembering that a^2c is constant,

$$\begin{aligned}\frac{d}{dc}\,\frac{\log\frac{8c}{a} + \frac{1}{4}}{c} &= -\frac{1}{c^2}\left(\log\frac{8c}{a} + \tfrac{1}{4}\right) + \frac{1}{c}\left(\frac{1}{c} + \frac{1}{a}\,\frac{a}{2c}\right)\\ &= -\frac{1}{c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right).\end{aligned}$$

Therefore,

$$\mathrm{M}\ddot{c} - \mathrm{M}c\dot{\phi}^2 + \frac{\gamma\mathrm{M}^2}{2\pi c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) = 0.$$

Thus the angular velocity in the position of relative equilibrium is given by

$$\begin{aligned}\dot{\phi}^2 &= \frac{\gamma\mathrm{M}^2}{2\pi\mathrm{M}c^3}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right)\\ &= \gamma\pi\,\frac{a^2}{c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) \quad \ldots\ldots\ldots \quad (26),\end{aligned}$$

agreeing with the results found, *ante*, p. 94.

For the small oscillation, write $c = c_0 + x$, $a = a_0 + y$, in the above equation.

Then,

$$a_0^{\,2}x + 2\,a_0c_0y = 0, \quad \text{or} \quad y = -\tfrac{1}{2}\,\frac{a_0}{c_0}\,x.$$

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* See note, p. 1060.

Therefore

$$M\ddot{x} - \frac{Mc^4\dot{\phi}^2}{(c_0 + x)^3} + \frac{\gamma M^2}{2\pi(c_0 + x)^2}\left(\log\frac{8(c_0 + x)}{(a_0 + y)} - \tfrac{5}{4}\right) = 0.$$

Therefore

$$\ddot{x} - \gamma\frac{a_0^2c_0^2\left(\log\frac{8c_0}{a_0} - \frac{5}{4}\right)}{(c_0 + x)^3} + \frac{\gamma\, 2\pi a_0^2 c_0}{2\pi(c_0 + x)^2}\left(\log\frac{8(c_0 + x)}{a_0 + y} - \tfrac{5}{4}\right) = 0,$$

or

$$\ddot{x} + 3\gamma\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} - \tfrac{5}{4}\right)x - 2\gamma\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} - \tfrac{5}{4}\right)x + \tfrac{3}{2}\gamma\frac{a_0^2}{c_0^2}x = 0,$$

or

$$\ddot{x} + \gamma\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} + \tfrac{1}{4}\right)x = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

giving for the time of a complete oscillation

$$\frac{2\pi c_0}{a_0}\Big/\sqrt{\left\{\gamma\left(\log\frac{8c_0}{a_0} + \tfrac{1}{4}\right)\right\}}.$$

The oscillations might be found more approximately, but this hardly seems worth while, as the ring will be proved unstable for disturbances of a different kind.

§ 16. *The effect of long beaded waves.*

As before, let U be the exhaustion of potential energy of the ring in its disturbed state. In this case the ring is disturbed so that its central circle remains an exact circle of radius c, the cross-section is always a circle, but the radius of this circle varies with the azimuth. Let it be given by

$$\rho = a\{1 + \Sigma(\alpha_n \sin n\phi + \beta_n \cos n\phi)\}.$$

Let V_i be the potential inside ; V_0, outside.

Then

$$U = \tfrac{1}{2}\int V_i\, dm$$

$$= \tfrac{1}{2}\int_0^{2\pi}\int_0^{\rho}\int_0^{2\pi} V_i\,(c - R\cos\chi)\, R\, dR\, d\chi\, d\phi.$$

If V_i be expressed in the form

$$v_0 + v_1\cos\chi + v_2\cos 2\chi + \ldots,$$

the only terms we need are v_0 and v_1,

Also, the only terms of the second order in the small quantities α and β, which need be retained are those independent of ϕ, as the others vanish on integration.

§ 17. To determine V_i we must first find V_0.

Let ϖ', z', ϕ' be the coordinates of any external point, ϖ, z, ϕ of any internal point. Then

$$\mathrm{V}_0 = \iiint \frac{\varpi \, d\varpi \, dz \, d\phi}{\sqrt{\{\varpi'^2 + \varpi^2 - 2\varpi\varpi' \cos(\phi' - \phi) + (z' - z)^2\}}}.$$

Let $\varpi = c - x$.

Then

$$\mathrm{V}_0 = \iiint \frac{(c - x) \, dx \, dz \, d\phi}{\sqrt{\{(c - x)^2 + \varpi'^2 - 2(c - x)\,\varpi' \cos(\phi' - \phi) + (z' - z)^2\}}}$$

$$= \iiint e^{-x(d/dc) - z(d/dz')} \, dx \, dz \, \frac{c \, d\phi}{\sqrt{\{c^2 + \varpi'^2 \, 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$= \int_0^{2\pi} \iint \Big\{ 1 - \left(x \frac{d}{dc} + z \frac{d}{dz'} \right)$$

$$+ \frac{1}{2\,!} \left(x \frac{d}{dc} + z \frac{d}{dz'} \right)^2 \ldots \Big\} \, dx \, dz \, \frac{c \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

where the double integral is taken over a circle of radius ρ.

Therefore

$$\mathrm{V}_0 = \int_0^{2\pi} \Big\{ \pi\rho^2 + \tfrac{1}{8} \pi\rho^4 \left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2} \right)$$

$$+ \tfrac{1}{192} \pi\rho^6 \left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2} \right)^2 + \ldots \Big\} \, \frac{c \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}.$$

Now

$$\rho^p = a^p \Big\{ 1 + p \, \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi)$$

$$+ \frac{p(p-1)}{2} \Sigma \left[\frac{\alpha_n^2 + \beta_n^2}{2} + (\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1}) \cos \phi + \ldots \right] \Big\}.$$

Substituting this, and writing $\dfrac{d^2}{dc^2} + \dfrac{d^2}{dz'^2} = \nabla^2$

$$\mathrm{V}_0 = \pi a^2 \int_0^{2\pi} \left\{ 1 + \frac{a^2}{8} \nabla^2 + \frac{a^4}{192} \nabla^4 \right\} \frac{c \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$+ \pi a^2 \int_0^{2\pi} \left\{ 2 + \frac{a^2}{2} \nabla^2 + \frac{a^4}{32} \nabla^4 \right\} \frac{c \, \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi) \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$+ \pi a^2 \int_0^{2\pi} \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \left\{ 1 + \frac{3a^2}{4} \nabla^2 + \frac{5a^4}{64} \nabla^4 \right\} \frac{c \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$= \pi a^2 \int_0^{2\pi} \left\{ 1 + \frac{a^2}{8} \nabla^2 + \frac{a^4}{192} \nabla^4 + \ldots \right\} \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ \pi a^2 \Sigma (\alpha_n \sin n\,\phi' + \beta_n \cos n\,\phi') \int_0^{2\pi} \left\{ 2 + \frac{a^2}{2} \nabla^2 \right\} \frac{c \cos n\phi\, d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ \pi a^2 \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \int_0^{2\pi} \left\{ 1 + \frac{3a^2}{4} \nabla^2 + \ldots \right\} \frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \quad . \quad . \quad (28).$$

§ 18. These integrals admit of simplification.

Denoting

$$\int_0^{2\pi} \frac{\cos n\phi\, d\phi}{\sqrt{\{c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2\}}} \text{ by I}$$

$$\frac{d^2\mathrm{I}}{d\varpi'^2} + \frac{1}{\varpi'}\frac{d\mathrm{I}}{d\varpi'} + \frac{d^2\mathrm{I}}{dz'^2} - \frac{n^2}{\varpi'^2}\mathrm{I} = 0,$$

since I cos $n\ \phi$ is a solution of LAPLACE's equation.

Symmetry shows that

$$\frac{d^2\mathrm{I}}{dc^2} + \frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{d^2\mathrm{I}}{dz'^2} - \frac{n^2}{c^2}\mathrm{I} = 0,$$

therefore

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) \mathrm{I}c = \frac{d\mathrm{I}}{dc} + \frac{n^2}{c}\mathrm{I},$$

and

$$\left(\frac{d^2}{dc} + \frac{d^2}{dz'^2}\right)^2 \mathrm{I}c = -\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right) - 2\frac{n^2}{c^2}\frac{d\mathrm{I}}{dc} + \frac{n^4}{c^3}\mathrm{I}.$$

Substituting, we obtain

$$\mathrm{V}_0 = 2\pi a^2 c \left\{ 1 + \tfrac{1}{8}\frac{a^2}{cdc} - \tfrac{1}{192}\left(\frac{a^2}{cdc}\right)^2 + \ldots \right\} \int_0^{\pi} \frac{d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ 4\pi a c\, \Sigma (\alpha_n \sin n\phi' + \beta_n \cos n\phi') \left\{ 1 + \frac{a^2 n^2}{4c^2} + \frac{a^4 n^4}{64 c^4} + \left(\tfrac{1}{4} - \frac{a^2 n^2}{32 c^2}\right) \frac{a^2}{cdc} \right.$$

$$\left. - \tfrac{1}{64}\left(\frac{a^2}{cdc}\right)^2 \right\} \int_0^{\pi} \frac{\cos n\phi\, d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ 2\pi a^2 c\, \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \left\{ 1 + \tfrac{3}{4}\frac{a^2}{cdc} - \tfrac{5}{32}\left(\frac{a^2}{cdc}\right)^2 \right\} \int_0^{\pi} \frac{d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}} \quad (29).$$

It is important to notice that at the surface of the ring differentiation with respect to c lowers the order of the terms once in a/c. Therefore the operation $\frac{a^2}{c} \cdot \frac{d}{dc}$ raises the order once in a/c. Provided that n is not greater than $\sqrt{(c/a)}$, that is, when the waves are long, the value of V_0 is given by § 17 correctly to the order $(a/c)^2$.

We shall, however, simplify the work by rejecting all but the most important terms, though it would be quite easy to retain terms of higher orders.

§ 19. Let the point ϖ', z', ϕ' be near the ring, and with the notation used before be the point R.χ.

The most important term in $\int_0^\pi \frac{d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c \cos\phi + c^2 + z'^2)}}$ is $\log\frac{8c}{R}$.

The integral $\int_0^\pi \frac{\cos n\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}}$, being of the form $\frac{1}{\sqrt{(2\varpi' c)}}\int_0^\pi \frac{\cos n\phi\, d\phi}{\sqrt{(q - \cos\phi)}}$ is

$$\frac{1}{\sqrt{(2\varpi' c)}}\frac{1}{\sqrt{2}}\log\frac{16(q+1)}{q-1} - \frac{2\sqrt{2}}{\sqrt{(2\varpi' c)}}\left(1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}\right)$$

(J. J. THOMSON, 'Motion of Vortex-rings,' p. 26)

$$= \log\frac{8c}{R} - 2f(n),$$

where

$$f(n) = 1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}.$$

Writing ϕ for the azimuth, instead of ϕ', we have, therefore, that near the ring outside

$$V_0 = 2\pi a^2\left\{1 + \Sigma\frac{\alpha_n^2 + \beta_n^2}{2}\right\}\log\frac{8c}{R}$$
$$+ 4\pi a^2\,\Sigma\,(\alpha_n \sin n\phi + \beta_n \cos n\phi)\left\{\log\frac{8c}{R} - 2f(n)\right\} \quad . \quad . \quad (30).$$

Inside the ring, therefore,

$$V_i = 2\pi a^2\left\{\log\frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right)\right\}$$
$$+ 4\pi a^2\,\Sigma\,(\alpha_n \sin n\phi + \beta_n \cos n\phi)\left\{\log\frac{8c}{a} - 2f(n)\right\}$$
$$+ C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where C is a constant of the second order in α_n, β_n, &c.

As in § 11, we find that at the surface of the ring,

$$\mathrm{V}_0 - \mathrm{V}_i = 2\pi a^2 \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \log \frac{8c}{a} + \frac{a^2}{2} \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi)^2 \, 4\pi$$
$$+ a \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi) [- 4\pi a \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi)] - \mathrm{C}.$$

Since V_0 and V_i are equal at the surface,

$$\mathrm{C} = 2\pi a^2 \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Now

$$\mathrm{U} = \tfrac{1}{2} \int \mathrm{V}_i \, dm$$
$$= \pi a^2 \int_0^{2\pi} \int_0^{\rho} \int_0^{2\pi} \left\{ \log \frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{\mathrm{R}^2}{a^2}\right) + 2 \Sigma \left(\log \frac{8c}{a} - 2f(n)\right) (\alpha_n \sin n\phi + \beta_n \cos n\phi) \right.$$
$$\left. + \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \right\} (c - \mathrm{R} \cos \chi) \, \mathrm{R} \, d\mathrm{R} \, d\chi \, d\phi$$
$$= 2\pi^2 a^2 c \int_0^{2\pi} \left\{ \left(\log \frac{8c}{a} + \tfrac{1}{2}\right) \frac{\rho^2}{2} - \frac{\rho^4}{8a^2} + 2 \Sigma \left(\log \frac{8c}{a} - 2f(n)\right) (\alpha_n \sin n\phi + \beta_n \cos n\phi) \frac{\rho^2}{2} \right.$$
$$\left. + \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \frac{\rho^2}{2} \right\} d\phi$$
$$= 2\pi^3 a^4 c \left\{ \log \frac{8c}{a} + \tfrac{1}{4} + \left(\log \frac{8c}{a} - 1\right) \Sigma (\alpha_n^2 + \beta_n^2) \right.$$
$$\left. + \left(2 \log \frac{8c}{a} - 4f(n)\right) \Sigma (\alpha_n^2 + \beta_n^2) \right\}.$$

Let a_0 be the mean value of a.
Then

$$\mathrm{U} = 2\pi^3 a_0^4 c \left\{ \log \frac{8c}{a_0} + \tfrac{1}{4} + 2 \Sigma \left(\log \frac{8c}{a_0} - 2f(n) - \tfrac{1}{4}\right) (\alpha_n^2 + \beta_n^2) \right\} \quad . \quad (33).$$

§ 20. When the ring is thin and n is small the change in the radius of the cross-section is slow, and an approximation to the disturbed motion is easily found by the method of parallel sections.

$$\pi \rho^2 v = \text{const.} = \mathrm{C}$$
$$2\mathrm{T} = \int_0^{2\pi} \pi \rho^2 c \, d\phi \, v^2$$
$$h = \int_0^{2\pi} \pi \rho^2 c \, d\phi \, cv = 2\pi c^2 \mathrm{C}.$$

Therefore

$$
\begin{aligned}
2\mathrm{T} &= \int_0^{2\pi} \frac{c\mathrm{C}^2}{\pi\rho^2}\, d\phi. \\
&= \frac{2c\mathrm{C}^2}{a^2}\left\{1 + \tfrac{3}{2}\Sigma\left(\alpha_n^2 + \beta_n^2\right)\right\} \\
&= \frac{2c\mathrm{C}^2}{a_0^2}\left\{1 + 2\,\Sigma\left(\alpha_n^2 + \beta_n^2\right)\right\} \\
&= \frac{h^2}{\mathrm{M}c^2}\left\{1 + 2\,\Sigma\left(\alpha_n^2 + \beta_n^2\right)\right\}.
\end{aligned}
$$

Let A_n be the component of momentum corresponding to the coordinate α_n. Then the whole kinetic may be expressed in the form

$$
\frac{h^2}{2\mathrm{M}c^2}\left\{1 + 2\,\Sigma\left(\alpha_n^2 + \beta_n^2\right)\right\} + \tfrac{1}{2}\Sigma k_n \mathrm{A}_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),
$$

where the terms in the summation are necessarily positive. Thus

$$
\begin{aligned}
\mathrm{T} - \mathrm{U} = \tfrac{1}{2}\Sigma k_n \mathrm{A}_n^2 &+ \frac{h^2}{2\mathrm{M}c^2}\left\{1 + 2\,\Sigma\left(\alpha_n^2 + \beta_n^2\right)\right\} \\
&- \frac{2\mathrm{M}^2}{\pi c}\left\{\mathrm{L} + \tfrac{1}{4} + 2\,\Sigma\left[\mathrm{L} - 2f(n) - \tfrac{1}{4}\right]\left(\alpha_n^2 + \beta_n^2\right)\right\} \quad . \quad . \quad (35).
\end{aligned}
$$

HAMILTON'S equations will give

$$
\frac{d}{dt}\left(k_n \mathrm{A}_n\right) + \frac{2h^2}{\mathrm{M}c^2}\alpha_n - \frac{2\mathrm{M}^2}{\pi c}\left[\mathrm{L} - 2f(n) - \tfrac{1}{4}\right]\alpha_n = 0 \quad . \quad . \quad . \quad . \quad (36).
$$

The steady motion will be stable or unstable as

$$
\frac{2h^2}{\mathrm{M}c^2} - \frac{2\mathrm{M}^2}{\pi c}\left[\mathrm{L} - 2f(n) - \tfrac{1}{4}\right] \text{ is } > \text{ or } < 0.
$$

That is, according as

$$
-\left[\mathrm{L} - 4f(n) + \tfrac{3}{4}\right] \text{ is } > \text{ or } < 0.
$$

Now, when $n = 1$

$$
\mathrm{L} - 4f(n) + \tfrac{3}{4} = \log\frac{8c}{a_0} - 3\tfrac{1}{4}.
$$

When $c > 3a_0$, this is positive: therefore for any ring whose cross-section is smaller than this, beaded waves cause instability.

§ 21. Next consider a disturbance which leaves the cross-section of the ring by any plane through the axis an exact circle of radius a, but the centre of the section is displaced to the point whose cylindrical coordinates are $c - \xi, \zeta, \phi$, where

$$\xi = c \, \Sigma \left(\alpha_n \sin n\phi + \beta_n \cos n\phi\right),$$

and

$$\zeta = c \, \Sigma \left(\gamma_n \sin n\phi + \delta_n \cos n\phi\right),$$

ξ and ζ are taken to be small quantities compared with a.

Let $c - \xi - x$, $z + \zeta$, ϕ be coordinates of any point of the ring, and ϖ', z', ϕ' of any external point.

The potential of the ring at $\varpi', z'. \phi'$ is given by

$$\begin{aligned}
\mathrm{V}_0 &= \iiint \frac{(c - x - \xi)\, dx\, dz\, d\phi}{\sqrt{\{\varpi'^2 - 2(c - x - \xi)\varpi' \cos(\phi' - \phi) + (e - x - \xi)^2 + (z' - z - \zeta)^2\}}} \\
&= \iiint e^{-(x+\xi)\, d/dc - (z+\zeta)\, d/dz'}\, dx\, dz \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \int_0^{2\pi} e^{-\xi(d/dc) - \zeta(d/dz')} \iint \left\{1 - x\frac{d}{dc} - z\frac{d}{dz'}\right. \\
&\qquad \left. + \ldots\right\} dx\, dz \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \int_0^{2\pi} e^{-\xi(d/dc) - \zeta(d/dz')}\, \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\right. \\
&\qquad \left. + \ldots\right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \int_0^{2\pi} \left\{1 - \xi\frac{d}{dc} - \zeta\frac{d}{dz'} + \frac{\xi^2}{2}\frac{d^2}{dc^2}\right. \\
&\qquad \left. + \ldots\right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \quad . \quad (37).
\end{aligned}$$

Retaining only those terms of the second order in the small quantities $\alpha, \beta, \gamma, \delta$, which do not involve ϕ',

$$\begin{aligned}
\mathrm{V}_0 = {} & \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \int_0^{\pi} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\
& - \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \left\{\Sigma\left(\alpha_n \sin n\phi' + \beta_n \cos n\phi'\right)\frac{d}{dc}\right. \\
& \left. + \Sigma\left(\gamma_n \sin n\phi' + \delta_n \cos n\phi'\right)\frac{d}{dz'}\right\} \int_0^{\pi} \frac{c \cos n\phi\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\
& + \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \left\{\frac{\alpha_n^2 + \beta_n^2}{4}\frac{d^2}{dc^2} + \frac{\alpha_n\gamma_n + \beta_n\delta_n}{2}\frac{d^2}{dc\, dz}\right. \\
& \left. + \frac{\gamma_n^2 + \delta_n^2}{4}\frac{d^2}{dz'^2}\right\} \int_0^{2\pi} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (38).
\end{aligned}$$

Write

$$\left.\begin{aligned}\xi' &= \alpha_n \sin n\phi' + \beta_n \cos n\phi' \\ \zeta' &= \gamma_n \sin n\phi' + \delta_n \cos n\phi'\end{aligned}\right\}$$

Then

$$\begin{aligned}\mathrm{V}_0 = \pi a^2 \Bigg\{ 1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) \\ + \ldots \Bigg\} \left\{1 - \xi' \frac{d}{dc} - \zeta' \frac{d}{dz'} + \frac{\xi'^2}{2}\frac{d^2}{dc^2} + \ldots \right\} \int_0^{2\pi} \frac{c \cos\phi \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\ + \pi a^2 \left\{ 1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots \right\} \left\{\xi' \frac{d}{dc} + \zeta' \frac{d}{dz'}\right\} \int_0^{2\pi} \frac{(1 - \cos n\phi)\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} .\end{aligned}$$

Let the point ϖ', z', ϕ' be on the surface of the ring.

Let

$$\begin{aligned}\varpi' &= c - \mathrm{R}' \cos\chi' & \text{and} \qquad z' &= \mathrm{R}' \sin\chi' \\ &= c - a \cos\chi_1 - \xi' & &= a \sin\chi_1 + \zeta'.\end{aligned}$$

Retaining only the most important terms,

$$\begin{aligned}\mathrm{V}_0 &= 2\pi a^2 \left(1 - \xi' \frac{d}{dc} - \zeta' \frac{d}{dz'} + \ldots\right) \left\{\log\frac{8c}{\mathrm{R}'} + \&c. \right\} \\ &\quad + 2\pi a^2 \left(\xi' \frac{d}{dc} + \zeta' \frac{d}{dz'}\right) \left\{2f(n) + f(n)\frac{\mathrm{R}' \cos\chi'}{c} + \ldots\right\} \\ &= 2\pi a^2 \left\{\log\frac{8(c - \xi')}{a} + \frac{\xi'}{c} f(n)\right\} \\ &= 2\pi a^2 \left\{\log\frac{8c}{a} + \frac{\xi'}{c}[f(n) - 1] - \frac{\xi'^2}{2c^2}\right\} \\ &= 2\pi a^2 \left\{\log\frac{8c}{a} + \Sigma[f(n) - 1](\alpha_n \sin n\phi' + \beta_n \cos n\phi') - \Sigma\frac{\alpha_n^2 + \beta_n^2}{4}\right\} . \quad (39).\end{aligned}$$

§ 22. Inside the ring the potential is V_i, where

$$\mathrm{V}_i = 2\pi a^2 \left\{\log\frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{\mathrm{R}^2}{a^2}\right) + \Sigma[f(n) - 1](\alpha_n \sin n\phi' + \beta_n \cos n\phi') - \Sigma\frac{\alpha_n^2 + \beta_n^2}{4}\right\} ,$$

where R_1 is the distance from the curve of centres.

The exhaustion of potential energy

$$\mathrm{U} = \tfrac{1}{2}\int \mathrm{V}_i \, dm,$$

thus

$$\frac{\mathrm{U}}{\pi a^2} = \iiint \frac{\mathrm{V}i}{2\pi a^2}(c - x - \xi')\, dx\, dz\, d\phi',$$

or

$$U = 2\pi^3 a^4 c \left\{ \log \frac{8c}{a} + \tfrac{1}{4} - \Sigma \frac{\alpha_n^2 + \beta_n^2}{4} - \Sigma [f(n) - 1] \frac{\alpha_n^2 + \beta_n^2}{2} \right\}$$

$$= \frac{M^2}{2\pi c} \left\{ \log \frac{8c}{a} + \tfrac{1}{4} - \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} [f(n) - \tfrac{1}{2}] \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

For

$$M = \iiint (c - x - \xi')\, dx\, dz\, d\phi'$$

$$= 2\pi^2 a^2 c.$$

§ 23. The exhaustion of potential energy is therefore diminished when the ring is displaced so that the central circle does not remain circular. For this kind of disturbance therefore, the ring is stable, even when the fluid does not rotate. The effect of rotation is to increase this stability.

We have therefore the following results.

The annular form of equilibrium of rotating gravitating fluid is stable for disturbances symmetrical about the axis, and for disturbances which alter the shape of the central curve, but is unstable for long beaded disturbances.

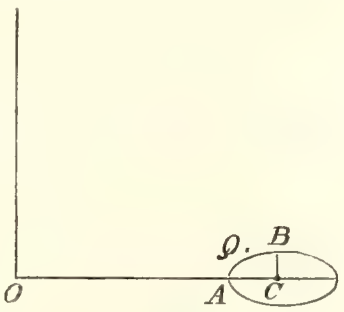
This result was, perhaps, to be expected, as by means of beaded waves, the mass would naturally be broken up into spheroidal masses.

SECTION III.

§ 24. The methods given in my paper, *ante*, may be used to find the potential of any ring whose cross-section does not deviate far from a circle. They will not, however, apply to a ring whose cross-section is very elliptic, any more than the potential of an elliptic cylinder can be obtained as an approximation from one whose section is circular. In this section, the potential of a ring of elliptic cross-section is obtained by taking the known result for an elliptic cylinder as a first approximation. *The value of the potential obtained applies only to points not far from the surface of the ring.* The potential at internal points may be derived from this, while the potential at other points may be found easily by other methods.

Consider a ring, whose cross-section is elliptic, the major axis of the ellipse being perpendicular to the axis of the ring.

Let the figure represent a section through the axis of the ring.



Let OC $= c$: CA $= a$: CB $= b$.

Let x and z be the coordinates of any point in this section of the ring referred to CA and CB as axes.

The potential of the ring at any external point ϖ', z', is

$$\iiint \frac{(c-x)\,dx\,dz\,d\phi}{\sqrt{\{(c-x)^2 + \varpi'^2 - 2\varpi'(c-x)\cos\phi + (z'-z)^2\}}} \quad . \quad . \quad . \quad . \quad . \quad (41),$$

$$= \iint e^{-x(d/dc) - z(d/dz')}\,dx\,dz \int_0^{2\pi} \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad . \quad . \quad (42).$$

Writing D for d/dc, and D′ for d/dz', and taking the double integral over the area of the ellipse, we obtain for the potential at an external point,

$$\mathrm{V}_0 = 4\pi ab\left\{\tfrac{1}{2} + \tfrac{1.1}{2.4}\,\frac{a^2\mathrm{D}^2 + b^2\mathrm{D}'^2}{2\,!} + \tfrac{1.1.3}{2.4.6}\,\frac{(a^2\mathrm{D}^2 + b^2\mathrm{D}'^2)^2}{4\,!}\right.$$
$$\left. + \ldots\right\} \int_0^{\pi} \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

§ 25. When the cross-section is circular, the formula may be simplified. Calling

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}},\ \mathrm{I}.$$

$$\frac{d^2\mathrm{I}}{dc^2} + \frac{d^2\mathrm{I}}{dz'^2} + \frac{1}{c}\,\frac{d\mathrm{I}}{dc} = 0.$$

Therefore

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) c\mathrm{I} = 2\,\frac{d\mathrm{I}}{dc} - \frac{d\mathrm{I}}{dc} = \frac{d\mathrm{I}}{dc}$$

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^2 c\mathrm{I} = \frac{d}{dc}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\mathrm{I} = -\,\frac{d}{dc}\left(\frac{1}{c}\,\frac{d\mathrm{I}}{dc}\right)$$

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^3 c\mathrm{I} = -\,\frac{d}{dc}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\left(\frac{1}{c}\,\frac{d\mathrm{I}}{dc}\right)$$

$$= -\,\frac{d}{dc}\left\{\frac{1}{c}(\mathrm{D}^2 + \mathrm{D}'^2\,\frac{d\mathrm{I}}{dc} - \frac{2}{c}\,\frac{d}{dc}\left(\frac{1}{c}\,\frac{d\mathrm{I}}{dc}\right)\right\}$$

$$= +\,1\,.\,3\,\frac{d}{dc}\left(\frac{1}{c}\,\frac{d}{dc}\right)^2 \mathrm{I}.$$

So that when the cross-section is circular, we arrive at the formula given in my former paper, p. 59.

$$\mathrm{V}_0 = \frac{\mathrm{M}}{\pi c}\left\{1 + \frac{a^2}{8c}\,\frac{d}{dc} - \frac{a^4}{192}\left(\frac{1}{c}\,\frac{d}{dc}\right)^2 - \ldots\right\}\int_0^{\pi} \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad (44).$$

§ 26. Let the external point at which the potential is required be near the ring; let its coordinates referred to CA and CB be $\mathrm{R}\cos\chi$ and $\mathrm{R}\sin\chi$.

The integral

$$\int_0^\pi \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}}$$

may be expanded in the form

$$\log\frac{8c}{\mathrm{R}} + \frac{\log\frac{8c}{\mathrm{R}} - 1}{2}\,\frac{\mathrm{R}\cos\chi}{c} + \left(\frac{2\log\frac{8c}{\mathrm{R}} - 3}{16} + \frac{3\log\frac{8c}{\mathrm{R}} - 4}{16}\cos 2\chi\right)\frac{\mathrm{R}^2}{c^2} + \text{\&c.} \qquad (45).$$

Now

$$\left\{\tfrac{1}{2} + \tfrac{1.1}{2.4}\,\frac{a^2\mathrm{D}^2 + b^2\mathrm{D}'^2}{2\,!} + \text{\&c.}\right\}\log\frac{8c}{\mathrm{R}}$$

$$= \tfrac{1}{2}\log 8c - \tfrac{1.1}{2.4}\,\tfrac{1}{2}\,\frac{a^2}{c^2}\,\tfrac{1.1.3}{2.4.6}\,\frac{a^4}{4c^4} - \text{\&c.}$$

$$- \tfrac{1}{2}\log\mathrm{R} + \tfrac{1.1}{2.4}\,\frac{a^2 - b^2}{2}\,\frac{\cos^2\chi}{\mathrm{R}^2} + \tfrac{1.1.3}{2.4.6}\,\frac{(a^2 - b^2)^2}{4}\,\frac{\cos 4\chi}{\mathrm{R}^4} + \text{\&c.} \quad . \quad . \quad (46).$$

Let

$$\frac{\mathrm{R}e^{i\chi}}{\sqrt{(a^2 - b^2)}} = y\,;$$

the second series is the real part of

$$- \tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y + \tfrac{1.1}{2.4}\,\frac{1}{2y^2} + \tfrac{1.1.3}{2.4.6}\,\frac{1}{4y^4} + \ldots$$

Calling this series S,

$$\frac{d\mathrm{S}}{dy} = -\left\{\tfrac{1}{2}\,\frac{1}{y} + \tfrac{1.1}{2.4}\,\frac{1}{y^3} + \tfrac{1.1.3}{2.4.6}\,\frac{1}{y^5} + \ldots\right\}$$

$$= \sqrt{y^2 - 1} - y.$$

Therefore

$$\mathrm{S} = \frac{y\sqrt{y^2 - 1}}{2} - \tfrac{1}{2}\log\left(y + \sqrt{y^2 - 1}\right) - \frac{y^2}{2} + \text{const.}$$

Now, when y is very large

$$\mathrm{S} = -\tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y.$$

Therefore

$$\frac{y^2}{2} - \tfrac{1}{4} - \tfrac{1}{2}\log 2y - \frac{y^2}{2} + \text{const.} = -\tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y.$$

Therefore

$$\text{Const.} = \tfrac{1}{4} + \tfrac{1}{2}\log 2 - \tfrac{1}{2}\log\sqrt{a^2 - b^2}.$$

Put

$$d^2 = a^2 - b^2,$$

$$y = \cosh w = \cosh(u + w).$$

Then

$$\begin{aligned} \mathrm{S} &= \frac{\cosh w(\sinh w - \cosh w)}{2} - \tfrac{1}{2}\log(\cosh w + \sinh w) + \tfrac{1}{4} - \tfrac{1}{2}\log\frac{d}{2} \\ &= -\tfrac{1}{2}w - \tfrac{1}{4}e^{-2w} - \tfrac{1}{2}\log\frac{d}{2}. \end{aligned}$$

Taking the real part of this, we find for the first term of the potential

$$\tfrac{1}{2}\log\frac{16c}{d} - \frac{u}{2} - \tfrac{1}{4}e^{-2u}\cos 2v - \frac{a^2}{16c^2}. \quad \text{. . . . .} \quad (46a).$$

Calling the operator

$$\tfrac{1}{2} + \tfrac{1.1}{2.4}\frac{a^2\mathrm{D}^2 + b^2\mathrm{D}'^2}{2\,!} + \&\text{c.}, \; f(\mathrm{D}, \mathrm{D}').$$

Since

$$f(\mathrm{D}, \mathrm{D}')\,uv = v f(\mathrm{D}, \mathrm{D}')\,u + \mathrm{D}\,v\frac{\partial f}{\partial \mathrm{D}}u + \mathrm{D}'\,v\frac{\partial f}{\partial \mathrm{D}'}u + \&\text{c.}$$

Therefore

$$\begin{aligned} &f(\mathrm{D}, \mathrm{D}')\left(\log\frac{8c}{\mathrm{R}} - 1\right)\frac{\mathrm{R}\cos\chi}{2c} \\ &\quad = \frac{\mathrm{R}\cos\chi}{2c} f(\mathrm{D}, \mathrm{D}')\left(\log\frac{8c}{\mathrm{R}} - 1\right) \\ &\quad\quad + \left(\frac{1}{2c} - \frac{\mathrm{R}\cos\chi}{2c^2}\right)\frac{\partial f}{\partial \mathrm{D}}\left(\log\frac{8c}{\mathrm{R}} - 1\right) \\ &\quad\quad + \&\text{c.} \end{aligned}$$

It is easily shown that

$$\frac{\partial f}{\partial \mathrm{D}}\left(\log\frac{8c}{\mathrm{R}} - 1\right) = \frac{a^2}{12d}\left(e^{-3u}\cos 3v - 3e^{-u}\cos v\right) + \frac{a^2}{8c} \quad \text{. . . .} \quad (47).$$

Retaining only the terms of the highest order

$$\begin{aligned} f(\mathrm{D}, \mathrm{D}')\left\{\left(\log\frac{8c}{\mathrm{R}} - 1\right)\frac{\mathrm{R}\cos\chi}{2c}\right\} &= \frac{d\cosh u\cos v}{2c}\left\{\tfrac{1}{2}\log\frac{16c}{d} - \frac{u}{2} - \tfrac{1}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right\} \\ &\quad + \frac{a^2}{24cd}\left\{e^{-3u}\cos 3v - 3e^{-u}\cos v\right\} \\ &= \frac{d}{c}\left\{\left(\frac{\log\frac{16c}{d} - u - 1}{4}\cosh u - \tfrac{1}{16}\cosh u\, e^{-2u} - \frac{a^2}{8d^2}e^{-u}\right)\cos v\right. \\ &\quad \left. + \left(-\tfrac{1}{16}\cosh u\, e^{-2u} + \frac{a^2}{24d^2}e^{-3u}\right)\cos 3v\right\} \quad \text{. . . . . .} \quad (47a). \end{aligned}$$

Let

$$\log \frac{16c}{d} - u = \mathrm{K}.$$

Then the potential of an elliptic ring at outside points not far from the ring is

$$2\pi ab \left\{ \mathrm{K} - \tfrac{1}{2} e^{-2u} \cos 2v + \frac{d}{c} \left[\left\{ \frac{\mathrm{K} - 1}{2} \cosh u - \tfrac{1}{8} \cosh u e^{-2u} - \tfrac{1}{4} \cosh^2 u_0 \, e^{-u} \right\} \cos v \right.\right.$$
$$\left.\left. + \left\{ - \tfrac{1}{8} \cosh u e^{-2u} + \tfrac{1}{12} \cosh^2 u_0 \, e^{-3u} \right\} \cos 3v \right] \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

At the surface of the ring $u = u_0$ and $\mathrm{K} = \mathrm{K}_0 = \log (16c/d) - u_0$, and

$$\mathrm{V} = 2\pi ab \left\{ \mathrm{K}_0 - \tfrac{1}{2} e^{-2u_0} \cos 2v + \frac{d}{c} \left[\left(\frac{\mathrm{K}_0 - 1}{2} - \frac{1 + 2e^{-2u_0}}{8} \right) \cosh u_0 \cos v \right.\right.$$
$$\left.\left. - \frac{2 - e^{-2u_0}}{24} e^{-2u_0} \cosh u_0 \cos 3v \right] \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

The complexity of the expressions renders it useless to find the potential any further, in the general case: but for a very flat ring, *i.e.*, where b is negligible compared with a, the expressions take much simpler forms.

§ 27. For a very flat ring

$$\mathrm{V} = 4\pi\, ab \left\{ \tfrac{1}{2} + \tfrac{1 \cdot 1}{2 \cdot 4} \frac{a^2 \mathrm{D}^2}{2} + \tfrac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \frac{a^4 \mathrm{D}^4}{4\,!} + \ldots \right\} \left\{ \log \frac{8c}{\mathrm{R}} + \frac{\log \dfrac{8c}{\mathrm{R}} - 1}{2} \frac{\mathrm{R} \cos \chi}{c} + \ldots \right\} \quad (50)$$

$$f\,(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \tfrac{1}{2} \left(\log \frac{16c}{a} - u \right) - \tfrac{1}{4} e^{-2u} \cos 2v - \frac{a^2}{16c},$$

$$f'\,(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \frac{a}{12} \left(e^{-3u} \cos 3v - 3e^{-u} \cos v \right) + \frac{a^2}{8c},$$

$$\tfrac{1}{2} f''\,(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \frac{a^2}{16} \left(\log \frac{16c}{a} - u - \tfrac{1}{4} e^{-4u} \cos 4v \right)$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{V}}{4\pi ab} = {} & \frac{\mathrm{K}}{2} - \tfrac{1}{4}e^{-2u}\cos 2v - \frac{a^2}{16c^2} \\
& + \left(\frac{\mathrm{K}-1}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right)\frac{a\cosh u\cos v}{2c} \\
& + \left[\frac{a}{12}\left(e^{-3u}\cos 3v - 3e^{-u}\cos v\right) + \frac{a^2}{8c}\right]\left[\frac{1}{2c} - \frac{a\cosh u\cos v}{2c^2}\right] \\
& - \frac{a^2}{32c^2}\left(\mathrm{K} - \tfrac{1}{4}e^{-4u}\cos 4v\right) \\
& + \frac{a^2}{c^2}\left(\frac{\mathrm{K}}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right)\frac{5\cosh^2 u\cos^2 v - \sinh^2 u\sin^2 v}{16} \\
& - \frac{a^2}{32c^2}\left(11\cosh^2 u\cos^2 v - 5\sinh^2 u\sin^2 v\right) \\
& + \frac{a^2}{12c^2}\left(e^{-3u}\cos 3v - 3e^{-u}\cos v\right)\frac{10\cosh u\cos v}{16} - \frac{11a^2}{128c^2} \\
& + \frac{a^2}{16c^2}\left(\mathrm{K} - \tfrac{1}{4}e^{-4u}\cos 4v\right)\tfrac{10}{16}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathrm{V}/2\pi ab = {} & \mathrm{K} - \tfrac{1}{2}e^{-2u}\cos 2v \\
& + \frac{a}{c}\left\{\cos v\left[\frac{\mathrm{K}}{2}\cosh u - \frac{4e^{2u} + 9 + e^{-2u}}{16}e^{-u}\right] - \frac{3e^{u} - e^{-u}}{48}e^{-2u}\cos 3v\right\} \\
& + \frac{a^2}{c^2}\left\{\mathrm{K}\,\frac{4\cosh 2u + 7}{64} - \tfrac{3}{16}e^{2u} - \tfrac{339}{256} - \tfrac{7}{32}e^{-2u} - \tfrac{3}{256}e^{-4u}\right\} \\
& + \frac{a^2}{c^2}\cos 2v\left\{\mathrm{K}\,\frac{3\cosh 2u + 2}{32} + \tfrac{1}{2}e^{2u} + \tfrac{11}{32} + \tfrac{107}{192}e^{-2u} + \tfrac{1}{96}e^{-4u}\right\} \\
& + \frac{a^2}{c^2}\cos 4v\left\{-\tfrac{3}{256} - \tfrac{1}{96}e^{-2u} - \tfrac{1}{96}e^{-4u}\right\} \quad \ldots\ldots\ldots \quad (51).
\end{aligned}
$$

At the surface of the ring

$$u = 0, \quad \mathrm{K} = \log\frac{16c}{a},$$

and

$$
\begin{aligned}
\mathrm{V} = 2\pi ab\, & \left\{\mathrm{K}\left(1 + \tfrac{11}{64}\frac{a^2}{c^2}\right) - \tfrac{323}{128}\frac{a^2}{c^2} + \cos v\left(\frac{\mathrm{K}}{2} - \tfrac{7}{8}\right)\frac{a}{c}\right. \\
& \left. + \cos 2v\left[-\tfrac{1}{2} + \left(\frac{5\mathrm{K}}{32} + \tfrac{271}{192}\right)\frac{a^2}{c^2}\right] - \frac{a}{24c}\cos 3v - \tfrac{25}{768}\frac{a^2}{c^2}\cos 4v\right\} \quad . \quad . \quad (52).
\end{aligned}
$$

§ 28. Let us take Saturn to be a sphere of radius r and of density ρ. Assume that the ring is fluid rotating with angular velocity ω, of density σ; let its section be supposed elliptic (semi-axes a and b), b being much smaller than a: and let its mean radius be c.

It is necessary that

$$2\pi\sigma ab\left\{\mathrm{K}+\ldots+\frac{a}{c}\left(\frac{\mathrm{K}}{2}-\tfrac{7}{8}\right)\cos v+\left[-\tfrac{1}{2}+\left(\frac{5\mathrm{K}}{12}+\tfrac{271}{192}\right)\frac{a^2}{c^2}\right]\cos 2v\right.$$
$$\left.-\frac{a}{24c}\cos 3v-\tfrac{25}{768}\frac{a^2}{c^2}\cos 4v\right\}$$
$$+\tfrac{4}{3}\pi\rho r^3\left\{\frac{1}{c}+\frac{a\cos v}{c^2}+\frac{a^2\cos^2 v}{c^3}+\ldots\right\}$$
$$+\frac{\omega^2}{2}\{c^2-2ac\cos v+a^2\cos^2 v\}\quad\ldots\ldots\ldots\ldots\quad(53)$$

should be constant at the surface of the ring.

Therefore

$$\pi\sigma\frac{a^2b}{c}\frac{4\mathrm{K}-7}{4}+\tfrac{4}{3}\pi\rho\frac{ar^3}{c^2}-ac\omega^2=0$$

and

$$\left.\begin{aligned}&\\&2\pi\sigma ab\left\{-\tfrac{1}{2}+\frac{80\mathrm{K}+271}{192}\frac{a^2}{c^2}\right\}+\tfrac{2}{3}\pi\rho\frac{a^2r^3}{c^3}+\frac{a^2\omega^2}{4}=0\end{aligned}\right\}\quad\ldots\quad(54).$$

Thus

$$\frac{\omega^2}{\pi}=\frac{4\mathrm{K}-7}{4}\frac{b}{a}\frac{a^2}{c^2}\sigma+\tfrac{4}{3}\frac{r^3}{c^3}\rho$$

and also

$$=4\quad 1-\frac{80\mathrm{K}+271}{96}\frac{a^2}{c^2}\Big\}\sigma-\tfrac{8}{3}\frac{r^3}{c^3}\rho$$

As a rough approximation to Saturn's rings, take

$$\frac{r}{c}=\tfrac{1}{2}\text{ and }\frac{a}{c}=\tfrac{1}{4}.\quad\text{Then }\mathrm{K}=\log_e 64=4\cdot15$$

and

$$\frac{\omega^2}{\pi}=\cdot15\frac{b}{a}\sigma+\tfrac{1}{6}\rho$$

and also

$$=2\cdot44\frac{b}{a}\sigma-\tfrac{1}{3}\rho\quad\ldots\ldots\ldots\ldots\quad(55),$$

therefore

$$\rho/\sigma=4\cdot58\,b/a.$$

For a ring, about the thickness of Saturn's, we shall have roughly $\rho/\sigma=\frac{1}{100}$; or the ring would need to be 100 times as dense as the planet, if it were a continuous fluid mass.

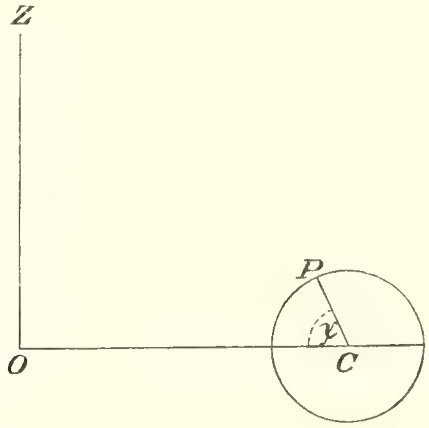
SECTION IV.

§ 29. The motion of two or more vortex rings on the same axis is most easily discussed by means of the stream-line function. The determination of this function is easily reduced to the determination of the potential of a distribution of gravitating matter, so that the preceding methods apply to this problem.

The Steady Motion of a Vortex ring of Finite Cross-section.

As before, let the figure represent a section through the axis of the ring.

Let O be the centre of the ring, Oz the axis, C the centre of gravity of the cross-section.



Let OC = c: the cross-section is nearly circular; let its equation be

$$R = a\,(1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \&c.) \quad . \quad . \quad . \qquad . \quad . \quad (56),$$

where β_1, β_2, &c., are small quantities.

Since C is the centre of gravity of the cross-section

$$\int_0^{2\pi}\int_0^{R} R\cos\chi\, R\, dR\, d\chi = 0.$$

Therefore

$$\int_0^{2\pi} (1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \ldots)^3 \cos\chi\, d\chi = 0,$$

or

$$\beta_1 + \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_4 + \ldots = 0 \quad . \quad . \quad . \quad . \quad . \qquad (57).$$

Now it will be shown that

β_2 is of the second order in a/c;

β_3 of the third, &c.

Thus β_1 will be of the fifth order.

Therefore, *correctly to the fourth order*, the equation of the cross-section is

$$\mathrm{R} = a\,\{1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi\} \quad . \quad . \quad . \quad . \quad (58).$$

§ 30. Let ω be the molecular rotation, and Ψ the stream-line function.

Then

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0$$

outside the ring, and

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + 2\varpi\omega = 0$$

inside the ring, while Ψ and $d\Psi/dn$ are continuous at the surface.

Let the *vorticity* be constant throughout the ring.

Then $\omega = (\mathrm{M}/2)\,\varpi$, where M is a constant.

Write $\Psi = \chi\varpi$, and the above equations become

and

$$\left.\begin{aligned} &\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} = 0 \\ &\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} + \mathrm{M}\varpi = 0 \end{aligned}\right\} .$$

Therefore $\chi \cos\phi$ is the potential of matter of density $(\mathrm{M}\varpi\cos\phi)/4\pi$ occupying the same space as the ring.

Therefore, at any external point, ϖ', z', ϕ',

$$\chi\cos\phi' = \frac{\mathrm{M}}{4\pi}\cos\phi' \iiint \frac{\varpi^2\cos\phi\, d\varpi\, dz\, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi\varpi'\cos\phi + \varpi^2 + (z'-z)^2\}}} .$$

Therefore

$$\Psi = \frac{\mathrm{M}}{4\pi}\varpi' \iiint \frac{\varpi^2\cos\phi\, d\varpi\, dz\, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi\varpi'\cos\phi + \varpi^2 + (z'-z)^2\}}} \quad . \quad . \quad . \quad . \quad (59),$$

the integration extending all over the ring.

(The value of Ψ at an internal point may be found as the solution of

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + \mathrm{M}\varpi^2 = 0,$$

which gives values continuous at the surface).

Let $\varpi = c - x$.

Then

$$\Psi = \frac{\mathrm{M}}{4\pi}\varpi' \iiint \frac{(c-x)^2 \cos\phi\, dx\, dz\, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi'(c-x)\cos\phi + (c-x)^2 + (z'-z)^2\}}}$$

$$= \frac{\mathrm{M}}{4\pi}\varpi' \iint e^{-x(d/dc) - z(d/dz')}\, dx\, dz \int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{\{\varpi'^2 - 2c\varpi'\cos\phi + c^2 + z'^2\}}} \quad . \quad . \quad . \quad (60),$$

where the double integral is to be taken all over the area of the cross-section $\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$.

§ 31. While finding this double integral we may treat d/dc and d/dz' as constants. Put

$$\frac{d}{dc} = \nabla \cos\alpha \quad \frac{d}{dz'} = \nabla \sin\alpha.$$

Then

$$\nabla^2 = \frac{d^2}{dc^2} + \frac{d^2}{dz'^2}.$$

The double integral

$$\iint e^{-\nabla(x\cos\alpha + z\sin\alpha)}\, dx\, dz$$

taken over the area of

$$\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$$

may be conveniently calculated in three parts.

(1) Over the circle of radius a;

(2) Round the circumference of this circle to obtain the first powers of β_2, β_3, β_4;

(3) To find separately the coefficient of β_2^2.

Part (1)

$$= \iint e^{-\nabla(x\cos\alpha + z\sin\alpha)}\, dx\, dz,$$

where

$$x^2 + z^2 < a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$= a^2 \iint e^{-\nabla\zeta}\, d\xi d\zeta,$$

where

$$\xi^2 + \zeta^2 < 1,$$

$$= \pi a^2 \int_{-1}^{1} e^{-\nabla\zeta} \sqrt{1-\zeta^2}\, d\zeta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} e^{-a\nabla\sin\theta} \cos^2\theta\, d\theta$$

$$= \pi a^2 \left\{1 + \frac{a^2\nabla^2}{2^2.2} + \frac{a^4\nabla^4}{2^2.4^2.3} + \ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

Part (2)

$$= a^2 \int_0^{2\pi} e^{-a\nabla \cos(\chi - \alpha)} (\beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)\, d\chi$$

$$= a^2 \int_0^{2\pi} \{\beta_2 \cos 2\alpha \cos 2\psi + \beta_3 \cos 3\alpha \cos 3\psi + \beta_4 \cos 4\alpha \cos 4\psi\}\, e^{-a\nabla \cos\psi}\, d\psi$$

$$= \pi a^4 \frac{\beta_2}{4}\left(\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right)\left\{1 + \frac{a^2\nabla^2}{12} + \frac{a^4\nabla^4}{384} + \ldots\right\}$$

$$- \pi a^5 \frac{\beta_3}{12}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)\left(1 + \frac{a^2\nabla^2}{16}\right)$$

$$+ \pi a^6 \frac{\beta_6}{192}\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (63).$$

Part (3) is easily found to be

$$\pi a^2 \frac{\beta_2^{\,2}}{2} + \tfrac{5}{16}\pi a^2 \beta_2^{\,2} \frac{a^4}{4\,!}\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \ldots\ldots \quad (64).$$

At any external point the stream-function is, therefore, the effect of the operation

$$\pi a^2\left\{1 + \frac{\beta_2^{\,2}}{2} + \frac{a^2\nabla^2}{2^2.2} + \frac{a^4\nabla^4}{2^2.4^2.3} + \ldots\right\} + \pi a^4 \frac{\beta_2}{4}\left\{\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right\}\left\{1 + \frac{a^2\nabla^2}{12} + \frac{a^4\nabla^4}{384} + \ldots\right\}$$

$$- \pi a^5 \frac{\beta_3}{12}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)\left(1 + \frac{a^2\nabla^2}{16} + \ldots\right)$$

$$+ \pi a^6\left(\frac{\beta_4}{192} + \frac{5\beta_2^{\,2}}{16\,\underline{|4}}\right)\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \ldots\ldots \quad (65)$$

on

$$\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}}\,.$$

§ 32. Now at the surface of the ring

$$\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}} \text{ is of the order } \frac{c^2}{a},$$

$$a^n \frac{d^n}{dc^n}\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}} \text{ is of the order } a^n \frac{c^2}{a^{n+1}} \text{ i.e., } \frac{c^2}{a},$$

$$a^{2n}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^n \int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}}$$

will be shown to be of the order

$$a^{2n}\frac{c^2}{ac^{n+1}\,c^n}\quad i.e.,\quad \frac{c^2}{a}\left(\frac{a}{c}\right)^n.$$

We have assumed β_2 to be of the second order, β_3 of the third, &c. Therefore, in the above formula, all terms as far as $\left(\frac{a}{c}\right)^4$ have been retained.

§ 33. Let

$$\mathrm{I}=\int_0^\pi\frac{c\cos\phi\,d\phi}{\sqrt{(\varpi'^2-2\varpi' c\cos\phi+c^2+z'^2)}}.$$

Then

$$\frac{d^2\mathrm{I}}{dz'^2}+\frac{d^2\mathrm{I}}{dc^2}-\frac{1}{c}\frac{d\mathrm{I}}{dc}=0.$$

Let

$$\mathrm{U}=\mathrm{I}\,.\,c.$$

Then

$$\left(\frac{d^2}{dc^2}+\frac{d^2}{dz'^2}\right)\mathrm{U}=c\left(\frac{d^2\mathrm{I}}{dc^2}+\frac{2}{c}\frac{d\mathrm{I}}{dc}+\frac{d^2\mathrm{I}}{dz'^2}\right).$$

Or,

$$\nabla^2\mathrm{U}=3\frac{d\mathrm{I}}{dc},$$

$$\nabla^4\mathrm{U}=3\frac{d}{dc}\nabla^2\mathrm{I}$$

$$=3\,.\,1\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right),$$

$$\nabla^6\mathrm{U}=3\,.\,1\frac{d}{dc}\nabla^2\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)$$

$$=3\,.\,1\frac{d}{dc}\left\{\frac{1}{c}\nabla^2\frac{d\mathrm{I}}{dc}-\frac{2}{c}\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)\right\},$$

$$=3\,.\,1\,(-1)\frac{d}{dc}\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{I}.$$

Hence

$$\nabla^{2n}\mathrm{U}=3\,.\,1\,(-1)\ldots(-2n+5)\frac{d}{dc}\left(\frac{1}{c}\frac{d}{dc}\right)^{n-1}\mathrm{I}\quad .\quad .\quad .\quad (65\text{A}).$$

§ 34. Therefore at a point ϖ', z', ϕ', the stream-function is given by

$$\begin{aligned}\Psi = \frac{\mathrm{M}a^2c}{2}\varpi' &\left\{1 + \frac{\beta_2^{\,2}}{2} + \frac{3a^2}{8}\frac{1}{c}\frac{d}{dc} + \frac{a^4}{64}\left(\frac{1}{c}\frac{d}{dc}\right)^2 - \frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^8}{81920}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right] \mathrm{I} \\ &+ \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^2\beta_2}{4}\left(\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right)c\left\{\mathrm{I} + \frac{a^2}{4}\frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{a^4}{128}\left(\frac{1}{c}\frac{d}{dc}\right)^2\right]\mathrm{I} \\ &- \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^3\beta_3}{24}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)c\left(\mathrm{I} + \frac{3a^2}{16}\frac{1}{c}\frac{d\mathrm{I}}{dc}\right) \\ &+ \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^4}{4\,!}\left(\frac{\beta_4}{8} + \frac{5\beta_2^{\,2}}{16}\right)\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right)c\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66).\end{aligned}$$

Using the fact that $\frac{d^2\mathrm{I}}{dc^2} - \frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{d^2\mathrm{I}}{dz'^2} = 0$, and neglecting terms of order higher than $\left(\frac{a}{c}\right)^4$ since $\left(\frac{1}{c}\frac{d}{dc}\right)^n \mathrm{I}$ is of order $\frac{1}{a^nc^n}$ at the surface of the ring, we find that

$$\begin{aligned}\frac{2\Psi}{\mathrm{M}a^2c\varpi'} = &\left\{1 + \frac{3a^2}{8}\frac{1}{c}\frac{d}{dc} + \frac{a^4}{64}\left(\frac{1}{c}\frac{d}{dc}\right)^2 - \frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^8}{81920}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right]\mathrm{I} \\ &+ \beta_2\left\{\frac{3a^2}{4}\frac{1}{c}\frac{d}{dc} + \frac{(8c^2 + 5a^2)\,a^2}{16}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^4c^2}{8}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^6c^2}{256}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right]\mathrm{I} \\ &- \beta_3\left\{\frac{5a^3c}{8}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^3c^3}{6}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^5c^3}{32}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right]\mathrm{I} \\ &+ \left(\frac{\beta_4}{24} + \frac{5\beta_2^{\,2}}{48}\right)a^4c^4\left(\frac{1}{c}\frac{d}{dc}\right)^4\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (67).\end{aligned}$$

Therefore

$$\Psi = \frac{\mathrm{M}a^2c}{2}\varpi'\left\{a_0\mathrm{I} + a_1ac\frac{1}{c}\frac{d\mathrm{I}}{dc} + a_2a^2c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{I} + \ldots\right\} \quad . \quad . \quad . \qquad (68),$$

where

$$\left.\begin{aligned} a_0 &= 1 + \frac{\beta_2^{\,2}}{2} \\ a_1 &= \frac{3\sigma}{8} + \frac{3\sigma}{4}\beta_2 \\ a_2 &= \frac{\sigma^2}{64} + \beta_2\left(\tfrac{1}{2} + \frac{5\sigma^2}{16}\right) - \beta_3\frac{5\sigma}{8} \\ a_3 &= -\frac{\sigma^3}{3\,.\,2^{10}} + \beta_2\frac{\sigma}{8} - \frac{\beta_3}{6} \\ a_4 &= \frac{\sigma^4}{5\,.\,2^{14}} + \beta_2\frac{\sigma^2}{256} - \beta_3\frac{\sigma}{32} + \frac{\beta_4}{24} + \frac{5\beta_2^{\,2}}{48} \\ \text{and} \qquad \sigma &= \frac{a}{c} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69).$$

§ 35. Let the point ϖ', z', ϕ' be on the surface of the ring, so that

$$\varpi' = c - \mathrm{R}\cos\chi, \qquad z' = \mathrm{R}\sin\chi.$$

Writing l for $\log(8c/\mathrm{R}) - 2$, and s for R/c, we have

$$\left.\begin{aligned}
\mathrm{J} = l &+ \frac{l-1}{2}s\cos\chi + \left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)s^2 \\
&+ \left(\frac{33l+1}{64}\cos\chi + \frac{15l-23}{192}\cos 3\chi\right)s^3 \\
&+ \left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right)s^4 + \dots \\
&\hspace{16em}\text{(}\textit{Suprà}\text{, p. 53).} \\
&\text{We obtain, by differentiation,} \\
\frac{1}{c}\frac{d\mathrm{J}}{dc} = &-\frac{1}{c^2 s}\left\{\cos\chi - \left(\frac{2l+1}{4} - \frac{\cos 2\chi}{4}\right)s - \left(\frac{20l+5}{32}\cos\chi - \tfrac{3}{32}\cos 3\chi\right)s^2\right. \\
&\left.\qquad - \left(\frac{36l+3}{128} + \frac{24l+3}{64}\cos 2\chi - \tfrac{5}{128}\cos 4\chi\right)s^3 + \dots\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{J} = &\frac{1}{c^4 s^2}\left\{\cos 2\chi + \left(\frac{\cos\chi}{4} + \frac{\cos 3\chi}{4}\right)s + \left(-\frac{12l+5}{32} + \tfrac{3}{32}\cos 4\chi\right)s^2 + \dots\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3\mathrm{J} = &-\frac{1}{c^6 s^3}\{2\cos 3\chi + (2\cos 2\chi + \tfrac{1}{2}\cos 4\chi)s + \dots\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4\mathrm{J} = &\frac{1}{c^8 s^4}\{6\cos 4\chi + \dots\}
\end{aligned}\right\}(70).$$

Multiplying by ϖ' or $c\,(1 - s\cos\chi)$

$$\left.\begin{aligned}
\mathrm{J}\frac{\varpi'}{c} = l &- \frac{l+1}{2}s\cos\chi + \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\chi\right)s^2 \\
&+ \left(\frac{3l+5}{64}\cos\chi - \frac{3l-1}{192}\cos 3\chi\right)s^3 \\
&+ \left(\frac{12l+11}{2048} + \frac{12l+17}{768}\cos 2\chi - \frac{15l-8}{3072}\cos 4\chi\right)s^4 + \dots \\
\frac{1}{c}\frac{d\mathrm{J}}{dc}\frac{\varpi'}{c} = &+\frac{1}{c^2 s}\left\{-\cos\chi + \left(\frac{2l+3}{4} + \frac{\cos 2\chi}{4}\right)s\right. \\
&+ \left(\frac{4l+1}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2 \\
&\left.+ \left(-\frac{4l+7}{128} + \frac{4l+1}{64}\cos 2\chi + \tfrac{1}{128}\cos 4\chi\right)s^3 + \dots\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{J}\frac{\varpi'}{c} = &\frac{1}{c^4 s^2}\left\{\cos 2\chi - \left(\frac{\cos\chi}{4} + \frac{\cos 3\chi}{4}\right)s\right. \\
&\left.- \left(\frac{12l+9}{32} + \tfrac{1}{4}\cos 2\chi + \tfrac{1}{32}\cos 4\chi\right)s^2 \dots\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3\mathrm{J}\frac{\varpi'}{c} = &-\frac{1}{c^6 s^3}\{2\cos 3\chi + (\cos 2\chi - \tfrac{1}{2}\cos 4\chi)s + \dots\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4\mathrm{J}\frac{\varpi'}{c} = &+\frac{1}{c^8 s^4}\{6\cos 4\chi + \dots\}.
\end{aligned}\right\}.\ (71)$$

At the surface of the ring

$$\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \&\text{c.}).$$

Substituting for s, $\sigma(1 + \beta_2 \cos 2\chi + \ldots)$ and writing $\log \frac{8c}{a} - 2 = \lambda$,

$$\left.\begin{aligned}
\mathrm{J}\frac{\varpi'}{c} &= \lambda + \frac{2\lambda + 5}{16}\sigma^2 + \frac{12\lambda + 11}{2048}\sigma^4 - \frac{2\lambda - 1}{32}\sigma^2\beta_2 + \frac{\beta_2^2}{4} \\
&\quad + \sigma\cos\chi\left(-\frac{\lambda + 1}{2} + \frac{3\lambda + 5}{64}\sigma^2 - \frac{\lambda}{4}\beta_2\right) \\
&\quad + \cos 2\chi\left(-\frac{\lambda\sigma^2}{16} + \frac{12\lambda + 7}{768}\sigma^4 - \beta_2 - \frac{\lambda}{4}\sigma\beta_3 + \frac{\lambda + 2}{4}\sigma^2\beta_2\right) \\
&\quad + \cos 3\chi\left(-\frac{3\lambda - 1}{192}\sigma^3 - \beta_3 - \frac{\lambda\sigma\beta_2}{4}\right) \\
&\quad + \cos 4\chi\left(-\frac{15\lambda - 8}{3072}\sigma^4 - \beta_4 - \frac{\lambda}{4}\sigma\beta_3 - \frac{2\lambda - 1}{32}\sigma^2\beta_2 + \frac{\beta_2^2}{4}\right) + \ldots \\
\frac{1}{c}\frac{d\mathrm{J}}{dc}\frac{\varpi'}{c} &= \frac{1}{c^2\sigma}\left\{\frac{2\lambda + 3}{4}\sigma - \frac{4\lambda + 7}{128}\sigma^3 + \cos\chi\left(-1 + \frac{4\lambda + 1}{32}\sigma^2 + \tfrac{1}{2}\beta_2\right)\right. \\
&\quad + \cos 2\chi\left(\frac{\sigma}{4} + \frac{4\lambda + 1}{64}\sigma^3 - \tfrac{1}{2}\sigma\beta_2 + \tfrac{1}{2}\beta_3\right) \\
&\quad \left. + \cos 3\chi\left(\frac{\sigma^2}{32} + \tfrac{1}{2}\beta_2\right) + \cos 4\chi\left(\frac{\sigma^3}{128} + \frac{\beta_3}{2}\right)\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2 \mathrm{J}\frac{\varpi'}{c} &= \frac{1}{c^4\sigma^2}\left\{-\frac{12\lambda + 9}{32}\sigma^2 - \beta_2 - \frac{\sigma}{4}\cos\chi + \cos 2\chi\left(1 - \frac{\sigma^2}{4}\right)\right. \\
&\quad \left. - \frac{\sigma}{4}\cos 3\chi - \cos 4\chi\left(\frac{\sigma^2}{32} + \beta_2\right)\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3 \mathrm{J}\frac{\varpi'}{c} &= \frac{1}{c^6\sigma^3}\left\{-\sigma\cos 2\chi - 2\cos 3\chi - \tfrac{1}{2}\sigma\cos 4\chi\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4 \mathrm{J}\frac{\varpi'}{c} &= \frac{1}{c^8\sigma^4} 6\cos 4\chi
\end{aligned}\right\} \quad (72).$$

§ 36. Now, in the steady motion, if V be the velocity of the ring,

$$\Psi - \tfrac{1}{2}\mathrm{V}\varpi^2 = \text{constant at the surface of the ring} \quad . \quad . \quad . \quad . \quad (73).$$

Therefore

$$\Psi - \tfrac{1}{2}\mathrm{V}\left(c^2 - 2c\mathrm{R}\cos\chi + \frac{\mathrm{R}^2}{2} + \frac{\mathrm{R}^2}{2}\cos 2\chi\right)$$

is constant when

$$\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \ldots).$$

Therefore

$$\begin{aligned}\Psi - \frac{\mathrm{V}c^2}{2}\Big\{1 + \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\beta_2 + \cos\chi\,(-2\sigma - \sigma\beta_2) + \cos 2\chi\left(\frac{\sigma^2}{2} + \sigma^2\beta_2 - \sigma\beta_3\right)\\ - \sigma\beta_2\cos 3\chi + \cos 4\chi\left(-\sigma\beta_3 + \sigma^2\frac{\beta_2}{2}\right)\Big\}\end{aligned} \qquad (74).$$

is constant.

Equating the coefficients of $\cos\chi$, $\cos 2\chi$, $\cos 3\chi$, $\cos 4\chi$ to zero,

$$\left.\begin{aligned}
&a_0\sigma\left(-\frac{\lambda+1}{2} + \frac{3\lambda+5}{64}\sigma^2 - \frac{\lambda}{4}\beta_2\right) + a_1\left(-1 + \frac{4\lambda+1}{32}\sigma^2 + \frac{\beta_2}{2}\right)\\
&\qquad - a_2\frac{\sigma}{4} + \frac{\mathrm{V}}{\mathrm{M}a^2}(2\sigma + \sigma\beta_2) = 0\\
&a_0\left(-\frac{\lambda\sigma^2}{16} + \frac{12\lambda+7}{768}\sigma^4 - \beta_2 - \frac{\lambda}{4}\sigma\beta_3 + \frac{\lambda+2}{4}\sigma^2\beta_2\right)\\
&\qquad + a_1\left(\frac{\sigma}{4} + \frac{4\lambda+1}{64}\sigma^3 - \frac{\sigma}{2}\beta_2 + \tfrac{1}{2}\beta_3\right) + a_2\left(1 - \frac{\sigma^2}{4}\right)\\
&\qquad - a_3\sigma - \frac{\mathrm{V}}{\mathrm{M}a^2}\left(\frac{\sigma^2}{2} + \sigma^2\beta_2 - \sigma\beta_3\right) = 0\\
&a_0\left(-\frac{3\lambda-1}{192}\sigma^3 - \beta_3 - \frac{\lambda\sigma}{4}\beta_2\right) + a_1\left(\frac{\sigma^2}{32} + \frac{\beta_2}{2}\right)\\
&\qquad - a_2\frac{\sigma}{4} - 2a_3 + \frac{\mathrm{V}}{\mathrm{M}a^2}\sigma\beta_2 = 0\\
&a_0\left(-\frac{15\lambda-8}{3072}\sigma^4 - \beta_4 - \frac{\lambda}{4}\sigma\beta_3 - \frac{2\lambda-1}{32}\sigma^2\beta_2 + \tfrac{1}{4}\beta_2^2\right)\\
&\qquad + a_1\left(\frac{\sigma^3}{128} + \frac{\beta_3}{2}\right) + a_2\left(-\frac{\sigma^2}{32} - \beta_2\right)\\
&\qquad + a_3\frac{\sigma}{2} + 6a_4 + \frac{\mathrm{V}}{\mathrm{M}a^2}\left(\sigma\beta_3 - \sigma^2\frac{\beta_2}{2}\right) = 0
\end{aligned}\right\} \qquad (75).$$

Now, $\omega = \frac{1}{2}\mathrm{M}\varpi$; therefore, if m be the strength of the vortex,

$$\begin{aligned} m = \iint \omega\, d\sigma &= \tfrac{1}{2}\mathrm{M}\iint (c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi\\ &= \tfrac{1}{2}\mathrm{M}\,\pi a^2 c\left(1 + \frac{\beta_2^2}{2}\right)\end{aligned} \qquad (76).$$

Substituting then for $\mathrm{M}a^2$, and solving the above equations:

$$\left.\begin{aligned}
\mathrm{V} &= \frac{m}{\pi c}\left\{\frac{4\lambda+7}{8} - \frac{12\lambda+9}{64}\sigma^2\right\}\\
\beta_2 &= -\frac{12\lambda+7}{32}\sigma^2 + \frac{72\lambda+77}{3\,.\,2^{10}}\sigma^4\\
\beta_3 &= -\frac{168\lambda+63}{1024}\sigma^3\\
\beta_4 &= \frac{84\lambda^2+111\lambda+41}{512}\sigma^4
\end{aligned}\right\} \qquad (77).$$

These equations give the velocity of the ring in steady motion, and the form of its cross-section. The cross-section is slightly elongated in the direction of the motion of the ring. The quantities β_3 and β_4 are very small; for example, when $\sigma = {\cdot}3$, the case of a very thick ring,

$$V = 2{\cdot}96 \times \frac{m}{2\pi c}, \quad \beta_2 = -{\cdot}063, \quad \beta_3 = -{\cdot}006, \quad \beta_4 = {\cdot}005.$$

§ 37. *Fluted oscillations of a Circular Vortex-Ring.*

Let the cross-section of the ring when disturbed be given by

$$R = a\{1 + \Sigma(\alpha_n \sin n\chi + \beta_n \cos n\chi)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

At an external point, not far from the ring, the stream-line function is given by

$$\Psi = \frac{mc}{\pi}\left\{\log\frac{8c}{R} - 2 - \frac{\log\frac{8c}{R} - 1}{2}\frac{R\cos\chi}{c} - \frac{3a^2}{8}\frac{\cos\chi}{cR}\right\}$$
$$+ \frac{mc}{\pi}\Sigma\frac{a^n}{nR^n}(\alpha_n \sin n\chi + \beta_n \cos n\chi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

Let the central circle of the ring have moved a distance z_0 from the plane of x, y. Then

$$\dot{R} + \frac{\partial R}{\partial z_0}\dot{z}_0 + \frac{\partial R}{\partial c}\dot{c} = \dot{a} + \Sigma[(\dot{a}\alpha_n + a\dot{\alpha}_n)\sin n\chi + (\dot{a}\beta_n + a\dot{\beta}_n)\cos n\chi]$$
$$+ a\,\Sigma n(\alpha_n \cos n\chi - \beta_n \sin n\chi)\left(\dot{\chi} + \frac{\partial\chi}{\partial z_0}\dot{z}_0 + \frac{\partial\chi}{\partial c}\dot{c}\right) \quad . \quad . \quad (80).$$

Now

$$\frac{\partial R}{\partial c} = \cos\chi, \qquad \frac{\partial\chi}{\partial c} = -\frac{\sin\chi}{R},$$
$$\frac{\partial R}{dz_0} = -\sin\chi, \qquad \frac{\partial\chi}{\partial z_0} = -\frac{\cos\chi}{R},$$

and

$$\dot{R} = \frac{1}{\varpi}\frac{d\Psi}{Rd\chi}, \qquad \dot{\chi} = -\frac{1}{\varpi}\frac{d\Psi}{RdR}.$$

Therefore

$$\dot{R} = \frac{m}{\pi(c - R\cos\chi)}\left\{\frac{\log\frac{8c}{R} - 1}{2}\frac{1}{c}\sin\chi + \frac{3a^2}{8}\frac{1}{cR^2}\sin\chi\right\}$$
$$+ \frac{m}{\pi(c - R\cos\chi)}\frac{1}{R}\Sigma\frac{a^n}{R^2}(\alpha_n \cos n\chi - \beta_n \sin n\chi) \quad . \quad . \quad . \quad . \quad (81),$$

and

$$\dot{\chi} = \frac{m}{\pi(c - \mathrm{R}\cos\chi)}\frac{1}{\mathrm{R}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82).$$

Writing R = a, and neglecting a in comparison with c

$$\frac{m}{2\pi c}\left(\log\frac{8c}{\mathrm{R}} - \frac{1}{4}\right)\sin\chi - \dot{z}_0\sin\chi + \dot{c}\cos\chi + \frac{m}{\pi a}\Sigma(\alpha_n\cos n\chi - \beta_n\sin n\chi)$$

$$= \dot{a} + a\Sigma(\dot{\alpha}_n\sin n\chi + \dot{\beta}_n\cos n\chi) + \frac{m}{\pi a}\Sigma n(\alpha_n\cos n\chi - \beta_n\sin n\chi) \quad . \quad (83).$$

This equation gives

$$\dot{a} = 0, \quad \dot{c} = 0 \qquad \dot{z}_0 \text{ or } \mathrm{V} = \frac{m}{2\pi c}\left(\log\frac{8c}{a} - \frac{1}{4}\right)$$

$$\left.\begin{aligned}\dot{\alpha}_n - \frac{m}{\pi a^2}(n-1)\beta_n &= 0\\ \dot{\beta}_n + \frac{m}{\pi a^2}(n-1)\alpha_n &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

Therefore

$$\ddot{\alpha}_n + \frac{m^2}{\pi^2 a^4}(n-1)^2\alpha_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

Therefore the time of an oscillation of the type

$$\alpha_n\sin n\chi + \beta_n\cos n\chi$$

is

$$\frac{2\pi}{\dfrac{m}{\pi a^2}(n-1)} = \frac{2\pi^2 a^2}{m(n-1)}.$$

The steady motion and small fluted oscillations of a single vortex ring have been worked out by Mr. HICKS by means of toroidal functions. Only the simplest case, that of a vortex ring in a fluid of equal density, with no added circulations, is considered in this paper; the same methods might be applied to other cases. The steady motion given above agrees with Mr. HICKS' result, in the velocity V, and the value of β_n to the first order.

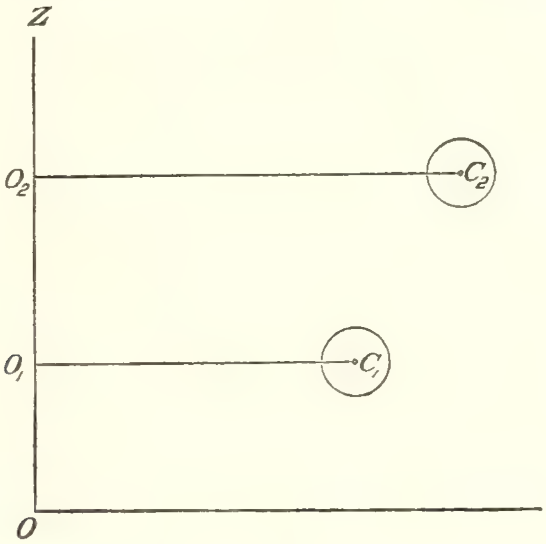
SECTION V.

The motion of any number of fine Vortex rings on the same axis.

§ 38. *The stream-line function.*

The cross-sections of the rings are approximately circles.

In each ring, let the vorticity be constant. Then $\omega/\varpi =$ constant, over each ring.



Let $\omega = \frac{1}{2}M_1\varpi, \frac{1}{2}M_2\varpi, \frac{1}{2}M_3\varpi$, &c., in the several rings.

The stream-function Ψ satisfies

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0, \text{ outside the rings ;}$$

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + M_1\varpi^2 = 0, \text{ inside the ring } C_1 \text{;}$$

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + M_2\varpi^2 = 0, \text{ inside the ring } C_2, \text{ \&c.;}$$

while Ψ and $d\Psi/dn$ are continuous everywhere.

The value of Ψ at any point ϖ', z', ϕ', outside the rings is given by

$$\Psi = \varpi' \Sigma \frac{M_1}{4\pi} \iiint \frac{\varpi^2 \cos\phi \, dz \, d\varpi \, d\phi}{\sqrt{\{(z'-z)^2 + \varpi'^2 - 2\varpi\varpi' \cos\phi + \varpi^2\}}} \quad . \quad . \quad . \quad . \quad (86),$$

where the integrals are taken throughout the volume of each ring.

Consider the integral over the ring whose mean radius is c_1, and cross-section a_1, and whose central circle is distant z_1 from the plane of xy.

Let

$$\left.\begin{aligned} \varpi &= c_1 - f \\ z &= z_1 + h \end{aligned}\right\}.$$

Due to this ring alone

$$\begin{aligned}\Psi &= \frac{\mathrm{M}_1\varpi'}{4\pi}\iint c^{-f(d/dc_1)+h(d/dz_1)}\,df\,dh\int_0^{2\pi}\frac{c_1^2\cos\phi\,d\phi}{\sqrt{\{(z'-z_1)^2+\varpi'^2-2c_1\varpi'\cos\phi+c_1^2\}}}\\ &= \tfrac{1}{2}\mathrm{M}_1a_1^2\,\varpi'\int_0^{\pi}\frac{c_1^2\cos\phi\,d\phi}{\sqrt{\{(z'-z_1)^2+\varpi'^2-2c_1\varpi'\cos\phi+c_1^2\}}}\quad .\ .\ .\ .\ .\ .\ .\ . \quad (87)\end{aligned}$$

Take a point on the surface of the first ring. Let its coordinates be

$$c_1 - a_1\cos\chi,\qquad z_1 + a_1\sin\chi.$$

The part of Ψ due to the ring itself is

$$\frac{\mathrm{M}_1a_1^2c_1^2}{2}\left\{\log\frac{8c_1}{a_1} - 2 + \frac{\log 8c_1/a_1 - \frac{1}{4}}{2}\frac{a_1}{c_1}\cos\chi + \ldots\right\}\quad .\ .\ .\ .\ . \quad (88).$$

The part due to the ring C_2 is

$$\frac{\mathrm{M}_2a_2^2}{2}\int_0^{\pi(c_1-a_1\cos\chi)}\frac{c_2^2\cos\phi\,d\phi}{\sqrt{\{(z_1+a_1\sin\chi-z_2)^2+(c_1-a_1\cos\chi)^2-2c_2(c_1-a_1\cos\chi)\cos\phi+c_2^2\}}}\,.$$

Let the integral

$$\int_0^{\pi}\frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_1-z_1)^2+c_1^2-2c_1c_2\cos\phi+c_2^2\}}}$$

be called I_{12}.

Then the part of Ψ due to the ring C_2 is

$$\frac{\mathrm{M}_2a_2^2c_2}{2}\left\{\mathrm{I}_{12} + a_1\sin\chi\frac{d\mathrm{I}_{12}}{dz_1} - a_1\cos\chi\frac{d\mathrm{I}_{12}}{dc_1}\right\}\quad .\ .\ .\ .\ .\ . \quad (89).$$

a^2c_1 is constant: write

$$\tfrac{1}{2}\mathrm{M}_1\,a_1^2c_1 = \frac{m_1}{\pi}\,.$$

Then at a point $c_1 - a_1\cos\chi$, $z_1 + a_1\sin\chi$, on the surface of the ring C_1, we find

$$\left.\begin{aligned}\Psi = \quad &\frac{m_1}{\pi}\left\{c_1\log\frac{8c_1}{a_1} - 2 - \frac{\log 8c_1/a_1 - \frac{1}{4}}{2}\frac{a_1}{c_1}\cos\chi + \ldots\right\}\\ &+\frac{m_2}{\pi}\left\{\mathrm{I}_{12} + a_1\sin\chi\frac{d\mathrm{I}_{12}}{dz_1} - a_1\cos\chi\frac{d\mathrm{I}_{12}}{dc_1} + \ldots\right\}\\ &+\frac{m_3}{\pi}\left\{\mathrm{I}_{13} + a_1\sin\chi\frac{d\mathrm{I}_{13}}{dz_1} - a_1\cos\chi\frac{d\mathrm{I}_{13}}{dc_1} + \ldots\right\}\\ &+\ldots\end{aligned}\right\}\quad .\ .\ . \quad (90).$$

§ 39. *The equations of Motion.*

Let the ring C_1 be moving forward with velocity $\dot{z}_1$

Let its radius increase with velocity $\dot{c}_1$. The radius of the cross-section will change, but since $a_1^2 c_1 = \text{const.}$,

$$\dot{a}_1 = -\tfrac{1}{2}\frac{a_1}{c_1}\dot{c}_1,$$

and therefore (a_1/c_1 being small), $\dot{a}_1$ is negligible compared with $\dot{c}_1$.

The normal velocity of any point on the ring's surface is

$$\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1.$$

But the resolved part of the velocity along the normal to the ring is

$$\frac{1}{\varpi}\frac{d\Psi}{d\varpi}\frac{d\varpi}{ds} + \frac{1}{\varpi}\frac{d\Psi}{dz}\frac{dz}{ds}.$$

Therefore

$$\frac{d\Psi}{ds} = \varpi\left(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1\right) \quad \text{. . . . . (91)}$$

at the surface of the ring.

Therefore at the surface of the ring C_1,

$$\begin{aligned}\Psi &= \int\left(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1\right)\varpi\, ds\\ &= \int\left(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1\right)\left(c_1 - a_1\cos\chi\right) a_1\, d\chi\\ &= -a_1 c_1 \dot{z}_1 \cos\chi - a_1 c_1 \dot{c}_1 \sin\chi + \text{terms of higher orders} \quad \text{. . (92).}\end{aligned}$$

Comparing this with the value already found for Ψ at the surface of the ring, we obtain

$$\left.\begin{aligned}c_1 \dot{z}_1 &= \frac{m_1}{2\pi}\left(\log\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + \frac{m_2}{\pi}\frac{dI_{12}}{dc_1} + \frac{m_3}{\pi}\frac{dI_{13}}{dc_1} + \dots\\ \text{and } -c_1\dot{c}_1 &= \frac{m_2}{\pi}\frac{dI_{12}}{dz_1} + \frac{m_3}{\pi}\frac{dI_{13}}{dz_1} + \dots\end{aligned}\right\} \quad \text{. . . (93)}$$

If we write

$$\Sigma\frac{m_1 m_2}{\pi} I_{12} = U,$$

then this equation may be written

$$\left.\begin{aligned}m_1 c_1 \dot{z}_1 &= \frac{m_1^2}{2\pi}\log\left(\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + \frac{dU}{dc_1}\\ \text{and}\qquad -m_1 c_1 \dot{c}_1 &= \frac{dU}{dz_1}\end{aligned}\right\} \quad \text{. . . . . . . (94).}$$

Similar equations hold for each of the other rings.

§ 40. *Two integrals of the equations.*

Since U is a function of $z_1 - z_2$, $z_1 - z_3$, $z_2 - z_3$, &c.,

$$\frac{d\mathrm{U}}{dz_1} + \frac{d\mathrm{U}}{dz_2} + \ldots = 0.$$

Thus $\Sigma\, m_1 c_1 \dot{c}_1 = 0$.

Therefore

$$\Sigma\, m_1 c_1^2 = \text{const.} \qquad (95).$$

Again, multiply the equations by $\dot{c}_1$, $\dot{z}_1$, $\dot{c}_2$, $\dot{z}_2$, &c., and add.

Therefore

$$\Sigma \frac{\mathrm{M}_1^2}{2\pi}\left(\log \frac{8c_1}{a_1} - \tfrac{1}{4}\right)\dot{c}_1 + \Sigma\left(\frac{d\mathrm{U}}{dc_1}\dot{c}_1 + \frac{d\mathrm{U}}{dz_1}\dot{z}_1\right) = 0. \qquad (96).$$

Remembering that $a_1^2 c_1 = \text{const.}$; this equation gives on integration

$$\Sigma \frac{\mathrm{M}_1^2}{2\pi c_1}\left(\log \frac{8c_1}{a_1} - \tfrac{7}{4}\right) + \mathrm{U} = \text{const.} \qquad (97).$$

§ 41. These integrals are the equations expressing the constancy of linear momentum and of energy in the system.

They are easily obtained directly.

The momentum

$$= \iiint \frac{1}{\varpi} \frac{d\Psi}{d\varpi}\, d\varpi\, dz\, d\phi \qquad (98),$$

the integral being taken all over space.

Since Ψ and $d\Psi/d\varpi$ are continuous everywhere, this

$$= 2\pi \int_{-\infty}^{\infty} (\Psi_\infty - \Psi_0)\, dz \qquad (99),$$

where Ψ_∞ = value of Ψ at a great distance from the axis, and Ψ_0 = value of Ψ at the axis.

Now

$$\begin{aligned}\Psi_\infty &= \Sigma \frac{\mathrm{M}_1}{\pi} \int_0^\pi \frac{\varpi c_1 \cos\phi\, d\phi}{\sqrt{(z - z_1)^2 + \varpi^2 - 2\,\varpi c_1 \cos\phi + c_1^2}} \\ &= \Sigma \frac{\mathrm{M}_1}{2} \frac{\varpi^2 c_1^2}{\{(z - z_1)^2 + \varpi^2\}^{\frac{3}{2}}} \qquad (100),\end{aligned}$$

and

$$\Psi_0 = 0.$$

Therefore the momentum

$$= 2\pi \Sigma \frac{m_1}{2} \int_{-\infty}^{\infty} \frac{c_1{}^2 \varpi^2 \, dz}{\{(z - z_1)^2 + \varpi^2\}^{\frac{3}{2}}}$$

$$= 2\pi \Sigma m_1 c_1{}^2 \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{\sec^3 \theta}, \text{ where } \tan \theta = \frac{z - z_1}{\varpi}$$

$$= 2\pi \Sigma m_1 c_1{}^2 \quad \ldots \ldots \quad (101).$$

§ 42. The kinetic energy is given by

$$\mathrm{T} = \pi \iint \frac{1}{\varpi} \left\{ \left(\frac{d\Psi}{d\varpi}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2 \right\} d\varpi \, dz$$

$$= -2\pi \iint \frac{\Psi}{\varpi} \left(\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi} \frac{d\Psi}{d\varpi} \right) d\varpi \, dz \quad \ldots \ldots \quad (102).$$

Now, outside the ring

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi} \frac{d\Psi}{d\varpi} = 0,$$

and inside the ring

$$= - \frac{2 m_1 c_1}{\pi a_1{}^2}.$$

At the surface of the ring

$$\Psi = \frac{m_1 c_1}{\pi} \left(\log \frac{8 c_1}{a_1} - 2 \right) + \Sigma \frac{m_2}{\pi} \mathrm{I}_{12} \quad \ldots \ldots \quad (103).$$

Therefore, inside the ring C_1,

$$\Psi = \frac{m_1 c_1}{\pi} \left(\log \frac{8 c_1}{a_1} - 2 \right) + \Sigma \frac{m_2}{\pi} \mathrm{I}_{12} + \frac{m_1 c_1}{2\pi} \left(1 - \frac{\mathrm{R}^2}{a_1{}^2} \right) . \quad \ldots \quad (104).$$

Integrating throughout this ring, the part of the kinetic energy arising from this is

$$\left\{ \Sigma m_1{}^2 c_1 \left(\log \frac{8 c_1}{a_1} - \tfrac{7}{4} \right) + m_1 (m_2 \mathrm{I}_{12} + m_3 \mathrm{I}_{13} + \ldots) \right\} \quad \ldots \quad (105).$$

Adding the integrals arising from the other rings, the kinetic energy

$$= 4\pi \left\{ \Sigma m_1{}^2 c_1 \left(\log \frac{8 c_1}{a_1} - \tfrac{7}{4} \right) + 2 \Sigma m_1 m_2 \mathrm{I}_{12} \right\} . \quad \ldots \ldots \quad (106).$$

The equations of motion may be written in the interesting forms,

$$\left.\begin{aligned} m_1 c_1 \dot{z}_1 &= \frac{1}{8\pi} \frac{\partial \mathrm{T}}{\partial c_1} \\ - m_1 c_1 \dot{c}_1 &= \frac{1}{8\pi} \frac{\partial \mathrm{T}}{\partial z_1} \end{aligned}\right\} \quad \ldots \ldots \quad (107).$$

The preceding work exemplifies an interesting fact about vortex motion. When the configuration of the vortices are known, their various velocities are obtained by linear equations; just as in the case of gravitating matter, linear equations give the accelerations of the various parts.

§ 43. When there are only two rings, the equations of motion are—

$$\left.\begin{aligned} \pi c_1 \dot{z}_1 &= \frac{m_1}{2}\left(\log\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + m_2 \frac{dI}{dc_1} \\ \pi c_1 \dot{c}_1 &= \qquad\qquad\qquad\quad - m_2 \frac{dI}{dz_1} \\ \pi c_2 \dot{z}_2 &= \frac{m_2}{2}\left(\log\frac{8c_2}{a_2} - \tfrac{1}{4}\right) + m_1 \frac{dI}{dc_2} \\ \pi c_2 \dot{c}_2 &= \qquad\qquad\qquad\quad - m_1 \frac{dI}{dz_2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (108),$$

where

$$I = \int_0^\pi \frac{c_1 c_2 \cos\phi \, d\phi}{\sqrt{\{(z_2 - z_1)^2 + c_1^2 - 2c_1 c_2 \cos\phi + c_2^2\}}} .$$

The two integrals of these equations, already found, are

$$m_1 c_1^2 + m_2 c_2^2 = \text{constant} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (109),$$

$$\frac{m_1^2}{2} c_1 \left(\log\frac{8c_1}{a_1} - \tfrac{7}{4}\right) + \frac{m_2^2}{2} c_2 \left(\log\frac{8c_2}{a_2} - \tfrac{7}{4}\right) + m_1 m_2 I = \text{constant} \quad . \quad (110).$$

Suppose the ring (C_1) in front of the ring (C_2).
Then $z_1 - z_2$ is positive, and

$$\frac{dI}{dz} = - \int_0^\pi \frac{c_1 c_2 (z_1 - z_2) \cos\phi \, d\phi}{\sqrt{\{(z_1 - z_2)^2 + c_1^2 - 2c_1 c_2 \cos\phi + c_2^2\}}} .$$

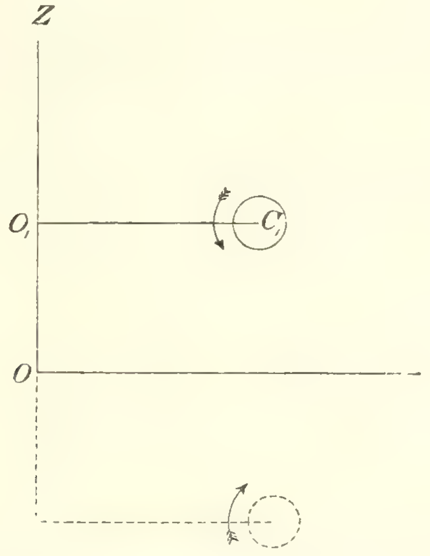
The integral is easily seen to be positive.

Therefore the radius of the front ring increases, and of the back ring diminishes. These changes cause decrease and increase of the velocities, so that it may happen that the second ring will overtake the first.

That the motion is of this character was shown by HELMHOLTZ, 'CRELLE'S Journal,' vol. 55, pp. 54, 55.

§ 44. *Vortex Ring approaching a Fixed Plane.*

This question is at once reduced to the case of two vortex rings on the same axis, by taking the image of the ring in the plane.



Let z be its distance from the plane at any moment.

The equation of energy gives

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-\int_0^\pi\frac{c^2\cos\phi\,d\phi}{\sqrt{\{4z^2+4c^2\sin^2(\frac{1}{2}\phi)\}}}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right),$$

where c_0, a_0, are the values of c and a when the ring is at a great distance from the plane.

Put

$$\phi=\frac{\pi}{2}-2\psi.$$

Then

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-\sqrt{z^2+c^2}\,(\mathrm{F}-\mathrm{E})+\frac{c^2}{\sqrt{z^2+c^2}}\,\mathrm{F}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right)\quad . \quad (111),$$

where $\dfrac{c}{\sqrt{c^2+z^2}}$ is the modulus of the elliptic functions.

This equation gives the relation between the radius of the ring and its distance from the plane.

Put

$$\frac{c}{\sqrt{c^2+z^2}}=\sin\theta.$$

Then

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-2c\operatorname{cosec}\theta\,(\mathrm{F}-\mathrm{E})+c\sin\theta\,.\,\mathrm{F}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right). \quad (112).$$

In addition to this $a^2c = a_0^2c_0$.

When the ring is so near the plane that z is much smaller than c, though greater than a,

$$\mathrm{F}=\log\frac{4\sqrt{c^2+z^2}}{z}\log\frac{4c}{z}\text{ . approximately.}$$

$$\mathrm{E}=1.$$

Therefore,

$$c\left(\log\frac{2z}{a}+\tfrac{1}{4}\right)=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113).$$

When the ring is near the plane, the equations of motion are

$$\dot{c}=\frac{m}{2\pi z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114),$$

$$c\dot{z}=-\frac{m}{2}\left(\log\frac{2z}{a}+\tfrac{3}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

From these, or more simply from the equation of energy

$$\dot{c}=\frac{m}{\pi a_0}e^{\frac{1}{4}}\sqrt{\frac{c}{c_0}}\,.\,e^{-c_0/c\,[\log(8c_0/a_0)-\frac{7}{4}]}.$$

Therefore

$$\frac{me^{\frac{1}{4}}}{\pi a_0}(t_2-t_1)=\int_{c_1}^{c_0}\sqrt{\frac{c}{c_0}}\,.\,e^{c_0/c\,[\log(8c_0/a_0)-\frac{7}{4}]}\,dc.$$

Let

$$c/c_0=x^2,$$

and

$$k=\log\frac{8c_0}{a_0}-\tfrac{7}{4}.$$

Then

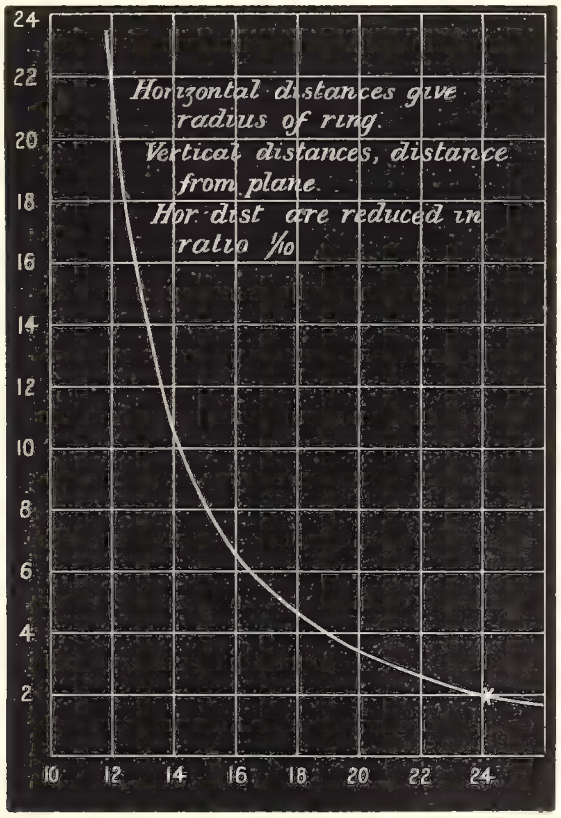
$$\frac{me^{\frac{1}{4}}}{2a_0c_0}(t_2-t_1)=\int_{x_1}^{x_2}\frac{e^{-kx^2}}{x}\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

The following Table shows the change in the radius of a ring in which $c_0/a_0 = 100$. When the ring is near the plane, the approximate formula

$$c\left(\log \frac{2z}{a} + \tfrac{1}{4}\right) = c_0\left(\log \frac{8c_0}{a_0} - \tfrac{7}{4}\right)$$

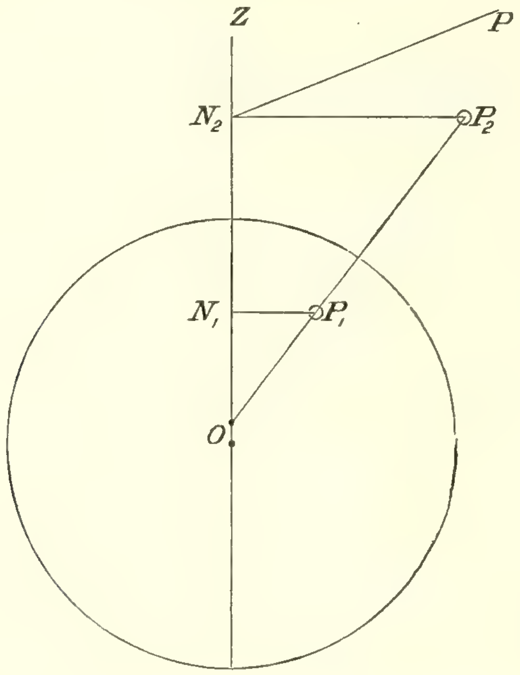
was used, but in other cases the exact formula.

θ.	$\frac{z}{a}$.	$\frac{z}{a_0}$.	$\frac{c}{c_0}$.
45°		104	1·04
60		61·8	1·07
70		40·0	1·10
75		30·5	1·14
80		21·3	1·21
83		16·0	1·30
85		12·1	1·38
86		10·1	1·44
	10	8·12	1·52
	8	6·26	1·63
	6	4·67	1·80
	5	3·60	1·93
	4	2·75	2·12
	3	1·93	2·42



§ 45. *Vortex ring passing over a Fixed Spherical Obstacle.*

When a vortex ring lies on a sphere concentric with a sphere in the fluid, the method of images may be applied, as is shown by Mr. LEWIS, 'Q.J.M.,' vol. 16, p. 338.



Let m_2 be the strength of the external ring
  c_2 its radius,
  a_2 the radius of its cross-section,
  z_2 the distance of its centre along OZ.
Let m_1, c_1, a_1, z_1, be corresponding quantities for the image
Then

$$m_1 = - m_2 \sqrt{\frac{c_2}{c_1}},$$

$$\frac{c_1}{a} = \frac{c_2}{a_2}.$$

At any external point P (coor. z, ϖ, ϕ),

$$\Psi = m_2 \int_0^\pi \frac{c_2 \varpi \cos\phi \, d\phi}{\sqrt{\{(z - z_2)^2 + \varpi^2 - 2\varpi c_2 \cos\phi + c_2^2\}}}$$

$$- m_2 \sqrt{\frac{c_2}{c_1}} \int_0^\pi \frac{c_1 \varpi \cos\phi \, d\phi}{\sqrt{\{(z - z_1)^2 + \varpi^2 - 2\varpi c_1 \cos\phi + c_1^2\}}}.$$

The equation of energy becomes

$$\frac{m_2{}^2c_2}{2}\left(\log\frac{8c_2}{a_2} - \tfrac{7}{4}\right) + \frac{m_2{}^2c_2}{c_1}\frac{c_1}{2}\log\left(\frac{8c_1}{a_1} - \tfrac{7}{4}\right)$$

$$- m_2{}^2 \sqrt{\frac{c_2}{c_1}}\int_0^\pi \frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_2 - z_1)^2 + c_2{}^2 - 2c_1c_2\cos\phi + c_1{}^2\}}}$$

$$= m_2{}^2c_0\left(\log\frac{8c_0}{a_0} - \tfrac{7}{4}\right) \quad \ldots\ldots\ldots\ldots \quad (117),$$

where a_0 and c_0 are the values of a and c, when the ring and sphere are a long way apart.

Let the radius of the sphere be k.

Changing to polar coordinates, let

$$c_2 = r\sin\theta. \qquad c_1 = \frac{k^2}{r}\sin\theta.$$

$$z_2 = r\cos\theta. \qquad c_2 = \frac{k^2}{r}\cos\theta.$$

Then

$$(z_2 - z_1)^2 + c_2{}^2 - 2c_1c_2\cos\phi + c_1{}^2 = r^2 + \frac{k^4}{r^2} - 2k^2\cos^2\theta - 2k^2\sin^2\theta\cos^2\phi$$

$$= \left(r - \frac{k^2}{r}\right)^2 + 4k^2\sin\theta\sin^2\frac{\phi}{2}\,.$$

Therefore

$$r\sin\theta\left(\log\frac{8r\sin\theta}{a} - \tfrac{7}{4}\right) - rk\sin^2\theta\int_0^\pi \frac{\cos\phi\,d\phi}{\sqrt{\{(r - k^2/r)^2 + 4k^2\sin^2\theta\sin^2(\frac{1}{2}\phi)\}}}$$

$$= r_0\sin\theta_0\left(\log\frac{8r_0\sin\theta_0}{a_0} - \tfrac{7}{4}\right).$$

Also

$$a^2r\sin\theta = a_0{}^2r_0\sin\theta_0.$$

Substituting a from the second equation, we have the path of the ring from infinity up to the sphere, *i.e.*, the path of any point in the ring.

Let

$$r_0\sin\theta_0 = c_0 \quad \text{and} \quad c_0/a_0 = n.$$

Then the above equation becomes

$$r\sin\theta\left\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{r\sin\theta}{c_0}\right\} - kr\sin^2\theta\int_0^\pi \frac{\cos\phi\,d\phi}{\sqrt{\{(r - k^2/r)^2 + 4k^2\sin^2\theta\sin^2(\frac{1}{2}\phi)\}}}$$

$$= c_0(\log 8n - \tfrac{7}{4}) \quad \ldots\ldots\ldots\ldots \quad (118).$$

When the centre of the ring coincides with the centre of the sphere, let $r\sin\theta = \varpi$, $\sin\theta = 1$; and an easy transformation of the integral gives

$$\varpi\left\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right\} - 2k^3\int_0^{\pi/2}\frac{\sin^2\psi\,d\psi}{\sqrt{\varpi^4 - k^4\sin^2\psi}} = c_0\left(\log 8n - \tfrac{7}{4}\right).$$

Therefore

$$\frac{\varpi}{c_0}\left\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right\} - 2\frac{\varpi}{k}\frac{\varpi}{c_0}(\mathrm{F} - \mathrm{E}) - \left(\log 8n - \tfrac{7}{4}\right) = 0,$$

where the modulus of the elliptic function is k^2/ϖ^2.

Therefore

$$\frac{\mathrm{F} - \mathrm{E}}{\sqrt{\sin\alpha}} = \tfrac{1}{2}\left(\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right) - \tfrac{1}{2}\frac{c_0}{\varpi}\left(\log 8n - \tfrac{7}{4}\right) \quad . \quad . \quad . \quad (119),$$

where $\sin\alpha = k^2/\varpi^2$.

Let c_0/a_0 or $n = 100$.

Then

$$\frac{\mathrm{F} - \mathrm{E}}{\sqrt{\sin\alpha}} = 2\cdot467\left(1 - \frac{c_0}{\varpi}\right) + \tfrac{3}{4}\log_{10}\frac{\varpi}{c_0}\,.$$

It is easy to construct the following table :—

α.	ϖ.	c_0.	k.
20°	1	·98	·58
30	1	·90	·71
45	1	·82	·84
60	1	·70	·93

From which, taking c_0 the value of the radius at infinity as the unit, we see that when k (radius of sphere) = ·6, ·8, 1, 1·3; ϖ, the radius of the ring when passing the middle of the sphere, = 1·02, 1·1, 1·3, 1·4.

[* § 46. *Two coaxal rings of equal strength and volume moving in the same direction.* Let the radii of the rings be c_1 and c_2, and the radii of their cross-sections a_1 and a_2. Then

$$a_1^2c_1 = a_2^2c_2 = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120).$$

The equation of momentum is

$$c_1^2 + c_2^2 = 2\kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121),$$

and the equation of energy is

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* This and the next paragraph have been altered since the paper was read, in consequence of a very valuable suggestion of one of the Referees.—July, 1893.

$$\frac{c_1}{2}\left(\log\frac{8c_1}{a_1}-\tfrac{7}{4}\right)+\frac{c_2}{2}\left(\log\frac{8c_2}{a_2}-\tfrac{7}{4}\right)+\int_0^\pi\frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_1-z_2)^2+c_1{}^2-2c_1c_2\cos\phi+c_2{}^2\}}}=\text{constant (122)},$$

and if m be the strength of each ring,

$$\dot{z}_1-\dot{z}_2=\frac{m}{\pi}\left\{\frac{1}{2c_1}\left(\log\frac{8c_1}{a_1}-\tfrac{1}{4}\right)-\frac{1}{2c_2}\left(\log\frac{8c_2}{a_2}-\tfrac{1}{4}\right)+\frac{1}{c_1}\frac{d\mathrm{I}}{dc_1}-\frac{1}{c_2}\frac{d\mathrm{I}}{dc_2}\right\}.$$

Let us write $z_1-z_2=z$.
Then

$$\begin{aligned}
&\frac{1}{c_1}\frac{d\mathrm{I}}{dc_1}-\frac{1}{c_2}\frac{d\mathrm{I}}{dc_2}\\
&\quad=\int_0^\pi\frac{(c_2{}^2-c_1c_2\cos\phi+z^2)\,c_2\cos\phi\,d\phi}{c_1\,(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}-\int_0^\pi\frac{(c_1{}^2-c_1c_2\cos\phi+z^2)\,c_1\cos\phi\,d\phi}{c_1\,(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}\\
&\quad=\frac{c_2{}^2-c_1{}^2}{c_1c_2}\int_0^\pi\frac{(c_1{}^2-c_1c_2\cos\phi+c^2+z^2)\cos\phi\,d\phi}{(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}\\
&\quad=\frac{c_2{}^2-c_1{}^2}{c_1c_2}\int_0^\pi\frac{\cos\phi\,d\phi}{(c_1{}^2-2c_1c^2\cos\phi+c_2{}^2+z^2)^{\frac{1}{2}}}+(c_2{}^2-c_1{}^2)\int_0^\pi\frac{(1-\sin^2\phi)\,d\phi}{(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}\\
&\quad=\int_0^\pi\frac{(c_2{}^2-c_1{}^2)\,d\phi}{(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}.
\end{aligned}$$

Therefore

$$\dot{z}=\frac{m}{\pi}\left\{\frac{1}{2c_1}\left(\log\frac{8c_1}{a_1}-\tfrac{1}{4}\right)-\frac{1}{2c_2}\left(\log\frac{8c_2}{a_2}-\tfrac{1}{4}\right)+(c_2{}^2-c_1{}^2)\int_0^\pi\frac{d\phi}{(c_1{}^2-2c_1c_2\cos\phi+c_2{}^2+z^2)^{\frac{3}{2}}}\right\}\quad(123).$$

Now the rings are at their greatest distance apart when $\dot{z}_1=\dot{z}_2$ or when $\dot{z}$ vanishes. Now, equation (123) shows that this takes place when $c_1=c_2$.

When $c_1=c_2$ let the value of each be κ, and let a_1 and a_2 each $=\alpha$. Also let κ_1, $\alpha_1:\kappa_2$, α_2, be the values of c_1, $a_1:c_2$, a_2, when $z_1=z_2$, *i.e.*, when the rings are in the same plane.

Then these quantities are connected by the equations

$$\alpha_1{}^2\kappa_1=\alpha_2{}^2\kappa^2=\alpha^2\kappa,$$
$$\kappa_1{}^2+\kappa_2{}^2=2\kappa^2,$$

and

$$\begin{aligned}
&\frac{\kappa_1}{2}\left(\log\frac{8\kappa_1}{a_1}-\tfrac{7}{4}\right)+\frac{\kappa_2}{2}\left(\log\frac{8\kappa_2}{\alpha_2}-\tfrac{7}{4}\right)+\int_0^\pi\frac{\kappa_1\kappa_2\cos\phi\,d\phi}{\sqrt{\{\kappa_1{}^2-2\kappa_1\kappa_2\cos\phi+\kappa_2{}^2\}}}\\
&\qquad=\kappa\left(\log\frac{8\kappa}{\alpha}-\tfrac{7}{4}\right)+\int_0^\pi\frac{\kappa^2\cos\phi\,d\phi}{\sqrt{\{2\kappa^2+z^2-2\kappa^2\cos\phi\}}}.
\end{aligned}$$

Thus, when we are given κ_1 and κ_2, we have two equations to determine κ and z; or given z and κ, the greatest distance apart of the two rings and their radius at that time, we have equations to find κ_1 and κ_2.

One of the quantities, α_1, α_2, α, is arbitrary, being the cross-section of the ring, when it has a definite radius.

Let us take

$$\alpha = \frac{\kappa}{n}.$$

Also let

$$\kappa_1 = \kappa\sqrt{2}\sin\theta_0 \qquad \text{and} \qquad \kappa_2 = \kappa\sqrt{2}\cos\theta_0.$$

Then the equation connecting z and θ_0 is

$$\frac{\sin\theta_0}{\sqrt{2}}\left[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\left(\sqrt{2}\sin\theta_0\right)\right] + \frac{\cos\theta_0}{\sqrt{2}}\left[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\left(\sqrt{2}\cos\theta_0\right)\right]$$
$$+ \frac{1}{\sqrt{2}}\int_0^\pi \frac{\sin 2\theta_0 \cos\phi \, d\phi}{\sqrt{(1 - \sin 2\theta_0\cos\phi)}}$$
$$= \log 8n - \tfrac{7}{4} + \int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\left(2 + \frac{z^2}{\kappa^2} - 2\cos\phi\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (124).$$

This equation can only be satisfied when θ_0 is between certain limits β and $(\pi/2) - \beta$.

The limiting values of θ_0 give $z = \infty$.

When θ_0 is between β and $(\pi/2) - \beta$, the equation gives a real value of z; this value becomes smaller as θ_0 approaches $\pi/4$.

We have, therefore, the following theorem.

If $\kappa\sqrt{2}\sin\theta_0$ *and* $\kappa\sqrt{2}\cos\theta_0$ *be the radii of two coaxal vortex rings of equal strength and volume when the rings are in the same plane* (θ_0 *being* $< \pi/4$, *so that* $\kappa\sqrt{2}\sin\theta_0$ *is the radius of the inner ring*)*: then these two rings will continue to thread one another in turns, or will separate to an infinite distance according as* θ_0 *is* $>$ *or* $<$ β, *where* β *is determined by the equation*

$$\frac{\sin\beta}{\sqrt{2}}\left[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\left(\sqrt{2}\sin\beta\right)\right] + \frac{\cos\beta}{\sqrt{2}}\left[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\left(\sqrt{2}\cos\beta\right)\right]$$
$$+ \frac{1}{\sqrt{2}}\int_0^\pi \frac{\sin 2\beta\cos\phi\, d\phi}{\sqrt{(1 - \sin 2\beta\cos\phi)}} = \log 8n - \tfrac{7}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (125).$$

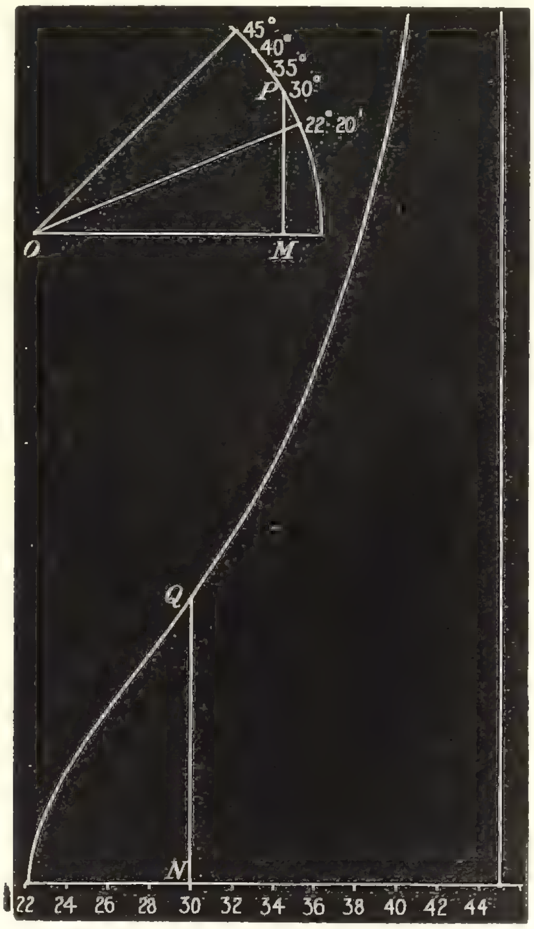
The following Table gives the values of β, and thence of κ_1 and κ_2, for different values of n.

n.	β.	κ_1/κ.	κ_2/κ.
8	18°	·437	1·345
11	19°	·460	1·337
20	20°	·484	1·329
100	22° 20′	·537	1·308
3300	26°	·620	1·271

Roughly speaking, when n is between 8 and 100, the ratio of $\kappa_1 : \kappa_2$ for this limiting case is between $\frac{1}{3}$ and $\frac{2}{5}$.

The following diagram calculated from the equations (121) and (122) gives, for $n=100$, simultaneous values of the radii of the rings and their distance apart in the limiting case in which the inner ring just goes to an infinite distance in front of the outer one.

The radius of the front ring is given by an abscissa of the circle in the diagram, the radius of the back ring by the corresponding ordinate, and the distance apart by the abscissa of the curve. Thus NQ is the distance apart when the ratio of the radii is tan 30°, or when the rings have the radii OM and MP.



§ 47. When θ_0 is greater than the limiting value β, the greatest distance apart to which the vortices can go is given by the equation (124). When θ_0 is less than the limiting value β, the inner ring goes to an infinite distance from the outer one, increasing in radius towards the limiting value $\kappa\sqrt{2}\sin\theta_\infty$, the radius of the outer ring diminishing towards the limiting value $\kappa\sqrt{2}\sin\theta_\infty$.

The equation determining θ_∞ is

$$\sin\theta_0\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\theta_0)\} + \cos\theta_0\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\theta_0)\} + \int_0^\pi \frac{\sin 2\theta_0\cos\phi\,d\phi}{\sqrt{(1-\sin 2\theta_0\cos\phi)}}$$
$$= \sin\theta_\infty\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\theta_\infty)\} + \cos\theta_\infty\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\theta_\infty)\} \quad (126).$$

The table in § 46 gives 22° 20′ as the value for β for rings in which $n = 100$.

The tables below give the greatest distance the rings will separate for values of $\theta_0 > 22° 20'$; and the radii at infinity of the rings for values of $\theta_0 < 22° 20'$. κ_1 and κ_2 are the radii of the rings when they are in the same plane; κ_1', κ_2' are their radii when at an infinite distance apart, if they separate; and z is their greatest distance apart if they remain together.

θ_0.	κ_1.	κ_2.	θ_∞.	κ_1'.	κ_2'.
5°	·123	1·408	5° 50′	·144	1·407
10	·246	1·392	12·49	·309	1·379
15	·366	1·366	22·20	·537	1·308
20	·484	1·329	38·30	·880	1·056

θ_0.	κ_1.	κ_2	z.
25°	·598	1·282	1·044
30	·707	1·225	·500
35	·811	1·159	·245
40	·909	1·083	·084

XXI. *On the Annual and Semi-Annual Seismic Periods.*

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Communicated by Professor J. H. POYNTING, *F.R.S.*

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1. IN 1834 the existence of an annual period of earthquake frequency was pointed out by MÉRIAN, and his discovery has been fully confirmed by the work of later seismologists. No result of equal importance was, however, added to our knowledge until fifty years afterwards, when a valuable memoir[*] was presented by Dr. C. G. KNOTT to the Seismological Society of Japan. As the present paper follows somewhat closely in the lines laid down by Dr. KNOTT, I will first give a short account of the results obtained by him, and in the next section will describe his method of investigation.

One of the objects of DR. KNOTT's work was to ascertain whether any trace could be detected of a six-monthly, as well as of an annual, period. He examined for this purpose lists of earthquakes in Japan from 1872 to 1880 (MILNE), Europe and the adjacent parts of Asia and Africa from 306 to 1843 (PERREY), the Grecian Archipelago from 1859 to 1873 (SCHMIDT), the East Indian Archipelago from 1873 to 1881 (BERGSMA), New Zealand from 1869 to 1879, and Chili from 1873 to 1881 (VERGARA). The chief results obtained from this discussion are the following: (1) In five of the six districts selected there is a well-marked annual period, the maximum occurring about December or January in Europe, the Grecian Archipelago, and Japan, and between June and September in New Zealand and Chili; *i.e.*, during the winter season in every case: while in the East Indies, lying close to and on either side of the equator, no such period could be discerned. (2) With the exception again of the East Indies there seems to be in the earthquakes of each district a semi-annual period. the maxima occurring about January[†] and June or July in Europe, about March and September in the Grecian Archipelago, about June and November in Japan, about February and August in New Zealand, and about April (for the first maximum) in Chili, the second not being clearly defined. In the case of Europe, the semi-annual period is not very pronounced, and the smoothness of the curve representing it as compared with the others "almost suggests," says Dr. KNOTT, "that this semi-annual characteristic is almost accidental, and that with a sufficient number of observations it might vanish altogether"; "and yet," he adds, "it seems too general to be accidental."

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[*] "Earthquake Frequency" [1884], 'Japan, Seismol. Soc. Trans.,' vol. 9, part 1, 1886, pp. 1–20.

[†] These dates differ from those given by Dr. KNOTT. The reason for the change is stated below in § 3.

It appears to me, therefore, desirable to examine the seismic records of other countries besides those which were at the disposal of Dr. KNOTT in 1884, and this I shall endeavour to do in the present paper. The relation of seismic periodicity to intensity will also be considered, though the materials are somewhat inadequate for the purpose. In the closing paragraphs, a summary is given of the principal conclusions arrived at, and an attempt is made to show that the annual variation in barometric pressure may be the cause of the annual seismic period.

With regard to the origin of the semi-annual period, I regret that I am unable to offer any definite suggestion.

METHOD OF INVESTIGATION.

2. *Dr.* KNOTT'S *Method.*—If $f(\theta)$ be a periodic function of a variable θ, we may write

$$f(\theta) = a_0 + a_1 \cos(\theta + \alpha_1) + a_2 \cos(2\theta + \alpha_2) + \ldots + a_n \cos(n\theta + \alpha_n) + \ldots,$$

from which it follows that

$$\frac{1}{2\phi}\int_{\theta-\phi}^{\theta+\phi} f(\theta)\,d\theta = a_0 + \frac{a_1 \sin\phi}{\phi}\cos(\theta + \alpha_1) + \frac{a_2 \sin 2\phi}{2\phi}\cos(2\theta + \alpha_2) + \ldots + \frac{a_n \sin n\phi}{n\phi}\cos(n\theta + \alpha_n) + \ldots;$$

the latter equation giving the mean value of $f(\theta)$ through an interval ϕ on either side of θ.

Let T and S be the values of the last series when ϕ is put equal to $\pi/4$ and $\pi/2$ respectively, then

$$\mathrm{T} = a_0 + \frac{2\sqrt{2}a_1}{\pi}\cos(\theta + \alpha_1) + \frac{2a_2}{\pi}\cos(2\theta + \alpha_2) + \ldots + \frac{4a_n \sin\frac{1}{4}n\pi}{n\pi}\cos(n\theta + \alpha_n) + \ldots,$$

$$\mathrm{S} = a_0 + \frac{2a_1}{\pi}\cos(\theta + \alpha_1) - \frac{2a_3}{3\pi}\cos(3\theta + \alpha_3) + \ldots + \frac{2a_n \sin\frac{1}{2}n\pi}{n\pi}\cos(n\theta + \alpha_n) + \ldots,$$

and

$$S - \frac{T}{\sqrt{2}} = a_0\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{a_2\sqrt{2}}{\pi}\cos\left(2\theta + \alpha_2\right) + \dots$$

$$+ \frac{2a_n}{n\pi}\left(\sin\tfrac{1}{2}n\pi - \sqrt{2}\sin\tfrac{1}{4}n\pi\right)\cos\left(n\theta + \alpha_n\right) + \dots;$$

where in T all terms are absent which involve multiples of 4θ, in S all those involving multiples of 2θ, while, in the third expression, the terms involving θ and all multiples of 4θ disappear.

If, then, we take 2π to represent a period of one year, T represents the result of taking three-monthly means, and the effect of this is to eliminate the three-monthly period, if there be one, and all periods which are fractions of three months, and also to diminish the amplitudes of all other periods relatively to that of the annual period. In like manner, S represents the result of taking six-monthly means, the effect being to eliminate the six-monthly period, if there be one, and all periods which are fractions of six months, and also to diminish the amplitudes of all other periods relatively to that of the annual period. The result of the third operation is to eliminate the annual period, and to give special prominence to the six-monthly and four-monthly periods, if such exist.

Now, if we have a record of all the earthquakes felt during a given interval (containing an exact number of years) over a definite area, and if we group together all the earthquakes felt during each month of the year, we may regard the numbers so obtained as representing the average number for each month felt on the middle day of that month. The mean of the numbers for the months of December, January, and February, will be the three-monthly mean corresponding to the middle of January; and the mean of the numbers for the months of November, December, January, February, March and April, will be the six-monthly mean corresponding to the end of January. If three-monthly and six-monthly means be calculated in this manner for each month, and if the results be plotted in the usual way, the first of the curves so obtained will show no trace of a three-monthly period, and the second none of a six-monthly period. To eliminate the annual period, arithmetic means are interpolated between each pair of numbers in the two series, the means of the first series giving the three-monthly means corresponding to the end of each month, and those of the second series the six-monthly means corresponding to the middle of each month. Multiplying each number of the first series by $1/\sqrt{2}$, or ·707 . . . , and subtracting it from the corresponding number of the second series, the curve obtained by plotting the resulting differences will show no trace of an annual period, and will give special prominence to the six-monthly and four-monthly periods, if such exist. Lastly, if each of the twenty-four differences be divided by the average of all, the curves so obtained for different intervals and for different areas of observation will be more directly comparable with one another.

3. If the third and succeeding terms in the expression for $S - T/\sqrt{2}$ may be neglected in comparison with the first and second, it is evident that $S - T/\sqrt{2}$ will be a maximum when the second term (which determines the six-monthly period) is a minimum, and *vice versâ*. By a slight oversight, Dr. KNOTT has reversed the sign of this term, and consequently the epochs which he obtains for the maxima of the six-monthly period are in reality those of the minima, and *vice versâ.*

4. *Method adopted in this Paper.*—I have described in some detail the method employed by Dr. KNOTT, for, in principle, it is the same as that adopted in this Paper. The chief differences are that I have used the same method in investigating both the annual and semi-annual periods,* and have also taken into account the variable length of the month.

Whenever possible I have grouped the earthquakes in half-monthly, instead of in monthly intervals, the first half of February containing fourteen days, and of all other months fifteen days. The numbers so obtained are then reduced to equal half-monthly intervals of fifteen days.

In investigating the annual period, the reduced figures for the halves of each month are added together, giving the numbers of earthquakes corresponding to equal monthly intervals of thirty days. Six-monthly means are then taken of these numbers, as in Dr. KNOTT's method, and the mean thus obtained for the end of each month is divided by the average of all twelve means.

For the semi-annual period, the reduced figures for the first half of January and the first half of July are added together, and so on. Six-half-monthly means are then taken of these numbers, and the mean thus obtained for the end of each half-month is divided by the average of all twelve means.

5. Seismologists generally group earthquakes in monthly intervals, and in some cases I have only been able to make use of such tables. In a few others, also, for reasons which will be obvious, I have used them though having access to the catalogues on which they were founded. The results obtained in this way are not of equal value for purposes of comparison, as they probably depend on different definitions of the unit earthquake.

In all these cases the monthly numbers are first reduced to equal intervals of thirty days. The discussion for the annual period is then continued as before.

For the semi-annual period the numbers for January and July are added together, and so on. Three-monthly means are then taken of these six numbers, and the mean thus obtained for the middle of each month is divided by the average of all six means. The method, of course, chiefly differs from the preceding in giving fewer points upon the curve.

6. Again, in taking six-monthly means for the annual period, and dividing each by the average of all twelve, it is evident, from the expression for S in § 2, that every

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* In investigating the semi-annual period by this method, it is obvious that the annual period is eliminated (see § 10).

ordinate in the curve so obtained is reduced in the ratio $2 : \pi$. It is also further reduced by the method of grouping the earthquakes in monthly intervals. For, instead of using the actual ordinate for $\cos\theta$, we use the average ordinate over an interval $\pi/6$, that is

$$\frac{6}{\pi}\int_{\theta-\pi/12}^{\theta+\pi/12}\cos\theta\,d\theta, \qquad \text{or} \qquad \frac{12}{\pi}\sin\frac{\pi}{12}\cos\theta.$$

Now, the value of $12\sin\frac{1}{12}\pi/\pi$ is ·9886. Hence the difference between each reduced six-monthly mean and unity must be multiplied by the factor $\frac{1}{2}\pi/{\cdot}9886$ or 1·589. The same remark applies to the case of taking six-half-monthly means for the semi-annual period. In like manner, in investigating the latter period by means of monthly intervals and taking three-monthly means, the corresponding augmenting factor is 1·645.

7. *Comparison of the Methods of Investigating the Semi-Annual Period.*—The method here employed for investigating the semi-annual period seems to me to possess several advantages. (1) It is less laborious to apply than Dr. KNOTT's method; (2) it avoids the interpolation of arithmetic means in the series of three-monthly and six-monthly means; and (3) it eliminates, or diminishes more completely, the effect of shorter periods, if any such should exist. The latter point requires a brief examination.

Taking Dr. KNOTT's method first, we have to evaluate the ratio

$$\frac{\dfrac{2a_n}{n\pi}\left(\sin\dfrac{n\pi}{2} - \sqrt{2}\,\sin\dfrac{n\pi}{4}\right)}{\dfrac{a_2\sqrt{2}}{\pi}} : \frac{a_n}{a_2}$$

for different values of n. If $n = 3$, the value of this ratio is ·943; if $n = 4$, it is zero; if $n = 5$, it is ·566; if $n = 6$, it is $\frac{1}{3}$. Hence, compared with that of the six-monthly period, the amplitude of the four-monthly period is slightly decreased, that of the period of one-fifth of a year is diminished by nearly one-half, while that of the two-monthly period is diminished by two-thirds.

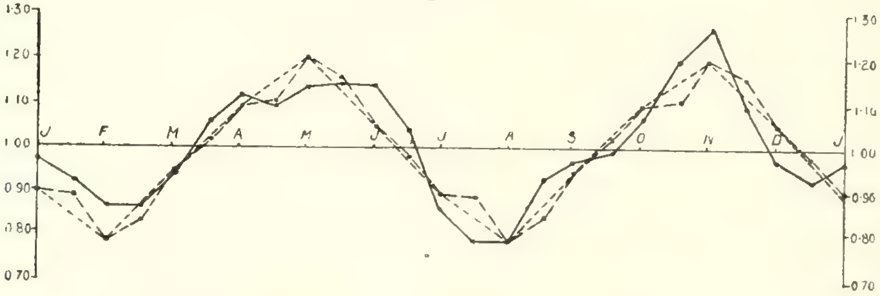
In the method adopted in this paper we have to evaluate the ratio

$$\frac{\dfrac{2a_n\sin\frac{1}{2}n\pi}{n\pi}}{2a_1/\pi} : \frac{a_n}{a_1}$$

for different values of n. If $n = 2$, its value is zero; if $n = 3$, it is $\frac{1}{3}$. Hence, the three-monthly period is eliminated, and, compared with that of the six-monthly period, the amplitude of the two-monthly period is diminished by two-thirds. Moreover, as will appear from § 10, both the four-monthly period and the period whose length is one-fifth of a year are eliminated, provided the interval over which the seismic record extends consists of an integral number of years.

8. The curves in fig. 1 exhibit the close agreement in the results obtained by the different methods of determining the semi-annual period, the seismic record chosen being that by Professor Milne of the earthquakes felt in Tokio during the years 1873–1881 and 1883–1890. The continuous curve is that obtained by Dr. Knott's method, every ordinate being first, however, multiplied by $\pi(\sqrt{2} - 1)/2$, or ·65, so as not to alter the ratio of the amplitude to that of the annual period. The broken curve is that obtained by the method used in this paper, the earthquakes being grouped in half-months, while the dotted curve is that obtained by the same method when the earthquakes are grouped in months.

Fig. 1.



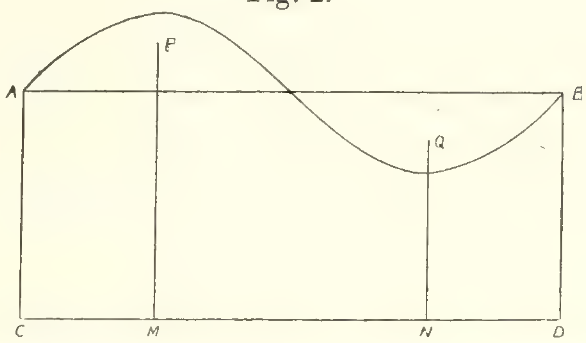
9. *Connection between the Amplitude of the Annual Period and the Winter-to-Summer Ratio in the Number of Earthquakes.*—In discussions on earthquake frequency, it is usual to estimate the ratio between the number of shocks felt during the six winter months (October to March) to the number felt during the six summer months (April to September). The ratio so obtained is, of course, only valuable when, as is frequently the case, the maximum of the annual period occurs either at the end of June or the end of December. Greater uniformity would be secured by evaluating the ratio of the number of shocks felt during the six months bisected by the maximum epoch to the number felt during the six months bisected by the minimum epoch. In the present paper, I have substituted the amplitude of the annual period for the last-named ratio, and it seems desirable to point out briefly the connection between them.

In fig. 2 let the distance AC represent the average number of earthquakes per month, and let the curve represent the annual seismic period. Let PM and QN represent the average length of the ordinates during the maximum and minimum halves of the year respectively. Then the ratio of the number of shocks felt during the maximum half to the number felt during the minimum half of the year is equal to PM : QN, *i.e.*, to the ratio of $1 + 2a/\pi$ to $1 - 2a/\pi$, where a is the amplitude of the annual period, the average number of shocks felt during each month of the year being taken as unity.

For instance, in the case of the European earthquakes of 1865–1884 as tabulated by Fuchs (see § 19), the maximum of the annual period occurs at the end of

December, the number of shocks felt during the six winter months is 3357, the number during the six summer months 2142, and the ratio of one to the other 1·57. The amplitude of the annual period is ·35, and the value of the ratio $1 + 2a/\pi : 1 - 2a/\pi$ is 1·56.

Fig. 2.



10. *Duration of the Seismic Record.*—It has been assumed in the preceding paragraphs that the curve obtained by plotting the number of earthquakes felt in each month is not affected by periods of longer duration than one year, or by periods which are not exact fractions of a year. The assumption may be tested by considering a period of $12m + n$ months, where m is either zero or a positive integer. The number of earthquakes (F_r) belonging to this period felt during the r^{th} month of each year may then be represented by

$$a \cos \alpha + a \cos (\alpha + 12\beta) + \ldots + a \cos [\alpha + (\lambda - 1) . 12\beta],$$

where $\beta = 2\pi/(12m + n)$, and λ is the number of years over which the seismic record extends. Then

$$F_r = a \cos [\alpha + (\lambda - 1) . 6\beta] \sin 6\lambda\beta/\sin 6\beta ;$$

and the value of F_r is zero whatever be the value of α if $6\lambda\beta = k\pi$, where k is an integer, *i.e.*, when $\lambda = k(12m + n)/12$. Hence, F_r is zero for all values of r, and consequently our method of investigation is not affected by the period considered, provided the interval over which the seismic record extends includes an exact number of these periods.

But whatever be the number of such periods included in the seismic record, the absolute error introduced by a given uneliminated portion of a period will be constant, and, therefore, the relative error will be the less the greater the interval over which the record extends.

On another account, also, it is desirable that this interval should be as long as possible. For, owing chiefly to orogenic causes, there are often series of frequently recurring shocks, lasting for one or more months, which, if the record be a short one or imperfect, would unduly increase the numbers felt during those months. The record should therefore be an extensive one in order to eliminate the effects of occasional variability in the rate of orogenic action and to bring into special prominence those of a periodic character.

11. *Definition of the Unit Earthquake.*—The difficulty of deciding what constitutes an earthquake attends all work on seismic periodicity, and, unless some uniform and natural system is followed, our conclusions cannot possess much value. For example, every great earthquake is preceded and followed by a large number of minor or secondary shocks, which are no doubt intimately associated with the principal shock. In such a case, are we to count every shock that is recorded, however slight, or are we to regard the whole series as a single earthquake? Again, the distinction between an earthquake and earth-tremor is a purely arbitrary one. Are we to include every earth-tremor recorded by a most delicate instrument, and rank it equally with the most destructive shock; or, if not, where are we to draw the line between the two?

It must be admitted that these are serious difficulties, but I do not think they would be met, as has been proposed, by taking the intensity of the shocks into account as well as their frequency. Even supposing it possible to estimate the intensity correctly, it may be doubted whether the results obtainable in the finite time at our disposal would possess much value, at any rate from the periodicity point of view. For, whatever be the causes of seismic periodicity, it seems highly probable that they are merely auxiliary, and determine the epoch when an earthquake shall take place rather than that there shall be an earthquake at all. The intensity is probably determined almost entirely by orogenic causes, the frequency partly by the same causes, and partly by certain others which are possibly variable and periodic in their action.

I believe, then, that, in the present state of our knowledge, it is desirable to regard each shock felt and recorded as one earthquake. Moreover, an approximate uniformity in estimating the minimum intensity of an earthquake will be attained if we consider special districts of comparatively small size—countries, for example, like Italy and Switzerland, rather than continents or hemispheres.

There is still room, however, for some arbitrariness of treatment, owing to the imperfection of nearly all our seismic records. The following principles have, therefore, been adhered to throughout this paper: (1) whenever the entry in a catalogue is "several shocks about this time," or some equivalent expression, without any attempt being made to give the time of occurrence of each shock, I have always reckoned such as a single earthquake; (2) when the shocks have been considered important enough to be entered separately, the time of occurrence of each being given, every such shock has been counted, provided the interval between any two or more was not less than five minntes; and (3) shocks entered as doubtful and earthquake-sounds unaccompanied by any shocks have been omitted.

12. *Explanation of the Tables, Figures, &c.*—Table I. contains the number of earthquakes for the different districts investigated during each half-month or month. The reduced six-monthly means obtained in discussing the annual period, as described

in §§ 4 and 6 are given in Table II.; and in Table III. the reduced six half-monthly means, or three-monthly means, obtained in investigating the semi-annual period.

At the end of the section relating to each record examined, are given the maximum epoch and amplitude of each period. The epoch mentioned is the end of the month or half-month indicated, the letters *a* and *b* denoting the first and second halves of the month. The date of the maximum and the amplitude have been taken as those of the greatest mean. It would not have been difficult to have given the result more minutely, but it is doubtful whether the appearance of greater accuracy would have any value, considering the unavoidable imperfections of most seismic records.

In the same place, I have also given the date of the maximum epoch of the annual barometric period over the greater part of the district considered, excluding high-level stations. For this purpose I have made use of Dr. BUCHAN'S great work on Atmospheric Circulation.* The method of investigation is the same as that employed for the annual seismic period, except that its application is much simpler in this case.

An asterisk placed before the name of a record, either in the Tables or detailed discussion, indicates that the results obtained are probably more reliable than in other cases. The series of six-monthly and six half-monthly means calculated from these records are illustrated by the curves in figs. 5–15, the continuous line representing the annual, and the dotted line the semi-annual, period.

THE REALITY OF THE ANNUAL PERIOD.

13. It is well known that many more earthquakes are recorded as felt during the night than during the day, and that this is due partly to the stillness of the night, but more, perhaps, to the position of the body during repose being more favourable for the detection of slight tremors. Now, the maximum of the annual period occurring during winter, it may seem possible that the periodicity may be apparent only, and due to the greater number of shocks observed during the longer nights of winter. The question is one that deserves examination, and the following facts seem to me conclusive in establishing the reality of the annual period.

In the case of the earthquakes recorded at Tokio by the Gray-Milne seismograph, there can, of course, be no doubt. But this is the only case treated in this paper in which the shocks are registered entirely by instrumental means.

The hours of rest may be taken roughly at between 10 P.M. and 8 A.M., and we may therefore divide the day into two parts, reckoning day-time from 9 A.M. to 9 P.M., and night-time from 9 P.M. to 9 A.M. Using FUCHS' valuable 'Statistik der Erdbeben,' I have counted the number of shocks, whose time of occurrence is recorded, in Austria,

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* "Report on Atmospheric Circulation, based on the Observations made on board H.M.S. Challenger during the years 1873–1876, and other Meteorological Observations." 'Report of the Scientific Results of the Voyage of H.M.S. Challenger, &c.; Physics and Chemistry,' vol. 2, 1889, pp. 1–263.

Hungary, and Switzerland during the years 1865–1884, and in Italy during the years 1865–1883. The total number is 2253. In the six summer months (April to September) the number of shocks felt during the night was 545, and during the day 358. In the six winter months (October to March) the number felt during the night was 821, and during the day 529. Thus, the ratio of the number of shocks observed during the night to that observed during the day was 1·52 in summer and 1·55 in winter. These ratios agree so closely that we may feel confident as to the reality of any marked annual period observable in these countries.

Again, the annual period is well marked in several tropical districts where the length of the night is nearly constant throughout the year. And, also, as will presently appear, there is an annual period, though of slight amplitude, in earthquakes which produce damage to buildings, and shocks of intensity so great could not possibly escape observation at any time of the night or day.

SEISMIC PERIODICITY IN RELATION TO INTENSITY.

14. If the causes of seismic periodicity are in the main auxiliary, as has been suggested above, it seems possible that the periodicity might be more marked in the case of very slight, than of very severe, earthquakes. An attempt is therefore made in the present section to classify earthquakes according to their intensity, so far as it is possible to do so with the incomplete materials at our disposal. For this purpose I have in the first place used MALLET's well-known catalogue,* and have selected from the shocks recorded by him: (1) those which may be regarded as slight, and (2) those which are strong enough to damage buildings; the latter class corresponding to intensities 8 to 10 of the Rossi-Forel scale, and the former, probably, to intensities 3 and 4. Both classes contain earthquakes felt in the northern hemisphere only.

Slight Earthquakes.—The shocks included under this heading are those which, from MALLET's description,† can have been hardly more than perceptible even at the places where the intensity was greatest. When two or more such shocks are felt within the same area on the same day, these have been omitted, for they may be regarded as equivalent to a single shock of greater intensity. And, for the same reason, I have excluded isolated slight shocks which are members of a long and continuous series of earthquakes of variable intensity. This has been done in four cases, namely, the series felt at Tiflis in Georgia in 1804, at Pignerol and other places in Piedmont during 1808 and following years, at Lunröe in Norway, beginning in 1819, and at Pesaro in Italy during the year 1826. The period investigated is that embraced by the third part of MALLET's catalogue—the years 1785–1842 inclusive.

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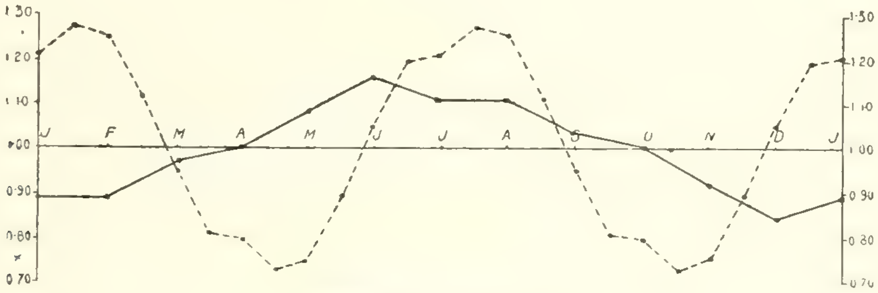
* "Catalogue of recorded Earthquakes," 'Brit. Assoc. Rep.,' 1852, pp. 1–176; 1853, pp. 117–212; 1854, pp. 1–326.

† The following are the terms employed by MALLET which I have taken to characterize a slight shock:—A tremor, a trembling, a slight trembling, a slight tremor, a slight tremulous shock, a trembling shock, a very feeble shock, a feeble vibration, a very slight shock, a slight shock, a slight vibration, a slight earthquake, a scarcely perceptible shock, a slight vibratory shock, rather slight, of trifling importance, a feeble shock, a trifling shock, a little shock, a slight undulatory motion, a very slight subterranean commotion, a slight undulatory shock, a slight oscillating motion.

Fig. 3.



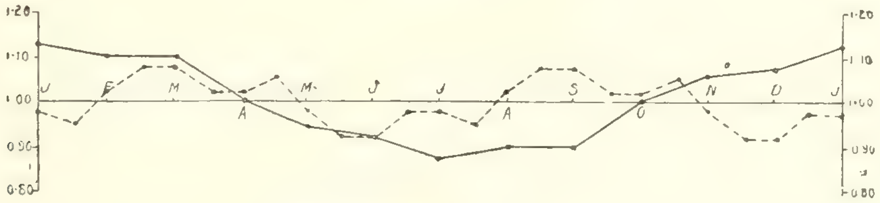
Slight earthquakes.

It seemed desirable to limit it thus closely in order to avoid any considerable variability in the estimation of a "slight" shock. The total number of such shocks is 187. The curves of the annual and semi-annual periods are shown in fig. 3.

Period.	Maximum.	Amplitude.
Annual.	June, *b*	·16
Semi-annual	January, *b* July, *b*	·27

Destructive Earthquakes.—Only those earthquakes have been counted in which it is expressly stated that some damage was done to buildings. The whole of the Christian era is included, but the first earthquake on which the month and day are definitely stated occurred in the year 528, and in several succeeding centuries the year is frequently the only date given. The total number of destructive earthquakes recorded by MALLET is 641. The curves of the annual and semi-annual period are shown in fig. 4.

Fig. 4.



Destructive earthquakes.

Period.	Maximum.	Amplitude.
Annual	January, *b*	·13
Semi-annual	February, *b*—March, *a* August, *b*—September, *a*	·08

It follows from this examination, imperfect as it is, (1) that in slight earthquakes both periods are more marked than in destructive ones, but that this is especially the case with the semi-annual period, the amplitude of the annual period being increased 1·2 times, and that of the semi-annual period 3·4 times; and (2) that the maximum epochs of the annual period are almost reversed.

15. *Japanese Earthquakes of* 1885–1889.—The organization of earthquake statistics in Japan is under the direction of the Imperial Meteorological Bureau. Reports are issued yearly, and abridged translations of these for the years 1885–1889, have been published by Professor S. Sekiya and Professor J. Milne (see § 51). Professor Milne has classified the earthquakes in monthly groups according to the areas disturbed by them. The results are given below, but it should be remembered that the duration of the record is short; also that the areas given are land areas only, and many Japanese earthquakes originate beneath the sea.

(1.) Earthquakes disturbing an area less than 100 square ri (1 square ri = 5·9 square miles). Number of earthquakes, 2256.

Period.	Maximum.	Amplitude.
Annual	September, *b*—October, *b*	·14
Semi-annual	March, *a* September, *a*	·12

(2.) Earthquakes disturbing an area less than 1000 square ri and greater than 100 square ri. Number of earthquakes, 565.

Period.	Maximum.	Amplitude.
Annual	February, *b*—March, *b*	·17
Semi-annual	June, *a* December, *a*	·16

(3.) Earthquakes disturbing an area greater than 1000 square ri. Number of earthquakes, 176.

Period.	Maximum.	Amplitude.
Annual.	March, *b*	·17
Semi-annual	March, *a* September, *a*	·12

Thus, (1) the amplitudes of both periods in all three classes are nearly the same, though somewhat less for the slight shocks than the others ; and (2) in the case of the annual period, the maximum epoch for the slight shocks agrees nearly with the minimum epoch for the more severe ones.

16. In nearly every case examined below, in which we have two or more records for the same area, it will be seen that the amplitude increases with the number of registered shocks per annum. Now, two records can only differ in this respect by the omission or inclusion of slight shocks, for severe ones are almost certain to find a place in every list. Thus, in these cases, the slighter the earthquakes the greater is the amplitude of each period.

This is especially evident when, with the same catalogue, we use different definitions of the unit earthquake. This has been done in three cases, namely, Zante (§§ 45, 46), California, &c., (§§ 59, 60), and New Zealand (§§ 76, 77).

Again, for Zante, the somewhat scanty list compiled from the catalogues of Schmidt and Fuchs (§ 47) gives the maximum of the annual period in December, and the amplitude ·29 ; the much fuller catalogue of Barbiani (§§ 45, 46) gives the maximum in August, and the amplitude ·10, or, using Barbiani's definition of the unit earthquake, the maximum in August, and the amplitude ·29. Thus, the effect of the first inclusion of slight shocks is to reverse approximately the maximum and minimum epochs, but to reduce the amplitude ; the effect of including a larger number of still slighter shocks is to maintain the reversal of epoch and to again increase the amplitude.

17. *Summary of Results.*—(1) There appear to be two distinct classes of slight shocks, one having the maximum of its annual period in winter, the other in summer. Of the two classes, the former probably contains the stronger shocks, for their effect is evident in the results obtained from the less incomplete of two imperfect catalogues ; while the effect of the latter is only visible in the results obtained from the fullest records, for example, those of the Imperial Meteorological Bureau for Japan, and of M. Barbiani for Zante, or from the list drawn up from Mallet's catalogue in which only very slight shocks are included. (2) In both classes of slight shocks the annual period is more marked than for strong shocks, for the effect of including the first class of slight shocks is to keep the winter maximum epoch unaltered and to increase the amplitude ; that of including some only of the second class is to interchange the maximum and minimum epochs approximately, the

amplitude of the period being diminished; while the effect of including a large number of slight shocks of the second class is to keep the epochs interchanged and to increase the amplitude. (3) With regard to the semi-annual period, the results, as a rule, show that strong shocks and slight shocks have the same maximum epochs, but that the amplitude for the latter is somewhat greater than for the former.

Seismic Periodicity in the Northern Hemisphere.

Northern Hemisphere.

18. R. **Mallet**, "Catalogue of Recorded Earthquakes from 1606 B.C. to A.D. 1850." 'Brit. Assoc. Rep.,' 1852, pp. 1–176; 1853, pp. 118–212; 1854, pp. 1–326; 1858, pp. 46 and 57.

Duration of record, A.D. 223–1850. Number of earthquakes, 5879.

Mallet's great catalogue includes several earthquakes which occurred before the Christian era, but the first in which the month is stated was felt in the year A.D. 223. It closes with the end of the year 1842, though he originally intended to bring it down to the end of 1850. M. **Perrey**'s catalogues, which rendered the continuation unnecessary, were, however, used by **Mallet** in his final discussion in his "Fourth Report." It was found necessary to make some convention with regard to the long-continued series of slight shocks or tremors felt at several places (Pignerol, Comrie, &c.), and in these cases "the slight shocks recorded for each month of the disturbed period are grouped as forming one earthquake at the locality." It will be seen from Table II. that the six-monthly means for November, December and January are the same. I have taken the intermediate month as the date of the annual maximum.

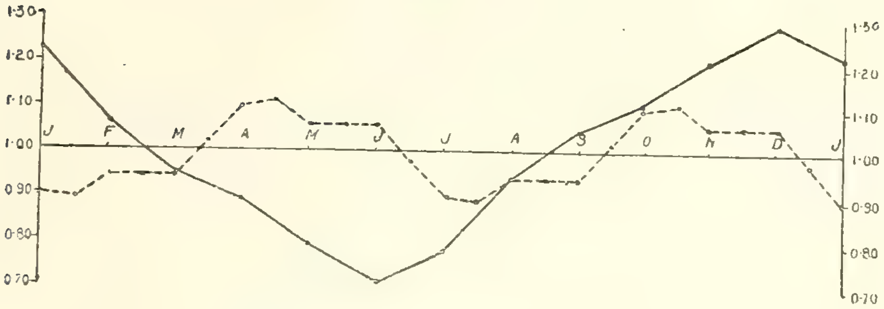
Period.	Maximum.	Amplitude.
Annual	December, *b*	·11
Semi-annual	February, *a* August, *a*	·07

19. *C. W. C. **Fuchs**, 'Statistik der Erdbeben,' pp. 13–286, 315–321, 357–408.

Duration of record, 1865–1884. Number of earthquakes, 8133 (fig. 5).

This number includes all the earthquakes recorded by **Fuchs** as occurring in the northern hemisphere, with the exception of those felt in the northern part of the Malay Archipelago. I have also omitted a few shocks in places which I am unable to identify, but the number of these is so small that it cannot possibly affect the result. The same remark applies to the discussion for **Fuchs**' catalogues for the southern hemisphere.

Fig. 5.



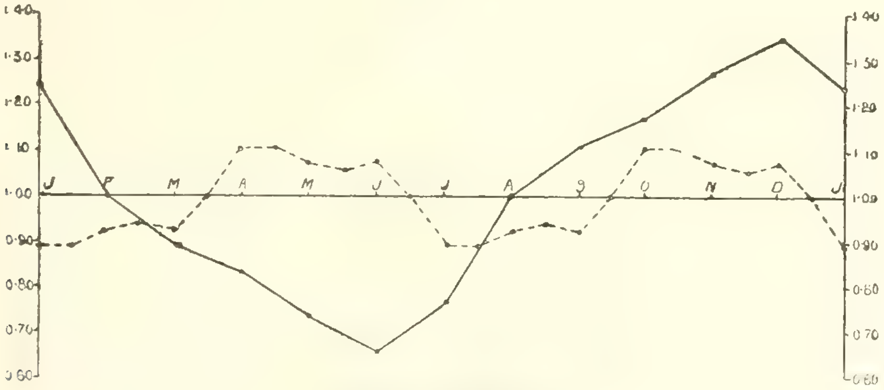
Northern hemisphere (Fuchs).

Period.	Maximum.	Amplitude.
Annual	December, *b*	·29
Semi-annual	April, *b* October, *b*	·11
Annual barometric . . .	November, *b*, or December, *b*	

Europe.

20. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 13–254 (fig. 6). Duration of record, 1865–1884. Number of earthquakes, 5499.

Fig. 6.



Europe (Fuchs).

Period.	Maximum.	Amplitude.
Annual	December, *b*	·35
Semi-annual	April, *a*—April, *b* October, *a*—October, *b*	·11
Annual barometric . . .	November, *b*	

21. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 28.

Duration of record, 306–1843. Number of earthquakes, 1961.

This is the table used in Dr. Knott's Memoir on "Earthquake Frequency," referred to at the beginning of this paper. The district includes also the adjacent parts of Asia and Africa.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·22
Semi-annual	January, *a* July, *a*	·12

Scandinavia and Iceland.

22. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 3.

Duration of record, 12th to 19th centuries. Number of earthquakes, 214.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·30
Semi-annual	February, *a* August, *a*	·22

The maximum epoch of the annual barometric period occurs in June, *b*, in the greater part of Norway, and in March, *b*, in the south of Norway, and nearly all over Sweden.

Great Britain.

23. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 5.

Duration of record, 11th to 19th centuries. Number of earthquakes, 217.

The results deduced from this table are of less value than those given below, especially those obtained from Mr. Roper's list. Perrey's district is more extensive than the others, being the "British Islands and Northern Isles."

Period.	Maximum.	Amplitude.
Annual	October, *b*	·32
Semi-annual	April, *a* October, *a*	·08

24. D. MILNE, "Notices of Earthquake-shocks felt in Great Britain, and especially in Scotland," &c. 'Edinburgh New Phil. Journ.,' 1841, vol. 31, pp. 95-122.

Duration of record, 1608–1838. Number of earthquakes, 205.

The figures for this district in Table I. differ somewhat from those given by Mr. MILNE on p. 289 of his paper. Some of the shocks counted by him are omitted on account of the rules laid down in § 11, others, because they were felt in Jersey and Guernsey.

Period.	Maximum.	Amplitude.
Annual	November, *b*	·49
Semi-annual	March, *a* September, *a*	·16

25. W. ROPER, "A List of the more Remarkable Earthquakes in Great Britain and Ireland during the Christian Era," 1889. (Published by T. BELL, "Observer" Office, Lancaster.)

Duration of record, 1739–1888. Number of earthquakes, 297.

This list is compiled from PERREY's and MALLET's catalogues and various other sources. Before the year 1739 the exact day of occurrence is often imperfectly given. Slight shocks, like those felt at Comrie, are omitted from the list, and this deprives it of much of its value for statistical purposes. I have excluded all earthquakes whose origin probably lay outside the area of Great Britain.

Period.	Maximum.	Amplitude.
Annual	January, *b*	·29
Semi-annual.	March, *a* September, *a*	·11

The maximum epoch of the annual barometric period occurs in June, *b*, in the north of Scotland; May, *b*, in the south of Scotland; and March, *b*, over most of England.

France, &c.

26. C. W. C. FUCHS, 'Statistik der Erdbeben,' pp. 117-127.

Duration of record, 1865–1884. Number of earthquakes, 193.

The number of earthquakes is small, and, during November 26–29, 1873, there was a series of twenty-four shocks at Bagnères de Bigorre. On these accounts the results obtained are hardly satisfactory.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·41
Semi-annual	None	

27. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 11.

Duration of record, 5th to 19th centuries. Number of earthquakes, 656.

Perrey's region includes France, Belgium, and Holland, the limits of which, Mallet remarks, are fixed somewhat arbitrarily.

Period.	Maximum.	Amplitude.
Annual	January, *b*	·33
Semi-annual	January, *a* July, *a*	·10

The maximum epoch of the annual barometric period generally occurs in November, *b*, in France, and in March, *b*, in Belgium and Holland. Over a large part of France, however, the six-monthly means are irregular.

Spain and Portugal.

28. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 8.

Duration of record, 12th to 19th centuries. Number of earthquakes, 201.

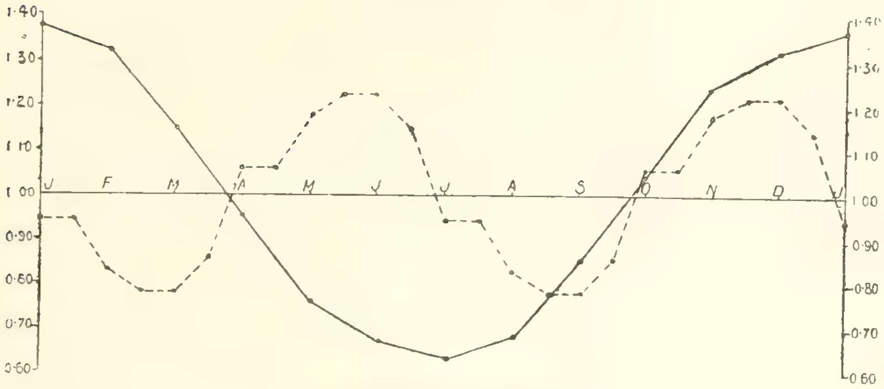
Period.	Maximum.	Amplitude.
Annual	December, *b*	·21
Semi-annual	Probably none	
Annual barometric . . .	November, *b*	

Austria.

29. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 13–37.

Duration of record, 1865–1884. Number of earthquakes, 461 (fig. 7).

Fig. 7.



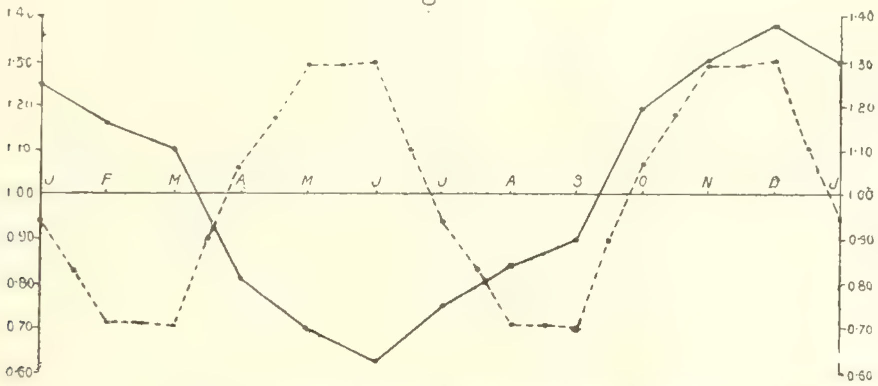
Austria (Fuchs).

Period.	Maximum.	Amplitude.
Annual	January, *b*	·37
Semi-annual	May, *b*—June, *a* November, *b*—December, *a*	·22
Annual barometric . . .	November, *b*	

Hungary, Croatia, and Transylvania.

30. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 47–65. Duration of record, 1865–1884. Number of earthquakes, 384 (fig. 8).

Fig. 8.



Hungary, Croatia, and Transylvania (Fuchs).

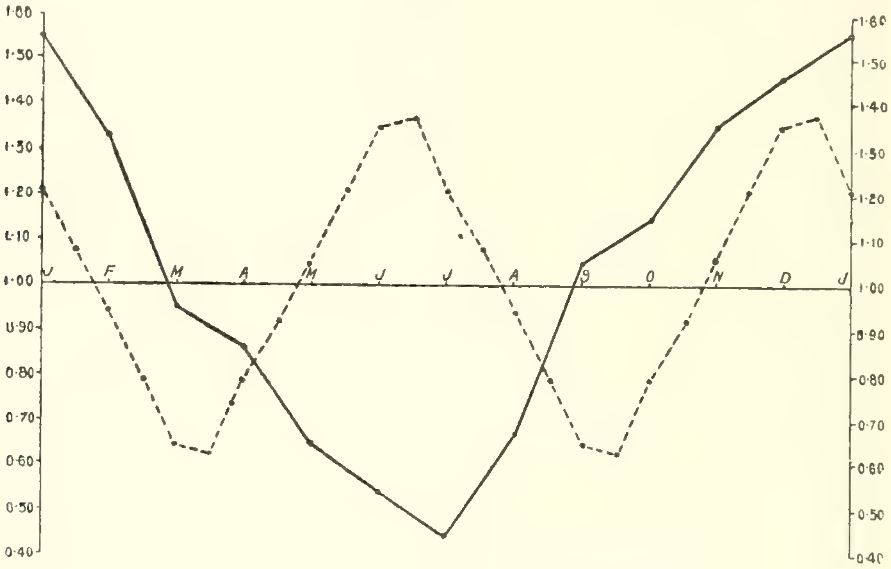
Period.	Maximum.	Amplitude.
Annual.	December, *b*	·31
Semi-annual	June, *a* December, *a*	·30
Annual barometric	November, *b*	

Switzerland and the Tyrol.

31. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 37–44, 99–117.

Duration of record: Switzerland, 1865–1884; Tyrol, 1865–1882. Number of earthquakes: Switzerland, 411; Tyrol, 113; total, 524 (fig. 9).

Fig. 9.



Switzerland and the Tyrol (Fuchs).

Period.	Maximum.	Amplitude.
Annual	January, *b*	·56
Semi-annual	June, *b* December, *b*	·37
Annual barometric . . .	November, *b*	

Basin of the Rhone.

32. A. Perrey, 'Brit. Assoc. Rep.,' 1848, p. 12.

Duration of record, 16th to 19th centuries. Number of earthquakes, 184.

Period.	Maximum.	Amplitude.
Annual	November, *b*	·46
Semi-annual	January, *a*—February, *a* July, *a*—August, *a*	·13

Basin of the Rhine and Switzerland.

33. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 13.
Duration of record, 9th to 19th centuries. Number of earthquakes, 529.

Period.	Maximum.	Amplitude.
Annual	January, *b*	·38
Semi-annual	June, *a* December, *a*	·15

Basin of the Danube.

34. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 14.
Duration of record, 5th to 19th centuries. Number of earthquakes, 268.

Period.	Maximum.	Amplitude.
Annual.	November, *b*	·14
Semi-annual	January, *a* July, *a*	·30

Italy.

35. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 16.
Duration of record, 5th to 19th centuries. Number of earthquakes, 984.

Perrey's district includes the whole of Italy, Sicily, Malta, and Sardinia, as well as the part of the Alpine region not included in Savoy or the basins of the Rhine, Rhone, and Danube.

Period.	Maximum.	Amplitude.
Annual.	March, *b*	·19
emi-annual	March, *a* September, *a*	·08

36. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 144–225.
Duration of record, 1865–1883. Number of earthquakes, 2350.

Fuchs classifies the Italian earthquakes under three headings: Italy without the district round Vesuvius, the district round Vesuvius, and Sicily. The results for the three separate regions are given below. The region here considered includes all three. The value of the result is affected by the large number of earthquakes felt in Sicily during March, 1883.

Period.	Maximum.	Amplitude.
Annual.	September, *b* and December, *b*	·14
Semi-annual	April, *b* October, *b*	·14
Annual barometric. . .	November, *b*	

37. *Italy, excluding Sicily and the district round Vesuvius.*—C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 144–197.

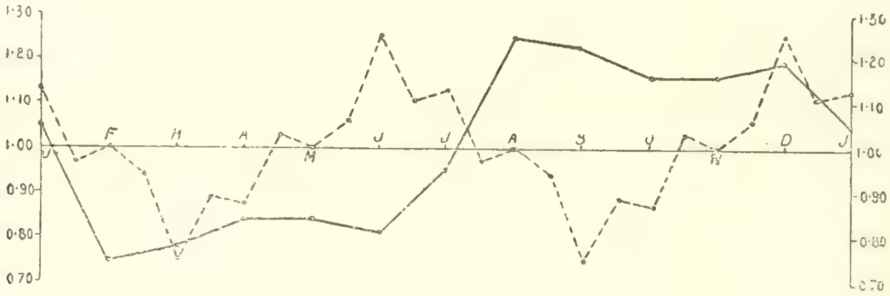
Duration of record, 1865–1883. Number of earthquakes, 1513.

Period.	Maximum.	Amplitude.
Annual	September, *b* and November, *b*	·21
Semi-annual	April, *a*—April, *b* October, *a*—October, *b*	·17

38. * *District round Vesuvius.*—C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 198–213.

Duration of record, 1865–1883. Number of earthquakes, 513.

Fig. 10.



District round Vesuvius (Fuchs).

Period.	Maximum.	Amplitude.
Annual	August, *b*	·25
Semi-annual	June, *a* December, *a*	·25
Annual barometric (Naples)	November, *b*	

39. *Sicily.*—C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 213–224.

Duration of record, 1865–1883. Number of earthquakes, 324.

The results, as pointed out above (§ 36), are not of great value, owing to the large number of shocks felt during March, 1883.*

Period.	Maximum.	Amplitude.
Annual	May, *b*	·67
Semi-annual	February, *b* August, *b*	·46
Annual barometric	November, *b*	

40. It may be interesting to compare with these results the observations made by P. T. Bertelli on the relative microseismic intensity at Florence. These are summarized for the interval December, 1872, to November, 1887, in his paper, "Delle variazioni dei valori d'Intensità Tromometrica relativa osservata nel Collegio alla Querce di Firenze dell' anno meteorico 1872–73 al 1886–87." Observations were made with two tromometers which are termed the "old tromometer" and the "normal tromometer"; with the first for the whole period of 15 years, and with the second for the last 11 years.

(1.) Old tromometer, December, 1872–November, 1887. 61,732 observations.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·49
Semi-annual	May, *a*—June, *a* November, *a*—December, *a*	·08

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* If we omit this year we get the following results, which agree more closely with those obtained from the Vesuvian district (the number of earthquakes being 242).

Period.	Maximum.	Amplitude.
Annual	July, *b*	·50
Semi-annual	June, *b* December, *b*	·19

(2.) Old tromometer, December, 1876–November, 1887. 38,546 observations.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·46
Semi-annual	May, *a* November, *a*	·04

(3.) Normal tromometer, December, 1876–November, 1887. 38,546 observations.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·49
Semi-annual	May, *a* November, *a*	·03

41. The results obtained above for Italy are somewhat discordant. We may conclude, however: (1) that the maximum of the annual seismic period occurs some time during the last three months of the year, and the maxima of the semi-annual period in April and October; the amplitudes of both periods being approximately equal; (2) that the maximum of the annual period in the two volcanic districts occurs during the summer months; and (3) that there is a marked annual microseismic period, with its maximum occurring at the end of December, the semi-annual microseismic period being comparatively very slight.

South-east of Europe, &c.

42. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 18.

Duration of record, 4th to 19th centuries. Number of earthquakes, 423.

Perrey's district includes the whole of the Balkan peninsula, Syria, the Ægean Islands, and the Levant. The seismic record is admittedly incomplete.

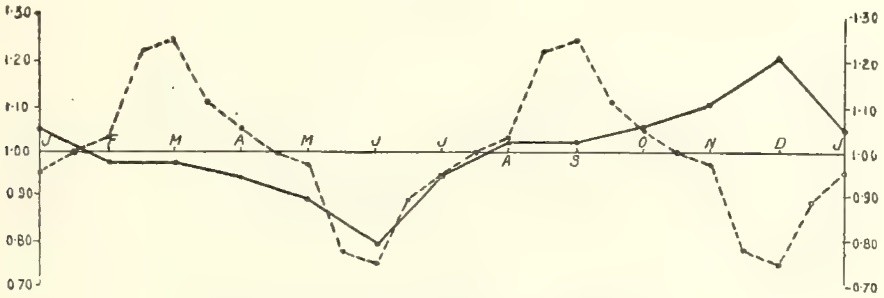
Period.	Maximum.	Amplitude.
Annual	Irregular	
Semi-annual	February, *a* August, *a*	·08

43. *J. F. J. Schmidt, 'Studien über Erdbeben' (Leipzig, 1879), pp. 183–355.

Duration of record, 1859–1877. Number of earthquakes, 3470 (fig. 11).

This district is also a wide one, including the Balkan peninsula and adjacent islands, and parts of Turkey in Asia.

Fig. 11.



South-east Europe, &c. (Schmidt).

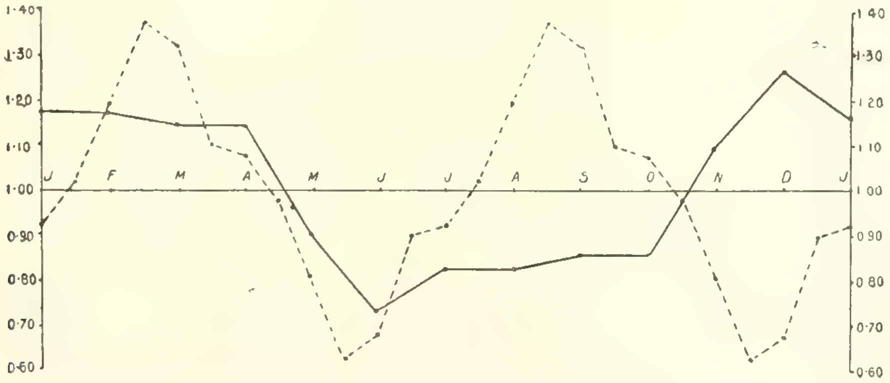
Period.	Maximum.	Amplitude.
Annual	December, *b*	·21
Semi-annual	March, *a* September, *a*	·25
Annual barometric . . .	November, *b* and December, *b*	

44. **Balkan Peninsula and adjacent Islands.*—C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 44–47, 225–251.

Duration of record, 1865–1884. Number of earthquakes, 624 (fig. 12).

This region contains Dalmatia as well as that included by Fuchs under the above heading.

Fig. 12.



Balkan peninsula, &c. (Fuchs).

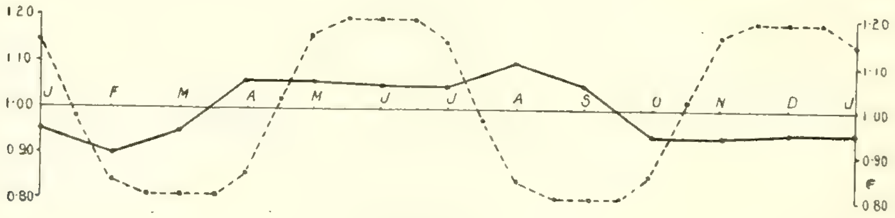
Period.	Maximum.	Amplitude.
Annual	December, *b*	·27
Semi-annual	February, *b* August, *b*	·37
Annual barometric . . .	November, *b*	

45. **Zante.*—D. G. and B. A. BARBIANI, "Mémoire sur les tremblements de terre dans l'île de Zante;" 'Mém. de l'Académie de Dijon,' vol. 11, 1863, pp. 1–112.

Duration of record, 1825–1863. Number of earthquakes, 1326 (fig. 13).

These figures show that, on an average, at least 34 earthquakes were felt every year. This number is, however, too small, for the number of shocks following the disastrous earthquake of October 30, 1840, was so great that M. BARBIANI found it impossible to record them all. According to the report issued by the Lord High Commissioner, Sir HOWARD DOUGLAS, 95 shocks, some very severe, were felt between October 30 and November 4. M. BARBIANI gives an estimate of the total number of shocks (p. 103), and in the next paragraph I have discussed the figures given by him.

Fig. 13.



Zante (BARBIANI).

Period.	Maximum.	Amplitude.
Annual	August, *b*	·10
Semi-annual	June, *a* December, *a*	·19

46. D. G. BARBIANI, p. 103.

Number of earthquakes, 1663.

M. BARBIANI's estimate of the monthly number of shocks differs from the preceding, owing to his including several slight shocks which he counted, without giving the time of occurrence of each, as well as to his estimate of the number of shocks in the early part of November, 1840.

Period.	Maximum.	Amplitude.
Annual	August, *b*	·29
Semi-annual	May, *a* November, *a*	·35

Thus, the effect of including these slighter shocks is to increase the amplitude of both periods, and to place the maximum epoch of the semi-annual period a month earlier.

47. *Zante.*—SCHMIDT and FUCHS.

Duration of record, 1859–1878. Number of earthquakes, 246.

The list on which this discussion is founded is compiled from those given by SCHMIDT and FUCHS for the south-east of Europe. The former extending from 1859 to 1878, the latter from 1865 to 1884, though not recording any shocks later than 1874. For the years 1859 to 1863, SCHMIDT'S records, with only seven exceptions, are all taken from BARBIANI'S catalogue. For the remainder of the time, SCHMIDT and FUCHS do not seem to have made use of the same authorities, for their dates agree in only a comparatively small number of cases.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·29
Semi-annual	March, *a* September, *a*	·33

Now, the average number of earthquakes felt in Zante every year is 34·8 or 43·8 according to BARBIANI, and 8·2 according to the list compiled from SCHMIDT'S and FUCHS' catalogues. M. BARBIANI'S catalogue may perhaps relate to a more than usually disturbed interval, but it was compiled by a resident in the island, and therefore probably includes a much greater number of slight shocks than the other list. This may account for the difference in the maximum epoch of the annual period (see § 17).

Algeria.

48. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 317–321.
Duration of record, 1865–1883. Number of earthquakes, 135.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·67
Semi-annual	June, *a*—June, *b* December, *a*—December, *b*	·30
Annual barometric . . .	November, *b*, and December, *b*	

Asia.

49. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 254–276.
Duration of record, 1865–1884. Number of earthquakes, 458.

The region considered includes Caucasia, which is treated separately (§ 50), and India, but excludes Japan, Formosa, and the Malay Archipelago. The record is probably incomplete.

Period.	Maximum.	Amplitude.
Annual	February, *b*	·33
Semi-annual	February, *a* August, *a*	·14
Annual barometric . .	December, *b*	

Caucasia.

50. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 257–263.
Duration of record, 1865–1884. Number of earthquakes, 152.

Period.	Maximum.	Amplitude.
Annual	January, *b*	·56
Semi-annual	February, *a* August, *a*	·38
Annual barometric . .	December, *b*	

Japan.

51. 'Trans. of Seismol. Soc. of Japan,' vol. 10, 1887, p. 61; vol. 13, Part 1, 1889, p. 93; vol. 15, 1890, p. 101; vol. 16, 1892, pp. 56 and 83.

Duration of record, 1885–1889. Number of earthquakes, 2997.

This record has already been discussed in § 17, with reference to seismic intensity. The early epoch of the annual period is due to the very large number of shocks disturbing an area less than 100 square ri.

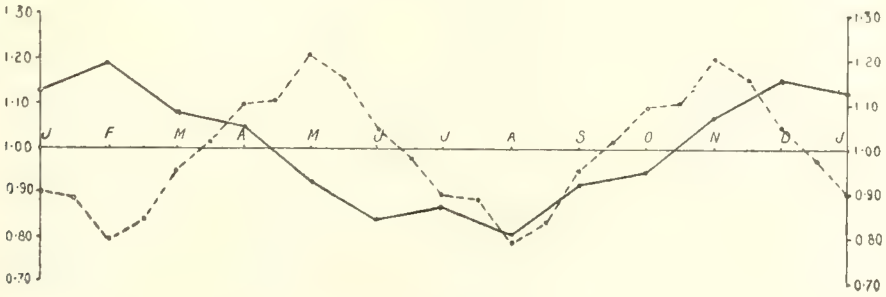
Period.	Maximum.	Amplitude.
Annual	October, *b*	·08
Semi-annual	March, *a* September, *a*	·07
Annual barometric . .	December, *b*	

52. **Tokio.*—Professor J. Milne, 'Trans. of Seismol. Soc. of Japan,' vol. 2, 1880, pp. 4–14, 39; vol. 6, 1883, pp. 32–35; vol. 8, 1885, pp. 100–108; vol. 10, 1887, pp. 97–99; vol. 15, 1890, pp. 127–134. Also (for the earthquakes recorded by the Gray-Milne seismograph) 'Brit. Assoc. Rep.,' 1886, pp. 414–415; 1887, pp. 212–213; 1888, pp. 435–437; 1889, pp. 295–296; 1890, pp. 160–162; 1891, pp. 123–124; 1892, pp. 93–95.

Duration of record, 1876–1881 and 1883–1891. Number of earthquakes, 1104 (fig. 14).

Professor Milne's record of earthquakes felt in Tokio, begins in September, 1872. From the end of 1875 until the present time, the earthquakes were all registered by instrumental means. Palmèri's seismograph was used until April, 1885, and the Gray-Milne seismograph after that date. At the end of the year 1882, observations were suspended during the removal of the instrument; this year is accordingly omitted.

Fig. 14.



Tokio (Milne).

Period.	Maximum	Amplitude.
Annual	February, *b*	·19
Semi-annual	May, *a* November, *a*	·21
Annual barometric (Tokio) .	December, *b*	

Omitting the year 1885, during which the change of seismographs was made, we have the following results, differing but slightly from the preceding. (Number of earthquakes, 1039.)

Period.	Maximum.	Amplitude.
Annual	February, *b*	·19
Semi-annual	May, *a* November, *a*	·22

53. *Tokio.*—Professor J. Milne, 'Trans. of Seismol. Soc. of Japan,' vol. 2, 1880, pp. 7–14, 39.

Duration of record, 1878–1881. Number of earthquakes, 246.

The results for this shorter interval are given for comparison with those from the next district.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·46
Semi-annual	January, *b* July, *b*	·19

54. *Yokohama.*—T. H. Streets, M.D. (U.S. Navy), "A Four Years' Record of Earthquakes in Japan, studied in their Relation to the Weather and Seasons": 'Amer. Journ. of Science', 3rd ser., vol. 25, 1883, pp. 361–367.

Duration of record, 1878–1881. Number of earthquakes, 130.

Dr. Streets' list is compiled from the weather statistics kept at the United States Naval Hospital at Yokohama, a station for taking international meteorological observations. It "includes every shock that could be appreciated without the aid of a seismometer." The results are interesting as showing the fairly close agreement in lists kept at places separated by a few miles, one with, and the other without, the aid of a seismograph; but they possess little value otherwise

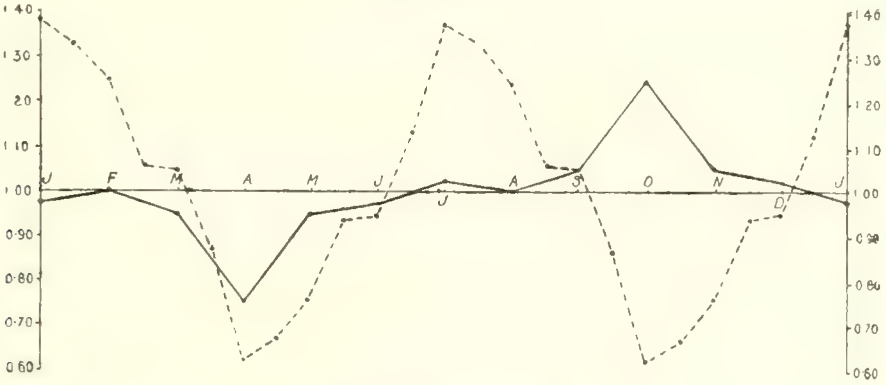
Period.	Maximum.	Amplitude.
Annual	December, *b* and March, *b*	·41
Semi-annual	January, *b* July, *b*	·29

India.

55. *T. Oldham, "A Catalogue of Indian Earthquakes from the Earliest Time to the End of A.D. 1869"; 'India Geol. Survey Mem.,' vol. 19, part 3, pp. 1–48.

Duration of record, 1803–1869. Number of earthquakes, 320 (fig. 15.)

Fig. 15.



India (Oldham).

Period.	Maximum.	Amplitude.
Annual	October, *b*	·25
Semi-annual	January, *a* July, *a*	·38

North America.

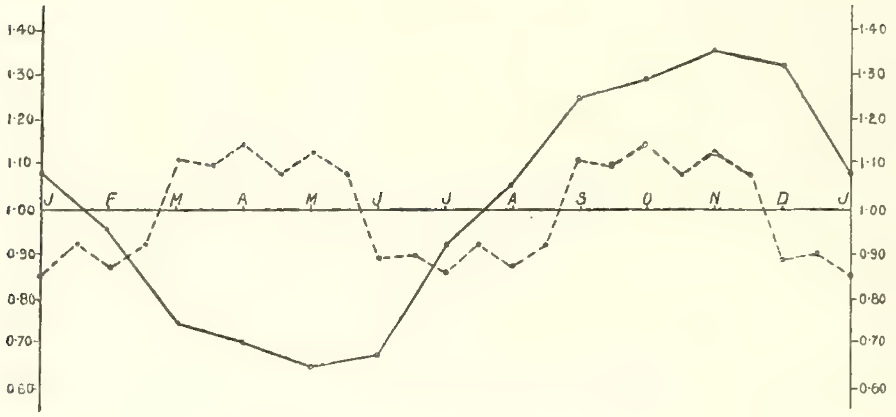
56. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 377–401.

Duration of record, 1865–1884. Number of earthquakes, 552 (fig. 16).

This district includes the whole of the continent north of Mexico.

Period.	Maximum.	Amplitude.
Annual	November, *b*	·35
Semi-annual	April, *a* October, *a*	·14
Annual barometric . .	November, *b* and December, *b*	

Fig. 16.



North America (Fuchs).

United States and Canada.

57. A. Perrey, 'Brit. Assoc. Rep.,' 1858, p. 23.

Duration of record, 17th to 19th centuries. Number of earthquakes, 134.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·46
Semi-annual	January, *a* July, *a*	·15
Annual barometric . . .	November, *b* and December, *b*	

New England.

58. W. T. Brigham, "Volcanic Manifestations in New England, being an Enumeration of the Principal Earthquakes from 1638 to 1869"; 'Boston Soc. Nat. Hist. Mem.,' vol. 2, 1873, pp. 1–28.

Duration of record, 1638–1869. Number of earthquakes, 212.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·51
Semi-annual	June, *b* December, *b*	·25
Annual barometric . . .	November, *b*	

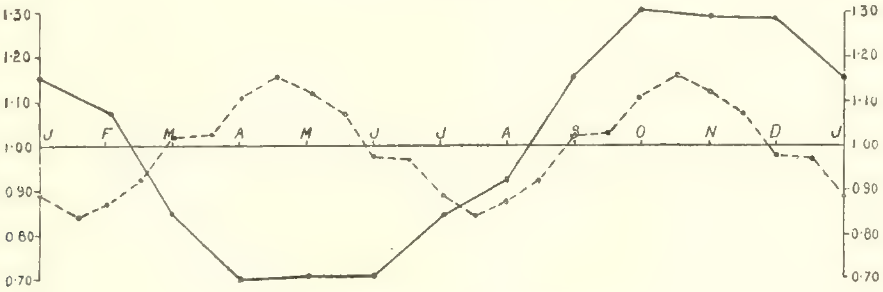
California, &c.

59. *E. S. Holden, "List of Recorded Earthquakes in California, Lower California, Oregon, and Washington Territory" (Sacramento, 1887).

Duration of record, 1850–1886. Number of earthquakes, 949 (fig. 17).

The six-monthly means for October, November, and December being very nearly equal, it is possible that the epoch of the maximum of the annual period may be placed too early.

Fig. 17.



California, &c. (Holden).

Period.	Maximum.	Amplitude.
Annual	October, *b*	·30
Semi-annual	April, *b* October, *a*	·16
Annual barometric . . .	December, *b* and January, *b*	

60. In Table A. of his memoir (p. 11) Professor Holden has given the number of earthquakes for each year grouped in monthly intervals. Using these figures, we have the following results (the number of earthquakes being 768):—

Period.	Maximum.	Amplitude.
Annual	October, *b*	·19
Semi-annual	April, *a* October, *a*	·16

Thus, while the maximum epochs are practically the same, as also the amplitude of the semi-annual period, the amplitude of the annual period increases with the number of earthquakes, that is, by including a greater number of slight shocks.

61. *San Francisco.*—E. S. HOLDEN, p. 13, Table B.
Duration of record, 1850–1886. Number of earthquakes, 254.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·41
Semi-annual	April, *a* October, *a*	·21
Annual barometric (San Francisco)	January, *b*	

62. *San José and Santa Clara.*—E. S. HOLDEN, p. 15, Table C.
Duration of record, 1850–1886. Number of earthquakes, 54.

Professor HOLDEN remarks that the record for these two places is very incomplete; he thinks that probably, if the record had been well kept, the number would have been as great as at San Francisco. As it is, it is too small to yield reliable results.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·56
Semi-annual	March, *a* September, *a*	·33

Mexico.

63. C. W. C. FUCHS, 'Statistik der Erdbeben,' pp. 372–376.
Duration of record, 1865–1882 Number of earthquakes, 86.
The annual period, if one exists, is very irregular.

Period.	Maximum.	Amplitude.
Annual	December, *b*	·43
Semi-annual	April, *a* October, *a*	·79

The maximum of the annual barometric period occurs in January at places near the sea-level, and in September and October on the high ground which occupies so large a part of the country.

Central America.

64. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 357-363.
Duration of record, 1865–1884. Number of earthquakes, 190.

Period.	Maximum.	Amplitude.
Annual	April, *b*	·32
Semi-annual	April, *a* October, *a*	·25
Annual barometric	January, *b*—April, *b*	

West Indies.

65. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 364-371.
Duration of record, 1865-1883. Number of earthquakes, 205.

In November, 1867, a large number of earthquakes were felt at St. Thomas, more than four hundred between the 19th and 28th insts. Except in the few cases where the exact times are given, the shocks of each day are counted as a single earthquake.

Period.	Maximum.	Amplitude.
Annual	October, *b*	·59
Semi-annual	June, *b* December, *b*	·59
Annual barometric	January, *b*—April, *b*	

66. A. Perrey, 'Brit. Assoc. Rep.,' 1858, pp. 24, 25.
Duration of record, 16th to 19th centuries. Number of earthquakes, 221.

Mallet remarks that Perrey was obliged to adopt some convention with regard to some of the prolonged earthquake-series. He reckoned "each month of such shocks as equivalent to one great earthquake."

Period.	Maximum.	Amplitude.
Annual	August, *b*	·11
Semi-annual	March, *a* September, *a*	·16

Sandwich Islands.

67. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 404–408.
Duration of record, 1865–1882. Number of earthquakes, 245.

Period.	Maximum.	Amplitude.
Annual	June, *b*	·33
Semi-annual	February, *b* August, *b*	·25

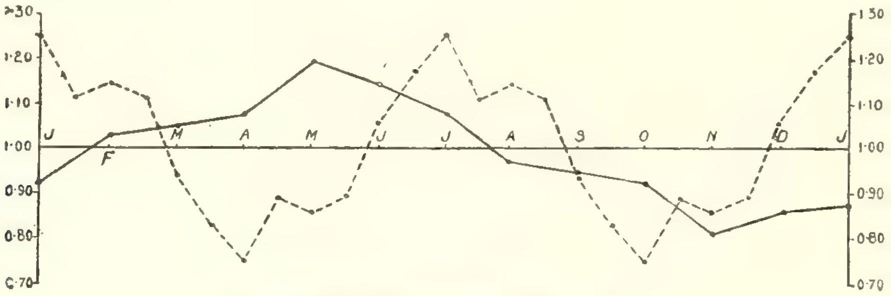
Buchan gives the results of two series of barometric observations at Honolulu. The first series gives the maximum epoch of the annual period in May, *b*; the second in April, *b*; the amplitudes, or semi-ranges, being ·025 and ·033 inch, respectively.

Seismic Periodicity in Countries near the Equator.

Malay Archipelago.

68. *C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 286–308.
Duration of record, 1865–1884. Number of earthquakes, 598.

Fig. 18.



Malay Archipelago (Fuchs).

Period.	Maximum.	Amplitude.
Annual	May, *b*	·19
Semi-annual	January, *a* July, *a*	·25

The maximum of the annual barometric period occurs in March, *b*, at Buitenzorg; in August, *b*, at Batavia; and in September, *b*, at Padang.

69. C. G. KNOTT, 'Trans. of Seismol. Soc. of Japan,' vol. 9, pt. 1, 1886, Table A.*
Duration of record, 1873–1881. Number of earthquakes, 515.

Period.	Maximum.	Amplitude.
Annual	Irregular	
Semi-annual	March, *a* September, *a*	·11 but irregular

New Granada and Venezuela.

70. C. W. C. FUCHS, 'Statistik der Erdbeben,' pp. 347–356.
Duration of record, 1865–1884. Number of earthquakes, 272.

Period.	Maximum.	Amplitude.
Annual	February, *b*	·64
Semi-annual	March, *a*—March, *b* September, *a*—September, *b*	·27

The maximum epoch of the annual barometric period is very variable in different parts of the district, occurring as early as April in some parts and as late as September in others. The range is, however, very small.

SEISMIC PERIODICITY IN THE SOUTHERN HEMISPHERE.

Southern Hemisphere.

71. R. MALLET, 'Brit. Assoc. Rep.,' 1858, p. 57. (See also reference and note given in § 18.)
Duration of record, 1578–1850. Number of earthquakes, 223.

Period.	Maximum.	Amplitude.
Annual	November, *b*	·24
Semi-annual	May, *a* November, *a*	·33

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* Dr. KNOTT quotes from a list by BERGSMA. The reference he gives is 'Natuurkundig Tijdschrift voor Nederlandsch-Indie.' I have not had an opportunity of seeing this work.

72. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 312–317, 322–347, 356–357, 401–404.

Duration of record, 1865–1884. Number of earthquakes, 751.

The semi-annual period is not well-defined.

Period.	Maximum.	Amplitude.
Annual	August, *b*	·37
Semi-annual	January, *a*—March, *a* July, *a*—September, *a*	·06
Annual barometric . . .	April, *b*—July, *b*	

New South Wales, Victoria, and South Australia.

73. G. Hogben, "Report of the Committee . . . appointed to investigate and report upon Seismological Phenomena in Australasia," 'Trans. Austral. Assoc. for the Adv. of Science,' vol. 4, 1892, pp. 8–27.

Duration of records: New South Wales, 1880–1891; Victoria, 1884–1891; South Australia, 1882–1891. Number of earthquakes: New South Wales, 24; Victoria, 60; South Australia, 75; total, 159.

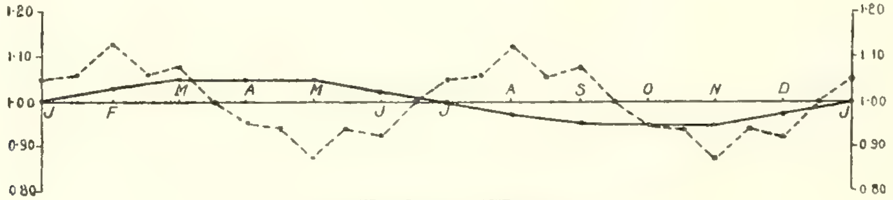
Period.	Maximum.	Amplitude.
Annual	May, *b*	·48
Semi-annual	February, *a* August, *a*	·43
Annual barometric . . .	May, *b* and June, *b*	

New Zealand.

74. *Sir J. Hector, "Report of the Committee . . . appointed to investigate and report upon Seismological Phenomena in Australasia," 'Trans. Austral. Assoc. for the Adv. of Science,' vol. 3, 1891, pp. 505–532.

Duration of record, 1868–1890. Number of earthquakes, 641 (fig. 19).

Fig. 19.



New Zealand (Hector).

Period.	Maximum.	Amplitude.
Annual	March, *b*—May, *b*	·05
Semi-annual	February, *a* August, *a*	·13
Annual barometric	April, *b*	

75. In discussing the geographical distribution of the New Zealand earthquakes, Sir J. HECTOR has classified the earthquakes in monthly groups in six districts. The following are the results obtained from the numbers given by him :—

District I. North part of the North Island, including Hokianga, Kaipara, and Cambridge. Number of earthquakes, 2.

District II. Central part of the North Island, including Tauranga, Taupo, Taranaki, and Manawatu. Number of earthquakes, 184.

Period.	Maximum.	Amplitude.
Annual	April, *b*	·22
Semi-annual	March, *a* September, *a*	·30

District III. East Coast of the North Island, including Gisborne, Napier, and Wellington. Number of earthquakes, 188.

Period.	Maximum.	Amplitude.
Annual	May, *b*	·11
Semi-annual	February, *a* August, *a*	·26

District IV. West Coast of the South Island, including Marlborough, Nelson, and Grey. Number of earthquakes, 88.

Period.	Maximum.	Amplitude.
Annual	July, *b*	·32
Semi-annual	March, *a* September, *a*	·20

District V. East Coast of the South Island, including Bealey, Christchurch, South Canterbury, Dunedin, and Southland. Number of earthquakes, 98.

Period.	Maximum.	Amplitude.
Annual	Irregular	
Semi-annual	March, *a* September, *a*	·28

District VI. South part of the South Island, including Central Otago. Number of earthquakes, 30.

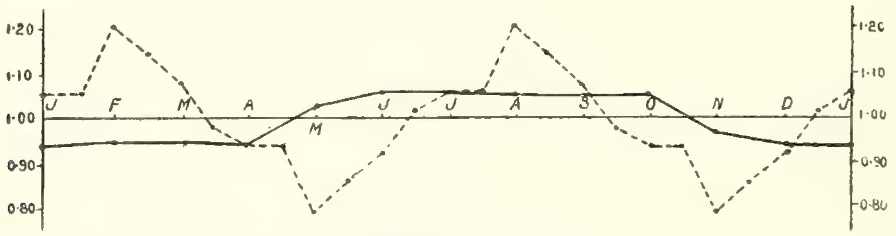
The number of earthquakes is far too small to yield trustworthy results.

Period.	Maximum.	Amplitude.
Annual	Irregular	
Semi-annual	May, *a* November, *a*	·56

76. *G. Hogben, "The Earthquakes of New Zealand." 'Trans. of the Austral. Assoc. for the Adv. of Science,' vol. 3, 1891, pp. 37–57; also, "Report of the Committee . . . appointed to investigate and report upon Seismological Phenomena in Australasia," vol. 4, 1892, pp. 28–33.

Duration of record, 1848–1891. Number of earthquakes, 737 (fig. 20).

Fig. 20.



New Zealand (Hogben).

Period.	Maximum.	Amplitude.
Annual	Between June, *b* and October, *b*	·06
Semi-annual	February, *a* August, *a*	·21

77. G. Hogben, *ibid.*, vol. 3, 1891, p. 44.

Duration of record, 1848–1890. Number of earthquakes, 745.

Counting slighter shocks, which would be reckoned together with the definition of the unit earthquake adopted in this paper, Mr. Hogben has given a table of the number of shocks felt in each month. The total number for the same years, obtained by myself, is 695. The result is to increase the amplitudes of both periods.

Period.	Maximum.	Amplitude.
Annual	July, *b*	·14
Semi-annual	February, *a* August, *a*	·23

Chili.

78. A. Perrey, 'Brit. Assoc. Rep.', 1858, p. 27.

Duration of record, 16th to 19th centuries. Number of earthquakes, 178.

Perrey's district consists of Chili and La Plata, or, in Mallet's words, "the region lying between the western slope of the Andes and the sea, from the 24° to the 45° south latitude, between the Desert of Atacama on the north, and the Archipelago of Chonos on the south." The number of earthquakes recorded by Perrey is small, considering that, as Mallet remarks, the region is one "in which shocks of greater or less intensity are almost of daily occurrence."

Period.	Maximum.	Amplitude.
Annual	August, *b*	·14
Semi-annual	Irregular	

79. C. W. C. Fuchs, 'Statistik der Erdbeben,' pp. 322–331.

Duration of record, 1865–1883. Number of earthquakes, 316.

For the first seven years (1865–1871) Fuchs' record is a full one. During the remainder of the time it is evidently very incomplete, there being five years (1875, 1877, 1878, 1881, 1882), without any disturbance recorded. This deprives the list of much of its value for statistical purposes.

The figures for both periods are very irregular. The maximum six-monthly mean (1·27) occurs at the end of August, but at the end of December it is 1·25, with much smaller values intervening. In like manner for the semi-annual period, the maximum six-half-monthly mean (1·11) occurs in the middle of April and October, but at the end of May and November it is 1·10.

80. C. G. KNOTT, 'Trans. of Seismol. Soc. of Japan,' vol. 9, pt. 1, 1886, Table A.*
Duration of record, 1873–1881. Number of earthquakes, 212.

Period.	Maximum.	Amplitude.
Annual	August, *b*	·48
Semi-annual	June, *a* December, *a*	·17
Annual barometric . .	July, *b*	

Peru, Bolivia, and Quito.

81. C. W. C. FUCHS, 'Statistik der Erdbeben,' pp. 331–347.
Duration of record, 1865–1884. Number of earthquakes, 350.

Period.	Maximum.	Amplitude.
Annual	July, *b*	·48
Semi-annual	January, *a* July, *a*	·24
Annual barometric . .	July, *b*	

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* Dr. KNOTT quotes VERGARA as his authority, and gives as the reference 'Observaciones Meteorologicas' (Santiago), 1884. I have not been able to consult this work.

Table I.—Number of Earthquakes.

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Slight earthquakes	1785–1842	Mallet	8 14	7 7	7 5	6 9	4 9	8 7	12 12	11 9	10 6	6 7	6 6	6 5
Destructive earthquakes . . .	528–1842	,,	33 32	29 29	22 33	36 28	21 26	38 11	16 28	24 28	32 17	26 31	21 31	22 27
Japanese earthquakes disturbing an area greater than 1000 sq. ri	1885–1889	Imp. Meteor. Bureau	15	21	11	21	16	13	11	10	17	13	16	12
Japanese earthquakes disturbing an area less than 1000 sq. ri and greater than 100 sq. ri	,,	,,	48	54	38	42	77	53	42	41	36	39	40	55
Japanese earthquakes disturbing an area less than 100 sq. ri	,,	,,	154	200	176	150	205	127	158	286	197	192	215	196
Northern hemisphere	223–1850	Mallet	627	539	503	489	438	428	415	488	463	516	473	500
* ,, ,, ...	1865–1884	Fuchs	342 409	344 338	403 405	286 269	345 285	242 256	269 321	237 244	252 318	343 422	487 505	437 374
*Europe	,,	,,	222 285	243 199	269 290	166 170	171 181	158 184	177 229	153 162	169 222	243 297	388 369	303 249
Europe and adjacent parts of Asia and Africa	306–1843	Perrey	228	189	172	147	126	131	148	147	147	176	148	202
Scandinavia and Iceland . . .	12th to 19th centuries	,,	33	20	21	13	16	10	17	13	18	17	19	17
British Islands and Northern Isles	11th to 19th centuries	,,	21	16	19	16	16	10	9	19	24	17	22	28
Great Britain	1608–1838	D. Milne	7 12	15 11	6 8	4 8	6 4	7 5	5 2	6 11	11 13	10 9	12 16	6 12
,, ,, ...	1739–1888	Roper	20 16	12 16	5 26	9 16	5 7	11 14	5 6	13 12	13 12	11 13	9 19	13 14

TABLE I.—Number of Earthquakes (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
France	1865–1884	FUCHS	2 13	9 5	12 9	8 3	1 10	1 7	3 8	9 3	11 8	9 5	6 31	7 13
France, Belgium, and Holland	5th to 19th centuries	PERREY	83	64	53	55	42	36	47	40	50	48	60	78
Spain and Portugal	12th to 19th centuries	,,	25	14	16	18	9	14	18	16	12	23	22	14
*Austria	1865–1884	FUCHS	19 34	28 13	32 10	13 22	24 25	15 14	13 12	5 12	12 15	11 22	22 32	28 28
*Hungary, Croatia, and Transylvania	,,	,,	21 21	17 11	21 15	7 9	23 17	12 28	2 6	6 10	14 13	16 15	26 24	37 13
*Switzerland and the Tyrol	,,	,,	26 36	26 24	24 22	13 24	13 18	11 15	21 26	12 8	12 13	11 11	26 42	45 45
Basin of the Rhone	16th to 19th centuries	PERREY	26	20	16	10	11	11	9	9	19	15	14	24
Basin of the Rhine and Switzerland	9th to 19th centuries	,,	62	54	44	37	36	30	35	30	36	36	58	71
Basin of the Danube	5th to 19th centuries	,,	31	31	14	16	23	19	26	25	16	23	18	26
Italy	,,	,,	101	99	98	84	80	86	63	77	63	92	64	77
,,	1865–1883	FUCHS	93 89	75 67	103 152	75 79	73 78	89 89	93 138	81 78	80 119	141 118	121 115	121 83
Italy, without the district round Vesuvius and Sicily	,,	,,	49 69	54 37	74 63	57 60	45 49	44 50	74 71	36 40	60 78	103 95	78 79	91 57
*District round Vesuvius	,,	,,	32 14	16 22	23 17	9 9	20 7	30 19	14 45	20 20	5 30	25 17	41 32	25 21
Sicily	,,	,,	12 6	5 8	6 72	9 10	8 22	15 20	5 22	25 18	15 11	13 6	2 4	5 5

TABLE I.—Number of Earthquakes (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
South-east of Europe, &c. . . .	4th to 19th centuries	PERREY	40	35	31	30	37	35	35	40	40	34	33	33
* ,, ,, . . .	1859–1877	SCHMIDT	88 174	233 133	165 173	161 113	119 92	119 118	91 125	201 144	107 124	231 220	170 114	110 141
*Balkan Peninsula and adjacent islands	1865–1884	FUCHS	16 36	56 34	50 41	34 17	18 31	19 17	23 28	30 17	15 34	43 32	25 22	23 20
*Zante	1825–1863	BARBIANI	37 40	54 47	41 56	43 44	53 63	72 88	59 62	61 52	44 41	41 52	72 58	75 71
,, ...	,,	BARBIANI'S monthly figures	80	107	105	89	131	176	141	131	91	165	279	168
,, ...	1859–1878	SCHMIDT and FUCHS	3 3	19 13	14 15	10 12	12 8	13 6	4 9	2 11	7 9	22 16	9 9	7 17
Algeria	1865–1883	FUCHS	25 11	11 5	5 5	5 2	10 1	2 2	6 2	1 2	4 2	6 6	3 16	3 0
Asia	1865–1884	,,	27 25	9 42	17 26	18 29	25 15	10 14	17 15	22 14	16 20	9 10	18 17	30 13
Caucasia	,,	,,	4 15	0 28	2 12	5 14	6 2	2 4	7 4	6 2	6 8	1 2	2 8	10 2
Japan	1885–1889	Imp. Met. Bur. of Japan	217	275	225	213	298	193	211	337	250	244	271	263
*Tokio	1876–1881 and 1883–1891	J. MILNE	33 52	41 57	53 43	37 59	61 61	42 35	32 41	34 31	34 28	45 68	39 59	52 67

TABLE I.—Number of Earthquakes (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Tokio	1876–1881 1883–1884 and 1886–1891	J. MILNE	30 48	35 54	51 37	36 56	59 60	38 33	32 41	34 28	31 23	39 64	38 57	48 64
,,	1878–1881	,,	10 21	14 16	20 9	7 7	13 7	12 6	5 13	4 8	4 2	7 10	13 10	11 17
Yokohama	,,	STREETS	3 13	8 10	12 8	4 5	6 3	5 4	4 8	4 3	1 1	3 8	6 3	2 6
*India	1803–1869	OLDHAM	27 18	15 9	6 14	10 13	7 9	13 12	13 11	14 35	11 10	13 14	6 8	9 23
*North America	1865–1884	FUCHS	26 23	27 13	22 21	23 19	14 17	12 13	25 16	14 19	21 26	36 52	28 23	19 43
United States and Canada	17th to 19th centuries	PERREY	14	14	12	6	6	4	10	14	8	10	19	17
New England	1638–1869	BRIGHAM	13 15	18 11	8 8	3 5	3 11	10 3	11 4	8 5	8 2	8 11	17 10	11 9
*California, &c.	1850–1886	HOLDEN	49 52	38 16	29 51	48 34	28 33	28 19	39 18	30 33	38 41	55 66	43 40	54 67
,,	,,	HOLDEN'S monthly figures	68	45	66	71	56	51	45	53	85	88	57	83
San Francisco	,,	HOLDEN	25	22	26	15	17	17	13	11	21	35	30	22
San José and Santa Clara	,,	,,	2	7	5	3	2	1	4	4	4	9	8	5
Mexico	1865–1882	FUCHS	4 1	2 4	5 6	1 3	13 7	1 1	0 1	1 2	0 1	1 23	4 1	2 2

TABLE I.—Number of Earthquakes (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Central America	1865–1884	FUCHS	6 5	8 7	15 13	14 7	6 8	13 6	8 9	5 5	9 9	8 3	9 9	1 7
West Indies	1865-1883	,,	5 17	5 7	3 7	9 1	2 5	10 4	6 5	7 9	6 9	1 5	6 30	39 7
,, ...	16th to 19th centuries	PERREY	15	16	23	17	16	16	20	23	22	20	18	15
Sandwich Islands	1865–1882	FUCHS	6 7	8 14	6 11	20 6	13 20	11 6	3 15	16 13	16 10	5 7	7 11	9 5
*Malay Archipelago	1865–1884	,,	28 27	23 24	21 32	20 23	18 36	36 21	20 45	30 42	15 26	21 13	15 16	27 29
,, ,, ...	1873–1881	BERGSMA	47	41	51	39	31	52	36	52	36	54	36	40
New Granada and Venezuela . .	1865–1884	FUCHS	6 8	13 20	38 13	14 22	10 15	9 9	7 5	5 5	6 9	9 4	9 5	17 14
Southern hemisphere	1578–1850	MALLET	19	14	9	17	20	19	18	12	17	25	32	21
,, ,, ...	1865–1884	FUCHS	35 25	24 23	23 35	23 23	22 20	40 45	33 43	41 47	27 23	48 53	31 33	18 16
New South Wales, Victoria, and South Australia	1880–1891	HOGBEN	3 3	3 6	9 17	5 3	6 4	9 7	14 10	13 7	8 8	6 6	3 3	4 2
*New Zealand	1868–1890	HECTOR	29 23	36 25	34 29	19 30	31 21	31 22	20 33	33 29	27 28	18 27	22 24	28 22
,, ,, District II. . . .	,,	,,	12	19	26	17	10	12	20	14	13	19	11	11

TABLE I.—Number of Earthquakes (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
New Zealand, District III. . .	1868–1890	HECTOR	19	15	19	16	11	17	18	20	18	6	14	15
,, District IV. . .	,,	,,	5	5	7	7	8	5	10	10	10	10	4	7
,, ,, District V. . . .	,,	,,	8	10	9	7	7	10	3	11	9	11	5	8
,, ,, District VI. . .	,,	,,	3	1	0	6	2	2	0	5	1	3	5	2
*New Zealand	1848–1891	HOGBEN	31 38	28 27	32 37	23 27	29 29	28 25	30 35	42 41	29 46	25 26	24 28	33 24
,, ,, ...	1848–1890	HOGBEN'S monthly figures.	67	52	70	47	56	67	67	79	88	51	50	51
Chili	16th to 19th centuries	PERREY	15	12	16	8	21	12	16	16	16	10	27	9
,, ...	1865–1883	FUCHS	15 13	16 12	12 17	4 12	11 8	14 14	11 16	13 11	10 9	15 28	26 18	4 7
,, ...	1873–1881	VERGARA	13	6	7	13	18	20	26	25	23	20	24	17
Peru, Bolivia, and Quito . . .	1865–1884	FUCHS	15 11	5 10	10 15	16 9	10 11	17 27	19 27	21 30	15 13	23 15	3 9	13 6

TABLE II.—Annual Seismic Period.

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Slight earthquakes	1785–1842	MALLET	0·89	0·89	0·97	1·00	1·08	1·16	1·11	1·11	1·03	1·00	0·92	0·84
Destructive earthquakes . . .	528–1842	"	1·13	1·10	1·10	1·00	0·94	0·92	0·87	0·90	0·90	1·00	1·06	1·08
Japanese earthquakes, disturbing an area greater than 1000 sq. ri	1885–1889	Imp. Met. Bur. of Japan	1·16	1·16	1·17	1·11	0·87	0·98	0·84	0·84	0·83	0·89	1·13	1·02
Japanese earthquakes, disturbing an area less than 1000 sq. ri and greater than 100 sq. ri	"	"	0·98	1·17	1·17	1·14	1·05	1·03	1·02	0·83	0·83	0·86	0·95	0·97
Japanese earthquakes, disturbing an area less than 100 sq. ri	"	"	0·97	0·95	0·86	0·86	0·94	0·98	1·03	1·05	1·14	1·14	1·06	1·02
Northern hemisphere	223–1850	MALLET	1·11	1·08	1·05	0·94	0·89	0·89	0·89	0·92	0·95	1·06	1·11	1·11
* " "	1865–1884	FUCHS	1·22	1·06	0·95	0·89	0·79	0·71	0·78	0·94	1·05	1·11	1·21	1·29
*Europe	"	"	1·24	1·00	0·89	0·83	0·73	0·65	0·76	1·00	1·11	1·17	1·27	1·35
Europe, and adjacent parts of Asia and Africa	306–1843	PERREY	1·19	1·14	1·03	0·90	0·81	0·78	0·83	0·86	0·97	1·10	1·19	1·22
Scandinavia and Iceland . . .	12th to 19th centuries	"	1·25	1·19	1·10	0·87	0·73	0·70	0·75	0·81	0·90	1·14	1·27	1·30
British Islands and Northern Isles	11th to 19th centuries	"	1·21	1·11	0·86	0·68	0·70	0·79	0·79	0·89	1·14	1·32	1·30	1·21
Great Britain	1608–1838	D. MILNE	1·22	0·94	0·86	0·68	0·51	0·67	0·78	1·06	1·14	1·32	1·49	1·33
" "	1739–1888	ROPER	1·29	1·11	1·10	0·84	0·78	0·73	0·71	0·89	0·90	1·16	1·22	1·27
France	1865–1884	FUCHS	1·37	0·92	0·73	0·67	0·62	0·59	0·63	1·08	1·27	1·33	1·38	1·41
France, Belgium, and Holland .	5th to 19th centuries	PERREY	1·33	1·24	1·03	0·87	0·73	0·72	0·67	0·76	0·97	1·13	1·27	1·29

TABLE II.—Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Spain and Portugal	12th to 19th centuries	PERREY	1·14	0·94	0·94	0·83	0·84	0·79	0·86	1·06	1·06	1·17	1·16	1·21
*Austria	1865–1884	FUCHS	1·37	1·32	1·14	0·95	0·76	0·67	0·63	0·68	0·86	1·05	1·24	1·33
*Hungary, Croatia, and Transylvania	,,	,,	1·25	1·16	1·10	0·81	0·70	0·62	0·75	0·84	0·90	1·19	1·30	1·38
*Switzerland and the Tyrol . .	,,	,,	1·56	1·33	0·95	0·86	0·65	0·54	0·44	0·67	1·05	1·14	1·35	1·46
Basin of the Rhone	16th to 19th centuries	PERREY	1·32	1·25	1·05	0·76	0·54	0·60	0·68	0·75	0·95	1·24	1·46	1·40
Basin of the Rhine and Switzerland	9th to 19th centuries	,,	1·38	1·24	1·00	0·84	0·67	0·63	0·62	0·76	1·00	1·16	1·33	1·37
Basin of the Danube	5th to 19th centuries	,,	1·05	1·10	1·02	0·97	0·86	0·89	0·95	0·90	0·98	1·05	1·14	1·11
Italy	,,	,,	1·11	1·16	1·19	1·08	0·97	0·87	0·88	0·84	0·81	0·92	1·03	1·10
,, ...	1865–1883	FUCHS	1·00	0·89	0·86	0·92	0·92	0·86	1·00	1·11	1·14	1·08	1·08	1·14
Italy, without the district round Vesuvius and Sicily	,,	,,	1·03	0·89	0·79	0·84	0·79	0·81	0·97	1·11	1·21	1·16	1·21	1·19
*District round Vesuvius . . .	,,	,,	1·05	0·75	0·78	0·84	0·84	0·81	0·95	1·25	1·22	1·16	1·16	1·19
Sicily	,,	,,	0·81	1·05	1·30	1·38	1·67	1·19	1·19	0·95	0·70	0·62	0·33	0·81
Florence, microseismic intensity (old tromometer)	Dec. 1872—Nov. 1887	BERTELLI	1·48	1·35	1·06	0·83	0·63	0·51	0·52	0·65	0·94	1·17	1·37	1·49
Florence, microseismic intensity (old tromometer)	Dec. 1876—Nov. 1887	,,	1·44	1·30	1·05	0·79	0·63	0·54	0·56	0·70	0·95	1·21	1·37	1·46
Florence, microseismic intensity (normal tromometer)	,,	,,	1·43	1·27	0·98	0·73	0·59	0·51	0·57	0·73	1·02	1·27	1·41	1·49
South-east of Europe, &c. . .	4th to 19th centuries	PERREY	0·94	0·97	0·98	0·95	0·97	1·03	1·06	1·03	1·02	1·05	1·03	0·97

TABLE II.—Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
*South-east of Europe, &c.	1859–1877	SCHMIDT	1·05	0·98	0·98	0·94	0·89	0·79	0·95	1·02	1·02	1·06	1·11	1·21
*Balkan Peninsula and adjacent islands	1865–1884	FUCHS	1·17	1·17	1·14	1·14	0·90	0·73	0·83	0·83	0·86	0·86	1·10	1·27
*Zante	1825–1863	BARBIANI	0·95	0·90	0·95	1·06	1·06	1·05	1·05	1·10	1·05	0·94	0·94	0·95
,,	,,	BARBIANI'S monthly numbers	1·02	0·71	0·73	0·86	0·86	0·84	0·98	1·29	1·27	1·14	1·14	1·16
,,	1859–1878	SCHMIDT and FUCHS	1·10	1·11	1·06	1·14	0·87	0·71	0·90	0·89	0·94	0·86	1·13	1·29
Algeria	1865–1883	FUCHS	1·57	1·38	1·40	0·75	0·41	0·33	0·43	0·62	0·60	1·25	1·59	1·67
Asia	1865–1884	,,	1·30	1·33	1·21	1·06	0·94	0·90	0·70	0·67	0·79	0·94	1·06	1·10
Caucasia	,,	,,	1·56	1·51	1·38	1·22	0·78	0·79	0·44	0·49	0·62	0·78	1·22	1·21
Japan	1885–1889	Imp. Met. Bur. of Japan	0·98	1·00	0·94	0·92	0·97	1·00	1·02	1·00	1·06	1·08	1·03	1·00
*Tokio	1876–1881 and 1883–1891	J. MILNE	1·13	1·19	1·08	1·05	0·92	0·84	0·87	0·81	0·92	0·95	1·08	1·16
,,	1876–1881 1883–1884 and 1886–1891	,,	1·13	1·19	1·08	1·06	0·95	0·86	0·87	0·81	0·92	0·94	1·05	1·14
,,	1878–1881	,,	1·43	1·38	1·27	1·10	0·84	0·54	0·57	0·62	0·73	0·90	1·16	1·46
Yokohama	,,	STREETS	1·38	1·38	1·41	1·32	1·02	0·59	0·62	0·62	0·59	0·68	0·98	1·41
*India	1803–1869	OLDHAM	0·98	1·00	0·95	0·75	0·95	0·98	1·02	1·00	1·05	1·25	1·05	1·02

Table II.—Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
*North America	1865–1884	Fuchs	1·08	0·95	0·75	0·71	0·65	0·68	0·92	1·05	1·25	1·29	1·35	1·32
United States and Canada . .	17th to 19th centuries	Perrey	1·36	1·05	0·75	0·67	0·64	0·54	0·64	0·95	1·25	1·33	1·36	1·46
New England	1638–1869	Brigham	1·35	1·14	1·05	0·86	0·59	0·49	0·65	0·86	0·95	1·14	1·41	1·51
*California, &c.	1850–1886	Holden	1·16	1·08	0·84	0·70	0·71	0·71	0·84	0·92	1·16	1·30	1·29	1·29
,, ...	,,	Holden's monthly figures	1·03	1·02	0·90	0·81	0·83	0·90	0·97	0·98	1·10	1·19	1·17	1·10
San Francisco	,,	Holden	1·17	1·00	0·95	0·79	0·63	0·59	0·83	1·00	1·05	1·21	1·37	1·41
San José and Santa Clara . . .	,,	,,	1·21	0·83	0·60	0·71	0·51	0·44	0·79	1·17	1·40	1·29	1·49	1·56
Mexico	1865–1882	Fuchs	0·73	1·29	1·21	1·06	0·94	0·57	1·27	0·71	0·79	0·94	1·06	1·43
Central America	1865–1884	,,	1·11	1·03	1·22	1·32	1·22	1·06	0·89	0·97	0·78	0·68	0·78	0·94
West Indies	1865–1883	,,	1·52	1·06	0·57	0·41	0·46	0·54	0·48	0·94	1·43	1·59	1·54	1·46
,, ,, ...	16th to 19th centuries	Perrey	0·92	0·89	0·90	0·97	1·05	1·05	1·08	1·11	1·10	1·03	0·95	0·95
Sandwich Islands	1865–1882	Fuchs	0·86	1·03	1·08	1·14	1·21	1·33	1·14	0·97	0·92	0·86	0·79	0·67
*Malay Archipelago	1865–1884	,,	0·92	1·03	1·05	1·08	1·19	1·14	1·08	0·97	0·95	0·92	0·81	0·86
,, ,, ...	1873–1881	Bergsma	0·98	0·95	1·03	0·97	1·02	0·92	1·02	1·05	0·97	1·03	0·98	0·92
New Granada and Venezuela .	1865–1884	Fuchs	1·51	1·64	1·49	1·48	1·17	0·76	0·49	0·36	0·51	0·52	0·83	1·24
Southern hemisphere	1578–1850	Mallet	1·02	0·84	0·83	0·81	0·76	0·87	0·98	1·16	1·17	1·19	1·24	1·13

TABLE II.—Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Southern hemisphere	1865–1884	Fuchs	0·73	0·63	0·86	0·92	1·06	1·05	1·27	1·37	1·14	1·08	0·94	0·95
New South Wales, Victoria, and South Australia	1880–1891	Hogben	0·63	0·71	0·92	1·27	1·48	1·29	1·37	1·29	1·08	0·73	0·52	0·71
*New Zealand	1868–1890	Hector	1·00	1·03	1·05	1·05	1·05	1·02	1·00	0·97	0·95	0·95	0·95	0·98
,, ,, District II. . .	,,	,,	1·08	1·06	1·08	1·22	1·11	0·89	0·92	0·94	0·92	0·78	0·89	1·11
,, ,, ,, III. . .	,,	,,	1·08	1·02	1·06	1·05	1·11	1·10	0·92	0·98	0·94	0·95	0·89	0·90
,, ,, ,, IV. . .	,,	,,	0·68	0·83	0·76	0·94	1·10	1·22	1·32	1·17	1·24	1·06	0·90	0·78
,, ,, ,, V. . .	,,	,,	0·95	1·00	1·08	0·92	0·92	0·94	1·05	1·00	0·92	1·08	1·08	1·06
,, ,, ,, VI. . .	,,	,,	1·22	0·90	0·90	0·59	1·00	1·11	0·78	1·10	1·10	1·41	1·00	0·89
*New Zealand	1848–1891	Hogben	0·94	0·95	0·95	0·94	1·03	1·06	1·06	1·05	1·05	1·06	0·97	0·94
,, ,, ...	1848–1890	Hogben's monthly numbers	0·86	0·87	0·95	0·95	1·05	1·13	1·14	1·13	1·05	1·05	0·95	0·87
Chili	16th to 19th centuries	Perrey	0·98	0·86	0·92	0·94	0·98	1·00	1·02	1·14	1·08	1·06	1·02	1·00
,, ...	1865–1883	Fuchs	1·00	0·73	0·92	0·90	0·84	0·75	1·00	1·27	1·08	1·10	1·16	1·25
,, ...	1873–1881	Vergara	0·62	0·52	0·57	0·76	1·03	1·29	1·38	1·48	1·43	1·24	0·97	0·71
Peru, Bolivia, and Quito . . .	1865–1884	Fuchs	0·52	0·60	0·84	1·02	1·32	1·37	1·48	1·40	1·16	0·98	0·68	0·63

Table III.—Semi-Annual Seismic Period.

District.	Duration of record.	Catalogue.	Jan.–July.		Feb.–Aug.		Mar.–Sept.		April.–Oct.		May.–Nov.		June.–Dec.	
			a.	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*
Slight earthquakes	1785–1842	Mallet	1·21	1·27	1·25	1·11	0·95	0·81	0·79	0·73	0·75	0·89	1·05	1·19
Destructive earthquakes	528–1842	,,	0·98	0·95	1·02	1·08	1·08	1·02	1·02	1·05	0·98	0·92	0·92	0·98
Japanese earthquakes disturbing an area greater than 1000 sq. ri	1885–1889	Imp. Met. Bur. of Japan	0·90		0·95		1·12		1·10		1·05		0·88	
Japanese earthquakes disturbing an area less than 1000 sq. ri and greater than 100 sq. ri	,,	,,	1·07		0·87		0·84		0·93		1·13		1·16	
Japanese earthquakes disturbing an area less than 100 sq. ri	,,	,,	0·98		1·07		1·12		1·02		0·93		0·88	
Northern hemisphere	223–1850	Mallet	1·03		1·07		1·05		0·97		0·93		0·95	
* ,, ,, ...	1865–1884	Fuchs	0·90	0·89	0·94	0·94	0·94	1·02	1·10	1·11	1·06	1·06	1·06	0·98
*Europe	,,	,,	0·89	0·89	0·92	0·94	0·92	1·00	1·11	1·11	1·08	1·06	1·08	1·00
Europe and adjacent parts of Asia and Africa	306–1843	Perrey	1·12		1·10		1·02		0·88		0·90		0·98	
Scandinavia and Iceland	12th to 19th centuries	,,	1·05		1·22		0·94		0·95		0·78		1·06	
British Islands and Northern Isles	11th to 19th centuries	,,	0·92		1·00		1·05		1·08		1·00		0·95	
Great Britain	1608–1838	D. Milne	0·94	1·00	1·06	1·10	1·16	1·10	1·06	1·00	0·94	0·90	0·84	0·90
,, ...	1739–1888	Roper	1·05	0·98	1·08	1·03	1·11	0·98	0·95	1·02	0·92	0·97	0·89	1·02
France	1865–1884	Fuchs	0·73	0·97	0·94	1·13	0·94	0·73	1·27	1·03	1·06	0·87	1·06	1·27
France, Belgium, and Holland . .	5th to 19th centuries	Perrey	1·10		1·05		0·94		0·90		0·95		1·06	
Spain and Portugal	12th to 19th centuries	,,	1·02		1·00		0·98		0·98		1·00		1·02	
*Austria	1865–1884	Fuchs	0·94	0·94	0·83	0·78	0·78	0·86	1·06	1·06	1·17	1·22	1·22	1·14

TABLE III.—Semi-Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.–July.		Feb.–Aug.		Mar.-Sept.		April–Oct.		May–Nov.		June–Dec.	
			a.	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*
*Hungary, Croatia, and Transylvania	1865–1884	FUCHS	0·94	0·83	0·71	0·71	0·70	0·90	1·06	1·17	1·29	1·29	1·30	1·10
*Switzerland and the Tyrol . . .	,,	,,	1·21	1·08	0·94	0·79	0·65	0·63	0·79	0·92	1·06	1·21	1·35	1·37
Basin of the Rhone	16th to 19th centuries	PERREY	1·13		1·13		0·95		0·87		0·87		1·03	
Basin of the Rhine and Switzerland	9th to 19th centuries	,,	1·12		0·98		0·85		0·88		1·02		1·15	
Basin of the Danube	5th to 19th centuries	,,	1·30		1·12		0·90		0·70		0·88		1·10	
Italy	,,	,,	1·05		1·03		1·08		0·95		0·97		0·92	
,, ...	1865–1883	FUCHS	0·90	0·86	0·98	1·03	1·00	1·05	1·10	1·14	1·02	0·97	1·00	0·95
Italy without the district round Vesuvius and Sicily	,,	,,	0·83	0·83	0·89	0·97	1·02	1·08	1·17	1·17	1·11	1·03	0·98	0·92
*District round Vesuvius	,,	,,	1·13	0·97	1·00	0·94	0·75	0·89	0·87	1·03	1·00	1·06	1·25	1·11
Sicily	,,	,,	0·86	0·87	1·41	1·46	1·35	1·14	1·14	1·13	0·59	0·54	0·65	0·86
Florence, microseismic intensity (old tromometer)	Dec. 1872—Nov. 1887	BERTELLI	0·98		0·97		1·00		1·02		1·03		1·00	
,, ,, ,,	Dec. 1876—Nov. 1887	,,	0·97		0·96		1·00		1·03		1·04		1·00	
Florence (normal tromometer) . .	,,	,,	1·00		0·92		0·92		1·00		1·08		1·08	
South-east of Europe, &c. . . .	4th to 19th centuries	PERREY	1·05		1·08		1·00		0·95		0·92		1·00	
* ,, ,, ,, . . .	1859–1877	SCHMIDT	0·95	1·00	1·03	1·22	1·25	1·11	1·05	1·00	0·97	0·78	0·75	0·89
*Balkan Peninsula and adjacent islands	1865–1884	FUCHS	0·92	1·02	1·19	1·37	1·32	1·10	1·08	0·98	0·81	0·73	0·68	0·90
*Zante	1825–1863	BARBIANI	1·14	0·98	0·84	0·81	0·81	0·81	0·86	1·02	1·16	1·19	1·19	1·19

TABLE III.—Semi-Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.–July.		Feb.–Aug.		Mar.–Sept.		April–Oct.		May–Nov.		June–Dec.	
			a.	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*
Zante	1825–1863	BARBIANI'S monthly numbers	0·95		0·65		0·72		1·05		1·35		1·28	
,,	1859–1878	SCHMIDT and FUCHS	0·78	0·79	0·81	1·13	1·33	1·32	1·22	1·21	1·19	0·87	0·67	0·68
Algeria	1865–1883	FUCHS	1·06	1·16	1·27	0·81	0·70	0·70	0·94	0·84	0·73	1·19	1·30	1·30
Asia	1865–1884	,,	1·06	1·02	1·14	1·03	1·03	1·11	0·94	0·98	0·86	0·97	0·97	0·89
Caucasia	,,	,,	1·17	1·10	1·38	1·27	1·24	1·27	0·83	0·90	0·62	0·73	0·76	0·73
Japan	1885–1889	Imp. Met. Bur. of Japan	1·00		1·02		1·07		1·00		0·98		0·93	
*Tokio	1876–1881 and 1883–1891	J. MILNE	0·90	0·89	0·79	0·84	0·95	1·02	1·10	1·11	1·21	1·16	1·05	0·98
,,	1876–1881 1883–1884 and 1886–1891	,,	0·90	0·89	0·78	0·83	0·92	1·00	1·10	1·11	1·22	1·17	1·08	1·00
,,	1878–1881	,,	1·17	1·19	1·05	1·03	0·83	0·92	0·83	0·81	0·95	0·97	1·17	1·08
Yokohama	,,	STREETS	1·13	1·29	1·25	1·25	1·08	1·06	0·87	0·71	0·75	0·75	0·92	0·94
*India	1803–1869	OLDHAM	1·38	1·33	1·24	1·06	1·05	0·87	0·62	0·67	0·76	0·94	0·95	1·13
*North America	1865–1884	FUCHS	0·86	0·92	0·87	0·92	1·11	1·10	1·14	1·08	1·13	1·08	0·89	0·90
United States and Canada	17th to 19th centuries	PERREY	1·15		1·13		0·93		0·85		0·87		1·07	
New England	1638–1869	BRIGHAM	1·19	1·11	1·10	0·89	0·86	0·75	0·81	0·89	0·90	1·11	1·14	1·25
*California, &c.	1850–1886	HOLDEN	0·89	0·84	0·87	0·92	1·02	1·03	1·11	1·16	1·13	1·08	0·90	0·97

TABLE III.—Semi-Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.–July.		Feb.–Aug.		Mar.–Sept.		April–Oct.		May–Nov.		June–Dec.	
			a.	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*
California, &c.	1850–1886	HOLDEN'S monthly numbers	0·84		0·92		1·12		1·16		1·08		0·88	
San Francisco	,,	,,	0·79		0·88		1·05		1·21		1·12		0·95	
San José and Santa Clara	,,	,,	0·77		0·97		1·33		1·23		1·03		0·67	
Mexico	1865–1882	FUCHS	0·21	0·28	0·43	0·35	1·21	1·73	1·79	1·72	1·57	1·65	0·79	0·27
Central America	1865–1884	,,	0·75	0·92	1·06	1·21	1·14	1·17	1·25	1·08	0·94	0·79	0·86	0·83
West Indies	1865–1883	,,	1·29	0·65	0·73	0·71	0·49	0·41	0·71	1·35	1·27	1·29	1·51	1·59
,,	16th to 19th centuries	PERREY	0·93		1·13		1·16		1·08		0·87		0·84	
Sandwich Islands	1865–1882	FUCHS	0·89	0·90	1·03	1·25	1·14	1·08	1·11	1·10	0·97	0·75	0·86	0·92
*Malay Archipelago	1865–1884	,,	1·25	1·11	1·14	1·11	0·94	0·83	0·75	0·89	0·86	0·89	1·06	1·17
,, ,,	1873–1881	BERGSMA	1·06		1·03		1·11		0·94		0·97		0·89	
New Granada and Venezuela	1865–1884	FUCHS	0·79	1·02	1·00	1·11	1·27	1·27	1·21	0·98	1·00	0·89	0·73	0·73
Southern hemisphere	1578–1850	MALLET	0·87		0·67		0·75		1·13		1·33		1·25	
,, ,,	1865–1884	FUCHS	1·06	1·03	1·02	1·03	1·06	1·00	0·94	0·97	0·98	0·97	0·94	1·00
New South Wales, Victoria, and South Australia	1880–1891	HOGBEN	1·03	1·13	1·43	1·30	1·24	1·08	0·97	0·87	0·57	0·70	0·76	0·92
*New Zealand	1868–1890	HECTOR	1·05	1·06	1·13	1·06	1·08	1·00	0·95	0·94	0·87	0·94	0·92	1·00
,, District II.	,,	,,	0·93		1·21		1·30		1·07		0·79		0·70	
,, ,, III.	,,	,,	1·18		1·26		1·02		0·82		0·74		0·98	

TABLE III.—Semi-Annual Seismic Period (continued).

District.	Duration of record.	Catalogue.	Jan.-July.		Feb.-Aug.		Mar.-Sept.		April-Oct.		May-Nov.		June-Dec.	
			a.	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*	*a.*	*b.*
*New Zealand, District IV. . . .	1868–1890	HECTOR	0·92		1·12		1·20		1·08		0·88		0·80	
,, ,, V. . . .	,,	,,	1·05		1·05		1·28		0·95		0·95		0·72	
,, ,, VI. . . .	,,	,,	0·67		0·44		1·12		1·23		1·56		0·88	
* ,, ...	1848–1891	HOGBEN	1·06	1·06	1·21	1·14	1·08	0·98	0·94	0·94	0·79	0·86	0·92	1·02
,, ...	1848–1890	HOGBEN'S monthly mumbers	1·05		1·23		1·08		0·95		0·77		0·92	
Chili.	16th to 19th centuries	PERREY	0·84		1·03		0·80		1·16		0·95		1·20	
,, ...	1865–1883	FUCHS	0·89	0·94	0·97	0·90	1·02	1·08	1·11	1·06	1·03	1·10	0·98	0·92
,, ...	1873–1881	VERGARA	1·02		0·90		0·83		0·98		1·10		1·17	
Peru, Bolivia, and Quito	1865–1884	FUCHS	1·24	1·19	1·14	1·19	1·06	0·94	0·76	0·81	0·86	0·81	0·94	1·06

Summary of Results.

82. *Annual Period.*—The number of seismic records examined is 62, 45 belonging to the northern hemisphere, 14 to the southern, and 3 to equatorial countries. Only five fail to indicate a fairly well-marked annual period, namely, South-east Europe (Perrey), East Indies (Bergsma), New Zealand, Districts V. and VI. (Hector), and Chili (Fuchs); and in all these cases the failure may, I think, be attributed to incompleteness in the seismic record. Different lists for the same district do not always give the same date for the maximum epoch, though in most cases they agree closely enough. The principal exceptions are Italy, Zante, Japan, West Indies, and the Southern Hemisphere; and in the three first of these districts the discrepancy may, perhaps, be explained. Excluding these somewhat doubtful cases, the distribution in time of the maximum epochs is given in the following table:—

	Northern hemisphere.	Equatorial countries.	Southern hemisphere.
January	6	0	0
February	1	1	0
March	0	0	0
April	1	0	2
May	1	1	2
June	1	0	0
July	0	0	3
August	1	0	2
September	0	0	0
October	3	0	0
November	4	0	0
December	16	0	0

These figures strongly support the view that *the maximum of the annual period occurs during the local winter in each hemisphere.* The evidence for equatorial countries is somewhat scanty, but it shows that in them, also, there is a well-marked annual period. If we were to include the two districts of India and Peru, Bolivia and Quito as equatorial, this conclusion will be further strengthened, the date of the maximum epoch being variable.

83. In amplitude, the results given by different lists for the same districts are more variable; and this is due in a great measure, I believe, to different definitions of the unit earthquake. This is especially shown in three cases in which the periodicity has been investigated both with my own definition and with that adopted by the cataloguer (§§ 45, 46, 59, 60, 76, 77).

The amplitude of the annual period ranges from ·05 (New Zealand) to ·67 (Sicily and Algeria). The average amplitude obtained from 57 records is ·33; the corresponding ratio of the number of shocks in the maximum and mininum halves of the year is 1·53, or, roughly, 3 : 2.

84. It will be seen, on examining the above tables, that the amplitude of the annual period is small in several countries which are generally regarded as earthquake countries. In some cases, this is no doubt due to the definition of the unit earthquake. Again, when the area is a wide one, the amplitude may be small, owing to the maximum epoch for different districts occurring at different dates. Partly to this, perhaps, is due the apparently small amplitude in the Northern Hemisphere (Mallet), Italy (Fuchs), and New Zealand (Hector and Hogben). Lastly, the amplitude is small in some insular districts, for example, Japan (·08) and New Zealand (·05 or ·06).

85. *Semi-Annual Period.*—Of the 62 records examined, only three fail to show a fairly well-marked semi-annual period, namely, France (Fuchs), Spain and Portugal (Perrey), and Chili (Fuchs); these failures also being probably due to incompleteness in the seismic record. Different records for the same district do not always agree in giving the same epoch for the maximum of the semi-annual period; for example, those of the Northern and Southern Hemispheres, Europe and the West Indies. As a rule, however, the agreement is fairly close. And this is also especially true in the case of districts adjoining and overlapping one another; for example, Austria, Hungary, and Switzerland and the Tyrol (Fuchs); South-east Europe (Perrey, Schmidt) and the Balkan Peninsula (Fuchs); California, San Francisco, and San José and Santa Clara (Holden); India (Oldham) and the Malay Archipelago (Fuchs); the different districts of New Zealand (Hector); and also Tokio (Milne) and Yokohama (Streets), for the short period 1878–1881. Excluding doubtful cases, the distribution in time of the maximum epochs is given in the following table:—

	Northern hemisphere.	Equatorial countries.	Southern hemisphere.
January, *a*, and July, *a* . . .	4	1	1
January, *b*, and July, *b* . . .	0	0	0
February, *a*, and August, *a* . .	4	0	4
February, *b*, and August *b* . .	3	0	0
March, *a*, and September, *a* . .	5	1	3
March, *b*, and September, *b* . .	0		0
April, *a*, and October, *a* . . .	5	0	0
April, *b*, and October, *b* . . .	2	0	0
May, *a*, and November, *a* . . .	0	0	1
May, *b*, and November, *b* . . .	0	0	0
June, *a*, and December, *a* . .	3	0	1
June, *b*, and December *b* . . .	2	0	0

In New Zealand and the south-east of Australia, the maximum epoch falls most frequently in February or March, and August or September; in North America it usually occurs in March or April and September or October. I have not been able to find any general rule in other cases, and it is possible that the materials are too imperfect to allow any law to be deduced.

86. The amplitude of the semi-annual period ranges from ·06 (Southern Hemisphere, FUCHS) to ·79 (Mexico), the average value being ·24.

In fifteen cases the amplitude of the semi-annual period is greater than that of the annual period, and in three others the amplitudes are equal; the amplitude of the annual period in the former cases ranging from ·05 to ·43, on an average being ·19. It is worthy of notice that different catalogues for the same district generally agree in this respect; especially is this the case with the Balkan Peninsula, Zante, and the south-east of Europe, and in New Zealand.

Moreover, eleven of these fifteen records include the following insular districts, which are among the most well-marked seismic regions in the world, namely, the Grecian Archipelago, Japan, the Malay Archipelago, New Zealand, and the West Indies. The average amplitude of the annual period in these cases is ·16, and that of the semi-annual period ·24; that is, the average amplitude of the annual period is just half that of all the districts examined, while, in the case of the semi-annual period, the two average amplitudes are exactly the same.*

ORIGIN OF THE ANNUAL SEISMIC PERIOD.

87. The most probable theory of the origin of the majority of non-volcanic earthquakes is that they are due to the impulsive friction occasioned by the slips which are the elements in the growth of faults. These slips are due, in the first instance, to endogenic causes, and, were there no external phenomena of a periodic nature to affect them, would take place equally at all times of the year.

In nearly every part of the world, however, there is a well-marked annual barometric period, the maximum epoch over the land-areas being somewhat variable, but occurring as a rule during winter. The range, or double amplitude, is small, the maximum being about one inch, and in some districts less than one-tenth of an inch.

The distortion of the earth's surface by parallel waves of barometric elevation and depression has been investigated by Professor G. H. DARWIN.† The vertical displacement is

$$\frac{gwh}{2v} b \left(1 + \frac{x}{b}\right) e^{-x/b} \cos \frac{z}{b},$$

where the axis of x is vertically downwards and the axis of z horizontal and perpendicular to the barometric undulations, g gravity, v the modulus of rigidity, w the

* This phenomenon shows that the annual and semi-annual periods must have entirely different origins.

† "On Variations in the Vertical due to Elasticity of the Earth's Surface," 'Brit. Assoc. Report,' 1882, pp. 106–119; 'Phil. Mag.,' vol. 14, 1882, pp. 409–427.

specific gravity of mercury, h the maximum range of the barometer, and $2\pi b$ the wave-length of the barometric undulation. To obtain an estimate of the maximum displacement at the surface, namely, $gwhb/2v$, Professor DARWIN takes $w = 13{\cdot}6$, $h = 5$ centims., $2\pi b = 5000$ miles $= 4{\cdot}8 \times 10^8$ centims., nearly, and $v/g = 3 \times 10^8$, which is equivalent to assuming that the superficial layers of the earth are more rigid than the most rigid glass. With these data he finds the maximum displacement of the surface to be 4·5 centims., so that the ground is 9 centims. higher under the barometric depression than under the elevation.

The ratio of the vertical displacement at depth x to that at the surface is

$$(1 + x/b)\, e^{-x/b},$$

or

$$1 - x^2/2b^2 + x^3/3b^3.$$

Thus, the displacement at first diminishes very slowly as the depth increases, and is independent of the modulus of rigidity and the barometric range; being, with the above value of b, only one-fifth per cent. less at a depth of fifty miles than it is at the surface.

Now, since most or all earthquakes probably originate at a depth much less than this, and since the fault-slip which produces even a moderately strong shock must be very small, for even in the most violent earthquakes it is rarely perceptible at the surface; also, since the work to be done is, not the compression of solid rock, but the slight depression of a fractured mass of rock whose support is very nearly, but not quite, withdrawn; on these accounts it seems possible that the annual variation in barometric pressure may be competent to produce the annual variation in seismic frequency.

It is not to be expected that the amplitude of the annual seismic period in different districts should be proportional to that of the annual barometric period; but if there is a connection between the two periods the epoch of the seismic maximum should coincide with, or follow closely after, that of the barometric maximum.

88. Over the greater part of the northern hemisphere, the annual barometric maximum occurs as a rule either in November or December, except at lofty stations or in high latitudes. In Europe the maximum occurs nearly everywhere in November, except along the western border in latitudes north of about 45°, over which the Atlantic conditions prevail for some distance inland. In Asia the maximum is nearly always in December. In North America it occurs in November and December, and along part of the western coast in January.

In the seismic districts of the Southern Hemisphere, the maximum occurs as a rule in the corresponding months:—in New Zealand in April, in the south-east of Australia in May or June, in South America in June or July, with some few exceptions in each case, depending either on the altitude or neighbourhood to the coast.

89. Comparisons between the dates of the maximum epochs of the seismic and barometric annual periods can be made in thirty-one of the districts treated in this paper. The epoch of the seismic maximum approximately coincides with that of the barometric maximum in ten districts, and follows it by about one month in nine, and by about two months in four districts. In four others (Japan, Tokio, India, California), the seismic epoch precedes the barometric by about two months ; but, in the first two of these at least, this may be due to the inclusion of slight shocks having their maximum in summer. To the same cause, perhaps, may be due the four exceptional cases (Scandinavia and Iceland, Great Britain, the district round and including Vesuvius, and Sicily), though many of the Vesuvian and Sicilian earthquakes are obviously of volcanic origin, and are probably more frequent with a low barometric pressure. Thus, as a general rule, the epoch of the seismic maximum either coincides with that of the barometric maximum, or follow it by a month or two.

If the epoch of the barometric or seismic maximum were constant in either hemisphere, this general agreement might not possess much significance, but it continues when the epochs vary. For example, in the southern hemisphere, we have the following dates :—

District.	Barometric maximum.	Seismic maximum.
South-east Australia . .	May and June	May
New Zealand	April	March–May
Chili	July	August
Peru, Bolivia, and Quito .	July	July

Again, while in both North and South America the maximum epochs of both periods fall in the winter months, in the Sandwich Islands (lat. 20° N) the epoch of the seismic maximum occurs in June, and that of the barometric maximum at Honolulu in April or May.

90. Lastly, supposing the view here suggested to be correct, the annual variation in barometric pressure cannot have any influence on seismic frequency when the earthquakes originate entirely beneath the sea ; for, as the barometric pressure changes, the sea will have time to take up its equilibrium position, and the total pressure on the sea-bottom will be unaltered. Hence, if some of the earthquakes felt in a district originate beneath the sea and some beneath the land, the amplitude of the annual period will be less than it would be if all were to originate underneath the land. Since some of the foci of the earthquakes felt in Japan and New Zealand are known to lie beneath the ocean, it seems not improbable that the small amplitude of the annual period in these two, as well as in other, insular districts may be thus accounted for.

XXII. *On the Differential Covariants of Plane Curves, and the Operators employed in their Development.*

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The subject of the earlier parts of this paper is immediately allied with the late M. Halphen's investigations in his Thesis, "Sur les Invariants Différentiels," (Gauthier-Villars, Paris, 1878), and in his Memoir, "Sur les Invariants Différentiels des courbes gauches" ('Journal de l'École Polytechnique,' vol. 28, cahier 47, 1880), and in the latter paper the word covariant is applied to a function having the character of the functions which are generally treated of in this paper.

The connection is perhaps greater with a previous, and, I think, less familiar paper of the same author, "Sur la recherche des points d'une courbe algébrique plane, qui satisfont à une condition exprimée par une équation différentielle algébrique, et sur les questions analogues dans l'espace" ('Journal de Math.,' 3rd series, vol. 2, 1876). I am not able to point out any paper which touches upon the actual investigation attempted in this paper, and the mode of investigation can not claim any resemblance to any of Halphen's work, and does not offer in itself any element of novelty.

The subject of Differential Invariants has been further treated by Sophus Lie, "Über Differentialinvarianten" ('Mathem. Annalen,' vol. 24, p. 564, 1884), and by Sylvester, in his lectures "On the Theory of Reciprocants" ('Amer. Journ. Math.,' vols. 8, 9, and 10, 1886 and 1887).

The literature on the cognate subject of Reciprocants has become extensive, several papers by Elliott, Forsyth, Hammond, Leudesdorf, and Rogers have appeared in the 'Proceedings of the Mathematical Society' (vols. 17, 18, and 19), and in the 'Messenger of Mathematics' (vols. 17, 18, and 19). Of these papers, those of which the analytical method is most similar to mine are by Mr. Forsyth, "Homographic Invariants and Quotient Derivatives" ('Mess. of Maths.,' vol. 17, No. 10, p. 154, 1888); "On a class of Functional Invariants" ('Phil. Trans.,' A, 1889, p. 71). I may also mention, as cognate matter, papers by Mr. Elliott "On the Interchange of Variables in Certain Linear Differential Operators" ('Phil. Trans.,' A, 1890, p. 19), by Professor L. J. ROGERS, "Conjugate Annihilators" ('Mess. of Maths.,' vol. 18, p. 153–158, 1889), and by Major MACMAHON, "On Multilinear Partial Differential Operators" ('Lond. Math. Soc. Proc.,' vols. 18 and 19, 1887 and 1888).

Starting with the mode of formation of Differential Covariants, I attempt to give greater prominence to the geometrical character of the problem of invariancy, which seems to have been prominent in HALPHEN's view of the subject, and hope that while the results are partly coextensive with the work of Professor SYLVESTER and others, they may still be complementary, and may be found applicable in investigating the characteristics of higher plane curves.

§ 1. *General Statement of the Problem Considered, as far as relates to Plane Curves.*

If a plane curve is transformed by the general homographic transformation, the transformed curve remains of the same order and possesses the same perspective singularities. If we introduce no other complications we may say that HALPHEN's Differential Invariants indicate properties which, if they exist at any point of the original curve, exist at the same point of the transformed curve. Any function of the differential coefficients at any point of the curve which, a definite factor *près*, retains its form under the general homographic transformation is a differential invariant. I shall write these differential coefficients $y_1, y_2, y_3 \ldots.$ In this paper I introduce a curve, the current coordinates of which I write ξ, η, and of which the equation consists of an algebraic relation between ξ, η and $x, y, y_1, y_2, y_3 \ldots.$ (x, y being the coordinates of a point on the curve which I shall call the standard curve). Then, if this equation is such that, a definite factor *près*, its form is unaltered under the general homographic transformation, I call it a differential covariant of the standard curve.

Besides the sense in which a differential invariant has here been spoken of as indicating a perspective singularity at a point in a curve, it may, equated to zero, be the differential equation to a curve, each point of which has that singularity. (I here consider a sextactic point and other points of similar characteristics, as singular points for homographic projection.)

A conic is one of the curves which, by a general homographic transformation, can be transformed into itself, so that any one arbitrary point can be replaced by any other. Hence, if $V = 0$ is the differential condition for a sextactic point, it is the differential invariant equation of conics, and if $f(\xi, \eta, y_1, y_2, y_3, y_4) = 0$ is a covariant equation of the second degree, such that when $\xi = x$ and $\eta = y$ we have $\eta_1 = y_1$, $\eta_2 = y_2$, $\eta_3 = y_3$, $\eta_4 = y_4$, it is the equation to the conic osculating the standard curve at (x, y). Also regarding ξ, η as the variables and $x, y, y_1 \ldots$ as arbitrary constants, it is the complete integral of $V = 0$. Generally the differential covariants will include the integrals of the differential-invariantal equations, provided these integrals are rational, integral, algebraic functions.

Having found a covariant curve, its polars, as well as its ordinary covariants, will be included among the differential-covariant curves.

To find the degree of contact with the standard curve, we substitute for the covariant $\xi - x = h$, $\eta - y = y_1 h + y_2 h^2/2\,! + \&c.$, if the equation is identically satisfied by $\eta_1 = y_1$, $\eta_2 = y_2 \ldots$, as far as the coefficient of h^n, the contact is of the n^{th} order. Equating to zero the coefficient of h^{n+1} we obtain the condition for contact of the $n + 1^{\text{th}}$ order. Since this condition is generally capable of being satisfied, and since the relation is unaltered by homographic transformation, the coefficient of h^{n+1} is a differential invariant.

The first section of this paper investigates the conditions which must be satisfied in order that (except as concerns a factor), the form of a function may be unaltered in consequence of any homographic transformation whatever, and finds the form of the factor.

The results are given in the equations (15).

In the second section, the relations of certain partial differential operators Ω_1, Ω_2, and Ω_3 with one another, and d/dx are considered. A method of eduction of covariants is established, and a method of development of a covariant from a source or matrix, as well as a process of eduction of matrices. The matrices are shown to be dependent upon differential invariants and two fundamental matrices u_4 and L_6, and from the mode in which these enter, the order of the covariant can be predicted.

This section prepares the way for the application of the theory in the two last sections.

The third section deals with the dual reciprocal relationship between the operators, and the deduction of reciprocal and contravariant functions.

The fourth section contains applications to the cubic of the methods which have been considered. The equation to the osculating cubic is determined when the cubic is non-singular, nodal, and cuspidal, and the differential equation in each case found.

The fifth section takes a new view of the subject. It is shown that the position of any point which is "homographically persistent" with regard to the standard curve (which, for example, is a point of intersection of two curves which are each covariant) is determined by the ratios of three differential-invariant functions. These can therefore be considered as invariantal coordinates of the point. It is also shown that there are similarly invariantal coordinates of a line, and the relations of these line and point coordinates are investigated, showing that the general methods of higher geometry are applicable to the novel system of coordinates.

The condition that the point should lie on a covariant curve is now given in terms of the invariantal coordinates by an equation in which the coefficients are the differential invariants which determine the character of the curve. For this reason I call it the intrinsic invariantal equation to the curve.

Also the invariantal coordinates are connected with the ordinary coordinates by relations having the form distinctive of a homographic transformation. And therefore, regarding the invariantal coordinates as ordinary coordinates, the curve represented by the intrinsic invariantal equation has all the perspective singularities of the original curve. No attempt is made to show the application of this method to the difficult case of the quartic, but several well known theorems regarding the cubic are shown to be arrived at without difficulty.

§ 1.

1. In general, the complete curve of the d^{th} algebraic degree is completely determined by $d(d+3)/2$ $(\equiv r)$ points, and if the coordinates of these points are known, the equation to the curve can be written down in the form of a determinant, thus

$$f(x, y, \ldots) \equiv \begin{vmatrix} \xi^d, & \xi^{d-1}\eta, & \ldots & 1 \\ x^d, & x^{d-1}y, & \ldots & 1 \\ x_1^{\,d}, & x_1^{\,d-1}y_1, & \ldots & 1 \\ \vdots & \vdots & & \vdots \end{vmatrix} = 0$$

If a general homographic transformation of the coordinates be now made, the form of the function will be unaltered, except by the introduction of a factor.

Denote the transformation by

$$\frac{x}{A_1X + B_1Y + C_1} = \frac{y}{A_2X + B_2Y + C_2} = \frac{1}{AX + BY + C} \quad . \quad . \quad . \quad . \quad (1)$$

with similar relations for ξ and η, except that we may retain the letters ξ, η, in the transformed equation without causing inconvenience.

Thus

$$f(x, y, \ldots) = D^{\frac{d(d+1)(d+2)}{3!}} (A\xi + B\eta + C)^{-d} (AX_1 + BY_1 + C)^{-d} \ldots$$
$$\times f(X, Y, \ldots).$$

We may also write

$$f(x, y, \ldots) \equiv \begin{vmatrix} (\xi - x)^d, & (\xi - x)^{d-1}(\eta - y), & \ldots & \xi - x, & \eta - y, \\ (x_1 - x)^d, & (x_1 - x)^{d-1}(y_1 - y), & \ldots & x_1 - x, & y_1 - y, \\ \vdots & \vdots & \ldots & \vdots & \vdots \end{vmatrix}.$$

Imagine the r points now moved up into coincidence at x, y, subject to some geometrical condition, such as that of moving along an arc, free from singularities, of a curve of degree not lower than the d^{th}.

The equation to the resulting curve, which will osculate the assumed curve at the chosen point may now be deduced. Most easily, write

$$\xi_1 - x = h, \quad \eta_1 - y = y_1 h + \frac{y_2}{1.2} h^2 + \frac{y_3}{3\,!} h^3 + \dots$$

For further simplicity, write $\pi - p$ for $\eta - y - y_1 (\xi - x)$, so that

$$(\pi - p)_1 = \frac{y_2}{1.2} h^2 + \frac{y_3}{3\,!} h^3 + \dots$$

and transform the determinant so that, in the first row $\eta - y$ may be replaced by $\pi - p$.

To fill up the several columns in the determinant we write under $(\xi - x)^n (\pi - p)^m$ the coefficients of $h^2, h^3, \dots$ in the expansion of $h^n (\pi - p)_1{}^m$. We thus obtain

$$f(x, y, \dots) = \begin{vmatrix} (\xi - x)^d, & (\xi - x)^{d-1} (\pi - p), & \dots & (\xi - x)(\pi - p), & \pi - p \\ 0, & 0, & \dots & 0, & \frac{y_2}{2\,!} \\ 0, & 0, & \dots & \frac{y_2}{2\,!}, & \frac{y_3}{2\,!} \end{vmatrix}$$
$$\times\, h^{\frac{d(d+1)(d+2)(d+3)}{8} - \frac{d(d+3)}{2}}.$$

The transformed equation is affected in the same way, so that

$$f(x, y, \dots) = \mathrm{D}^{d(d+1)(d+2)/3!} (\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C})^{-d} (\mathrm{AX} + \mathrm{BY} + \mathrm{C})^{-d^2(d+3)/2}$$
$$\times\, \mathrm{H}^{d(d+1)(d+2)(d+3)/8 - d(d+3)/2} \begin{vmatrix} (\xi - \mathrm{X})^d, & (\xi - \mathrm{X})^{d-1} (\pi - \mathrm{P}), & \dots & \pi - \mathrm{P} \\ 0, & 0, & 0, & \frac{y_2}{2\,!} \\ 0, & 0, & \frac{y_2}{2\,!}, & \frac{y_2}{2\,!} \end{vmatrix}.$$

Now write the determinants shortly

$$f(\xi - x, \pi - p, y_2 \dots) \quad \text{and} \quad f(\xi - \mathrm{X}, \pi - \mathrm{P}, \mathrm{Y}_2 \dots).$$

Also

$$\mathrm{H}/h = d\mathrm{X}/dx = \mu^2 \lambda^{-1}$$

where

$$\left.\begin{aligned} &\mathrm{AX} + \mathrm{BY} + \mathrm{C} = \mu. \\ &(\mathrm{AX} + \mathrm{BY} + \mathrm{C})(\mathrm{A}_1 + \mathrm{B}_1\mathrm{Y}_1) - (\mathrm{A}_1\mathrm{X} + \mathrm{B}_1\mathrm{Y} + \mathrm{C}_1)(\mathrm{A} + \mathrm{BY}_1) = \lambda \\ &\text{Also write} \\ &\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C} = \nu. \end{aligned}\right\} \quad (2).$$

Then

$$f(\xi - x, \pi - p, y_2, \ldots) \\ = \mathrm{D}^{d(d+1)(d+2)/3!} \lambda^{-d(d+1)(d+2)(d+3)/8+d(d+3)/2} \mu^{d(d+1)(d+2)(d+3)/4-d(d+3)-d^2(d+3)/2} \nu^{-d} \\ \times f(\xi - \mathrm{X}, \pi - \mathrm{P}, \mathrm{Y}_2 \ldots) \quad \text{(3a)}.$$

To put the multiplying factor in a more concise form, we will use HALPHEN's nomenclature. If a term contains the differential coefficients y_2, $y_3 \ldots$, we will call the number of differential coefficients in the term its degree, and write it d_x, and the sum of the indices of differentiation its weight, and write it w.

Considering the transformation to be made in the ultimate form of the equation, we see that

$$\pi - p = \mathrm{D}\lambda^{-1}\nu^{-1}(\pi - \mathrm{P})$$
$$y_2 = \mathrm{D}\mu^3\lambda^{-3}\mathrm{Y}_2$$
$$y_3 = \mathrm{D}\mu^5\lambda^{-4}\mathrm{Y}_3 + \&\text{c.}$$
$$y_k = \mathrm{D}\mu^{2k-1}\lambda^{-(k+1)}\mathrm{Y}_k + \&\text{c.}$$

From the mode of formation of the determinant, the degree of the terms in each coefficient of $\xi - x$ and $\pi - p$ is uniform, and their weight is uniform, so that we may consider the weight and degree of each coefficient.

Let d_x and w denote the degree and weight of the coefficient of the highest power of $\pi - p$ in the expanded determinant.

We thus determine the multiplying factor to be

$$\mathrm{D}^{d+d_x} \lambda^{-(d+d_x+w)} \mu^{2w-d_x} \nu^{-d}.$$

From this we conclude that

$$d + d_x = \frac{d(d+1)(d+2)}{3!}$$
$$w = \frac{d(d+1)(d+2)(3d+5)}{4!} - \frac{d(d+3)}{2},$$

results which can be verified by arranging the terms in the determinant in descending order and summing the quantities required.

It is also easy to see that if $d_x(m, n)$ and $w(m, n)$ are the degree and weight of the coefficient of $(\pi - p)^{d-m}(\xi - x)^n$

$$d_x(m. n) = d_x + m$$
$$w(m, n) = w - n,$$

or the determinant is a homogeneous function of $\pi - p$ and of the differential coefficients (independent of their order), and that the difference of the weight of the coefficient of a term and the index of the power of $\xi - x$ which it contains is uniform throughout.

These determine the character of the general equation of a curve of the d^{th} degree, osculating a given curve, the order of the highest differential coefficient being $d(d+3)/2 - 1$.

In its differential equation the degree is greater than this value by d, the weight by $\frac{1}{2}d(d+3)$, and the order by 1.

If the curve has any imposed condition, such as the possession of a double point, these criteria are altered.

2. *Determination of Differential Covariants of Plane Curves.*

Abandoning the condition, as yet imposed, of considering the equation to an osculating curve, the question proposed is to examine the conditions, both simple and in the form of linear partial differential equations, which $f(\xi - x, \eta - y, y_1, y_2, \ldots)$ must satisfy in order that it may become

$$D^{d+d_x} \lambda^{-(d+d_x+w)} \mu^{2w-d_x} \nu^{-d} f(\xi - X, \eta - Y, Y_1, Y_2, \ldots) \quad . \quad . \quad . \quad (36)$$

under the general linear homographic transformation stated above.

The condition that d_x and w are necessarily connected with d, by the relations just found, will not be required, but they will serve as criteria for recognising general cases.

The simple conditions are exactly analogous to those found by HALPHEN for Differential Invariants ('Thèse,' p. 21), and result from the same simple transformations.

1. From putting $x = X$, $\xi = \xi$, $y = B_2Y$, $\eta = B_2\eta$, we conclude, as before, that the function is homogeneous in $\eta - y$ and the several differential coefficients of y.

2. From putting $x = A_1X$, $\xi = A_1\xi$, $y = Y$, $\eta = \eta$, we conclude that the weight of the coefficient of each power of $\xi - x$ is uniform, and that this weight diminished by the index of the power of $\xi - x$ is uniform throughout the function.

3. We have chosen the form of the expression, so that we draw no further conclusion from a mere change of origin of coordinates. The form of the multiplying factor as well as that of the function might have been found from the method which follows, and its correctness is established from it.

To obtain the linear partial differential equations which the functions must satisfy, we consider infinitesimal transformations. Being infinitesimal we may consider them separately. The transformations are four in number. The first two would apply to a general infinitesimal Cartesian transformation, and so the results are the partial differential equations suitable to that case. The other two only concern our more general transformation, and, as we shall see, the results have a different character from those of the first two.

Write

$$x = \mathrm{X}, \quad \xi = \xi, \quad y = \mathrm{Y} + \epsilon \mathrm{X}, \quad \eta = \eta + \epsilon \xi,$$

where, ϵ being infinitesimal, we shall only retain its first power.

Then, by hypothesis,

$$f(\xi - x, \eta - y, y_1, y_2, \ldots) = f(\xi - \mathrm{X}, \eta - \mathrm{Y}, \mathrm{Y}_1, \mathrm{Y}_2, \ldots).$$

Also expanding by TAYLOR'S theorem

$$\begin{aligned} &f(\xi - x, \eta - y, y_1, y_2, \ldots) \\ &= f(\xi - \mathrm{X}, \eta - \mathrm{Y}, \mathrm{Y}_1, \mathrm{Y}_2, \ldots) + \epsilon \left\{ (\xi - \mathrm{X}) \frac{\partial f}{\partial (\eta - \mathrm{Y})} + \frac{\partial f}{\partial \mathrm{Y}_1} \right\}. \end{aligned}$$

Therefore

$$(\xi - \mathrm{X}) \frac{\partial f}{\partial (\eta - \mathrm{Y})} + \frac{\partial f}{\partial \mathrm{Y}_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence we come to the conclusion we arrived at before, that

$\eta - \mathrm{Y}$ and Y_1 only enter in the form $\eta - \mathrm{Y} - \mathrm{Y}_1 (\xi - \mathrm{X})$, or $\pi - \mathrm{P}$.

We shall now reintroduce $\pi - p$, and write the functions

$$f(\xi - x, \pi - p, y_2, y_3, \ldots), \text{ \&c.}$$

The method we have used here we shall use in the other cases. We evaluate the multiplying factor to obtain one expression for $f(\xi - x, \pi - p, \ldots)$, and obtain a second by expanding the function in accordance with the transformation considered.

3. Having now introduced $\pi - p$, we shall treat the other infinitesimal transformations by a more general method.

Consider the homographic transformation

$$\frac{x}{\mathrm{X} + \mathrm{B}_1 \mathrm{Y}} = \frac{y}{\mathrm{Y}} = \frac{1}{\mathrm{AX} + \mathrm{BY} + 1},$$

where B_1, A and B are indefinitely small quantities whose squares and products will be neglected.

Thus

$$\begin{aligned} \lambda &= 1 + \mathrm{B}_1 \mathrm{Y}_1 + \mathrm{B} (\mathrm{Y} - \mathrm{XY}_1) \\ \mu &= 1 + \mathrm{AX} + \mathrm{BY} \\ \nu &= 1 + \mathrm{A}\xi + \mathrm{B}\eta \end{aligned}$$

so that the multiplying factor

$$\begin{aligned}&= 1 - (d + d_x + w)\{B_1Y_1 + B(Y - XY_1)\}\\ &\quad + (2w - d_x)(AX + BY)\\ &\quad - d(A\xi + B\eta).\end{aligned}$$

4. *Determination of* Y_n.

The dependent variable is supposed to be connected with the independent by some relation, say

$$y = \phi(x),$$

so that $y_n/n!$ is the coefficient of h^n in the expansion of $\phi(x + h)$.

The homographic relations may for our present purpose be written

$$\begin{aligned}x &= X + B_1Y - X(AX + BY)\\ y &= Y - Y(AX + BY),\end{aligned}$$

whence

$$Y = \phi\{X + B_1Y - X(AX + BY)\} + Y(AX + BY)\ldots$$

where on the right-hand side Y never appears except when affected by a small coefficient.

Now write $X + h$ in place of X, and consequently

$$Y + Y_1h + \&c. + \frac{Y_m}{m!}h^m + \&c. \ (\equiv Y + \psi) \text{ for } Y,$$

the coefficient of h^n in the resulting expression on the right is then

$$Y_n/n!.$$

But this expression may be written

$$\begin{aligned}&\phi\{x + h + B_1\psi - (2AX + BY)h - BX\psi - Ah^2 - Bh\psi\}\\ &+ Y(AX + BY) + A(X\psi + Yh + h\psi) + B(2Y\psi + \psi^2).\end{aligned}$$

Hence

$$Y_1 = y_1\{1 + B_1Y_1 - BXY_1 - 2AX - BY\} + A(XY_1 + Y) + 2BYY_1,$$

and generally $Y_n/n!$ = the coefficient of h^n from

$$
\begin{aligned}
&\Sigma\, y_p h^{p-1} [h + p\,\{\mathrm{B}_1\psi - (2\mathrm{AX} + \mathrm{BY})\, h - \mathrm{BX}\psi - \mathrm{A}h^2 - \mathrm{B}h\psi\}]/p\,! \\
&\quad + \mathrm{A}\left\{\frac{\mathrm{XY}_n}{n!} + \frac{\mathrm{Y}_{n-1}}{n-1!}\right\} \\
&\quad + \mathrm{B}\left\{\frac{2\mathrm{YY}_n}{n!} + \Sigma\,\frac{\mathrm{Y}_p\mathrm{Y}_{n-p}}{p!\,n-p!}\right\} \\
&= y_n\,\{1 - n\,(2\mathrm{AX} + \mathrm{BY})\}/n\,! \\
&\quad + \mathrm{B}_1\,\Sigma\, y_p \mathrm{Y}_{n-p+1}/p - 1!\,n - p + 1! \\
&\quad + \mathrm{A}\left\{\frac{\mathrm{XY}_n}{n!} + \frac{\mathrm{Y}_{n-1}}{n-1!} - \frac{\mathrm{Y}_{n-1}}{n-2!}\right\} \\
&\quad + \mathrm{B}\left\{\frac{2\mathrm{YY}_n}{n!} - (p-1)\,\Sigma\,\frac{\mathrm{Y}_p\mathrm{Y}_{n-p}}{p!\,n-p!} - \mathrm{X}\,\Sigma\,\frac{y_p\mathrm{Y}_{n-p+1}}{p-1!\,n-p+1!}\right\}.
\end{aligned}
$$

In the small terms we may substitute Y for y, and thus obtain

$$
\begin{aligned}
y_n = \mathrm{Y}_n &- \mathrm{B}_1\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1} \\
&+ \mathrm{A}\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\} \\
&+ \mathrm{B}\,\{(n-2)\,\mathrm{YY}_n + \Sigma\,(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\} \quad . \quad . \quad (5),
\end{aligned}
$$

where by Σ we denote that all possible values of p are to be taken from 1 upwards.

5. *Determination of* $\pi - p$ *and* $\xi - x$.

$\pi - p$ stands in place of $\eta - y - y_1(\xi - x)$, and, upon transformation, $\eta - y$ becomes

$$
\begin{aligned}
\eta - \mathrm{Y} - \mathrm{A}\,\{(\xi - \mathrm{X})(\eta - \mathrm{Y}) + \mathrm{X}(\eta - \mathrm{Y}) + \mathrm{Y}(\xi - \mathrm{X})\} \\
- \mathrm{B}\,\{(\eta - \mathrm{Y})^2 + 2\mathrm{Y}(\eta - \mathrm{Y})\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),
\end{aligned}
$$

$\xi - x$ becomes

$$
\begin{aligned}
&\xi - \mathrm{X} + \mathrm{B}_1(\eta - \mathrm{Y}) \\
&\quad - \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}(\xi - \mathrm{X})\} \\
&\quad - \mathrm{B}\,\{(\xi - \mathrm{X})(\eta - \mathrm{Y}) + \mathrm{X}(\eta - \mathrm{Y}) + \mathrm{Y}(\xi - \mathrm{X})\} \quad . \quad . \quad . \quad (7),
\end{aligned}
$$

and y_1 becomes

$$
\mathrm{Y}_1 - \mathrm{B}_1\mathrm{Y}_1^2 + \mathrm{A}(\mathrm{XY}_1 - \mathrm{Y}) + \mathrm{BY}_1(\mathrm{XY}_1 - \mathrm{Y}) \quad . \quad . \quad . \quad . \quad . \quad (8),
$$

therefore $\pi - p$ becomes

$$
\begin{aligned}
&\pi - \mathrm{P} - \mathrm{B}_1\mathrm{Y}_1(\pi - \mathrm{P}) - \mathrm{A}\,\{(\xi - \mathrm{X})(\pi - \mathrm{P}) + \mathrm{X}(\pi - \mathrm{P})\} \\
&\qquad - \mathrm{B}\,\{(\pi - \mathrm{P})^2 + \mathrm{Y}_1(\pi - \mathrm{P})(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)(\pi - \mathrm{P})\} \quad . \quad (9),
\end{aligned}
$$

and $\xi - x$ becomes

$$
\begin{aligned}
&\xi - \mathrm{X} + \mathrm{B}_1\,\{(\pi - \mathrm{P}) + \mathrm{Y}'(\xi - \mathrm{X})\} \\
&\quad - \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}(\xi - \mathrm{X})\} \\
&\quad - \mathrm{B}\,\{(\xi - \mathrm{X})(\pi - \mathrm{P}) + \mathrm{Y}_1(\xi - \mathrm{X})^2 + \mathrm{X}(\pi - \mathrm{P}) + (\mathrm{Y} + \mathrm{XY}_1)(\xi - \mathrm{X})\} \quad . \quad (10).
\end{aligned}
$$

6. *Deduction of the conditions in the form of linear partial differential equations.*

For this purpose, it will be convenient to consider the independent parts of the conditions separately, and the headings of the paragraphs indicate the portions considered.

B_1 only.

Multiplying factor $= 1 - (d + d_x + w)\,B_1 Y_1$

$$y_n = Y_n - B_1 \Sigma_n C_{p-1} Y_p Y_{n-p+1},$$

$$\pi - p \text{ becomes } \pi - P - B_1 Y_1 (\pi - P);$$

$$\xi - x \text{ becomes } \xi - X + B_1 \{\pi - P + Y_1 (\xi - X)\}.$$

Therefore

$$-Y_1 (\pi - P) \frac{\partial f}{\partial (\pi - P)} + \{\pi - P + Y_1 (\xi - X)\} \frac{\partial f}{\partial (\xi - X)}$$
$$= \Sigma \Sigma_n C_{p-1} Y_p Y_{n-p+1} \frac{\partial f}{\partial Y_n} - (d + d_2 + w)\,Y_1 f.$$

Since Y_1 does not occur explicitly in the function, we must equate separately the parts which contain Y_1 and those which do not contain it. The symbol S will now be used instead of Σ to denote that Y_1 does not occur within the summation.

Thus

$$(\pi - P) \frac{\partial f}{\partial (\pi - P)} - (\xi - X) \frac{\partial f}{\partial (\xi - X)} = (d + d_x + w) f - S (n + 1)\,Y_n \frac{\partial f}{\partial Y_n}$$

$$(\pi - P) \frac{\partial f}{\partial (\xi - X)} = S S_n C_{p-1} Y_p Y_{n-p+1} \frac{\partial f}{\partial Y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Since f is homogeneous of degree $d + d_x$ in $\pi - P$ and the differential coefficients, we may write the first of these equations

$$\omega f + (\xi - X) \frac{\partial f}{\partial (\xi - X)} = S n Y_n \frac{\partial f}{\partial Y_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

which expresses analytically a condition already found.

There is no other condition necessary in order that a function may be a covariant for the ordinary Cartesian transformation, which leaves the relations of a curve with infinity unaltered, that is for a change of coordinates.

To proceed, we have

A only.

Multiplying factor

$$= 1 + (2w - d - d_x)\,\mathrm{AX} - d\mathrm{A}\,(\xi - \mathrm{X}),$$

$$y_n = \mathrm{Y}_n + \mathrm{A}\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\},$$

$$\xi - x = \xi - \mathrm{X} - \mathrm{A}\,\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\},$$

$$\pi - p = \pi - \mathrm{P} - \mathrm{A}\,\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{X}\,(\pi - \mathrm{P})\}.$$

Hence

$$\{(\xi - \mathrm{X})^2 + 2\mathrm{X}\,(\xi - \mathrm{X})\}\,\frac{\partial f}{\partial\,(\xi - \mathrm{X})} + \{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{X}\,(\pi - \mathrm{P})\}\,\frac{\partial f}{\partial\,(\pi - \mathrm{P})}$$

$$- \Sigma\,\{(2n-1)\,\mathrm{XY}_n + n\,(n-2)\,\mathrm{Y}_{n-1}\}\,\frac{\partial f}{\partial \mathrm{Y}_n}$$

$$= \{d\,(\xi - \mathrm{X}) - (2w - d - d_x)\,\mathrm{X}\}\,f.$$

The only new condition arising from these terms is

$$(\xi - \mathrm{X})\left\{(\xi - \mathrm{X})\frac{\partial f}{\partial\,(\xi - \mathrm{X})} + (\pi - \mathrm{P})\frac{\partial f}{\partial\,(\pi - \mathrm{P})} - df\right\} = \mathrm{S}n\,(n-2)\,\mathrm{Y}_{n-1}\frac{\partial f}{\partial \mathrm{Y}_n}\quad . \quad (13).$$

Lastly,

B only.

Multiplying factor

$$= 1 + \mathrm{B}\,\{(w - 2d - 2d_x)\,\mathrm{Y} + (w + d + d_x)\,\mathrm{XY}_1 - d\,(\eta - \mathrm{Y})\}$$

$$y_n = \mathrm{Y}_n + \mathrm{B}\,\{(n-2)\,\mathrm{YY}_n + \Sigma(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\}$$

$$\xi - x = \xi - \mathrm{X} - \mathrm{B}\,\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{Y}_1(\xi - \mathrm{X})^2 + (\mathrm{Y} + \mathrm{XY}_1)\,(\xi - \mathrm{X}) + \mathrm{X}\,(\pi - \mathrm{P})\}$$

$$\pi - p = \pi - \mathrm{P} - \mathrm{B}\,\{(\pi - \mathrm{P})^2 + \mathrm{Y}_1\,(\pi - \mathrm{P})\,(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)\,(\pi - \mathrm{P})\}$$

therefore

$$\{(\xi - \mathrm{X})\,(\pi - \mathrm{P}) + \mathrm{Y}_1\,(\xi - \mathrm{X})^2 + (\mathrm{Y} + \mathrm{XY}_1)\,(\xi - \mathrm{X}) + \mathrm{X}\,(\pi \quad \mathrm{P})\}\,\frac{\partial f}{\partial\,(\xi - \mathrm{X})}$$

$$+ \{(\pi - \mathrm{P})^2 + \mathrm{Y}_1\,(\pi - \mathrm{P})\,(\xi - \mathrm{X}) + (2\mathrm{Y} - \mathrm{XY}_1)\,(\pi - \mathrm{P})\}\,\frac{\partial f}{\partial\,(\pi - \mathrm{P})}$$

$$- \Sigma\,\{(n-2)\,\mathrm{YY}_n + \Sigma\,(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p} + \mathrm{X}\,\Sigma\,{}_n\mathrm{C}_{p-1}\mathrm{Y}_p\mathrm{Y}_{n-p+1}\}\,\frac{\partial f}{\partial \mathrm{Y}_n}$$

$$= \{d\,(\eta - \mathrm{Y}) + (2d + 2d_x - w)\,\mathrm{Y} - (w + d + d_x)\,\mathrm{XY}_1\}\,f.$$

We have again repetitions of previous conditions with the sole new condition

$$(\pi - \mathrm{P})\left\{(\pi - \mathrm{P})\frac{\partial f}{\partial\,(\pi - \mathrm{P})} + (\xi - \mathrm{X})\frac{\partial f}{\partial\,(\xi - \mathrm{X})} - df\right\} = \mathrm{SS}(p-1)\,{}_n\mathrm{C}_p\mathrm{Y}_p\mathrm{Y}_{n-p}\frac{\partial f}{\partial \mathrm{Y}_n}\quad (14).$$

The capital letters may now be conveniently replaced by the smaller type, and the equations of condition written

$$\left.\begin{array}{lr}
(\pi - p)\dfrac{\partial f}{\partial(\xi - x)} = \mathrm{SS}\,{}_n\mathrm{C}_{p-1}y_p y_{n-p+1}\dfrac{\partial f}{\partial y_n} \quad \dots\dots\dots\dots & (\mathrm{A}) \\[2ex]
(\xi - x)\left\{(\xi - x)\dfrac{\partial f}{\partial(\xi - x)} + (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} - df\right\} = \mathrm{S}n(n-2)\,y_{n-1}\dfrac{\partial f}{\partial y_n}\ . & (\mathrm{B}) \\[2ex]
(\pi - p)\left\{(\xi - x)\dfrac{\partial f}{\partial(\xi - x)} + (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} - df\right\} & \\[2ex]
\qquad\qquad = \mathrm{SS}\,(p-1)\,{}_n\mathrm{C}_p y_p y_{n-p}\dfrac{\partial f}{\partial y_n} \quad \dots\dots\dots\dots & (\mathrm{C}) \\[1ex]
\text{also} & \\[1ex]
wf + (\xi - x)\dfrac{\partial f}{\partial(\xi - x)} = \mathrm{S}ny_n\dfrac{\partial f}{\partial y_n} \equiv \mathrm{W}f \quad \dots\dots\dots\dots & (\alpha) \\[1ex]
\text{and} & \\[1ex]
(d + d_x)f = (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} + \mathrm{S}y_n\dfrac{\partial f}{\partial y_n} \equiv (\pi - p)\dfrac{\partial f}{\partial(\pi - p)} + \mathrm{H}f \quad \dots & (\beta)
\end{array}\right\}(15),$$

of which equations the last two will be identically satisfied by any functions which follow the laws of weight and degree which the simple conditions impose upon a covariant.

We may call these the form, weight, and degree conditions, respectively.

[A precisely analogous process may be applied to the covariants of surfaces, as is in this paper used for plane curves.

If we, in this case, write $\pi - p$ to stand for

$$\zeta - z - \frac{\partial z}{\partial x}(\xi - x) - \frac{\partial z}{\partial y}(\eta - y),$$

and $c_{m,n}$ for

$$\partial^{m+n}z/\partial x^m \partial y^n\, m!\, n!\,,$$

the set of conditions corresponding to (15) are

$$(\eta - y)\frac{\partial f}{\partial(\xi - x)} = \mathrm{S}_m\mathrm{S}_n\,(m+1)\,c_{m+1,n-1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\xi - x)\frac{\partial f}{\partial(\eta - y)} = \mathrm{S}_m\mathrm{S}_n\,(n+1)\,c_{m-1,n+1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\pi - p)\frac{\partial f}{\partial(\xi - x)} = \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,pc_{p,q}\,c_{m-p+1\ n-q}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\pi - p)\frac{\partial f}{\partial(\eta - y)} = \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,qc_{p,q}\,c_{m-p,n-q+1}\frac{\partial f}{\partial c_{m,n}}\,,$$

$$(\xi - x)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n(m + n - 2)\, c_{m-1,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\eta - y)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n(m + n - 2)\, c_{m,n-1}\frac{\partial f}{\partial c_{m,n}},$$

$$(\pi - p)\left\{(\xi - x)\frac{\partial f}{\partial(\xi - x)} + (\eta - y)\frac{\partial f}{\partial(\eta - y)} + (\pi - p)\frac{\partial f}{\partial(\pi - p)} - df\right\}$$

$$= \mathrm{S}_m\mathrm{S}_n\,\mathrm{S}_p\mathrm{S}_q\,(p + q - 1)\, c_{p,q}\, c_{m-p,n-q}\frac{\partial f}{\partial c_{m,n}},$$

and

$$(\pi - p)\frac{\partial f}{\partial(\pi - p)} = (d + d_z)f - \mathrm{S}_m\mathrm{S}_n\, c_{m,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\xi - x)\frac{\partial f}{\partial(\xi - x)} = -\, wf + \mathrm{S}_m\mathrm{S}_n\, m\, c_{m,n}\frac{\partial f}{\partial c_{m,n}},$$

$$(\eta - y)\frac{\partial f}{\partial(\eta - y)} = -\, wf + \mathrm{S}_m\mathrm{S}_n\, n\, c_{m,n}\frac{\partial f}{\partial c_{m,n}}.$$

The coefficient of the highest power of $\pi - p$ is not only isobaric in x and in y, but is homoioibaric, of weight w, in x and y. Also d_z denotes the degree of that coefficient, d denoting the algebraic degree of f.

We have, as in the case of plane curves, a theory of deduction from a matrix, and a theory of eduction (in the form of a Jacobian function) of both matrices and invariants.

The subject is quite similar at all steps to the investigation for plane curves in this paper, and offers little additional interest except in the solution of the differential equations which the matrices and invariants satisfy. By solution we can verify HALPHEN's statement in the last paragraph of his thesis (p. 60).

There is an invariant of the second and of the third order, two of the fourth and six of the fifth. There are three matrices, not invariants, of the fourth order, and a system of invariantal coordinates (§ 5 below) can be formed.

The theory of covariants of twisted curves is sufficiently different to merit a separate investigation, but the germ of it already exists in HALPHEN's paper (quoted above) "On the Invariants of Twisted Curves."

The conditions for covariants of surfaces given above, include those previously obtained by Mr. E. B. ELLIOTT, "On Ternary and n-ary Reciprocants," 'Proc. Math. Soc.,' No. 262, 1886.—*August* 2, 1893.]

§ 2. *On the Conditions of a Covariant Function, and on the modes of Eduction and Development.*

7. *The Form Conditions.*

The equations (A) (B) and (C) determine the form of the covariant functions; we will write them respectively

$$(O_1 - \Omega_1) f = 0 \quad \text{(A)}$$

$$(O_2 - \Omega_2) f = 0 \quad \text{(B)}$$

$$(O_3 - \Omega_3) f = 0 \quad \text{(C)},$$

where

$$O_1 \equiv (\pi - p) \frac{\partial}{\partial (\xi - x)} \qquad \Omega_1 \equiv \mathrm{SS}_n \mathrm{C}_{p-1} y_p y_{n-p+1} \frac{\partial}{\partial y_n}, \text{ \&c.}$$

Expanding the expressions for Ω_1, Ω_2, and Ω_3, we obtain

$$^{*}\,\Omega_1 = 3y_2^2 \frac{\partial}{\partial y_3} + 10 y_2 y_3 \frac{\partial}{\partial y_4} + (15 y_2 y_4 + 10 y_3^2) \frac{\partial}{\partial y_5} + (21 y_2 y_5 + 35 y_3 y_4) \frac{\partial}{\partial y_6}$$
$$+ (28 y_2 y_6 + 56 y_3 y_5 + 35 y_4^2) \frac{\partial}{\partial y_7} + \text{\&c.}$$

$$^{*}\,\Omega_2 = 3y_2 \frac{\partial}{\partial y_3} + 8 y_3 \frac{\partial}{\partial y_4} + 15 y_4 \frac{\partial}{\partial y_5} + 24 y_5 \frac{\partial}{\partial y_6} + 35 y_6 \frac{\partial}{\partial y_7} + \text{\&c.}$$

$$\Omega_3 = 6 y_2^2 \frac{\partial}{\partial y_4} + 30 y_2 y_3 \frac{\partial}{\partial y_5} + (60 y_2 y_4 + 40 y_3^2) \frac{\partial}{\partial y_6} + (105 y_2 y_5 + 175 y_3 y_4) \frac{\partial}{\partial y_7}$$
$$+ 42 (4 y_2 y_6 + 8 y_3 y_5 + 5 y_4^2) \frac{\partial}{\partial y_8} + \text{\&c.}$$

Relations between the Operators in the Form Conditions.

It is easy to prove that we have the relations

$$\left.\begin{aligned} (O_1 - \Omega_1)(O_2 - \Omega_2) - (O_2 - \Omega_2)(O_1 - \Omega_1) &= O_3 - \Omega_3 \\ (O_2 - \Omega_2)(O_3 - \Omega_3) - (O_3 - \Omega_3)(O_2 - \Omega_2) &= 0 \\ (O_3 - \Omega_3)(O_1 - \Omega_1) - (O_1 - \Omega_1)(O_3 - \Omega_3) &= 0 \end{aligned}\right\} \quad . \quad . \quad (16).$$

---

* The operators Ω_2 and Ω_1 are considered most conveniently in the form (25).

$\Omega_2 = 0$ is the partial differential equation which is satisfied by invariants in the theory of forms, and $\Omega_1 = 0$ was established for reciprocants by Professor SYLVESTER, in his lectures "On the Theory of Reciprocants" ('Amer. Jour. Math.,' vol. 8, p. 238).

I have not been able to consult a paper by SOPHUS LIE ('Mathem. Annalen.,' vol. 32), in which I am informed that this operator is applied to kindred purposes.—*August* 2, 1893.

8. *Relation between the operator and d/dx.*

In the functions under consideration, x and y only appear in $\xi - x$, $\pi - p$, and the differential coefficients.

Hence

$$\left.\begin{aligned} \frac{d}{dx} &= -\frac{\partial}{\partial(\xi - x)} - y_2(\xi - x)\frac{\partial}{\partial(\pi - p)} + \mathrm{S}y_{n+1}\frac{\partial}{\partial y_n} \\ &= \mathrm{D}_x + \partial_x, \text{ say,} \\ \text{where}\quad \mathrm{D}_x &= -\frac{\partial}{\partial(\xi - x)} - y_2(\xi - x)\frac{\partial}{\partial(\pi - p)} \\ \text{and}\quad \partial_x &= \mathrm{S}y_{n+1}\frac{\partial}{\partial y_n} \end{aligned}\right\} \quad . \quad . \quad . \quad (17).$$

None of the operators Ω_1, Ω_2, Ω_3, contain $\partial/\partial y_2$. Also we have to bear in mind that the functions do not contain y (except so far as it enters through $\pi - p$), and therefore, in dealing with general terms, we must exclude y_2 when it makes its appearance as if a derivative of y_1.

The division of d/dx into two parts makes the steps somewhat easier, and the final results are

$$\left.\begin{aligned} (\mathrm{O}_1 - \Omega_1)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_1 - \Omega_1) &= y_2(w + d + d_x) \quad . \\ (\mathrm{O}_2 - \Omega_2)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_2 - \Omega_2) &= -(2w - d_x) \\ (\mathrm{O}_3 - \Omega_3)\frac{d}{dx} - \frac{d}{dx}(\mathrm{O}_3 - \Omega_3) &= (\mathrm{O}_1 - \Omega_1) + y_2(\mathrm{O}_2 - \Omega_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (18).$$

Hence if f be a covariant, satisfying the necessary conditions, and if f' stand for df/dx, it follows that

$$\left.\begin{aligned} (\mathrm{O}_1 - \Omega_1)f' &= -y_2(w + d + d_x)f \\ (\mathrm{O}_2 - \Omega_2)f' &= -(2w - d_x)f \\ (\mathrm{O}_3 - \Omega_3)f' &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

Now f' satisfies the condition as to weight and degree necessary for being a covariant, simply d and d_x remain unaltered, while w is increased by unity.

Hence if the weight and degree of f are such that

$$w + d + d_x = 0$$
$$2w - d_x = 0,$$

then f' will also be a covariant.

We are thus led to the

9. *Theory of eduction of covariants.*

This theory is clear from what has preceded, but, to give it its simplest form, it is best to return to the preliminary conditions for a covariant.

If ϕ is a covariant, such that $w + d + d_x = 0$ and $2w - d_x = 0$, the multiplying factor is $D^{d+d_x}\, \nu^{-d}$, and we may write the relation

$$\phi = D^{d+d_x}\, \nu^{-d}\, \Phi.$$

Then

$$\frac{d\phi}{dx} = \frac{dX}{dx}\,\frac{d\phi}{dX} = D^{d+d_x}\, \lambda^{-1}\, \mu^2\, \nu^{-d}\, \frac{d\Phi}{dX}.$$

Hence $d\phi/dx$ is a covariant in which d and d_x are unaltered and w is increased by unity (3).

Before making use of this theory of eduction it is necessary to find a method of finding some covariant functions from which the eduction must be made, and the method now to be explained will in general supersede this mode of eduction.

10. *Method of development from a matrix.*

We have, however, another method for the formation of a covariant, namely, directly from the differential equations.

Let ${}_{d-m}\Phi$ stand for the collection of terms algebraically homogeneous, and of degree $d - m$, and let ${}_{d-m}\phi_0$, ${}_{d-m}\phi_1$, ${}_{d-m}\phi_2$ stand for the coefficients of terms containing $(\xi - x)^0$, $\xi - x$, $(\xi - x)^2$ among these homogeneous terms. Then, amongst these terms (15A) gives the law of derivation, namely,

$$(q + 1)\, \phi_{q+1} = \Omega_1 \phi_q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

a law which is independent of $d - m$, and for all values of $d - m$, $\Omega_1\, {}_{d-m}\phi_{d-m} = 0$.

Hence, if we know ${}_{d}\phi_0$, the coefficient of the highest power of $\pi - p$, we find all the terms of the d^{th} degree.

11. *Modes of derivation of homogeneous terms from those of a higher degree.*

The conditions (15B) and (15C) give us two modes of derivation, which imply the existence of certain relations and conditions.

We have from (15B) and (15C) respectively,

$$\begin{aligned}(\xi - x)\, \{{}_{d-1}\Phi + 2\, {}_{d-2}\Phi + \ldots \quad &+ (m + 1)\, {}_{d-m-1}\Phi + \ldots\} \\ + \Omega_2\, \{{}_{d}\Phi + {}_{d-1}\Phi + \ldots \quad &+ {}_{d-m}\Phi + \ldots\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (21),\end{aligned}$$

$$\begin{aligned}(\pi - p)\, \{{}_{d-1}\Phi + 2\, {}_{d-2}\Phi + \ldots \quad &+ (m + 1)\, {}_{d-m-1}\Phi + \ldots\} \\ + \Omega_3\, \{{}_{d}\Phi + {}_{d-1}\Phi + \ldots \quad &+ {}_{d-m}\Phi + \ldots\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (22).\end{aligned}$$

Or,

$$\begin{aligned}(m+1)(\xi - x)\,{}_{d-m-1}\Phi + \Omega_2\,{}_{d-m}\Phi &= 0,\\ (m+1)(\pi - p)\,{}_{d-m-1}\Phi + \Omega_3\,{}_{d-m}\Phi &= 0,\end{aligned}$$

and from these we again conclude

$$\left.\begin{aligned}(m+1)\,{}_{d-m-1}\phi_q + \Omega_2\,({}_{d-m}\phi_{q+1}) = 0\\ (m+1)\,{}_{d-m-1}\phi_q + \Omega_3\,({}_{d-m}\phi_q) = 0\end{aligned}\right\} \quad \ldots\ldots \quad (23).$$

From (20) we deduce that

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\phi_q = 0 \quad \ldots\ldots\ldots \quad (24).$$

That this is an identical relation among the coefficients appears from considering the operator.

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\Omega_1{}^q$$

which

$$\begin{aligned}&= \{\Omega_1\Omega_2 - q\,\Omega_3\}\,\Omega_1{}^q\\ &= \{\Omega_1\Omega_2\Omega_1 - q\,\Omega_3\Omega_1\}\,\Omega_1{}^{q-1}\\ &= \Omega_1\,\{\Omega_2\Omega_1 - q\,\Omega_3\}\,\Omega_1{}^{q-1}\\ &= \Omega_1\,\{\Omega_1\Omega_2 - (q-1)\,\Omega_3\}\,\Omega_1{}^{q-1}\\ &\quad \&\text{c.},\end{aligned}$$

and ultimately

$$= \Omega_1{}^{q+1}\Omega_2.$$

But

$$\Omega_2\phi_0 = 0,$$

hence

$$\{\Omega_2\Omega_1 - (q+1)\,\Omega_3\}\,\phi_q = 0.$$

There is then no further condition to be satisfied among the coefficients in the expansion. The importance of this will be seen later in the development of covariants.

12. The order of derivation is symbolised by a chart of the coefficients, with arrows showing the direction of derivation. Thus :—

$$\begin{tikzcd}
{}_{d}\phi_0 \arrow[rr] \arrow[dr] & & {}_{d}\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d}\phi_2 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d}\phi_3 \arrow[dl] & \cdots \\
& {}_{d-1}\phi_0 \arrow[rr] \arrow[dr] & & {}_{d-1}\phi_1 \arrow[rr] \arrow[dl] \arrow[dr] & & {}_{d-1}\phi_2 \arrow[dl] \arrow[dr] & & \cdots \\
& & {}_{d-2}\phi_0 \arrow[rr] & & {}_{d-2}\phi_1 \arrow[rr] & & {}_{d-2}\phi_2 & \cdots \\
& & & & & \cdot & & \cdot
\end{tikzcd}$$

where

→ denotes deduction by the operator Ω_1,

↙ ,, ,, ,, ,, Ω_2,

↘ ,, ,, ,, ,, Ω_3,

in which we can easily trace the relations (16)

$$\begin{aligned}\Omega_2\Omega_1 - \Omega_1\Omega_2 &= \Omega_3,\\ \Omega_2\Omega_3 &= \Omega_3\Omega_2,\\ \Omega_1\Omega_3 &= \Omega_3\Omega_1.\end{aligned}$$

According to this mode of procedure ${}_d\phi_0$ is the matrix from which the other coefficients in the covariant are developed. It is necessary that ${}_d\phi_0$ should be a homogeneous isobaric function of the differential coefficients satisfying the differential equation $\Omega_2 f = 0$. It is also essential that it shall not be an invariant as then no development ensues. Taking the solution of $\Omega_2 f = 0$, including only differential coefficients up to the fourth order, we obtain ${}_2\phi_0 = 3y_2y_4 - 4y_3^2$, and from it the general covariant of conics may be developed.

13. *General solution of* $\Omega_2 f = 0$, *and theory of eduction of matrices.*

Confining ourselves to the coefficients in the covariant functions, the equations (18) may be written

$$\left.\begin{aligned}\Omega_1\frac{d}{dx} - \frac{d}{dx}\Omega_1 &= -y_2(w + d_x)\\ \Omega_2\frac{d}{dx} - \frac{d}{dx}\Omega_2 &= (2w - d_x)\\ \Omega_3\frac{d}{dx} - \frac{d}{dx}\Omega_3 &= \Omega_1 + y_2\Omega_2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\text{A}).$$

Hence, if ϕ is a homogeneous isobaric function of the differential coefficients such that $\Omega_2\phi = 0$, then

$$\Omega_2\frac{d\phi}{dx} = (2w - d_x)\,\phi,$$

and if the weight and degree of ϕ are such that $2w = d_x$, then $d\phi/dx$ also satisfies the same differential equation, and it is homogeneous and isobaric.

The solutions of $\Omega_2 f = 0$ up to the fourth order are

$$y_2 \text{ and } 3y_2y_4 - 4y_3^2;$$

from these we form, so as to satisfy $2w - d_x = 0$,

$$\phi = (3y_2y_4 - 4y_3^2)^3\, y_2^{-10},$$

and obtain from it by differentiation the educt of the fifth order

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3,$$

which is the differential invariant of conics, and does not produce any covariant by development.

By this process of eduction, a solution can be found for each order, but it is possible to find a simpler series of solutions of $\Omega_2 f = 0$ than those thus obtained.

14. *Extension of the theory of eduction of matrices.*

If ϕ is a matrix, and if we write ϕ_1, ϕ_2, &c. for $d\phi/dx$, $d^2\phi/dx^2$ &c., it is easily seen that

$$\begin{aligned}\Omega_2\phi_2 &= \frac{d}{dx}\left(\Omega_2\frac{d\phi}{dx}\right) + (2w - d_x + 2)\,\phi_1,\\ &= 2\,(2w - d_x + 1)\,\phi_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_2\phi_3 &= 3\,(2w - d_x + 2)\,\phi_2, \text{ \&c.},\\ \Omega_2\phi_n &= n\,(2w - d_x + n - 1)\,\phi_{n-1},\end{aligned}$$

where w and d_x are the order and degree of ϕ.

Hence

$$(2w - d_x)\,\phi\phi_2 - (2w - d_x + 1)\,\phi_1^2$$

and

$$(2w-d_x)^2\,\phi^2\phi_3 - 3\,(2w-d_x)(2w-d_x+2)\,\phi\phi_1\phi_2 + 2\,(2w-d_x+1)(2w-d_x+2)\,\phi_1^3$$

are matrices, and are of the form of the matrix and differential invariant of conics.

Generally, if ϕ and ψ are two matrices, and if α and β are the respective values of $2w - d_x$ for each,

$$\beta\phi\psi_1 - \alpha\phi_1\psi$$

is a matrix.

Convenient forms of the differential equations and of the irreducible matrices.

In obtaining the differential equations of condition (15), and in considering the method of eduction by differentiation, there has been a convenience in retaining the differential coefficients y_2, y_3, &c., but the forms of the equations $\Omega f = 0$, and of their solutions, are simplified by replacing them by a_2, a_3, &c., where

$$a_n = y_n/n\,!.$$

We thus obtain (15)

$$\left.\begin{aligned}
\Omega_1 &= 2a_2^2\frac{\partial}{\partial a_3} + 5a_2a_3\frac{\partial}{\partial a_4} + (6a_2a_4 + 3a_3^2)\frac{\partial}{\partial a_5} + (7a_2a_5 + 7a_3a_4)\frac{\partial}{\partial a_6} \\
&\qquad + (8a_2a_6 + 8a_3a_5 + 4a_4^2)\frac{\partial}{\partial a_7} + \&c. \\
\Omega_2 &= a_2\frac{\partial}{\partial a_3} + 2a_3\frac{\partial}{\partial a_4} + 3a_4\frac{\partial}{\partial a_5} + \&c. \\
\Omega_3 &= a_2^2\frac{\partial}{\partial a_4} + 3a_2a_3\frac{\partial}{\partial a_5} + (4a_2a_4 + 2a_3^2)\frac{\partial}{\partial a_6} + (5a_2a_5 + 5a_3a_4)\frac{\partial}{\partial a_7} \\
&\qquad + (6a_2a_6 + 6a_3a_5 + 3a_4^2)\frac{\partial}{\partial a_8} + \&c.
\end{aligned}\right\} \quad . \quad (25).$$

The irreducible matrices of even order are easily seen to be

$$\left.\begin{aligned}
u_2 &= a_2 \\
u_4 &= a_2a_4 - a_3^2 \\
u_6 &= a_2a_6 - 4a_3a_5 + 3a_4^2 \\
u_8 &= a_2a_8 - 6a_3a_7 + 15a_4a_6 - 10a_5^2
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (26),$$

and, generally, when n is even,

$$u_n = a_2a_n - (n-2)\,a_3a_{n-1} + \frac{(n-2)\,(n-3)}{1\,.\,2}\,a_4a_{n-2} - \&c.,$$

the last coefficient being half of the value as appearing from this series.

The matrices of odd orders are found from these by the last form of the process of eduction and are

$$\left.\begin{aligned}
u_5 &= a_2^2a_5 - 3a_2a_3a_4 + 2a_3^3 \\
u_7 &= a_2^2a_7 - 5a_2a_3a_6 + 2a_2a_4a_5 + 8a_3^2a_5 - 6a_3a_4^2 \\
u_9 &= a_2^2a_9 - 7a_2a_3a_8 + 9a_2a_4a_7 - 5a_2a_5a_6 + 12a_3^2a_7 - 30a_3a_4a_6 + 20a_3a_5^2
\end{aligned}\right\} \quad . \quad (27).$$

From these it follows that

$$\left.\begin{aligned}
\Omega_3u_2 &= 0 \\
\Omega_3u_4 &= u_2^3 \\
\Omega_3u_5 &= 0 \\
\Omega_3u_6 &= 10u_2u_4 \\
\Omega_3u_7 &= 7u_2u_5 \\
\Omega_3u_8 &= 21u_2u_6 \\
\Omega_3u_9 &= 16u_2u_7 \\
&\&c.
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28),$$

and

$$\left.\begin{aligned} \Omega_1 u_2 &= 0 \\ \Omega_1 u_4 &= u_2^3 a_3 \\ \Omega_1 u_5 &= 0 \\ \Omega_1 u_6 &= -u_5 + 10u_4 a_3 \\ \Omega_1 u_7 &= -2u_2^2 u_6 + 10u_4^2 + 7u_5 a_3 \\ \Omega_1 u_8 &= -3u_7 + 21u_6 a_3 \\ u_2^2 \Omega_1 u_9 &= -4u_2^4 u_8 + 52u_2^2 u_4 u_6 - 30\,(u_5^2 + 4u_4^3) + 16u_2^2 u_7 a_3 \end{aligned}\right\} \quad . \quad . \quad (29).$$

15. *Reduction of the matrices to functions of differential invariants, and two fundamental matrices of orders* 4 *and* 6.

On the reduction which is now introduced depends the possibility of making the subject of this paper an instrument of research into the character of the higher curves.

For every order higher than the third, there is either a possible matrix or a differential invariant, and for every order higher than the sixth there is a differential invariant. Of the third order there is neither, and of the fourth and sixth orders there is no differential invariant. Every function of differential coefficients can be expressed as a function of a_3, matrices of orders 4 and 6, and differential invariants, and if the function is itself a matrix it will not contain a_3.

As a first step, I replace the irreducible matrices of the sixth and higher orders, by matrices of homogeneous covariants, that is, by functions of the irreducible matrices which satisfy $\Omega_3 f = 0$.

As far as the 9th order these are the irreducible solutions of

$$\frac{du_2}{0} = \frac{du_4}{u_2^2} = \frac{du_5}{0} = \frac{du_6}{10u_4} = \frac{du_7}{7u_5} = \frac{du_8}{21u_6} = \frac{du_9}{16u_7},$$

and are

$$\left.\begin{aligned} \mathrm{L}_6 &= u_2^2 u_6 - 5u_4^2 \\ \mathrm{L}_7 &= u_2^2 u_7 - 7u_4 u_5 \\ \mathrm{L}_8 &= u_2^4 u_8 - 21u_2^2 u_4 u_6 + 70u_4^3 \\ \mathrm{L}_9 &= u_2^4 u_9 - 16u_2^2 u_4 u_7 - 30u_2^2 u_5 u_6 + 206u_4^2 u_5 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (30).$$

From these again we find

$$\begin{aligned} \Omega_1 \mathrm{L}_6 &= -u_2^2 u_5 \\ \Omega_1 \mathrm{L}_7 &= -2u_2^2 \mathrm{L}_6 \\ \Omega_1 \mathrm{L}_8 &= -3u_2^2 \mathrm{L}_7 \\ \Omega_1 \mathrm{L}_9 &= -4u_2^2 \mathrm{L}_8 . \end{aligned}$$

From which we find the differential invariants of orders 7 and 8 conveniently in the forms

$$\left.\begin{aligned} \mathrm{U}_7 &= u_5\mathrm{L}_7 - \mathrm{L}_6^2 \\ \mathrm{U}_8 &= u_5^2\mathrm{L}_8 - 3u_5\mathrm{L}_6\mathrm{L}_7 + 2\mathrm{L}_6^3 \\ \mathrm{U}_9 &= u_5\mathrm{L}_9 - 4\mathrm{L}_6\mathrm{L}_8 + 3\mathrm{L}_7^2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (31),$$

which present the same forms as the irreducible matrices of order 4, 5, and 6.

16. *The differential coefficients of the irreducible matrices, and of the quantities by which they are replaced.*

In this section of the paper the relations between the matrices, &c., are being investigated for the purpose of use in the next section.

The differential coefficients of the irreducible matrices of even order are readily written down, and those of the matrices of odd orders with rather more difficulty. Thus

$$\left.\begin{aligned} & u_2\frac{du_4}{dx} = 5u_5 + 10u_4a_3 \\ & u_2\frac{du_6}{dx} = 7u_7 + 14u_6a_3 \\ & u_2\frac{du_8}{dx} = 9u_9 + 18u_8a_3 \\ & \text{and} \\ & u_2\frac{du_5}{dx} = 6\mathrm{L}_6 + 15u_5a_3 \\ & u_2^3\frac{du_7}{dx} = 8\mathrm{L}_8 + 40u_4\mathrm{L}_6 + 90u_5^2 + 19u_2^2u_7a_3. \\ & \text{Hence} \\ & u_2u_5\frac{d\mathrm{L}_6}{dx} = 7\mathrm{U}_7 + 7\mathrm{L}_6^2 - u_4u_5^2 + 20u_5\mathrm{L}_6a_3 \\ & u_2u_5\frac{d\mathrm{U}_7}{dx} = 8\mathrm{U}_8 + 16\mathrm{U}_7\mathrm{L}_6 + 55u_5^4 + 40u_5\mathrm{U}_7a_3 \\ & u_2u_5\frac{d\mathrm{U}_8}{dx} = 9u_5^3\mathrm{L}_9 - 21\mathrm{U}_7^2 - 12\mathrm{U}_8\mathrm{L}_6 - 54\mathrm{U}_7\mathrm{L}_6^2 - 9\mathrm{L}_6^4 + 60u_5\mathrm{U}_8a_3 \end{aligned}\right\} \; . \quad (32).$$

17. *Expressions for a_2, a_3, a_4, &c., in terms of u_4, L_6, differential invariants and a_3.*

These expressions are found consecutively by the aid of the formulæ which have just been established. Thus,

$$\left.\begin{aligned}
a_2 &= u_2 \\
a_3 &= a_3 \\
a_4 &= \frac{u_4 + a_3^2}{u_2} \\
a_5 &= \frac{u_5 + 3u_4a_3 + a_3^3}{u_2^2} \\
a_6 &= \frac{L_6 + 2u_4^2 + 4u_5a_3 + 6u_4a_3^2 + a_3^4}{u_2^3} \\
a_7 &= \frac{U_7 + L_6^2 + 5u_4u_5^2 + 5u_5(L_6 + 2u_4^2)\,a_3 + 10u_5^2a_3^2 + 10u_4u_5a_3^3 + a_3^5}{u_2^4u_5} \\
a_8 &= \frac{U_8 + 10u_5^4 + 3U_7L_6 + L^3_6 + 6u_4u_5^2L_6 + 5u_4^3u_5^2 + 6u_5(U_7 + L_6^2 + 5u_4u_5^2)\,a_3 + \&c.}{u_2^5u_5^2} \\
a_9 &= \frac{u_5^2U_9 - 3U_7^2 + \&c.}{u_2^6u_5^3}
\end{aligned}\right\} \quad (33).$$

In these formulæ, and in their applications, it must be remembered that they imply that u_2 and u_5 are not zero at the point to which they refer.

§ 3. *Correlative Forms.*

18. *The deduction of the equation to the reciprocal of a covariant from the equation to the original covariant.*

[The principle of duality, as applied to differential invariants, is explained by HALPHEN (p. 56) in his thesis. In the case of covariants, the relation is still more interesting.

The general investigation will come later, but let us begin by considering the connection in its most elementary form. Let x, y, be the coordinates on a covariant in either point- or line-coordinates, and X, Y, the corresponding coordinates on the reciprocal in line- or point-coordinates. We express the correlative transformation by

$$\left.\begin{aligned} X &= y_1 \\ Y &= xy_1 - y \end{aligned}\right\} \quad \text{and inversely} \quad \left\{\begin{aligned} x &= Y_1 \\ y &= XY_1 - Y \end{aligned}\right.$$

y_1 and Y_1 being understood to stand for dy/dx and dY/dX. Then we easily find

$$Y_2 = \frac{1}{y_2}, \quad Y_3 = -\frac{y_3}{y_2^3}, \quad Y_4 = -\frac{y_4}{y_2^4} + \frac{3y_3^2}{y_2^5}, \quad \text{and so on,}$$

with the inverse relationships.

These equations enable us to express any function of Y_2, Y_3, Y_4, &c., as an equivalent function of y_2, y_3, y_4, &c., and HALPHEN has shown the relationship between invariant functions.

Our concern now is with covariants and especially with matrices.

Obviously, the conditions to which a function is subject, in order that it may be a covariant function, are of the same form, whether the coordinates are point- or line-coordinates, and therefore the matrix of the reciprocal of a covariant, regarded as a function of Y_2, Y_3, &c., is a solution of $\Omega_2 F = 0$. It is proposed, in the first place, to deduce the equation to the reciprocal from the equation to the original covariant, and, found in this way, the coefficient which we have called the matrix will be a function of y_2, y_3, &c., and as a function of these quantities it will not be a solution of $\Omega_2 f = 0$.

To find the equation which it does satisfy, we will simplify the writing by taking the point x, y as origin, so that we may write π for $\pi - p$ and ξ for $\xi - x$. The equation to the covariant is a rational integral function in π and ξ.

Let γ and α be the corresponding coordinates on the reciprocal curve with the relation

$$\alpha\xi - \pi - \gamma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

where the third letters which are usually inserted to make the equation homogeneous are omitted as unnecessary and, for the purpose, undesirable.

Following the usual course (SALMON, 'Higher Plane Curves,' 2nd ed., p. 73), the equation to the covariant is now made homogeneous by using this equation, and the discriminant of the resulting equation, considered as a binary quantic in π and ξ, equated to zero, is the required equation to the reciprocal.

It is not necessary that the whole of this process should be performed, for all that is necessary is to obtain the coefficient of the highest power of γ in the resulting equation. We obtain it by putting $\alpha = 0$, and hence *the coefficient of the highest power of γ is the discriminant of the highest group of homogeneous term in the original equation in π and ξ, treated as a binary quantic.*

This coefficient is, of course, found as a function of $y_2, y_3, \ldots$, and when replaced by the corresponding value in terms of Y_2, Y_3, it will be the matrix of the reciprocal and a solution of $\Omega_2 F = 0$.

From the mode of formation of the coefficient of the highest power of γ, it appears that, as a function of $y_2, y_3, \ldots$, it is an invariant for changes of Cartesian coordinates, and therefore, as has been remarked in finding the condition resulting from the coefficient B_1 in the original homographic transformation (12), it will satisfy the equation $\Omega_1 f = 0$.

Before continuing the more general consideration of this relation we will illustrate this on the general osculating conic and its reciprocal.

The terms of the highest order in the covariant conic are

$$(a_2 a_4 - a_3^2)\,\pi^2 + a_2^2 a_3 \pi\xi + a_2^4 \xi^2,$$

and, therefore, the coefficient of γ^2 in the reciprocal will be, omitting a power of a_2,

$$4a_2a_4 - 5a_3^2,$$

and this is seen to satisfy $\Omega_1 f = 0$ (25).

On introduction of A_2, A_3, A_4 to replace Y_2, Y_3, Y_4, as a_2, a_3, a_4 replace y_2, y_3, and y_4, and omitting a power of A_2, we obtain

$$A_2A_4 - A_3^2$$

for the matrix of the reciprocal conic as a function of A_2, A_3, A_4, in which form it is, of course, a solution of $\Omega_2 F = 0$.—*Rewritten August* 2, 1893.]

19. *Dual relation between the operators* Ω_1 *and* Ω_2.

The methods of solution of $\Omega_1 f = 0$ and $\Omega_2 f = 0$, are alike, inasmuch as each depends upon a method of eduction starting from the solutions of the second and fourth orders. The general proposition is now stated, that if u be a solution of either $\Omega_1 f = 0$ or $\Omega_2 f = 0$, in which y_2, y_3, &c., or, what is the same thing, a_2, a_3, &c., are the arguments, and if in consequence of the substitution for these quantities of Y_2, Y_3, &c., or of A_2, A_3, &c., u becomes U, then U is a solution of the other corresponding equation, that is, of $\Omega_2 F = 0$ or $\Omega_1 F = 0$.

As far as the first three solutions of $\Omega_1 f = 0$ are concerned, it is readily verified that

$$u_2, \qquad u_4, \qquad \text{and} \qquad u_5,$$

become

$$U_2, \qquad 4A_2A_4 - 5A_3^2, \qquad \text{and} \qquad U_5,$$

divided respectively by U_2^2, $-U_2^6$, and $-U_2^9$ and numerical factors, where the index of the power of U_2 is the weight of the function which it affects.

If θ be a function of a_2, a_3, &c., of weight w and degree d_x, on substitution it will become Θ/U_2^w, where Θ has the weight and degree of θ. Writing W and D_x for the weight and degree of Θ/U_2^w we obtain

$$W = -w, \qquad D_x = d_x - w,$$

so that

$$W + D_x = -(2w - d_x) \quad \text{and} \quad -(2W - D_x) = w + d_x \, . \quad . \quad . \quad (35).$$

Bearing in mind the conditions (18A) for the eduction of solutions of $\Omega_1 f = 0$ and $\Omega_2 f = 0$, it follows that processes which educe solutions of $\Omega_2 f = 0$ from functions of the character of θ, will educe solutions of $\Omega_1 F = 0$ from functions of the character of Θ/U_2^w, and *vice versâ*.

As a consequence, in making a change from point-coordinates to the correlated line coordinates, or *vice versâ*, the operators Ω_1 and Ω_2 are interchanged. Or expressed in terms of a_2, a_3, &c., the coefficient of the highest power of γ in a contravariant is a solution of $\Omega_1 f = 0$.

The proof just given is connected with the theory of eduction, but it is perhaps more simple to notice that the relation

$$\alpha\xi - \pi - \gamma = 0$$

must hold not only for the original coordinates, but also for the coordinates in the general homographic transformations, and hence to deduce the dual relationship from the essential principles of this paper.

That is, if we make infinitesimal homographic transformation for both ξ, π, and α, γ, indicated by

$$\frac{\xi}{\xi' + B_1\pi'} = \frac{\pi}{\pi'} = \frac{1}{A\xi' + B\pi' + 1},$$

$$\frac{\alpha}{\alpha' + D_1\gamma'} = \frac{\gamma}{\gamma'} = \frac{1}{C\alpha' + D\gamma' + 1},$$

as in Art. 2,

then, in consequence of

$$\alpha\xi - \pi - \gamma = 0,$$

$$B_1\pi'\alpha' + D_1\xi'\gamma' = \pi'(C\alpha' + D\gamma') + \gamma'(A\xi' + B\pi'),$$

and

$$B_1 = C, \qquad A = D_1, \qquad D + B = 0.$$

Or A is replaced by (C or by) B_1, and B_1 by (D_1 or by) A, while B is replaced by (D or by) $-$ B.

Hence the conditions derived from A and B_1 are interchanged, while those derived from B remain unaltered (Art. 6). That is Ω_1 and Ω_2 are interchanged, and Ω_3 is unaltered.

20. *Relations of the differential invariants, and the solutions of* $\Omega_1 f = 0$ *and* $\Omega_2 f = 0$ *to the ordinary invariants of a curve.*

Differential invariants appear in the processes of eduction, that is ultimately as the result of differentiation, in three ways. Firstly, in a series of differential invariants only; secondly, from a series of solutions of $\Omega_1 f = 0$, *i.e.*, a series of differential invariants for change of coordinates or semi-invariants; and thirdly, from the series which we have called matrices, which are semi-invariants in the correlated system of coordinates.

The differential equation to a curve involves differential invariants only. If we find the series of first integrals (in the next section I illustrate, on the cubic, the general method of finding this series), we may form from among them the functions of differential invariants, of semi-invariants, and of matrices which are constant along the curve. These functions may be properly called (each in its proper sphere, as homographic or Cartesian) differential expressions for the invariants of the curve.

It is perhaps worthy of remark that in showing the mode of integrating the differential equation of conics, Mr. ELLIOTT* does, in fact, illustrate these methods of procedure.

§ 4. *Application of Results to the Cubic.*

21. *The differential equation of the general cubic.*

It has been previously shown that the order of this differential equation will be 9, the degree 10, and the weight 35. It is also a function of u_5, U_7, U_8, and U_9, since it is easily seen that u_2 cannot explicitly enter the equation.

Written as a determinant, the equation is

$$\begin{vmatrix} a_9, & a_8, & a_7, & 2a_3a_6+2a_4a_5, & a_4^2+2a_3a_5+2a_2a_6, & a_3^3+6a_2a_3a_4+3a_2^2a_5 \\ a_8, & a_7, & a_6, & a_4^2+2a_3a_5, & 2a_2a_5+2a_3a_4, & 3a_2^2a_4+3a_2a_3^2 \\ a_7, & a_6, & a_5, & 2a_3a_4, & a_3^2+2a_2a_4, & 3a_2^2a_3 \\ a_6, & a_5, & a_4, & a_3^2, & a_2a_3, & a_2^3 \\ a_5, & a_4, & a_3, & 0, & a_2^2, & 0 \\ a_4, & a_3, & a_2, & -a_2^2, & 0 & 0 \end{vmatrix} = 0.$$

From this we see that a_8 and a_9 enter only in the form $a_7a_9 - a_8^2$.

Hence we have the terms $u_5^2U_7U_9 - U_8^2 + \ldots$ Since the degree of these terms is 24, and weight 72, the differential invariant when complete must be divisible by $u_2^5u_5^3$.

The differential equation therefore takes the form

$$u_5^2U_7U_9 - (U_8^2 + 4U_7^3) + Eu_5^4U_8 + Tu_5^8 = 0.$$

Determining E and T, either by the condition of containing u_2^5 as further factors or by comparison with the determinant, we obtain

$$u_5^2U_7U_9 - U_8^2 - 4U_7^3 - 15U_5^4U_8 - 56U_5^8 = 0,$$

or

$$u_5^2U_7U_9 - V_8^2 - 4U_7^3 + u_5^4V_8 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

where

$$V_8 = U_8 + 8u_5^4.$$

22. *The first integrals of the differential equation of the general cubic.*

From the equation to the general cubic, found as a differential covariant, it is possible to find a complete set of the first integrals of the differential equation.

For the equation to the cubic being written in the form

$$\begin{gathered} b\,(\eta - y_1\,\xi - y + y_1x)^3 + 3b_1\,(\eta - y_1\,\xi - y + y_1x)^2\,(\xi - x) \\ + 3a_2\,(\eta - y_1\,\xi - y + y_1x)\,(\xi - x)^2 + a\,(\xi - x)^3 \\ + 3b_3\,(\eta - y_1\,\xi - y + y_1x)^2 + 6m\,(\eta - y_1\,\xi - y + y_1x)\,(\xi - x) \\ + 3a_3\,(\xi - x)^2 + 3c_2\,(\eta - y_1\,\xi - y + y_1x) = 0, \end{gathered}$$

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* 'Messenger of Mathematics,' vol. 19, p. 5, 1889.

and expressing it in terms of powers of ξ and η, we get

$$\begin{aligned} & b\eta^3 \\ & - 3(by_1 - b_1)\eta^2\xi \\ & + 3(by_1^2 - 2b_1y_1 + a_2)\eta\xi^2 + \&c. \\ & - 3\{b(y - y_1x) + b_1x - b_3\}\eta^2 + \&c. = 0\,; \end{aligned}$$

the coefficients being easily written down.

Now, whatever be the point $(x.y)$ taken upon the curve, we obtain the equation to the curve in an identical form, and therefore the ratios of corresponding coefficients in any two such equations are equivalent.

This being the case, we obtain nine first integrals of the differential equation to the cubic, to which we may give the forms

$$\begin{aligned} & y_1 - \frac{b_1}{b} = \text{constant}, \\ & y_1^2 - 2\frac{b_1}{b} + \frac{a_2}{b} = \text{constant}, \\ & y - y_1x + \frac{b_1}{b}x - \frac{b_3}{b} = \text{constant}, \\ & \quad \&c. \end{aligned}$$

On account of the obvious character of these expressions, I do not write them down, nor shall I perform the simple processes whereby they are simplified, but merely write down the final forms in which I propose to deal with them.

For this purpose, write

$$\left.\begin{aligned} & \frac{a_2}{b} - \frac{b_1^2}{b^2} = h_1 \\ & \frac{m}{b} - \frac{b_1b_3}{b^2} = h_2 \\ & \frac{c_2}{b} - \frac{b_3^2}{b^2} = h_3 \\ & \frac{a}{b} - 3\frac{a_2b_1}{b^2} + 2\frac{b_1^3}{b^3} = k_1 \\ & \frac{a_3}{b} - \frac{a_2b_3}{b^2} - 2\frac{b_1}{b}\left(\frac{m}{b} - \frac{b_1b_3}{b^2}\right) = k_2 \\ & \frac{2mb_3}{b^2} + \frac{b_1c_2}{b^2} - 2\frac{b_1b_3^2}{b^3} = k_3 \\ & 3\frac{c_2b_3}{b_2} - 2\frac{b_3^3}{b^3} = k_4. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (37).$$

Then the nine integrals take the form

$$\left.\begin{aligned} y_1 - \frac{b_1}{b} &= \text{constant} \\ y - y_1 x + \frac{b_1}{b} x - \frac{b_3}{b} &= \text{constant} \\ h_1 x - h_2 &= \text{constant.} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (38),$$

$$\left.\begin{aligned} h_1 &= \text{constant} \\ k_1 &= \text{constant} \\ h_2 k_1 - h_1 k_2 &= \text{constant} \\ h_1 h_3 - h_2^2 &= \text{constant} \\ k_1 k_3 + k_2^2 &= \text{constant} \\ k_1^2 k_4 - 3k_1 k_2 k_3 - 2k_3^2 &= \text{constant} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (39),$$

of which the last six do not contain x, y, or y_1.

To select from among these six first integrals those which are functions of the matrices only.

It is readily proved that

$$\begin{aligned} \Omega_2 h_1 &= -\ 2h_2, & \Omega_2 k_1 &= -\ 3k_2, \\ \Omega_2 h_2 &= -\ \ h_3, & \Omega_2 k_2 &= \quad 2k_3, \\ \Omega_3 h_3 &= \quad 0, & \Omega_2 k_3 &= -\ \ k_4, \\ & & \Omega_2 k_4 &= \quad 0, \end{aligned}$$

and forming the functional solutions of

$$\frac{dh_1}{2h_2} = \frac{dh_2}{h_3} = \frac{dh_3}{0} = \frac{dk_1}{3k_2} = \frac{dk_2}{-2k_3} = \frac{dk_3}{k_4} = \frac{dk_4}{0} \quad \ldots\ldots \quad (40)$$

we get

$$\left.\begin{aligned} & h_3 \\ & k_4 \\ & h_1 k_4 - h_3 k_3 \\ & h_1 h_3 - h_2^2 \\ & k_4 k_2 + k_3^2 \\ & k_4^2 k_1 - 3k_4 k_3 k_2 - 2k_2^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (41),$$

forms which are at once recognized as being obtained from those of the last six integrals (39), by interchange of the subscripts 1 and 3 of h, and of 1 and 4, 2 and 3 of k.

Hence the functions involved in the first integrals are the solutions of a set of differential equations analogous to (40), and it is the common solutions of these two sets of equations treated as simultaneous that we are seeking. It is easy to prove that the number of such simultaneous solutions is four, and they can easily be constructed as follows :—

$$\left.\begin{aligned}
P_1 &= h_1h_3 - h_2^2 \\
P_2 &= h_1(k_2k_4 + k_3^2) + h_2(k_2k_3 - k_1k_4) + h_3(k_1k_3 + k_2^2) \\
P_3 &= k_1^2k_4^2 - 6k_1k_2k_3k_4 - 4k_1k_3^3 - 4k_2^3k_4 - 3k_2^2k_3^2 \\
P_4 &= h_1(h_1k_4 - 2h_2k_3 - h_3k_2)^2 + 2h_2(h_1k_4 - 2h_2k_3 - h_3k_2)(h_3k_1 - 2h_2k_2 - h_1k_3) \\
&\qquad + h_3(h_3k_1 - 2h_2k_2 - h_1k_3)^2
\end{aligned}\right\} (42).$$

Of these P_1 can be shown to be Matrix of Hessian/(Matrix of General Cubic)^3.

23. *To find the first integral of the differential equation to the general cubic which is a differential invariant.*

For this purpose we are to find a function of the P's which is a solution of $\Omega_1 f = 0$. With rather more difficulty than in the case of Ω_2, we find

$$\begin{aligned}
\Omega_1 h_1 &= k_1 - 4\frac{b_1}{b}h, \\
\Omega_1 h_2 &= k_2 - 3\frac{b_1}{b}h_2 - \frac{b_3}{b}h_1, \\
\Omega_1 h_3 &= -k_3 - 2\frac{b_1}{b}h_3 - 2\frac{b_3}{b}h_2, \\
\Omega_1 k_1 &= -6h_1^2 - 6\frac{b_1}{b}k_1, \\
\Omega_1 k_2 &= -6h_1h_2 - 5\frac{b_1}{b}k_2 - \frac{b_3}{b}k_1, \\
\Omega_1 k_3 &= 2h_1h_3 + 4h_2^2 - 4\frac{b_1}{b}k_3 + 2\frac{b_3}{b}k_2, \\
\Omega_1 k_4 &= 6h_2h_3 - 3\frac{b_1}{b}k_4 - 3\frac{b_3}{b}k_3.
\end{aligned}$$

The problem is now to find functions of the P's, which satisfy simultaneously

$$\frac{dh_1}{k_1} = \frac{dh_2}{k_2} = \frac{dh_3}{-k_3} = \frac{dk_1}{-6h_1^2} = \frac{dk_2}{-6h_1h_2} = \frac{dk_3}{2h_1h_3 + 4h_2^2} = \frac{dk_4}{6h_2h_3},$$

and

$$\frac{dh_1}{4h_1} = \frac{dh_2}{3h_2} = \frac{dh_3}{2h_3} = \frac{dk_1}{6k_1} = \frac{dk_2}{5k_2} = \frac{dk_3}{4k_3} = \frac{dk_4}{3k_4},$$

the third equation being satisfied by each of the six first integrals.

The first of these equations is satisfied by

$$\left.\begin{aligned}&P_1^2 - P_2\\ \text{and}\qquad &2P_4 + 3P_3 - 10P_1P_2 + 14P_1^3\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

Hence

$$(P_1^2 - P_2)^3/(2P_4 + 3P_3 - 10P_1P_2 + 14P_1^3)^2 \quad . \quad . \quad . \quad . \quad . \quad (44),$$

is the function of differential invariants which gives a first integral of the differential equation, or in other words, is the absolute invariant of the cubic.

24. *The general form of the matrix of a cubic.*

In all cases the matrix is a function of u_4, L_6, and the differential invariants. In the most general case the matrix of the cubic will contain u_4^3, but if we take the cubic to touch the standard curve (which is actually to be the curve itself), the matrix would contain only u_4^2.

The coefficient of u_4^2 will also be simply a differential invariant, and, from the nature of the operators Ω_1 and Ω_3, the complete form is found to be

$$\psi + \psi_1 L_6 + \psi_2 L_6^2 + \psi_3 L_6^3 + \psi_4 L_6^4 + (\phi + \phi_1 L_6 + \phi_2 L_6^2)\, u_4 + f u_4^2,$$

with the condition which will be presently seen

$$f + u_5^2\phi_2 + u_5^4\psi_4 = 0.$$

Taking the point of contact with the standard curve as temporary origin, and writing the equation

$$b\pi^3 + 3b_1\pi^2\xi + 3a_2\pi\xi^2 + a\xi^3 + 3b_3\pi^2 + 6m\pi\xi + 3a_3\xi^2 + 3c_2\pi = 0,$$

we shall have

$$\left.\begin{aligned}
b\ &= \text{matrix, as above}\\
3b_1 &= -\,u_2^2u_5\,\{\psi_1 + 2\psi_2 L_6 + 3\psi_3 L_6^2 + 4\psi_4 L_6^3 + (\phi_1 + 2\phi_2 L_6)\,u_4\}\\
&\qquad + u_2^2\,\{\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2fu_4\}\,a_3\\
3a_2 &= \quad u_2^4u_5^2\,\{\psi_2 + 3\psi_3 L_6 + 6\psi_4 L_6^2 + \phi_2 u_4\}\\
&\qquad + u_2^4\,(\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2fu_4) - u_2^4u_5\,(\phi_1 + 2\phi_2 L_6)\,a_3 + u_2^4 f a_3^2\\
a\ &= -\,u_2^6u_5^3\,(\psi_3 + 4\psi_4 L_6) - u_2^6u_5\,(\phi_1 + 2\phi_2 L_6) + u_2^6\,(2f + u_5^2\phi_2)\,a_3\\
&\text{with the condition}\\
&\qquad\qquad f + u_5^2\phi_2 + u_5^4\psi_4 = 0.\\
&\text{Also}\\
&\qquad\qquad 3b_3 = -\,u_2^3\,(\phi + \phi_1 L_6 + \phi_2 L_6^2 + 2fu_4)\\
&\qquad\qquad 6m = \quad u_2^5u_5\,(\phi_1 + 2\phi_2 L_6) - 2u_2^5 f a_3\\
&\qquad\qquad 3a_3 = -\,u_2^7u_5^2\phi_2 - 2u_2^7 f\\
&\qquad\qquad 3c_2 = \quad u_2^6 f
\end{aligned}\right\} \quad (45).$$

We may now form the coefficient of the highest power of γ in the equation to the reciprocal curve. This coefficient is to be found, as has been shown, from the discriminant of the terms of the highest degree in this cubic, regarded as a binary cubic, and is therefore

$$b^2a^2 + 4\,ba_2^3 + 4\,b_1^3a - 6ab\,b_1a_2 - 3\,b_1^2a_2^2.$$

From this the equation to the reciprocal curve would be developed according to the previous rules but with an interchange of the operators Ω_1 and Ω_2. All the quantities in the expanded coefficient are, however, solutions of $\Omega_2 f = 0$, except a_3, and hence the degree of the reciprocal curve is indicated by the power of a_3 in the expanded expression.

25. *The matrix of the non-singular cubic, and its differential equation.*

The condition of intersecting the standard curve at a number of points coincident with the temporary origin is an invariantal relation, and in finding the condition we may treat the expressions as if they contained only differential invariants in their coefficients.

Thus we write the equation to the cubic

$$\begin{aligned}\psi\pi^3 - u_2^2u_5\psi_1\pi^2\xi + u_2^4(\phi + u_5^2\psi_2)\,\pi\xi^2 - u_2^6u_5(\phi_1 + u_5^2\psi_3)\,\xi^3\\ - u_2^3\phi\pi^2 + u_2^2u_5\phi_1\pi\xi - u_2^7(2f + u_5^2\phi_2)\,\xi^2 + u_2^6f\pi = 0,\end{aligned}$$

and, for the consecutive points on the standard curve, putting

$$\xi = h,$$

we have (33)

$$\pi = u_2h^2 + \frac{u_5}{u_2^2}h^5 + \frac{U_7}{u_2^4u_5}h^7 + \frac{U_8 + 10u_5^4}{u_2^5u_5^2}h^8 + \frac{u_5^2U_9 - 3U_7^2}{u_2^6u_5^3}h^9 + \&c.$$

Equating to zero the coefficients of the several powers of h, on substitution for π and ξ in the equation to the cubic, we get

$$\left.\begin{array}{lll}\text{from the coefficient of } h^2, & \psi_4 = 0 \quad \text{and} \quad f + u_5^2\phi_2 = 0\\ \quad ,, \qquad\quad ,, \qquad\quad h^3, & \psi_3 = 0\\ \quad ,, \qquad\quad ,, \qquad\quad h^4, & \psi_2 = 0\\ \quad ,, \qquad\quad ,, \qquad\quad h^5, & \psi_1 = f\\ \quad ,, \qquad\quad ,, \qquad\quad h^6, & \psi = -u_5^2\phi_1\\ \quad ,, \qquad\quad ,, \qquad\quad h^7, & u_5^2\phi = U_7f\\ \quad ,, \qquad\quad ,, \qquad\quad h^8, & u_5^2U_7\phi_1 = -V_8f\end{array}\right\} \quad . \quad . \quad (46).$$

These are sufficient to determine uniquely the coefficients in the matrix, and from the coefficient of h^9 we derive the differential equation to the non-singular cubic

$$\frac{u_5^2 U_9 - 3U_7^2}{u_5^3} f + \frac{V_8 + 2u_5^4}{u_5} \phi_1 - \frac{U_7}{u_5} \phi + 3u_5 \psi = 0,$$

which becomes

$$u_5^2 U_7 U_9 - V_8^2 - 4U_7^3 + u_5^4 V_8 = 0,$$

and is identical with that previously found in (36).

And the matrix of the non-singular cubic is

$$U_7 u_4^2 + \frac{U_7^2 - V_8 L_6 - U_7 L_6^2}{u_5^2} u_4 + V_8 + U_7 L_6.$$

26. *The matrix of the equation to the tangents to the cubic from the temporary origin.*

To find this equation, put $\pi = m\xi_1$, and form the discriminant of the resulting expression in ξ as a binary quantic. Equating this to zero, replace m by π/ξ. Since we need only the coefficient of the highest power of π, the simplest method of finding it is to put $\xi = 0$ in the matrix of the cubic, and form the discriminant of the resulting expression in π.

The matrix required is

$$9b_3^3 - 12bc_2,$$

or

$$(\phi + \phi_1 L_6 + \phi_2 L_6^2)^2 - 4f(\psi + \psi_1 L_6 + \psi_2 L_6^2 + \psi_3 L_6^3 + \psi_4 L_6^2) \quad . \quad . \quad (47),$$

and the degree of this in L_6 denotes the number of tangents which can be drawn.

The conditions that the cubic may be nodal or cuspidal are that this matrix, as a function of L_6, may have a linear factor twice or three times repeated.

27. *Case of nodal cubic.*

In this case we still determine uniquely all the ratios of the coefficients, except ψ/f, by the condition that the coefficients of h up to the seventh vanish identically, while the differential equation is found by equating the coefficient of h^8 to zero.

The further condition, to determine ψ/f, is (47) that the discriminant of

$$(fL_6^2 + \psi L_6 - U_7 f)^2 - 4u_5^4 f(fL_6 + \psi)$$

may vanish.

Writing k or $\psi/2f$, and x for $L_6 + k$, this equation becomes

$$x^4 - 2(k^2 + U_7)x^2 - 4u_5^4 x + (k^2 + U_7)^2 - 4u_5^4 k = 0 \quad . \quad . \quad . \quad (48).$$

With the usual notation (I and J) of theory of equations

$$\frac{3\mathrm{I}}{4} = (k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k,$$

$$-\frac{27\mathrm{J}}{8} = (k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^4,$$

and the equation giving k is

$$\{(k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k\}^3 - \{(k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^8\}^2 = 0,$$

a biquadratic for k.

To find the differential equation we have

$$\mathrm{U}_7\psi = \mathrm{V}_8 f,$$

and therefore

$$\begin{aligned}&\{(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3)^2 - 24u_5{}^4\mathrm{V}_8\mathrm{U}_7{}^3\}^3\\ &\qquad - \{(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3)^3 - 36u_5{}^4\mathrm{V}_8\mathrm{U}_7{}^3(\mathrm{V}_8{}^2 + 4\mathrm{U}_7{}^3) + 216u_5{}^8\mathrm{U}_7{}^6\}^2 = 0 \quad . \quad (49),\end{aligned}$$

which contains the factor $u_5{}^{12}$.

The biquadratic from which k is to be found is

$$\mathrm{U}_7k^4 - \tfrac{1}{2}u_5{}^4k^3 + 2\mathrm{U}_7{}^2k^2 - \tfrac{9}{2}u_5{}^4\mathrm{U}_7k + \mathrm{U}_7{}^3 + \tfrac{27}{16}u_5{}^8 = 0 \quad . \quad . \quad . \quad (50).$$

The discriminant of this is found to be

$$16u_5{}^8\{256\mathrm{U}_7{}^3 - 27u_5{}^8\}^3.$$

And if $256\mathrm{U}_7{}^3 - 27u_5{}^8 > 0$, all the roots are real.

In this case, four real nodal cubics can be drawn to have contact of the seventh order with the standard curve. If $256\mathrm{U}_7{}^3 - 27u_5{}^8 < 0$ only two. If $256\mathrm{U}_7{}^3 - 27u_5{}^8 = 0$, there will be two nodal cubics, the third being cuspidal, as will appear in the next section.

Equiharmonic and harmonic cubics can also be drawn to have contact of the seventh order, and it is readily seen from the Theory of Forms that k will in these cases be determined by

$$(k^2 + \mathrm{U}_7)^3 - \tfrac{9}{2}u_5{}^4k\,(k^2 + \mathrm{U}_7) + \tfrac{27}{8}u_5{}^8 = 0 \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and

$$(k^2 + \mathrm{U}_7)^2 - 3u_5{}^4k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

respectively,

the roots in the latter case being all real, provided

$$256U_7^3 - 243u_5^8 > 0.$$

28. *Case of cuspidal cubic.*

The ratio ϕ/f is now to be determined from the equations of condition, as well as ψ/f.

Writing $u_5^2\phi/f = q$, we may write down the conditions from that obtained for the nodal cubic, viz.,

$$(k^2 + q)^2 - 3u_5^4k = 0.$$

$$(k^2 + q)^3 - \tfrac{9}{2}u_5^4k\,(k^2 + q) + \tfrac{27}{8}u_5^8 = 0.$$

Whence,

$$\left.\begin{aligned} 256q^3 &= 27u_5^8 \\ 16k^3 &= 27u_5^4 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (53),$$

and the differential equation becomes

$$256U_7^3 - 27u_5^8 = 0 \quad \ldots\ldots\ldots \quad (54).$$

29. *To exemplify the use of the general forms of the equation to the cubic and of the matrix.*

The coordinates of the tangential of the origin are

$$\pi = 0, \quad \xi = -\frac{3a_3}{a},$$

and the matrix of the polar conic of the tangential becomes

$$(\phi_1 + 2\phi_2L_6)\,[fu_4 + (\phi + \phi_1L_6 + \phi_2L_6^2)] - f\psi_1.$$

Taking the general cubic this becomes

$$\frac{(V_8 + 2U_7L_6)}{u_5}\left[U_7u_4 + \frac{(U_7^2 - V_8L_6 - U_7L_6^2)}{u_5^2}\right] + u_5,$$

and the matrix (and conic) breaks up into factors, provided $u_5 = 0$, since $U_7^2 - V_8L_6 - U_7L_6^2$ contains the factor u_5^2.

That is, as is known, the tangent at a sextactic point passes through a point of inflexion.

The matrix of the nodal cubic is given by

$$b/u_5^2 = 2k + L_6 + \frac{U_7 - 2kL_6 - L_6^2}{u_5^2}u_4 + u_4^2.$$

The biquadratic for k may be written

$$\mathrm{U}_7 (k^2 + \mathrm{U}_7)^2 = \tfrac{1}{2} u_5^4 (k^3 + 9\mathrm{U}_7 k - \tfrac{27}{8} u_5^4),$$

and $k^2 + \mathrm{U}_7$ contains the factor u_5^2.

Hence, from the equation (48) for x, x or $k + \mathrm{L}_6$ contains the factor u_5, and $\mathrm{U}_7 - 2k\mathrm{L}_6 - \mathrm{L}_6^2$ or $\mathrm{U}_7 + k^2 - (k + \mathrm{L}_6)^2$ contains the factor u_5^2, so that u_5^2 is a factor of the matrix b.

The matrix of the polar conic of the tangential of the temporary origin is therefore

$$2 \frac{k + \mathrm{L}_6}{u_5} \left\{ u_4 + \frac{\mathrm{U}_7 - 2k\mathrm{L}_6 - \mathrm{L}_6^2}{u_5^2} \right\} + u_5,$$

and the tangent at a sextactic point will pass through the point of inflexion.

Evidently there can not exist any sextactic point on the cuspidal cubic.

§ 5. *Invariant Coordinates and Invariantal Equations.*

To make it possible to use the methods of this paper to investigate the properties of the higher plane curves, it is desirable to effect some gain in brevity and lucidity without any sacrifice in generality. This is the object of the system of invariant coordinates, which are introduced in this last section.

The matrix from which the equation to a covariant curve is developed contains u_4, L_6, and invariant functions. The order of the curve, in general, depends upon the mode in which u_4 and L_6 enter the matrix, and its characteristics upon the ratios between the invariant functions. These ratios have the properties which entitle them to be regarded as the invariantal coordinates of the curve whose equation is developed from the matrix. In especial, I shall show that this is the case in the equation to a covariant straight line, or line of homographic persistence, and, from the consideration of such lines, arrive at a definition of invariant coordinates of a point of homographic persistence. After showing that we obtain a dual invariant system of point and line coordinates, I ultimately develop a mode of representing a curve by an invariant equation, which will be much shorter than the covariant form of the equation which we have as yet considered. The coordinates to the curve will be the characteristic invariants of the curve, and none of the homographic peculiarities will be sacrificed by the abbreviation.

30. *Introduction of invariant coordinates of a homographically persistent point or curve, and of an intrinsic invariant equation to such a curve.*

The general equation to a covariant line takes the form

$$(u_5^2\phi_3u_4+\phi_1+\phi_2\mathrm{L}_6-\phi_3\mathrm{L}_6^2)\pi - u_2^2\{(\phi_2-2\phi_3\mathrm{L}_6)u_5-u_5^2\phi_3a_3\}\xi - u_2^3u_5^2\phi_3 = 0\ldots$$

where all the functions ϕ contain differential invariants only. Such a line may be called homographically persistent with regard to the standard curve.

Indicating a second homographically persistent line by a similar notation, but using dashed functions, we may find the coordinates of the point of intersection in the form

$$\left.\begin{aligned}\pi &= u_2^3\mathrm{A}/u_4\mathrm{A}+\mathrm{C}+\mathrm{B}a_3\\ \xi &= u_2\mathrm{B}/u_4\mathrm{A}+\mathrm{C}+\mathrm{B}a_3\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55),$$

where

$$\left.\begin{aligned}\mathrm{A} &= -u_5^2(\phi_2\phi_3'-\phi_2'\phi_3)\\ \mathrm{B} &= u_5\{(\phi_3\phi_1'-\phi_3'\phi_1)-(\phi_2\phi_3'-\phi_2'\phi_3)\mathrm{L}_6\}\\ \mathrm{C} &= (\phi_1\phi_2'-\phi_1'\phi_2)+2(\phi_3\phi_1'-\phi_3'\phi_1)\mathrm{L}_6-(\phi_2\phi_3'-\phi_2'\phi_3)\mathrm{L}_6^2\end{aligned}\right\},$$

or, more shortly,

$$\left.\begin{aligned}\mathrm{A} &= -u_5^2\lambda\\ \mathrm{B} &= u_5(\mu-\lambda\mathrm{L}_6)\\ \mathrm{C} &= \nu+2\mu\mathrm{L}_6-\lambda\mathrm{L}_6^2\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

We may call such a point a point of homographic persistence with regard to the standard curve.

Treating $(\lambda:\mu:\nu)$ as determining the position of a point, and $(\phi_1:\phi_2:\phi_3)$ the position of a line, the condition that $(\lambda:\mu:\nu)$ may lie on $(\phi_1:\phi_2:\phi_3)$ is

$$\lambda\phi_1+\mu\phi_2+\nu\phi_3=0.$$

Hence these form correlative systems of point- and line-coordinates.

If two lines $(\phi_1:\phi_2:\phi_3)$, $(\phi'_1:\phi'_2:\phi'_3)$ meet in $(\lambda:\mu:\nu)$

$$\frac{\lambda}{\begin{vmatrix}\phi_2, & \phi_3\\ \phi'_2, & \phi'_3\end{vmatrix}} = \frac{\mu}{\begin{vmatrix}\phi_3, & \phi_1\\ \phi'_3, & \phi'_1\end{vmatrix}} = \frac{\nu}{\begin{vmatrix}\phi_1, & \phi_2\\ \phi'_1, & \phi'_2\end{vmatrix}}$$

and if two points $(\lambda:\mu:\nu)$, $(\lambda':\mu':\nu')$ lie on $(\phi_1:\phi_2:\phi_3)$

$$\frac{\phi_1}{\begin{vmatrix}\mu, & \nu\\ \mu', & \nu'\end{vmatrix}} = \frac{\phi_2}{\begin{vmatrix}\nu, & \lambda\\ \nu', & \lambda'\end{vmatrix}} = \frac{\phi_3}{\begin{vmatrix}\lambda, & \mu\\ \lambda', & \mu'\end{vmatrix}}$$

If three points $(\lambda : \mu : \nu)$, $(\lambda' : \mu' : \nu')$, $(\lambda'' : \mu'' : \nu'')$ lie on a straight line

$$\begin{vmatrix} \lambda, & \mu, & \nu \\ \lambda', & \mu', & \mu' \\ \lambda'', & \mu'', & \nu'' \end{vmatrix} = 0,$$

and so on.

I define $(\lambda : \mu : \nu)$ as the invariantal coordinates of the point, and $(\phi_1 : \phi_2 : \phi_3)$ as the invariantal coordinates of the line.

The condition that $(\lambda : \mu : \nu)$ may lie upon a covariant curve of the n^{th} order will be an invariantal relation between λ, μ, ν and the coordinates of the curve, homogeneous of the n^{th} degree in λ, μ, ν.

It follows that if $f(\lambda, \mu, \nu) = 0$ expresses this relation, it is, in this system of coordinates, the equation of the curve. I shall call it the intrinsic invariant equation to the curve.

The coordinates of the tangent to the curve at $(\lambda : \mu : \nu)$ are

$$\frac{\partial f}{\partial \lambda} : \frac{\partial f}{\partial \mu} : \frac{\partial f}{\partial \nu}.$$

If $(\lambda' : \mu' : \nu')$ lies on the tangent to the curve at $(\lambda : \mu : \nu)$ then

$$\lambda' \frac{\partial f}{\partial \lambda} + \mu' \frac{\partial f}{\partial \mu} + \nu' \frac{\partial f}{\partial \nu} = 0.$$

Taken with $f(\lambda, \mu, \nu) = 0$, this gives the coordinates of the points of contact of tangents from $(\lambda' : \mu' : \nu')$, and, being of the $n - 1^{\text{th}}$ degree, it represents the first polar of $f(\lambda, \mu, \nu) = 0$ at $(\lambda' : \mu' : \nu')$.

Finding the first polar of this again, the second polar of $f(\lambda, \mu, \nu) = 0$ has the equation

$$\left(\lambda' \frac{\partial}{\partial \lambda} + \mu' \frac{\partial}{\partial \mu} + \nu' \frac{\partial}{\partial \nu}\right)^2 f = 0.$$

It is unnecessary to carry this further to determine how far the ordinary methods of geometry apply to this novel system of coordinates, since the character of the analogy is obvious.

31. *The values of the differential invariants at different points on the curve.*

The coordinates of any point on a covariant curve were, (55) and (56), given in the form

$$\pi = - u_2^3 u_5^2 \mathrm{Y} / - (u_4 u_5^2 + \mathrm{L}_6^2 + u_5 \mathrm{L}_6 a_3) \mathrm{Y} + (\mathrm{L}_6 + u_5 a_3) \mathrm{X} + 1,$$

$$\xi = - u_2 u_5 (\mathrm{L}_6 \mathrm{Y} - \mathrm{X}) / - (u_4 u_5^2 + \mathrm{L}_6^2 + u_5 \mathrm{L}_6 a_3) \mathrm{Y} + (\mathrm{L}_6 + u_5 a_3) \mathrm{X} + 1,$$

where X and Y have been written for μ/ν and λ/ν respectively. Here X and Y are the variables as we pass along a curve, and $d^2\mathrm{Y}/d\mathrm{X}^2$, &c., are to be found from the intrinsic invariant equation to the curve which we may write $f(\mathrm{X}, \mathrm{Y}) = 0$.

The relations between ξ, π and X, Y are of the form of a homographic transformation, and therefore any of the functions of $d^2\pi/d\xi^2$, &c., which we know to be differential invariants, will only differ from the similar functions of $d^2\mathrm{Y}/d\mathrm{X}^2$, &c., by a factor, of which we know the form, which will contain u_4, L_6, and a_3.

It will not be possible to express the value of a differential invariant at (X, Y) in terms of X, Y and differential invariants at the origin only, unless the said differential invariant is absolute, but the invariantal coordinates of a point of homographic singularity will be found by applying the condition immediately to $f(\mathrm{X}, \mathrm{Y}) = 0$.

Obviously, if the intrinsic invariantal equation is regarded as the equation to a curve in the ordinary sense, that curve will have the same homographic singularities as the covariant from which it is derived, and the coefficients in the equation are differential invariants. Hence we are led to the simpler method of finding the conditions for the existence of homographic singularities in terms of differential invariants, namely, the immediate application of the ordinary theory of forms to the intrinsic invariantal equation treated as a ternary quantic. For this reason, the equation in this form may be regarded as the canonical form of the equation to the curve, since the coefficients are the differential invariants which characterise the curve.

I proceed to illustrate this upon the conic and cubic.

To obtain the intrinsic invariant equation, we may omit all terms which are not purely invariant from the coefficients in the covariant form of the equation and substitute $-u_2^3u_5^2\lambda/\nu$ or $-u_2^3u_5^2\mathrm{Y}$ for π, and $u_2u_5\mu/\nu$ or $u_2u_5\mathrm{X}$ for ξ. I shall retain λ, μ, ν.

Thus, the intrinsic invariant equation to the osculating conic is

$$\lambda\nu + \mu^2 = 0 \qquad (57).$$

32. *Illustrations in the case of the cubic.*

The intrinsic invariantal equation to the general non-singular cubic is (omitting a factor containing u_2)

$$u_5^4\lambda^2(\mathrm{V}_8\lambda + \mathrm{U}_7\mu) + (\mathrm{U}_7^2\lambda - \mathrm{V}_8\mu + \mathrm{U}_2\nu)(\lambda\nu + \mu^2) = 0 \qquad (58),$$

and that of its Hessian is consequently

$$\begin{vmatrix} 6u_5^4\mathrm{V}_8\lambda+2u_5^4\mathrm{U}_7\mu+2\mathrm{U}_7^2\nu, & 2u_5^4\mathrm{U}_7\lambda+2\mathrm{U}_7^2\mu-\mathrm{V}_8\nu, & 2\mathrm{U}_7^2\lambda-\mathrm{V}_8\mu+2\mathrm{U}_7\nu \\ 2u_5^4\mathrm{U}_7^2\lambda+2\mathrm{U}_7^2\mu-\mathrm{V}_8\nu, & 2\mathrm{U}_7^2\lambda-6\mathrm{V}_8\mu+2\mathrm{U}_7\nu, & -\mathrm{V}_8\lambda+2\mathrm{U}_7\mu \\ 2\mathrm{U}_7^2\lambda-\mathrm{V}_8\mu+2\mathrm{U}_7\nu, & -\mathrm{V}_8\lambda+2\mathrm{U}_7\mu, & 2\mathrm{U}_7\lambda \end{vmatrix} = 0.$$

If $(\lambda_1 : \mu_1 : \nu_1)$ lies on the tangent at the origin, $U_7\lambda_1 = 0$, and for the tangential of the origin $\lambda_1 = 0$, $\mu_1 : \nu_1 = U_7 : V_8$.

Hence, on a non-singular cubic we can have neither $U_7 = 0$ nor $V_8 = 0$.

Also, on substitution in the equation to the Hessian, we obtain $u_5 = 0$, or, if the tangential is a point of inflexion, the origin must be a sextactic point, as is well known.

The equation to the conic of closest contact at the origin is

$$\lambda\nu + \mu^2 = 0,$$

and to the polar conic of the origin is

$$U_7^2\lambda^2 - V_8\lambda\mu + 2U_7\lambda\nu + U_7\mu^2 = 0.$$

So that

$$U_7^2\lambda - V_8\mu + U_7\nu = 0$$

is the equation to the common chord of these two conics, and it is the tangent at the tangential of the origin.

The second tangential of the origin lies on

$$V_8\lambda + U_7\mu = 0,$$

that is, lies on the common chord of the cubic and the conic of the closest contact.

The coordinates of the point at which this chord meets the cubic again, or the sixth point in which the osculating conic meets the cubic, are given by

$$\frac{\lambda}{U_7^2} = \frac{\mu}{-U_7V_8} = \frac{\nu}{-V_8^2},$$

and, therefore, $V_8\mu - U_7\nu = 0$ is the equation to the line joining this point to the tangential of the origin.

The cordinates of the third tangential of the origin are given by

$$\frac{\lambda}{U_7^2} = \frac{\mu}{-u_5^4U_7} = \frac{\nu}{-(U_7^3 + u_5^8)},$$

and are independent of V_8. This point is the corresidual of the eight consecutive points on each of the several cubics in which V_8 is arbitrary. This is also a known property, but the method allows us, with little difficulty, to prove properties of which other analytical proofs are laborious.

33. *General method of finding the form of the intrinsic invariant equation to a curve of any order.*

It will not be necessary to pass through all the steps which I have taken in developing this theory of intrinsic invariant equations.

If we have an equation to a covariant curve, say $f(\pi, \xi) = 0$, and if $F(\lambda : \mu : \nu) = 0$, or $F(X, Y) = 0$, is the corresponding intrinsic invariant equation where X and Y stand for μ/ν and λ/ν respectively, then the relations between π, ξ and X, Y are essentially of the character of a homographic transformation. Hence, if for points near the origin in the covariant curve π is written $a_2h^2 + a_5h^5 + a_7h^7 + \ldots$, and h is put for ξ, while in the intrinsic invariant equation Y is written $A_2h^2 + A_5h^5 + A_7h^7 + \ldots$, where h is put for X, and if a_2, a_5, a_7, &c., are written for the invariantive portions of a_2, a_5, a_7, &c., then A_2, A_5, A_7 differ from the values of a_2, a_5, a_7, &c., by factors of the character which we have considered in the earlier part of this paper.

Thus $\mathrm{a}_n = Dp^{2n-1}q^{-n-1}A_n$, where p and q stand for the expressions corresponding to those written μ and λ in (2).

In general $D = -u_2{}^4u_5{}^3$, and at the origin

$$q = u_2u_5, \quad p = 1.$$

Therefore,

$$\mathrm{a}_n = -u_2{}^4u_5{}^3(u_2u_5)^{-n-1}A_n, \text{ or } A_n = -u_2{}^{n-3}u_5{}^{n-2}\mathrm{a}_n,$$

therefore

$$\begin{aligned} A_2 &= -1 \\ A_5 &= -u_5{}^4 \\ A_7 &= -u_5{}^4U_7 \\ A_8 &= -u_5{}^4W_8, \text{ where } W_8 = V_8 + 2u_5{}^4, \end{aligned}$$

and, comparing with (33), generally

$$A_n = -u_5{}^4W_n.$$

Hence the value which, near the origin, is to replace Y is

$$-h^2 - u_5{}^4(h^5 + U_7h^7 + W_8h^8 + \ldots) \quad \ldots \ldots \quad (59).$$

Taking the general equation of the intrinsic invariant of conics as

$$\theta_5\lambda^2 + \theta_4\lambda\mu + (\theta_3 + \theta_2)\mu^2 + \theta_2\lambda\nu + \theta_1\mu\nu + \theta\nu^2 = 0,$$

and substituting for λ/ν and μ/ν, the values shown above, we find all the invariant coefficients vanish except θ_2, and thus

$$\lambda\nu + \mu^2 = 0$$

is, as before, the intrinsic invariant equation of osculating conics.

So we may, without further investigation, find the similar equation for osculating cubics as we have found it above.

The equation to the osculating curve of each order will aid us in finding that of the next; thus, the general form of a non-singular osculating quartic will be

$$\begin{gathered}\theta_1\lambda\,({u_5}^4\lambda^3 - \lambda\mu\nu - \mu^3) + \theta_2\lambda^2\,(\lambda\nu + \mu^2) + \theta_3\,(\lambda\nu + \mu^2)^2 \\ + (\theta_4\lambda + \theta_5\mu + \theta_6\nu)\{{u_5}^4\lambda^2\,(\mathrm{V}_8\lambda + \mathrm{U}_7\mu) + ({\mathrm{U}_7}^2\lambda - \mathrm{V}_8\mu + \mathrm{U}_7\nu)\,(\lambda\nu + \mu^2)\} = 0.\end{gathered}$$

The forms of the functions θ can be found, but they depend upon differential invariants of a higher order than those of which the values have been investigated.

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